

Estimates for the Roots of Polynomials Without Complex Analysis

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Abstract

In our precedence paper (Diouf, Diakhate, & Watt, 2013), we show the continuity of the zeros of a univariate polynomial which respect to the coefficients. Here we study the sizes of a polynomial and their bounds. The main originality of this paper is maybe a definition of the measure of a polynomial without any reference to the roots, this leads to a very elementary proof of bounds for the factors of polynomials, a subject which is also revisited here. Most of our proofs are extremely simple and all are quite elementary.

Keywords: polynomials, roots, size of a polynomial, separation of roots, measure of a polynomial

1. Bounds for the Roots

1.1 Bounds for All the Roots

We consider a field K equipped by a non-trivial absolute value, i.e a map $x \mapsto |x|$ from K to $\mathbb{R}$ such that

$$\forall x, |x| \geq 0, \quad |x| = 0 \Rightarrow x = 0, \quad \forall x, y, |x + y| \leq |x| + |y|, \quad |xy| = |x| |y|.$$

If there exists $x \in K^*$ with $|x| \neq 1$, then clearly the set of absolute values of elements of K contains arbitrary large and arbitrary non zero values.

We also need a lemma to bound for the roots of polynomials, this result is essentially due to Cauchy (1891).

Lemma 1 Let $P(X) = X^d + a_{d-1}X^{d-1} + \dots + a_0$ be a polynomial with coefficients in K . If a positive real number r satisfies the condition

$$r^d \geq |a_{d-1}| r^{d-1} + \dots + |a_0|,$$

and if $z \in K$ is a root of P then $|z| \leq r$.

Proof. For the proof of this lemma, see Diouf (2007). □

This lemma enables us to give many inequalities for the complex roots of a polynomial, most of which are due to Cauchy (1891). Here are two of them.

Theorem 1 Let $P(X) = X^d + a_1X^{d-1} + \dots + a_d$ be a monic polynomial with coefficients in K and let $z \in K$ be a root of P . Then z satisfies the following inequalities:

i) $|z| < 1 + \max |a_k|$; $1 \leq k \leq d$. If the coefficients of P are equal to zero for $j \notin \{j_1, j_2, \dots, j_k\}$ and if $\lambda_{j_1}, \dots, \lambda_{j_k}$ are positive real numbers which satisfy

$$\lambda_{j_1}^{-1} + \lambda_{j_2}^{-1} + \dots + \lambda_{j_k}^{-1} \leq 1,$$

then

$$\text{ii) } |z| \leq \max_{1 \leq h \leq k} \{(\lambda_{j_h} |a_{j_h}|)^{1/j_h}\}.$$

Proof. Let us prove the upper bound i).

Let A be the maximum of the quantities $|a_k|$ for $1 \leq k \leq d$. Without loss of generality, we may suppose that a_d is nonzero, thus $A > 0$. Let f be the function defined in the proof of Lemma 1. Then,

$$\begin{aligned}
f(1+A) &= (1+A)^d - |a_1|(1+A)^{d-1} - \cdots - |a_{d-1}|(1+A) - |a_d| \\
&\geq (1+A)^d - A(1+A)^{d-1} - \cdots - A(1+A) - A \\
&= (1+A)^d - A \frac{(1+A)^d - 1}{1+A-1} \\
&= 1.
\end{aligned}$$

Hence the result.

Consider now the property ii).

Let us call r the right hand-side of inequality ii). Then, by definition, the inequalities

$$|a_{j_h}| \leq \lambda_{j_h}^{-1} r^{j_h},$$

hold for $1 \leq h \leq k$.

Hence the upper bound

$$\sum_{h=1}^k |a_{j_h}| r^{d-j_h} \leq \sum_{h=1}^k r^d \lambda_{j_h}^{-1} \leq r^d.$$

Which can also be written as $f(r) \geq 0$. Hence the result, using again Lemma 1. $\square$

Corollary 1 Let $P(X) = a_0X^d + a_1X^{d-1} + \cdots + a_d$ be a polynomial in $K[X]$ with $a_d \neq 0$. Then every root $z \in K$ of P satisfies the inequality

$$|z| > \left(1 + \max \left\{ \left| \frac{a_0}{a_d} \right|, \left| \frac{a_1}{a_d} \right|, \dots, \left| \frac{a_{d-1}}{a_d} \right| \right\} \right)^{-1}.$$

Proof. Consider the polynomial $P^* = X^d P(X^{-1})$, called the *reciprocal polynomial* of P . Then the conclusion follows directly from i) applied to P^* , polynomial whose roots are the inverses of the roots of P . $\square$

Remark Of course, the argument used in the proof of the Corollary shows that every upper bound of the modules of the roots of P^* leads to a lower bound of those of P when $P(0) \neq 0$. We leave to the reader the similar application of inequality ii).

