Journal of Mathematics Research; Vol. 5, No. 3; 2013
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

An Extension of the Hadamard-Type Inequality for a Convex Function Defined on Modulus of Complex Integral Functions

Md Mainul Islam[1] & A. N. M. Rezaul Karim[2]

[1] Department of Electrical and Electronics Engineering, International Islamic University Chittagong, Chittagong, Bangladesh

[2] Department of Computer Science and Engineering, International Islamic University Chittagong, Chittagong, Bangladesh

Correspondence: Md Mainul Islam, Department of Electrical and Electronics Engineering, International Islamic University Chittagong, Chittagong, Bangladesh. E-mail: mainulmath83@yahoo.com

Received: July 15, 2013 Accepted: August 18, 2013 Online Published: August 22, 2013

doi:10.5539/jmr.v5n3p92 URL: http://dx.doi.org/10.5539/jmr.v5n3p92

Abstract

In this paper we extend the Hadamard's type inequalities for convex functions defined on the modulus of integral functions in complex field. Firstly, by using the Principal of maximum modulus theorem we show that $M(r)$ and $lnM(r)$ are continuous and convex functions for any non-negative values of r. Finally we derive two inequalities analogous to well known Hadamard's inequality by using elementary analysis.

Keywords: convex functions, Hermite-Hadamard integral inequality, modulus of complex function, principal of maximum and minimum modulus

1. Introduction

Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on the interval $I \subset \mathbb{R}$. If $a, b \in I$, and $a < b$, then the following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

holds. This is called the *Hermite-Hadamard inequality*. Since its discovery in 1893, Hadamard's inequality (Hadamard, 1893) has been proved to be one of the most useful inequalities in mathematical analysis. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalization and numerous applications (Hadamard, 1893; Heing & Maligranda, 1991/1992; Pachpatte, 2003; Mitrinovic, 1970; Tunc, 2012; Dragomir, 1990b) and reference cited therein. The main purpose of this paper is to establish some integral inequality involving the modulus of complex integral functions. Throughout this note, we write integral functions for complex integral functions. Here some necessary definitions and theorem are mentioned which are closely connected to our main result.

Definition 1.1 A function $f(x)$ is said to be convex on the closed interval $I \subset \mathbb{R}$ if and only if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$, for all $x, y \in I$ and $0 \leq \lambda \leq 1$.

Definition 1.2 If the derivative $f'(z)$ exists at all points z of a region $\mathcal{R}$, then $f(z)$ is said to be analytic in $\mathcal{R}$ and referred to as an analytic function in $\mathcal{R}$ or a function analytic in $\mathcal{R}$. The terms regular and holomorphic are sometimes used as synonyms for analytic.

Definition 1.3 If AB and BC are two rectifiable arcs of lengths l and l' respectively, which have only the point B in common, the arc AC is evidently also rectifiable, its lengths being $l + l'$. From this it follows that a Jordan arc which consists of a finite number of regular arcs is rectifiable, its length being the sum of the lengths of the regular arcs forming it. Such an arc we call a contour. Also a closed contour means a simple closed Jordan curve which consists of a finite number of regular arcs. Obviously a closed contour is rectifiable.

Definition 1.4 An integral function is a function which is analytic for all finite values of z. For example e^z, $cosz$, $sinz$, and all polynomials are integral functions.

Definition 1.5 The maximum and minimum modulus of an integral function usually denoted by $M(r)$ and $m(r)$ respectively, and defined by

$$M(r) = \max_{z\in D} |f(z)| \text{ and } m(r) = \min_{z\in D} |f(z)| . \tag{1.2}$$

Where D is a region bounded by a closed contour C.

Theorem 1.6 (The Principal of Maximum Modulus Theorem) *Let $f(z)$ is an analytic function, regular in a region D and on its boundary C, where C is a simple closed contour. Then $|f(z)|$ is continuous in D, since*

$$||f(z+h)| - |f(z)|| \le |f(z+h) - f(z)|,$$

and $|f(z+h) - f(z)| \to 0$ as $h \to 0$. Hence $|f(z)|$ has a maximum value, which is obtained at one or more points. In fact $|f(z)|$ reaches its maximum on the boundary C, and not at any interior point of D. We may claim that if $|f(z)| \le M$ on C, then the same inequality holds at all points of D.

