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Perturbation on Polynomials

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Abstract

For univariate polynomials with complex coefficients there are many estimates about the roots of polynomials. Moreover, the result corresponding to the "continuity of the zeroes which respect to the coefficients" is generally obtained as a corollary of Rouché's theorem and is rarely precise. Here we prove an explicit result for algebraically closed fields with an absolute value, in any characteristic.

Keywords: polynomials, roots, perturbation, separation of roots, Rouché's theorem

1. Introduction

The first motivation of this paper was the following: try to perturb slightly polynomials with integer coefficients with the hope of finding examples with very small root separation. At the beginning of the work we noticed that it is very difficult to find explicit statements of the theorem of "Continuity of the roots of a polynomial with respects to the coefficients". For this reason we decided to write the section 1 of this paper. Then we studied examples and we had the bad surprise to see that in each case a perturbation of a "good" example leads to a "poor" example.

Preliminaries We begin by a general lemma in algebra.

Lemma 1 *Let R be any ring and let $P \in R[X]$ be a polynomial of degree $n \geqslant 1$. Let us define the $k-$hyperderivative by the formula*

$$(X^m)^{<k>} = \binom{m}{k} X^{m-k}, \quad \text{where } \binom{m}{k} = 0 \text{ if } k > m,$$

and the linearity properties

$$(S+T)^{<k>} = S^{<k>} + T^{<k>}, \qquad (\lambda S)^{<k>} = \lambda S^{<k>},$$

when $\lambda \in R$. So that $P^{<k>} \in R[X]$ for all k. Then, for any $a \in R$, we have the formula

$$P(x+a) = P(X) + aP^{<1>}(X) + a^2 P^{<2>}(X) + \cdots + a^n P^{<n>}(X).$$

Moreover, if $R = K$ is a field and if P has all its roots, say $\alpha_1, \ldots, \alpha_n$, in some extension L of K, then for all $a \in K$ and all $k \in \mathbb{N}$, we have

$$P^{<k>}(a) = (-1)^{n-k} \sum_{1 \leqslant j_1 < \cdots < j_{n-k} \leqslant n} (\alpha_{j_1} - a) \cdots (\alpha_{j_{n-k}} - a).$$

Proof. The first assertion is well-known (in the case of a field of characteristic zero, this is just Taylor's formula for polynomials). To prove it, by linearity, it's enough to verify that it's true if $P(X) = X^m$ for $m \leqslant n$. Then, one notices that this case is just Newton's formula for the expansion of $(X+a)^m$.

For $a = 0$, the second relation is again well-known: this is Vieta's formula. The general case is obtained by the translation $X \mapsto X + a$. □

Remark 1 If K is of characteristic zero, then

$$P^{<k>}(X) = \frac{P^{(k)}(X)}{k!},$$

where $P^{(k)}(X)$ is the usual $k-$derivative of P.

2. Continuity of the Roots of Polynomials in Terms of the Coefficients

We want to prove the following result which is a generalization of a classical one when K is the field of complex numbers, whose proof is generally obtained as a corollary of Rouché's theorem.

Theorem 1 *Let K be an algebraically closed field with an absolute value and let $f \in K[X]$ be a fixed polynomial with the decomposition*

$$f(x) = a(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_k)^{m_k}, \quad a \neq 0,$$

where $\alpha_i \neq \alpha_j$ for $i \neq j$ and $m_1, m_2, \ldots, m_k$ are positive integers.

Let $n = m_1+m_2+\cdots+m_k$ be the degree of f. Let $0 < \varepsilon < \mathrm{sep}(P)$ a given real number then there exist a real number $\eta > 0$, such that if $g \in K[X]$ is a polynomial of degree n, satisfying $H(f-g) < \eta$ (where $H()$ is the height of a polynomial and $\mathrm{sep}(P) = \min_{i\neq j}\left|\alpha_i - \alpha_j\right|, i, j = 1,\ldots,k$). Then all the roots of g belong to the union for $1 \leqslant i \leqslant k$ of the disks $D_i = \{z \in K; |z-\alpha_i| < \varepsilon\}$. Moreover if ε is sufficiently small, then for $i = 1,\ldots,k$, there exists exactly m_i roots of g which belong to the disk D_i.