Corollary 2 Let $P(X) = X^d + a_1X^{d-1} + \cdots + a_d = (X - \alpha_1) \cdots (X - \alpha_d) \in K[X]$ be a monic polynomial which splits completely in K and put

$$K(P) = \max \left\{ |a_j|^{1/j}; 1 \leq j \leq d \right\} \quad \text{and} \quad \rho(P) = \max \left\{ |\alpha_j|; 1 \leq j \leq d \right\}.$$

Then

$$\rho(P) \leq 2K(P) \quad \text{and} \quad K(P) \leq d\rho(P).$$

Proof. The first inequality is a special case of inequality ii) of Theorem 1 (take $\lambda_j = 2^{-j}$ for $j = 1, 2, \dots, d$). The second inequality is an easy consequence of Vieta's formula. Indeed,

$$|a_j| = \left| \sum_{1 \leq i_1 < \cdots < i_j \leq d} \alpha_{i_1} \cdots \alpha_{i_j} \right| \leq \binom{d}{j} \rho(P)^j,$$

and the result follows from the trivial inequality

$$\binom{d}{j} \leq d^j.$$

$\square$

1.2 Bounds for Some Roots

In this section, we indicate two results dealing with several roots of a polynomial.

Proposition 1 Let $P = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_0 \in K[X]$ be a polynomial. Suppose that $\alpha_1, \dots, \alpha_k \in K$ are some roots of P and put

$$P = (X - \alpha_1) \cdots (X - \alpha_k) Q \quad \text{and} \quad H(P) = \max_{0 \leq j < d} |a_j|.$$

Then

$$\|\alpha_1 - 1\| \cdots \|\alpha_k - 1\| \cdot H(Q) < H(P),$$

where $H(Q)$ is the maximum of the absolute values of the coefficients of Q .

Proof. This result has been proved in Mignotte (1991). The proof is very simple. First, notice that this is enough to prove this inequality for $k = 1$, then the general case follows by induction. Assume that

$$P = (X - \alpha)Q, \quad \text{where} \quad Q = b_d X^d + b_{d-1} X^{d-1} + \cdots + b_0.$$

Suppose first that $|\alpha| \geq 1$ and let i be the minimal index such that $H(Q) = |b_i|$. The relation

$$a_i = b_{i-1} - \alpha b_i, \quad \text{where} \quad b_d = b_{-1} = 0,$$

implies

$$H(P) \geq |a_i| \geq |\alpha| H(Q) - |b_{i-1}| > (|\alpha| - 1)H(Q),$$

and the result follows.

When $|\alpha| < 1$ the proof is similar (consider now the greatest index j such that $H(Q) = |b_j|$), we omit the details. $\square$

Using the notion of compound matrices Specht (1938) proved the following result, for which we do not reproduce the proof. We just notice that the original proof was given for complex numbers but that it works in the case of a field K with an absolute value.

Proposition 2 Let $P = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_0 \in K[X]$ be a monic polynomial. Suppose that $\alpha_1, \dots, \alpha_k \in K$ are some roots of P . Then

$$|\alpha_1 \cdots \alpha_k| \leq kH(P) + 1, \quad \text{where} \quad H(P) = \max_{0 \leq j < d} |a_j|.$$

We also notice that the proof uses Gerschgorin bounds for the eigenvalues of a matrix and that these bounds still hold for matrices with coefficients in K (the proof being the same).