A more precise form of the theorem is as follows:

"Let $f(z)$ be an analytic function, regular within and on the closed curve C. Let M be the upper bound of $|f(z)|$ on C. Then the inequality $|f(z)| \le M$ holds everywhere within C. Moreover, $|f(z)| = M$, at a point within C if and only if $f(z)$ is constant".

Theorem 1.7 (The Principal of Minimum Modulus Theorem) *If $f(z)$ is a non-constant integral function without zeros within the region bounded by a closed contour C, then $|f(z)|$ obtained its minimum value at a point on the boundary of C, i.e. if m is the minimum value of $|f(z)|$ on C, then the inequality holds $|f(z)| \ge m$, for any z lies inside C.*

Theorem 1.8 (Hadamard's Three Circle Theorem) *Let $f(z)$ is an analytic function, regular for any z lies inside the annulus $r_1 \le |z| \le r_3$. If $r_1 < r_2 < r_3$ and M_1, M_2, M_3 are the maxima of $|f(z)|$, on the circle $|z| = r_1, r_2, r_3$ respectively, then we get*

$$M_2^{ln\left(\frac{r_3}{r_1}\right)} \le M_1^{ln\left(\frac{r_3}{r_2}\right)} M_3^{ln\left(\frac{r_2}{r_1}\right)} . \tag{1.3}$$

Proof. See (Titchmarsh, 1939, pp. 173) and (Copson, 1935, pp. 164). □

In Section 2, we are mentioned some necessary lemmas with proof, which are closely connected to our main results. Mainly here we established that $M(r)$ and $lnM(r)$ are continuous and convex functions for any non-negative real values of r. Finally in Section 3, we established two integral inequalities analogous to the well-known Hadamard's inequality (1.1).

2. Some Useful Lemmas

In order to establish the result, we need the following lemmas, some of them are discussed in (Titchmarsh, 1939; Polya, 1926).

Lemma 2.1 *If $f(z)$ is an integral function, and $M(r)$ denotes the maximum value of $|f(z)|$, on the region D: $|z| \le R$, then $M(r)$ is a steadily increasing continuous function of r.*

Proof. Let $0 \le r_1 \le r_2 \le R$ and $M(r_1)$, $M(r_2)$ denote the maximum modulus of $|f(z)|$ on the circles $|z| = r_1$ and $|z| = r_2$ respectively. Here $r_1 < r_2$ implies that the circle $|z| = r_1$ lies inside the circle $|z| = r_2$. Say $\left|f\left(r_1 e^{i\alpha_1}\right)\right| = M(r_1)$, hence $M(r_1)$ obtained at $z = r_1 e^{i\alpha_1}$. Now $z = r_1 e^{i\alpha_1}$ lies inside the circle $|z| = r_2$. So, by using the principal of maximum modulus theorem, we get

$$M(r_1) = \left|f\left(r_1 e^{i\alpha_1}\right)\right| \le M(r_2). \tag{2.1}$$

Therefore, $M(r)$ is a steadily increasing function of r.

Now we show that $M(r)$ is a continuous function of r. For any $\delta\,(> 0)$ there exits $\epsilon > 0$ such that $|f(z) - f(z_0)| < \epsilon$, whenever, $|z - z_0| < \delta$, since $f(z)$ is a continuous function. Also, if $|z - z_0| < \delta$, then we get

$$||f(z)| - |f(z_0)|| \le |f(z) - f(z_0)| < \epsilon, \tag{2.2}$$

which implies that $|f(z)|$ is continuous in $|z| \le R$. Let $0 < z_0 < r$ and $z_0 = r_0 e^{i\alpha_0}$, i.e. z_0 lies inside $|z| \le R$. Suppose $z = re^{i\alpha}$ be any point such that $|z - z_0| < \delta$, so this z satisfies the inequality (2.2). And for any values of θ, $z_0 = r_0 e^{i\theta}$

and $z = re^{i\theta}$ satisfies inequality (2.2), also z and z_0 lies inside $|z| = R$, and lies on $|z| = r$, $|z| = r_0$ respectively. Now for the annulus D: $r_1 \leq |z| \leq r$, the inequality (2.2) holds. So we get

$$\left|\left|f\left(re^{i\theta}\right)\right| - \left|f\left(r_o e^{i\theta_0}\right)\right|\right| < \epsilon, \tag{2.3}$$

$$\text{and} \quad |r - r_0| = \left|\left|re^{i\theta}\right| - \left|r_o e^{i\theta_0}\right|\right| \leq \left|re^{i\theta} - r_o e^{i\theta_0}\right| < \delta.$$