Proof. Without loss of generality, we may assume that f is monic and that f and f are of the same degree n. Let $H = H(f)$.

Let us first notice that, the roots $\alpha_i, i = 1,\ldots,n$, and any root β satisfy

$$|\alpha_i| < H+1 \quad \text{and} \quad |\beta| < H+2. \tag{1}$$

Put $r = f - g$. If we assume $\eta < 1$. Since $g(\beta) = 0$, we have

$$\prod_{i=0}^{k} |\beta-\alpha_i|^{m_i} = |f(\beta)| = |r(\beta)| \leqslant \eta(1+(H+2)+\cdots+(H+2)^n) = \eta\frac{[(H+2)^{n+1}-1]}{H+1}.$$

Thus, assuming also $\eta\dfrac{[(H+2)^{n+1}-1]}{H+1} < 1$, we get

$$\min_i\{|\beta-\alpha_i|\} \leqslant \left(\eta\frac{[(H+2)^{n+1}-1]}{H+1}\right)^{1/n}.$$

This shows that each root of g is "close" to some root of f . For small enough η, more precisely when

$$\eta < \min\{1, (\mathrm{sep}(f))^n\}\,\frac{H+1}{(H+2)^{n+1}-1} =: \eta_0. \tag{2}$$

The roots of g belong to the union of the disks D_i defined in the theorem. This proves the first assertion.

Now, to prove the second assertion, it is enough to prove that each D_i does not contain more than m_i roots of g. To simplify the notation, let α be a root of f and let m be its multiplicity.

Suppose that the disks $D_i = \{z \in K; |z-\alpha_i| < \mathrm{sep}(f)\}$ contains $m+1$ roots of g, say $\beta_1, ..., \beta_{m+1}$. First notice that

$$\left|g^{<m>}(\alpha)\right| \geqslant \frac{\mu}{2}, \quad \text{where} \quad \mu = \min\left\{\left|f^{<m_1>}(\alpha_1\right|,\ldots,\left|f^{<m_k>}(\alpha_k\right|\right\},$$

we add the condition $\eta < 2^{-n}\mu\dfrac{H}{(H+1)^{n-m}-1}$. But, by the lemma, we also have

$$g^{<m>}(\alpha) = \pm\sum_{1<j_1<\cdots<j_{n-m}<n}(\beta_{j_1}-\alpha)\cdots(\beta_{j_{n-m}}-\alpha),$$

and the assumption $\left|\beta_j - \alpha\right| < \varepsilon$ for $j = 1, 2, \ldots, m+1$ implies

$$\left|g^{<m>}(\alpha)\right| \leqslant \varepsilon\binom{n}{m}(2H+3)^{n-m}$$

so that, if ε is small enough, we obtain $|g^{<m>}(\alpha)| \leqslant \dfrac{\mu}{2}$. This gives a contradiction. Thus we have proved the second assertion. □

Remark 2 We have proved that

• for the first assertion we can take $\eta < \eta_0$, defined in (2),

• and for the second assertion, $\eta < \eta_1 = 2^{-n}\mu\dfrac{H}{(H+1)^{n-M}-1}$ where M is equal to $\min\{m_i, i = 1, \ldots, k\}$.

Notice that the argument works if $\eta < \min\{\eta_0, \eta_1\}$

3. Perturbation of the Coefficients of a Polynomial

Example 1 Let $P \in K[X]$ be the polynomial

$$P(X) = (aX - b)(X^2 - X - 1),$$

then $\alpha = (1 + \sqrt{5})/2$ is a root of P. Let $a = F_n$ and $b = F_{n+1}$, where $(F_n)_n$ is the Fibonacci sequence. Denote by H the height of P. We have,

$$P(x) = F_n x^3 - (F_n + F_{n+1})x^2 + (F_{n+1} - F_n)x + F_{n+1}.$$

The height satisfies $H = F_{n+1} + F_n = F_{n+2} \simeq \alpha^{n+2}/\sqrt{5}$. And we have

$$\frac{F_{n+1}}{F_n} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n} \quad \text{where} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Thus, $b/a = F_{n+1}/F_n = \alpha + O(\alpha^{-2n}) = \alpha + O(1/H^2)$, and $\operatorname{sep}(P) \simeq 1/H^2$.