2. Different Notions of “Size” for Polynomials

As in the previous section, K is a field equipped with an absolute value. We consider a polynomial

$$P(X) = a_0 X^d + a_1 X^{d-1} + \cdots + a_d \in K[X].$$

We define the (naïve) *height* of P by

$$H(P) = \max |a_j|; 0 \leq j \leq d,$$

as in Proposition 1 and 2 above. Then

$$H(P \cdot Q) \leq (d+1)H(P) \cdot H(Q), \quad \text{where} \quad d = \min \{\deg P, \deg Q\}. \quad (1)$$

We define the *length* of P by

$$L(P) = \sum_{j=0}^d |a_j|.$$

Then

$$L(P \cdot Q) \leq L(P) \cdot L(Q). \quad (2)$$

We define the *norm* of P by

$$N(P) = \|P\|_\infty = \sup \{|P(z)|; |z| = 1\}.$$

Then

$$N(P \cdot Q) \leq N(P) \cdot N(Q). \quad (3)$$

The following result is trivial but nevertheless it is useful.

Lemma 2 Let P, Q and R be polynomials in $K[X]$, with $P = QR$. Then

$$\inf \{ |Q(z)|; |z| = 1 \} \cdot N(R) \leq N(P).$$

Moreover, if $Q = (X - \alpha_1) \cdots (X - \alpha_k)$ splits in K , then

$$\|\alpha_1 - 1\| \cdots \|\alpha_k - 1\| \cdot N(R) \leq N(P).$$

These different sizes satisfy the following obvious inequalities for any $P \in K[X]$,

$$H(P) \leq L(P) \leq (1 + \deg P)H(P), \quad N(P) \leq L(P). \quad (4)$$

Moreover, if K contains primitive roots of unity of arbitrary large order n coprime to the characteristic of K then let $n > d$ be coprime with the characteristic of K and let ζ be a primitive n -th root of unity. Then it is easy to verify that if

$$P(X) = \sum_{k=0}^d a_k X^k$$

Then, since

$$\sum_{j=0}^{n-1} \zeta^{r_j} = \begin{cases} n, & \text{if } n|r \\ 0, & \text{otherwise,} \end{cases} \quad (5)$$

we get

$$na_k = \sum_{j=0}^{n-1} \zeta^{-kj} P(\zeta^j)$$

for any $k = 0, \dots, d$. Indeed,

$$\sum_{j=0}^{n-1} \zeta^{-kj} P(\zeta^j) = \sum_{h=0}^d ah \sum_{j=0}^{n-1} \zeta^{(h-k)j}$$

and the result follows from (5).

For example, in the case $K = \mathbb{C}$ this implies

$$H(P) \leq N(P).$$

We can obtain a similar, but weaker result, with more general hypotheses. Suppose that K contains more than d different elements of absolute value equal to 1 and consider a nonzero polynomial $P \in K[X]$ of degree d . Choose $d+1$ elements $x_0, x_1, \dots, x_d$ of K with $|x_i| = 1$ for all i . Then P can be interpolated at the points $x_0, \dots, x_d$ and its coefficients are given by Lagrange's formulas (for example). This study yields to the following result.

Proposition 3 Let d be a given positive integer. Suppose that K contains more than d different elements of absolute value equal to 1 and consider any nonzero polynomial $P \in K[X]$ of degree d . Then there exists a constant C_d independent of P such that

$$H(P) \leq C_d \|P\|_{\infty}.$$

Moreover, if K contains primitive roots of unity of order $n > d$, with n coprime to the characteristic of K , and let ζ be the primitive n -th root of unity. Then if $P = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0$, the coefficients a_k satisfies

$$na_k = \sum_{j=0}^{n-1} \zeta^{kj} P(\zeta^j)$$

for any $k = 0, \dots, d$. For example, in the case $K = \mathbb{C}$ this implies

$$H(P) \leq N(P).$$

3. The Dandelin-Graeffe Transformation

Definition 1 According to Diouf (2007), the principles of this method are well presented.

Let K be any field (we could even work in any ring) and $P \in K[X]$. We define formally the Dandelin-Graeffe Transformation as

$$P(X) \mapsto P(\sqrt{X})P(-\sqrt{X}).$$

Decompose P in its “even” part and its “odd” part, i.e.

$$P(X) = F(X^2) + XG(X^2).$$

Then we have the following result.

