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Excellent Extensions and Ding Projective Modules

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Abstract

In this paper, we study the invariant properties of global Ding projective dimensions under excellent extensions of rings. And the excellent extensions of Ding-Chen rings are also discussed.

Keywords: excellent extension, Ding-Chen ring, Ding projective module

1. Introduction

In studying the algebraic structure of group rings, Passman (1977) introduced the notion of the excellent extensions of rings (the name comes from Bonami, 1984). Such extensions of rings are vital because they include two important classes of rings, that is, finite matrix rings and skew group rings RG ($|G|^{-1} \in R$). Many authors have studied the invariant properties of rings under excellent extensions (see e.g. Bonami, 1984; Feng, 1997; Passman, 1977; Xue, 1996; Xiao, 1994). It has been known that many important homological properties, such as the (weak) global dimension of rings, the projectivity, injectivity and flatness of modules and so on, are invariant under excellent extensions (Xue, 1996).

Throughout this paper, all rings are associative with identity and all modules are unitary. M_R ($_RM$) denotes a right (left) R-module, and we freely use the terminology and notations of (Boami, 1984; Ding & Chen, 1993). A right R-module M is called *FP-injective* in (Stenström, 1970) if $\mathrm{Ext}^1_R(N, M) = 0$ for all finitely presented right R-modules N. The *FP-injective dimension* of N, denoted $\text{FP-id}_R N$, is defined similarly to the classical injective dimension. A ring R is called an *n-FC ring* if R is a left and right coherent ring with $\text{FP-id}_R R \leq n$ and $\text{FP-id} R_R \leq n$. These rings were introduced and studied by Ding and Chen (1993, 1996). Gillespie (2010) first called a ring *Ding-Chen* when it is n-FC for some n, which seen to have many properties similar to n-Gorenstein rings.

As a particular case of Gorenstein projective modules, Ding projective modules was introduced by Gillespie (2010) and further studied by many authors (see e.g., Wang & Liu; Yang, 2012, 2013). An R-module M is called *Ding projective* if there exists an exact sequence of projective R-modules

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $M = \mathrm{Ker}(P^0 \to P^1)$ and which remains exact after applying $\mathrm{Hom}_R(-, F)$ exact for each flat R-module F. This class of modules have been treated by different authors.

In this paper, we will study the excellent extension of Ding-Chen rings and global Ding projective dimensions of rings under excellent extensions.

In Section 2, we give some notations in our terminology and some preliminary results which are often used in this paper. In Section 3, we prove our main results.

Theorem 1.1 *Let $S \geq R$ be an excellent extension, then S is a Ding-Chen ring if and only if so is R.*

We used $\mathrm{r.Dpd}_R M$ to denoted the right Ding projective dimension of a module M in $\mathrm{Mod}\, R$, which defined as the smallest non-negative n such that there exists an exact sequence

$$0 \to D_n \to D_{n-1} \to \cdots \to D_1 \to D_0 \to M \to 0$$

with each D_i Ding projective. We used $\mathrm{r.gl.Dpd} S$ to denoted the right global Ding projective dimension of R, which defined as

$$\text{r. gl. Dpd}(S) = \sup\{\text{r. Dpd}(M) | M \in \text{Mod}\, R\}.$$

Theorem 1.2 *Let $S \geq R$ be an excellent extension, then* r. gl. Dpd(R) = r. gl. Dpd(S).

2. Preliminaries

We begin with the definition of excellent extensions of rings.

Definition 2.1 Let R be a subring of a ring S, such that S and R have the same identity. Then S is called a ring extension of R, and denoted by $R \leq S$. A ring extension $R \leq S$ is called an excellent extension, if

(1) S is right R-projective (Passman, 1977, p. 273), that is, if N_S is a submodule of M_S and if N_R is a direct summand of M_R, denoted by $N_R | M_R$, then $N_S | M_S$.

(2) S is a free normalizing extension of R, that is, there exist $b_1, b_2, \cdots, b_n$, such that and ${}_RS$ and S_R are free with a common basis $\{b_1, b_2, \cdots, b_n\}$ such that $b_iR = Rb_i$, for each i.

Example 2.2 (See Bonami, 1984; Passman, 1977)

(1) Let R be a ring, then $M_n(R)$ (the matrix ring of R of degree n) is an excellent extension of R.

(2) Let R be a ring and G a finite group. If $|G|^{-1} \in R$, then the skew group ring RG is an excellent extension of R.