Put $\tilde{P} = P - 1$. If $\alpha + \varepsilon$ is a root of $\tilde{P}$ where ε is small, then

$$\tilde{P}(\alpha + \varepsilon) = P(\alpha + \varepsilon) - 1 = \varepsilon P'(\alpha) + \frac{\varepsilon^2}{2}P''(\alpha) + \frac{\varepsilon^3}{3!}P'''(\alpha) - 1 = 0.$$

We have $P'(\alpha) \simeq a^{-1}$, $P'(b/a) \simeq a^{-1}$ and $P''(b/a) \simeq 2a(2b/a - 1) \simeq 2\sqrt{5}a$. Hence

$$a^2\sqrt{5}\varepsilon^2 + \varepsilon - a = 0 \Rightarrow \varepsilon \simeq \pm 1/\sqrt{a}.$$

Therefore, $\operatorname{sep}(\tilde{P}) \simeq 1/\sqrt{H}$.

Conclusion: When perturbating P by $\eta = 1$, the roots of perturbated polynomial deviate considerably.

Example 2 Let us consider the following polynomial

$$P(x) = (ax - 1)(x^n - ax + 1).$$

Denote H the height of P . Then $H = a^2 (a \gg 1)$. Then one root, say α, of the second factor of this polynomial is in the neighborhood of $1/a$. Indeed, $P(1/a) = 0$ and $a\alpha - 1 = \alpha^n \simeq a^{-n}$. Let ε sufficiently small such that $P(1/a + \varepsilon) = 0$. We have $P(1/a + \varepsilon) = \varepsilon a(1/a + \varepsilon)n - \varepsilon a$. Thus $\varepsilon \simeq a^{-(n+1)}$. Therefore,

$$\operatorname{sep}(P) \simeq a^{-(n+1)}.$$

Put $P_1(x) = P(x) + x^{n+1}$. Therefore,

$$P_1(1/a + \varepsilon) = (1/a + \varepsilon)^{n+1} + \varepsilon a((1/a + \varepsilon)^n - \varepsilon a).$$

For ε sufficiently small such that $P_1(1/a + \varepsilon) = 0$, we have the approximation:

$$a^{-(n+1)} + \varepsilon a^{-(n-1)} - \varepsilon^2 a^2 = 0.$$

Thus we have two possible roots, say ε_1 and ε_2:

$$\varepsilon_1 \simeq 2^{-1}a^{-\frac{(n-1)}{2}-2} \quad \text{and} \quad \varepsilon_2 \simeq 2^{-3}a^{-\frac{(n-1)}{2}-2}$$

Hence

$$\operatorname{sep}(P_1) \simeq a^{-\frac{(n-1)}{2}-2}.$$

Example 3 Let $P(z) = z^n - a$, where $a \in \mathbb{N},\ a \gg 0$. The roots of P are:

$$z_k = a^{1/n} exp(i2k\pi/n), \quad k = 0, 1, \ldots, n-1,$$

thus $\text{sep}(P) = 2a^{1/n}\sin(\pi/n)$ and $H = a$. Moreover,

$$\eta < \eta_0 = \min\{1, \text{sep}(P)^n\} \frac{H+1}{(H+2)^{n+1}-1} \simeq \min\{1, a(2\sin\pi/n)^n\} a^{-n}.$$

For $a = 2853$ and $n = 16$, $P(z) = z^{16} - 2853$, $\eta_0 = (2\sin\pi/16)^{16}(2853)^{-15}$, $\eta_0 \simeq 4.2710^{-59}$. We notice that in this case, for which the roots of P are very well separated, these roots remain very "stable" under perturbation of the coefficients, for example, the real roots of P are:

$$\pm 1.6442064998888648346270848409203190688386502495500672674046599829975 79$$

where the real roots of perturbated polynomial are:

$$\pm 1.6442064998888648346270848409203190688386502495500672674046599845366 53$$

We have the same remark with the other roots.