Proposition 4 *With the above notation,*

$$P_1(X) = P(\sqrt{X})P(-\sqrt{X})$$

is a polynomial with coefficients in K which satisfies

$$P_1(X) = F^2(X) - XG^2(X)$$

and if $\alpha_1, \alpha_2, \dots, \alpha_d$ are the roots of P in a field L containing K then the roots of P_1 in L are the squares of α_j 's.

Proof. Formally,

$$\begin{aligned} P_1(X) &= P(\sqrt{X})P(-\sqrt{X}) \\ &= (F(X) + \sqrt{X}G(X))(F(X) - \sqrt{X}G(X)) \\ &= F^2(X) - XG^2(X), \end{aligned}$$

which prove the above formula and – at the same time – that P_1 is indeed a polynomial with coefficients in K .

The third assertion follows from the relations

$$\begin{aligned} P_1(X) &= \pm a_d^2(\sqrt{X} - \alpha_1) \cdots (\sqrt{X} - \alpha_d)(\sqrt{X} + \alpha_1) \cdots (\sqrt{X} + \alpha_d) \\ &= \pm a_d^2(X - \alpha_1^2) \cdots (X - \alpha_d^2). \end{aligned}$$

□

For the transformation $P \mapsto P_1$ we also use the notation

$$P_1 = \mathcal{G}(P).$$

Notice that $\mathcal{G}$ is “multiplicative”:

$$\mathcal{G}(P \cdot Q) = \mathcal{G}(P)\mathcal{G}(Q).$$

It is easy to verify that

$$L(P_1) \leq L^2(P).$$

The fact that the roots of P_1 are the squares of the roots of P is the origin of the Dandelin-Graeffe's method.

Suppose that $P \in \mathbb{R}[X]$ is a polynomial of degree d with the roots $\alpha_1, \dots, \alpha_d \in \mathbb{C}$, and assume moreover that P has a unique dominant root, let α_1 . Consider the sequence of transformation $P_0 = P, P_1 = \mathcal{G}(P), \dots, P_k = \mathcal{G}(P_{k-1})$. Then the roots of P_k are the $\alpha_i^{2^k}$'s and $|\alpha_1|^{2^k}$ is much greater than any $|\alpha_i^{2^k}|$ for $i = 2, \dots, d$. This shows that, if we denote

$$P_k(X) = a_0^{(k)}X^d + a_1^{(k)}X^{d-1} + \dots$$

then

$$-\frac{a_1^{(k)}}{a_0^{(k)}} \simeq \alpha_1^{2^k},$$

thus

$$\left| \frac{a_1^{(k)}}{a_0^{(k)}} \right|^{2^{-k}} \xrightarrow{k \rightarrow \infty} |\alpha_1|.$$

We put $\rho(P) = |\alpha_1|$. In other words, the Dandelin-Graeffe method furnishes the value of $\rho(P)$, when P has a unique dominant root.

Now a glance shows that the previous proof holds in any field K with an absolute value, provided that the roots of P belong to K and that there is a unique dominant root.

3.1 A Computation of $\rho(P)$ in the General Case

Here we consider $P \in K[X]$, a field with an absolute value and we assume that P splits in K and that its roots are $\alpha_1, \dots, \alpha_d$ belong to K and satisfies $\alpha_1 \leq \dots \leq \alpha_d$, thus $\rho(P) = |\alpha_1|$.

In section 2 we have seen that

$$\rho(P) \leq 2K(P) \quad \text{and} \quad K(P) \leq d\rho(P),$$

where d is the degree of P . This implies

$$\frac{K(P)}{d} \leq \rho(P) \leq 2K(P).$$

If we apply these inequalities to $P_k = \mathcal{G}^k(P)$ we get

$$\frac{K(P_k)}{d} \leq \rho^{2^k}(P) \leq 2K(P_k),$$

which prove that

$$K(P_k)^{2^{-k}} \xrightarrow{k \rightarrow \infty} \rho(P).$$

In other words, the Dandelin-Graeffe's method can be used to compute $\rho(P)$ for $P \in K[X]$, even if there is no dominant root. For the case of polynomials with complex coefficients, the result was proved by Davenport-Mignotte (1990).