Case 1: If $r > r_0$, from inequality (2.3), we get

$$\left|f\left(re^{i\theta}\right)\right| - \epsilon < \left|f\left(r_0 e^{i\theta}\right)\right| < \left|f\left(re^{i\theta}\right)\right| + \epsilon, \tag{2.4}$$

which is true for any θ. Suppose $M(r) = \left|f\left(re^{i\beta}\right)\right|$, by using the inequality (2.4), we get

$$\begin{aligned} &\left|f\left(re^{i\beta}\right)\right| - \epsilon < \left|f\left(r_0 e^{i\beta}\right)\right| < \left|f\left(re^{i\beta}\right)\right| + \epsilon, \\ \Rightarrow \quad &M(r) - \epsilon < \left|f\left(r_0 e^{i\beta}\right)\right| < M(r) + \epsilon. \end{aligned} \tag{2.5}$$

Also $z = r_0 e^{i\beta}$ lies inside the circle $|z| = r$. So by using the principal of maximum modulus theorem, we get

$$\left|f\left(r_0 e^{i\beta}\right)\right| \leq M(r_0) \leq M(r) < M(r) + \epsilon, \tag{2.6}$$

From inequalities (2.5) and (2.6), we get

$$\begin{aligned} &M(r) - \epsilon < M(r_0) < M(r) + \epsilon, \\ &i.e. \quad \lim_{r\to r_0^+} M(r) = M(r_0). \end{aligned} \tag{2.7}$$

Case 2: If $r < r_0$, then from inequality (2.3), we get

$$\left|f\left(r_0 e^{i\theta}\right)\right| - \epsilon < \left|f\left(re^{i\theta}\right)\right| < \left|f\left(r_0 e^{i\theta}\right)\right| + \epsilon, \tag{2.11}$$

Consider $M(r_0) = \left|f\left(r_o e^{i\alpha_0}\right)\right|$, then from inequality (2.8), we get

$$\begin{aligned} &\left|f\left(r_0 e^{i\alpha_0}\right)\right| - \epsilon < \left|f\left(re^{i\alpha_0}\right)\right| < \left|f\left(r_0 e^{i\alpha_0}\right)\right| + \epsilon, \\ \Rightarrow \quad &M(r_0) - \epsilon < \left|f\left(re^{i\alpha_0}\right)\right| < M(r_0) + \epsilon. \end{aligned} \tag{2.9}$$

Also, $z = re^{i\alpha_0}$ lies inside $|z| = r_0$. Now by principal of maximum modulus theorem, we get

$$\left|f\left(re^{i\beta}\right)\right| \leq M(r) \leq M(r_0) < M(r_0) + \epsilon, \tag{2.10}$$

From inequalities (2.9) and (2.10), we get

$$\begin{aligned} &M(r_0) - \epsilon < M(r) < M(r_0) + \epsilon, \\ &i.e. \quad \lim_{r\to r_0^-} M(r) = M(r_0) \end{aligned} \tag{2.11}$$

From equations (2.7) and (2.11), we get

$$\lim_{r\to r_0} M(r) = M(r_0)$$

Therefore, $M(r)$ is continuous at r_0. Now r_0 is arbitrary, so we say that $M(r)$ is a continuous function for any non-negative real values of r. □

Lemma 2.2 *Let $f(z)$ is an integral function, and $M(r)$ is the maximum modulus of $|f(z)|$ on the region D: $|z| \leq r$. Then $|f(z)| \leq M(r)$, and $|f(r)| \leq M(r)$.*

Proof. By using the principal of maximum modulus theorem, we get $|f(z)| \leq M(r)$, for any z lies inside and on the circle $|z| = r$. Also, if we take $z = r + 0 \times i = r$, then this z lies on the circle $|z| = r$, hence for this $z = r$, we get $|f(r)| \leq M(r)$. This completes the proof. □

Lemma 2.3 *Let $f(z)$ is an integral function and $M(r)$ be the maximum modulus of $|f(z)|$ on the region D: $|z| \leq r$. Then $M(r)$ is a convex function for any non-negative real values of r.*