Example 4 Let $P(x) = (x-1)(x-2)(x-3)$, and $Q(x) = 0.009x^3 + P(x)$. We have $\text{sep}(P) = 1$. The roots of Q are: 0.9621055, $2.2712408 \pm 0.7552410i$ and $\text{sep}(Q) \simeq 1.500482$. For $\varepsilon \simeq 1/2$, D_2 and D_3 (see notation in the theorem) do not contain any root of Q. We have

$$\eta_0 = \frac{H+1}{(H+2)^4-1} < 0.000421.$$

Now, by taking $\eta = 0.000420 (< \eta_0)$, $Q(x) = \eta x^3 + P(x)$. And then, the roots of Q are

$$0.99979011430215892, \quad 2.00337844607166149, \quad \text{and} \quad 2.99431149002517172,$$

which are rather close to the roots of the initial polynomial P, and each disk D_i contain exactly one root of Q.

Remark 3 To keep integer coefficients, instead of the above Q, we could consider the polynomial $Q_1(x) = x^3 + aP(x)$ where a is a large positive integer.

Example 5 The polynomial $P(x) = \prod_{i=1}^{20}(x-i)$ of Wilkinson satisfies

$$H = 13803759753640704000 \quad \text{and} \quad \text{sep}(P) = 1.$$

By the theorem,

$$\eta < \eta_0 = \frac{H+1}{(H+2)^{21}-1} \simeq 1.5810^{-383}.$$

This value of η_0 , is extraordinarily small and seems very pessimistic. But, indeed, Wilkinson shows that a very small perturbation ($\eta = 10^{-9}$) causes a dramatic change of some of the roots for example the perturbated polynomial admits the complex roots $16.57173899 \pm 0.8833156071 \cdot I$.

Example 6 Let $P(x)$ be the polynomial defined by

$$P(x) = x^n - (ax-1)^2.$$

The height of this polynomial is $H = a^2$ as soon as $a > 2$.

Moreover, $P(1/a) = a^{-n}$. We consider $a \gg 1$. Let ε sufficiently small such that $P(1/a + \varepsilon) = 0$. Then

$$P(1/a + \varepsilon) = (1/a + \varepsilon)n - ((1/a + \varepsilon)a - 1)^2 = (1/a + \varepsilon)^n - \varepsilon^2 a^2.$$

Hence,

$$P(1/a + \varepsilon) \simeq a^{-n} - \varepsilon^2 a^2.$$

Then from $a^{-n} - \varepsilon^2 a^2 \simeq 0$ we obtain

$$\varepsilon \simeq a^{-(n+2)/2},$$

thus we obtain

$$\mathrm{sep}(P) \simeq 2a^{-(n+2)/2}.$$

Put $P_1(x) = P(x) + x^2$. We have $P_1(x) = x^n - (ax - 1)^2 + x^2$.

Let ε sufficiently small such that $P(1/a + \varepsilon) = 0$. Then

$$P_1(1/a + \varepsilon) = (1/a + \varepsilon)^n - \varepsilon^2 a^2 + (1/a + \varepsilon)^2 = 0.$$

Since $a \gg 1$ and $\varepsilon \ll 1$, we have

$$P_1(1/a + \varepsilon) \simeq -\varepsilon^2 a^2 + a^{-2}.$$

Thus $\varepsilon \simeq \pm a^{-2}$. Hence

$$\mathrm{sep}(P_1) \simeq 2a^{-2}.$$

If we consider the polynomial $P_2(x) = P(x) + x^n$, the height of P_2 is a^2 for $a \gg 1$. Let ε sufficiently small such that $P(1/a + \varepsilon) = 0$. We have,

$$2a^{-n} - \varepsilon^2 a^2 \simeq 0.$$

Thus $\varepsilon \simeq 2^{1/2} a^{-(n+2)/2}$, hence

$$\mathrm{sep}(P_2) \simeq 2\sqrt{2}a^{-(n+2)/2}.$$

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