Remark When $|a_d| > 0$, working with the polynomial P^* , this method gives also an approximate value of $|\alpha_d^{-1}|$, hence of $|\alpha_d|$.

3.2 A Computation of the $|\alpha_j|$'s in the General Case

It is easy to see that in the case of a polynomial whose roots $\alpha_1, \dots, \alpha_d$ satisfy $|\alpha_1| > |\alpha_2| > \dots > |\alpha_d|$ then the classical Dandelin-graeffe method permits to compute approximate values for all the $|\alpha_j|$'s. Here we want to generalize the method of the previous section to make this computation without any special hypothesis on the roots.

We suppose that $P \in K[X]$ is a monic polynomial (this assumption just to have simpler formulas) whose roots $\alpha_1, \dots, \alpha_d$ all belong to K and satisfy $|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_d|$.

Suppose that for some l we have $|\alpha_l| > |\alpha_{l+1}|$ then the l th elementary symmetric function of the roots of P is

$$\sigma_l = \sum_{1 \leq i_1 < \dots < i_l \leq d} \alpha_{i_1} \cdots \alpha_{i_l} = \alpha_1 \cdots \alpha_l + \text{"smaller terms"}.$$

This implies that

$$\left| a_l^{(k)} - (-1)^l (\alpha_1 \cdots \alpha_l)^{2^k} \right| \leq \binom{d}{l} |\alpha_1 \cdots \alpha_{l-1} \alpha_{l+1}|^{2^k},$$

hence

$$\left| a_l^{(k)} \right|^{2^{-k}} \xrightarrow{k \rightarrow \infty} |\alpha_1 \cdots \alpha_l|, \quad |\alpha_l| > |\alpha_{l+1}|.$$

For example:

$$\left| a_2^{(k)} \right|^{2^{-k}} \xrightarrow{k \rightarrow \infty} |\alpha_1 \cdots \alpha_l|, \quad |\alpha_2| > |\alpha_3|.$$

One can also notice that we always have

$$\left| a_j^{(k)} \right| \leq \binom{d}{j} |\alpha_1 \cdots \alpha_j|^{2^k},$$

which implies

$$\forall j, \quad |\alpha_1 \cdots \alpha_j| \leq \lim_{k \rightarrow +\infty} \sup \left| a_j^{(k)} \right|^{2^{-k}}.$$

One usefull question is

Question 1 How to recognize the case $|\alpha_l| > |\alpha_{l+1}|$?

It will help to improve the efficiency of the calculations. We'll try to give a detailed analysis and a possible answer to this question in another paper.

Example 1 Suppose that the degree of the polynomial P is 4 and that its roots satisfy $|\alpha_1| \geq |\alpha_2| \geq |\alpha_3| \geq |\alpha_4|$. By the method of the previous section, we can compute approximate values of $|\alpha_1|$ and $|\alpha_4|$ (see previous Remark). Now there are two cases:

- if $|\alpha_2| > |\alpha_3|$, then

$$|\alpha_2| = \frac{1}{|\alpha_1|} \lim_{k \rightarrow \infty} |a_2^{(k)}|^{2^{-k}}$$

and

$$|\alpha_3| = \frac{a_4}{\alpha_1 \alpha_2 \alpha_4},$$

- if $|\alpha_2| = |\alpha_3|$, then

$$|\alpha_2| = |\alpha_3| = \sqrt{\frac{a_4}{|\alpha_1 \alpha_4|}}.$$

3.3 The Measure of a Polynomial

For $P \in \mathbb{C}[X]$ the measure was defined in 1955 by K. Mahler as follows. If

$$P = a(X - \alpha_1) \cdots (X - \alpha_d)$$

then

$$M(P) = |a| \prod_{j=1}^d \max\{1, |\alpha_j|\}.$$

In Cerlienco-Mignotte-Piras (1987), it was proved that (with the notation of the previous section)

$$L(P_k)^{2^{-k}} \xrightarrow[k \rightarrow \infty]{} M(P).$$

Here, for $P \in K[X]$, we put, by the definition,

$$M(P) := \lim_{k \rightarrow \infty} L(P_k)^{2^{-k}}.$$

The fact that $|a_0| \leq L(P_1)^{1/2} \leq L(P)$, $|a_0| \leq L(P_2)^{1/4} \leq L(P)^{1/2}, \dots$ shows that the sequence $L(P_k)^{2^{-k}}$ is non-increasing and bounded below by $|a_0|$, this proves that this limit exists and that

$$|a_0| \leq M(P) \leq L(P).$$

In the case $K = \mathbb{C}$, the field of complex numbers, it is known that

$$M(P) \leq \|P\|,$$

where $\|P\| = \left(\sum_{j=0}^d |a_j|^2 \right)^{1/2}$ (reference: Landau, 1905).