(3) Let A be a finite-dimensional algebra over a field K, and let F be a finite separable field extension of K. Then $A \otimes_K F$ is an excellent extension of A.

(4) Let K be a field, and let G be a group and H a normal subgroup of G. If $[G : H]$ is finite and is not zero in K, then KG is an excellent extension of KH.

Lemma 2.3 *Let $S \geq R$ be an excellent extension. Then*

(1) (Xue, 1996, Lemma 1.1) *Let M be an S-module, then $M_S | (M \otimes_R S)_S$;*

(2) (Xue, 1996, Lemma 1.1) *Let N be a R-module, then $N_R | (N \otimes_R S)_R$;*

(3) (Xiao,1994), Theorem 2.3) *Let M be an S-module, then* $\text{pd}_S\, M = \text{pd}_R\, M = \text{pd}_S(M \otimes_R S)$.

3. Results

Lemma 3.1 *Let $S \geq R$ be an excellent extension.*

(1) *Let M be an S-module, then* FP-id$_R\, M$ = FP-id$_S\, M$;

(2) *Let N be an R-module, then* FP-id$_R\, N$ = FP-id$_S(\text{Hom}_R(S, N))$.

Proof. (1) Assume that Fp-id$_S\, M = m$, there is a finite FP-injective coresolution of S-module N :

$$0 \to N \to F^0 \to F^1 \to \cdots \to F^m \to 0,$$

which also an exact sequence in Mod R. By Mao and Ding (2005, Lemma 2.2), every F^i is a FP-injective R-module. And so Fp-id$_R N \leq m$, that is, FP-id$_R\, M \leq$ FP-id$_S\, M$.

On the other hand, if FP-id$_R\, M = n$, there is a finite FP-injective coresolution in Mod$R$

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^n \to 0. \tag{3.1}$$

Applying the functor $\text{Hom}_R(S, -)$ to (3.1), there exists an exact sequence in Mod S

$$0 \to \text{Hom}_R(S, M) \to \text{Hom}_R(S, E^0) \to \text{Hom}_R(S, E^1) \to \cdots \to \text{Hom}_R(S, E^n) \to 0$$

with $\text{Hom}_R(S, E^i)$ FP-injective for each i, by Mao and Ding (2005, Lemma 2.3). Hence, FP-id$_S\, \text{Hom}_R(S, M) \leq$ FP-id$_R M$. We get FP-id$_S\, N \leq$ FP-id$_R\, N$ from Lemma 2.3.

(2) The result follows from Lemma 2.1 and (1). □

Theorem 3.2 *If S is an excellent extension of R, then S is a Ding-Chen ring if and only if so is R.*

Proof. "Only if" Since R is a Ding-Chen ring, Fp-id$_{RR}R$ = Fp-id$_R R_R = n$ for some n. And we have Fp-id$_R S_R = n$, because that S_R is a finite free normalizing R-module. Hence, Fp-id$_S S_S = n$ by Lemma 3.1. Similarly, Fp-id$_{SS}S = n$. Therefore, S is a Ding-Chen ring.

"*If*" Since S is a Ding-Chen ring, FP-id$_S$ $_SS$ = FP-id$_S$ S_S = n. It follows that R is a Ding-Chen ring from Lemma 3.1. □

Lemma 3.3 *Let $S \geq R$ be an excellent extension.*

(1) *For any S-module N, N is a Ding projective R-module if and only if N is a Ding projective S-module.*

(2) *Let M be an R-module, then M is a Ding projective R-module if and only if $S \otimes_R M$ is a Ding projective S-module.*

Proof. Let M be a Ding projective R-module, there is a complete projective resolution

$$\mathcal{P}' = \cdots \to P'_1 \to P'_0 \to P'^0 \to P'^1 \to \cdots$$

with $M = \mathrm{Ker}(P'^0 \to P'^1)$, which left applying the functor $\mathrm{Hom}_R(-, F)$ exact for any flat module F. Applying the functor $- \otimes_R S$ to $\mathcal{P}'$, we get an exact complex $\mathcal{P}' \otimes_R S$:

$$\mathcal{P}' \otimes_R S = \cdots \to P'_1 \otimes_R S \to P'_0 \otimes_R S \to P'^0 \otimes_R S \to P'^1 \otimes_R S \to \cdots$$