Proof. By using lemma, we say that $M(r)$ is a increasing function for any non-negative real values of r. Choose, $r_1 \leq r \leq r_2$, lies on line segment between the points r_1 and r_2. And for $0 < \lambda < 1$, we get $r = \lambda r_1 + (1-\lambda) r_2$. Also the value $M(r)$ lies on the curve which is downward of the line segment between the points $M(r_1)$ and $M(r_2)$, and $\lambda M(r_1) + (1-\lambda) M(r_2)$ lies on this line segment. So we get $M(r) \leq \lambda M(r_1) + (1-\lambda) M(r_2)$. Therefore $M(r)$ is a convex function for any non-negative real values of r. This completes the proof. □

Lemma 2.4 *Let $f(z)$ is an analytic function in $|z| \leq R$, $f(0) \neq 0$, and $M(r)$ the maximum modulus of $|f(z)|$ on the region D: $|z| \leq r$. Then $lnM(r)$ is a convex function of lnr for any positive real values of r.*

Proof. Let $r_1 < r_2 < r_3 \leq R$ and $M_i(r_i)$ is the maximum modulus of $|f(z)|$ on the region bounded by the circles $|z| = r_i$, for $i = 1, 2, 3$, respectively. Then by using Hadamard's three circle theorem, we get

$$M_2^{\ln\left(\frac{r_3}{r_1}\right)} \leq M_1^{\ln\left(\frac{r_3}{r_2}\right)} M_3^{\ln\left(\frac{r_2}{r_1}\right)}. \tag{2.12}$$

The sign of equality will occur only if the function $f(z)$ is constant multiple of a power of z. Excluding this case, we get

$$M_2^{\ln\left(\frac{r_3}{r_1}\right)} < M_1^{\ln\left(\frac{r_3}{r_2}\right)} M_3^{\ln\left(\frac{r_2}{r_1}\right)}.$$

Taking logarithm on the both sides, we get

$$\begin{aligned} & ln\left(\tfrac{r_3}{r_1}\right) lnM_2 < ln\left(\tfrac{r_3}{r_2}\right) lnM_1 + ln\left(\tfrac{r_2}{r_1}\right) lnM_3, \\ \Rightarrow \quad & lnM_2 < \tfrac{lnr_3 - lnr_2}{lnr_3 - lnr_1} lnM_1 + \tfrac{lnr_2 - lnr_1}{lnr_3 - lnr_1} lnM_3. \end{aligned} \tag{2.13}$$

Now $lnM(r)$ is a continuous function of lnr, and so if we put $x = lnr$, then we get, $lnM(r) = \varphi(lnr) = \varphi(x)$. Also consider, $x_i = lnr_i$, for $i = 1, 2, 3$, then

$$lnM_i = lnM_i(r_i) = \varphi(lnr_i) = \varphi(x_i).$$

So we obtained the following inequality from (2.13)

$$\varphi(x_2) < \frac{x_3 - x_2}{x_3 - x_1}\varphi(x_1) + \frac{x_2 - x_1}{x_3 - x_1}\varphi(x_3) = \lambda\varphi(x_1) + (1-\lambda)\varphi(x_3),$$

where, $\lambda = (x_3 - x_2)/(x_3 - x_1) < 1$, since $x_1 < x_2 < x_3$.

Also here we choose x_2 arbitrarily. Hence we can say that $\varphi(x)$ is a convex function of x, i.e. $lnM(r)$ is a convex function of lnr. This completes the proof. □

3. Main Results

Theorem 3.1 *Let $f(z)$ is a complex integral function defined on any finite region D: $|z| = R$, then for any $a, b \in I \subset [0, \infty)$ with $a < b$, we get the following inequalities*

$$\left|f\left(\frac{a+b}{2}\right)\right| \leq M\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b M(r)\,dr \leq \frac{M(a)+M(b)}{2},$$

$$\text{and} \qquad \frac{|f(a)| + |f(b)|}{2} \leq \frac{M(a)+M(b)}{2}.$$

Proof. By using lemma, we get M: $I \subset [0, \infty) \to \mathbb{R}$ is a convex mapping defined on the interval I of real numbers, and $a, b \in I$ with $a < b$. Now by using the Hermite-Hadamard inequality on convex function, we derive the following double inequality

$$M\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b M(r)\,dr \leq \frac{M(a)+M(b)}{2}, \tag{3.1}$$

Also, by using Lemma 2.2, we get

$$\left|f\left(\frac{a+b}{2}\right)\right| \leq M\left(\frac{a+b}{2}\right), \text{ and } \frac{|f(a)| + |f(b)|}{2} \leq \frac{M(a)+M(b)}{2}. \tag{3.2}$$