Question 2 Is it true in general that $M(P) \leq \|P\|$?

We know that it is true in the complex field and we think that it still be true in every field equipped with an adequate norm. It is not very important for the scope of this paper. It's just a curiosity.

Notice that we also have, using (4),

$$H(P_k)^{2^{-k}} \xrightarrow[k \rightarrow \infty]{} M(P).$$

The measure satisfies also the elementary following property

$$M(P(X^k)) = M(P), \quad \text{for any integer } k \geq 2.$$

From the above results, it is clear that

$$M(P \cdot Q) \leq M(P) \cdot M(Q). \quad (6)$$

Trivially, if $a \in K$ then

$$M(a) = |a|.$$

Let us estimate $M(X - \alpha)$. By the previous study, in this case

$$P_k(X) = \pm(X - \alpha^{2^k}),$$

which shows that

$$M(X - \alpha) = \max\{1, |\alpha|\}.$$

Suppose now that

$$P = (X - \alpha)Q(X)$$

where $\alpha \in K$ (thus $Q(X) \in K[X]$). For any K we have

$$\mathcal{G}^k(P) = \mathcal{G}^k(X - \alpha) \cdot \mathcal{G}^k(Q) = \pm(X - \alpha^{2^k}) \cdot \mathcal{G}^k(Q).$$

Then Proposition 1 implies that

$$\left| |\alpha^{2^k}| - 1 \right| L(\mathcal{G}^k(Q)) \leq (1 + \deg Q) L(\mathcal{G}^k(Q)), \quad \text{for any } k \geq 1,$$

thus if $|\alpha| \neq 1$ we get

$$\max\{1, |\alpha|\} M(Q) \leq M(P),$$

which, combined with (6) gives

$$M((X - \alpha)Q(X)) = M(X - \alpha) \cdot M(Q).$$

The remaining case, namely $|\alpha| = 1$, can be treated using Lemma 3 below and leads to the the same result.

This leads to the following proposition.

Proposition 5 *If*

$$P = a(X - \alpha_1) \cdots (X - \alpha_k)R(X) \in K[X]$$

where $\alpha_1, \dots, \alpha_k$ belong to K , then

$$M(P) = |a| \cdot \prod_{j=1}^k \max\{1, |\alpha_j|\} \cdot M(R).$$

In particular, if

$$P = a(X - \alpha_1) \cdots (X - \alpha_d)$$

splits completely in K then we recover Mahler's definition

$$M(P) = |a| \prod_{j=1}^d \max\{1, |\alpha_j|\}.$$

Question 3 Is it always true that $M(Q \cdot R) = M(Q) \cdot M(R)$?

This question is also very important because it will help us to use the principle of “divide and conquer”!

Recall that we have

$$M(P) \leq L(P), \quad M(P) \leq L(P_1)^{1/2}, \quad M(P) \leq L(P_2)^{1/4}, \dots \quad (7)$$

Notice also that when P splits then

$$L(P) \leq 2^d M(P).$$

Remark We could also have defined the measure by the similar formula

$$M_\infty(P) = \lim_{k \rightarrow \infty} \|P_k\|_\infty^{2^{-k}},$$

where, as above, $P_k = \mathcal{G}^k(P)$. It is clear that the sequence $(\|P_k\|_\infty^{2^{-k}})$ is non-increasing, so that the limit exist, and we have

$$M_\infty(P) \leq M(P)$$

since $N(P_k) \leq L(P_k)$ for each k . When K contains at least $d+1$ elements of absolute value equal to 1, Proposition 3 shows that

$$M_\infty(P) = M(P)$$

and, in this case we have

$$M(P) \leq N(P), \quad M(P) \leq N(P_1)^{1/2}, \quad M(P) \leq N(P_2)^{1/4}, \dots \quad (8)$$

4. Bounds for the Factors of Polynomials

4.1 The Case of a Linear Factor

We first consider the case

$$P(X) = (X - \alpha)Q(X),$$

and follow Glesser (1990, pp. 71-75). Put

$$P = \sum_{i=0}^p b_i X^i \quad \text{and} \quad Q = \sum_{i=0}^{p-1} a_i X^i.$$

Then $b_j = a_{j-1} - \alpha a_j$, $0 \leq j \leq p$, where $a_p = a_{-1} = 0$.