and $M \otimes_R S = \mathrm{Ker}(P'^0 \otimes_R S \to P'^1 \otimes_R S)$. Note that $\mathrm{Hom}_R(S, F)$ is a flat R-module for any flat S-module F by (Feng, 1997), $\mathrm{Hom}_R(\mathcal{P}', \mathrm{Hom}_S({}_RS_S, F_S))$ is exact. For any right R-module N_R, we have the isomorphisms $\mathrm{Hom}_R(N_R, \mathrm{Hom}_S(S, F)) \cong \mathrm{Hom}_S(N \otimes_R S, F)$, and so $\mathrm{Hom}_S(\mathcal{P}' \otimes_R S, F_S)$ is exact. Hence, $M \otimes_R S$ is a Ding projective S-module.

(1) Assume that N_S is a Ding projective module. There is an exact sequence

$$\mathcal{P}_S = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective S-modules such that $N = \mathrm{Ker}(P^0 \to P^1)$, which left applying the functor $\mathrm{Hom}_S(-, F)$ exact for any flat S-module F. Clearly,

$$\mathcal{P}_R = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

is an exact sequence in Mod R. We claim that $\mathrm{Hom}_R(\mathcal{P}_R, Q)$ is exact for any flat R-module Q. In fact, $\mathrm{Hom}_R({}_SS_R, Q_R)$ is a flat S-module by Feng (1997), and so $\mathrm{Hom}_S(P_S, \mathrm{Hom}_R({}_SS_R, Q_R))$ is exact. For any S-module X_S, we have isomorphisms

$$\mathrm{Hom}_R(X_R, Q_R) \cong \mathrm{Hom}_R(X \otimes_S S, Q) \cong \mathrm{Hom}_S(X, \mathrm{Hom}_R(S, Q)),$$

and we get our claim. It follows that M_R is a Ding projective module.

Conversely, assume that M_R is Ding projective, then $M \otimes_R S$ is a Ding projective S-module. By Lemma 2.3, $M_S | (M \otimes_R S)_S$. And we get that M_S is a Ding-projective module from (Yang, 2013, Lemma 3.2).

(2) We have only to show that M_R is a Ding projective module if $M \otimes_R S$ is a Ding projective S-module. By (1), $(M \otimes_R S)_R$ is Ding projective. And this result follows from Lemma 2.3 and (Yang, 2013, Lemma 3.2). □

Theorem 3.4 *Let $S \geq R$ be an excellent extension, then* $\mathrm{r.gl.Dpd}(S) = \mathrm{r.gl.Dpd}(R)$.

Proof. Suppose that $\mathrm{r.gl.Dpd}(R) = m < \infty$. Let M_S be a non-zero S-module, then $\mathrm{r.Dpd}\, M_R \leq m$. So there exists a Ding projective resolution of M_R:

$$0 \to F_k \to \cdots \to F_1 \to F_0 \to M \to 0 \tag{3.2}$$

with each F_i Ding projective and $k \leq m$. By Lemma 3.3, every $F_i \otimes_R S$ is a Ding projective S-module. Applying the functor $- \otimes_R S$ to (3.2), we have an exact sequence

$$0 \to F_k \otimes_R S \to \cdots \to F_1 \otimes_R S \to F_0 \otimes_R S \to M \otimes_R S \to 0.$$

By Lemma 2.3, $M_S | (M \otimes_R S)_S$, and hence $\mathrm{r.Dpd}_S\, M \leq m$. It follows $\mathrm{r.gl.Dpd}(S) \leq m$, i.e. $\mathrm{r.gl.Dpd}(S) \leq \mathrm{r.gl.Dpd}(R)$.

Conversely, assume that $\mathrm{r.gl.Dpd}(S) = n$. Let N_R be a non-zero R-module, then $N \otimes_R S \in \mathrm{Mod}\, S$. The following sequence

$$0 \to E_L \to \cdots \to E_1 \to E_0 \to N \otimes_R S \to 0$$

is exact in Mod S, where each E_i is a Ding projective S-module as well as a Ding projective R-module, by Lemma 3.3. Because $N_R|(N \otimes_R S)_R$ by Lemma 2.3, we have $\mathrm{r.Dpd}_R N \leq n$. Hence $\mathrm{r.gl.Dpd}(R) \leq n$, that is $\mathrm{r.gl.Dpd}(S) \geq \mathrm{r.gl.Dpd}(R)$. □

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