The proof of this theorem is completed. □

Theorem 3.2 *Let $f(z)$ a complex integral function defined on any finite region D: $|z| \leq R$, $f(0) \neq 0$. Now for any $a, b \in I \subset [0, \infty)$, with $a < b$, we get the following inequalities*

$$ln\left|f(\frac{a+b}{2})\right| \leq lnM(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b lnM\,(r)dr \leq \frac{lnM\,(a) + lnM\,(b)}{2},$$

$$\text{and} \quad \frac{ln\,|f(a)| + ln\,|f(b)|}{2} \leq \frac{lnM\,(a) + lnM\,(b)}{2}.$$

Proof. By using lemma, we get lnM: $I \subseteq [0, \infty) \to \mathbb{R}$ is a convex mapping defined on the interval I of real numbers, and $a, b \in I$ with $a < b$. Now by using the Hermite-Hadamard inequality, we derive the following double inequality

$$lnM(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b lnM\,(r)dr \leq \frac{lnM\,(a) + lnM\,(b)}{2}. \tag{3.3}$$

Also, by using Lemma 2.2, we get

$$ln\left|f(\frac{a+b}{2})\right| \leq lnM\left(\frac{a+b}{2}\right),$$

$$\text{and} \quad \frac{ln\,|f(a)| \ + ln\,|f(b)|}{2} \leq \frac{lnM\,(a) \ + lnM\,(b)}{2}. \tag{3.4}$$

The proof of this theorem is completed. □

4. Conclusion

Inequalities (3.1) to (3.4) indicated our final results. These are the extension of the Hadamard's type inequalities (1.1) for the functions $M(r)$ and $lnM(r)$. In the next paper we will give the applications on these results by obtaining some Hadamard's-type inequality for Meromorphic functions in complex field. Specially we will try to obtain Hadamard's-type inequalities for $n(r, f)$ and $T(r, f)$, defined on Meromorphic functions.

References

Chen, Y. (2012). Multi-dimensional Hadamard's inequalities. *Tamkang Journal of Mathematics, 43*(1), 1-10. Spring.

Copson, E. T. (1935). *An introduction to the theory of functions of a complex variable*. Oxford University Press.

Dragomir, S. S. (1990a). Some refinements of Hadamard's inequality. *Gaz. Math. Metod. (Romania), 11*, 189-191.

Dragomir, S. S. (1990b). Two refinements of Hadamard's inequalities. *Coll. Of Sci. Pap. Of the Fac. of Sci., Kragujevac (Yougoslavia), 11*, 23-26.

Dragomir, S. S. (1992). On Hadamard's inequalities for convex functions. *Math. Balkanica, 6*(4), 251-222.

Hadamard, J. (1893). Etude sur les properties des fonctionsentieres en particulierd'unefonctionconsideree par Riemann. *J. Math. Pures Appl., 58*, 171-215.

Heing, H. P., & Maligranda, L. (1991/1992). Chebyshev inequality in function spaces. *Real Analysis Exchange, 17*, 211-47.

Mitrinovic, D. S. (1970). *Analytic Inequalities*. Berlin, New York: Springer-Verlag.

Pachpatte, B. G. (2003). On some inequalities for convex functions. *RGMIA Res. Rep. Coll., 6*(E).

Pecoarico, J. E., Proschan, F., & Tong, Y. L. (1992). *Convex Functions, Partial Orderings, and Statistical Applications*. Academic Press, Inc.

Polya, G. (1926). On the minimum modulus of integral functions. *Journal of the London Math. Soc., 1*, 78-86. http://dx.doi.org/10.1112/jlms/s1-1.2.78

Spiegel, M. R. *Complex variables with an introduction to conformal mapping and its application* (Sl (metric) Ed.). Schaum's Outline Series.

Titchmarsh, E. C. (1939). *The theory of functions* (2nd ed.). Oxford University Press.

Tunc, M. (2012). On some new inequalities for convex functions. *Turk J. Math., 36*, 245-251.

Valiron, G. (1949). *Lectures on the general theory of integral functions*. New York: Chelsea Pub. Co.

Yang, G. S., & Hong, M. C. (1997). A note on Hadamard'sinequality. *Tamkang. J. Math., 28*, 33-77.