This implies

$$|a_j| \leq |b_0| |\alpha|^{-j-1} + |b_1| |\alpha|^{-j} + \dots + |b_j| |\alpha|^{-1} \quad (9)$$

and

$$|a_j| \leq |b_p| |\alpha|^{p-j-1} + |b_{p-1}| |\alpha|^{p-j-2} + \dots + |b_{j+1}|. \quad (10)$$

In the case $|\alpha| \geq 1$, using (9), we get

$$|\alpha| |a_k| \leq \sum_{j=0}^k |b_j|,$$

and using (10), we get

$$\frac{|a_k|}{|\alpha|^{p-k-1}} \leq \sum_{j=k+1}^p |b_j|.$$

Combining these two results we get the following result.

Lemma 3 If $P(X) = (X - \alpha)Q(X)$ and $Q = \sum_{i=0}^{p-1} a_i X^i$ then, if $|\alpha| \geq 1$, we have

$$\left(|\alpha| + \frac{1}{|\alpha|^{p-k-1}} \right) |a_k| \leq L(P)$$

for $k = 0, 1, \dots, p-1$.

Theorem 2 If $P(X) = (X - \alpha)Q(X)$, then

$$H(Q) \leq \frac{\max\{1, |\alpha|^{p-1}\}}{1 + |\alpha|^p} L(P), \quad \text{where } p = \deg(P).$$

Proof. If $|\alpha| \geq 1$, this is an easy consequence of Lemma 3 above. Just notice this lemma implies

$$|a_k| \leq \frac{|\alpha|^{p-k-1}}{1 + |\alpha|^{p-k}} L(P)$$

and that

$$\frac{|\alpha|^{t-1}}{1 + |\alpha|^t} \leq \frac{|\alpha|^t}{1 + |\alpha|^{t+1}}$$

for all $t \geq 1$ when $|\alpha| \geq 1$.

If $|\alpha| < 1$, apply Lemma 3 to the polynomials P^* and Q^* . Then

$$P^* = (X - 1/\alpha)(-\alpha Q^*)$$

and we get

$$H(\alpha Q^*) = |\alpha| H(Q^*) \leq \frac{|\alpha|^{-(p-1)}}{1 + |\alpha|^{-p}} L(P^*) = \frac{1}{|\alpha|^{p-1} + |\alpha|^{-1}} L(P),$$

and the result follows. $\square$

Remark As notice by Ph. Glesser, Theorem 2 is sharp: if

$$Q(X) = X^{p-1} + \alpha X^{p-2} + \cdots + \alpha^{p-2} X + \alpha^{p-1}$$

then

$$P(X) = (X - \alpha)Q(X) = X^p - \alpha^p$$

and

$$H(Q) = \max\{1, |\alpha|^{p-1}\}, \quad \text{and} \quad L(P) = 1 + |\alpha|^p,$$

so that equality holds in Theorem 2 for these two polynomials.

Corollary 3 If $P(X) = (X - \alpha)Q(X)$, then

$$H(Q) \leq L(P).$$

Remark With the notation of Theorem 2, in the case of polynomials with complex coefficients, Durand (1980) proved that

$$N(Q) \leq \frac{p}{1 + |\alpha|} N(P).$$

Notice also that Proposition 1 above provides a variant of Theorem 2.

4.2 The Case of a Split Factor

By the previous study we have:

Theorem 3 If P is a monic polynomial and if $P = QR$ where Q and R are monic, and Q is completely split over K , then

$$L(Q) \leq 2^q M(Q) \leq 2^q M(P) \leq 2^q L(P), \quad \text{where } q = \deg(Q).$$

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