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Solution of the Inverse Eigenvalue Problem for Certain (Anti-) Hermitian Matrices Using Newton's Method

Francis T. Oduro[1]

[1] Department of Mathematics, Kwame Nkrumah University of Science and Technology Kumasi, Ghana

Correspondence: Francis T. Oduro, Department of Mathematics, Kwame Nkrumah University of Science and Technology Kumasi, Ghana. E-mail: francistoduro@yahoo.co.uk

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Abstract

Using distinct non zero diagonal elements of a Hermitian (or anti Hermitian) matrix as independent variables of the characteristic polynomial function, a Newton's algorithm is developed for the solution of the inverse eigenproblem given distinct nonzero eigenvalues. It is found that if a 2×2 singular Hermitian (or singular anti Hermitian) matrix of rank one is used as the initial matrix, convergence to an exact solution is achieved in only one step. This result can be extended to $n \times n$ matrices provided the target eigenvalues are respectively of multiplicities p and q with $p + q = n$ and $1 \leq p, q < n$. Moreover, the initial matrix would be of rank one and would have only two distinct corresponding nonzero diagonal elements, the rest being repeated. To illustrate the result, numerical examples are given for the cases $n = 2, 3$ and 4.

Keywords: Hermitian matrix, anti-Hermitian matrix, inverse eigenproblem, Newton's method

1. Introduction

The matrix inverse eigenvalue problem entails the reconstruction of a matrix from its eigenvalues. It has been of interest not only to algebraists but also to numerical analysts, control theorists, statisticians and engineers. Most research effort have been directed at solving the inverse eigenvalue problem for nonsingular symmetric matrices (Chu & Golub, 2005; Gladwell, 2004; Deakin & Luke, 1992; Chu, 1995). Recently, however, the case of singular symmetric matrices of arbitrary order and rank has been virtually solved provided linear dependency relations are specified (Gyamfi, Oduro, & Aidoo, 2013; Aidoo, Gyamfi, Ackora-Prah, & Oduro, 2013). It is plausible to expect solutions of the inverse eigenvalue problem in a sufficiently small neighborhood of any such Hermitian matrix. In this paper, therefore, we develop a Newton's method for the solution of the inverse eigenvalue problem for a (anti-) Hermitian matrix using a singular (anti-) Hermitian matrix as the initial approximation. In the case of a 2×2 (anti-) Hermitian matrix with non-zero diagonal elements, we show that an exact solution is obtained in only one step if the initial matrix is of rank one. This result is subsequently extended to $n \times n$ matrices provided the target eigenvalues are respectively of multiplicities p and q with $p + q = n$ and $1 \leq p, q < n$.

2. Preliminaries

In this section we review previous results obtained by Gyamfi et al. (2013) and Aidoo et al. (2013) in respect of the inverse eigenvalue problem for singular symmetric matrices which are here extended to include singular (anti-) Hermitian matrices as well.

Lemma 1 *There exists a one-to-one correspondence between the elements of a Hermitian or anti Hermitian matrix and its distinct nonzero eigenvalues if and only if the matrix is of rank 1* (Aidoo et al., 2013).

Proof. Let the given Hermitian or anti Hermitian matrix be of rank r, then, clearly, the number of independent elements is $\frac{r(r+1)}{2}$. Thus a one to one onto correspondence will exist between the elements of the matrix and its distinct nonzero eigenvalues if and only if $\frac{r(r+1)}{2} = r$. i.e., if and only if $r = 1$.

Proposition 1 *If the row dependence relations for a Hermitian or anti Hermitian matrix of rank 1 are specified as follows*

$$R_i = k_{i-1} R_1, \quad i = 2, \dots n - 1 \tag{1}$$

where R_i is the ith row and each k_i is a nonzero scalar. Then the matrix can be generated from its nonzero eigenvalue λ:

$$A = a_{11}\begin{bmatrix} 1 & \bar{k}_1 & \bar{k}_2 & \dots & \bar{k}_{n-1} \\ k_1 & |k_1|^2 & \bar{k}_1 k_2 & \dots & \dots \\ k_2 & k_1\bar{k}_2 & |k_2|^2 & \dots & \dots \\ \vdots & \vdots & \vdots & & \vdots \\ k_{n-1} & \bar{k}_1 k_{n-1} & \bar{k}_2 k_{n-1} & \dots & |k_{n-1}|^2 \end{bmatrix} \tag{2}$$

where

$$a_{11} = \frac{\lambda}{1 + |k_1|^2 + \cdots + |k_{n-1}|^2}$$

Proof. By induction on n. Similarly result has also been obtained for singular matrix of rank greater than 1. □

3. Main Results

3.1 Characteristic Function of Matrix Elements

Let $X = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ where $x_1, x_2, \dots, x_n$ are n elements of a Hermitian matrix A of order $n \times n$. For given distinct (target) eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, of another Hermitian matrix A_t, called the target matrix, we now define a characteristic function as a function of n elements of A given by

$$f_i(x_1, x_2, \dots, x_n) = \sum_{k=0}^{n} (-1)^{2n-k} I_k \lambda_i^{n-k} \tag{3}$$

Here, I_k the coefficient of λ_i^{n-k} is the kth principal invariant of the second order Hermitian tensor represented by A. As is well-known I_k can be written in terms of the eigenvalues of A, which we denote here by $\mu_1, \mu_2, \dots, \mu_n$.

$$I_k = \frac{1}{n!} \sum_{\pi \in S_n} \prod_{j=1}^{k} \mu_{\pi(j)} \tag{4}$$

where $\pi \in S_n$ is a permutation of the natural numbers and S_n is the symmetric group of order n. Moreover the I_k may also be expressed in terms of the trace of A and its powers. For our purposes it suffices to write down the first few terms:

$$I_0 = 1$$
$$I_1 = \mathrm{tr}A = \sum_{j=1}^{n} \mu_j$$
$$I_2 = \frac{1}{2}\left[(\mathrm{tr}A)^2 - (\mathrm{tr}A^2)\right] = \sum_{i<j} \mu_i \mu_j$$

Thus, $I_n = \det A = \prod_{j=1}^{n} \mu_j$

Note that when A coincides with A_t then each characteristic function reduces to a characteristic equation in λ_i

$$f_i(x_1, x_2, \dots, x_n) = 0 \qquad i = 1, \cdots, n \tag{5}$$

For our purposes it is convenient to choose $x_1 = a_{11}, x_2 = a_{22}, \dots, x_n = a_{nn}$ as the n elements of the matrix A which are also used as the n independent variables of each f_i.

$$f_i(a_{11}, a_{22}, \dots, a_{nn}) = 0 \qquad i = 1, \cdots, n \tag{6}$$

Definition 1 (An Inverse Eigenvalue Problem) An inverse eigenvalue problem can be formulated as follows: To find the diagonal elements a_{ii} of the target (anti-) Hermitian matrix A_t, given its non-diagonal elements, as specified in Equation (2) and its spectrum $\{\lambda_1, \dots, \lambda_n\}$.

Remark 1 The solution of the IEP stated above is then simply the solution of Equation (6). Now this equation is clearly solvable by Newton's method.

This will require the Jacobian matrix J of the smooth vector function with components f_i which is given by

$$J = \left[\frac{\partial f_i}{\partial a_{jj}}\right] \tag{7}$$

The entries of J may be computed as follows.

Theorem 1

$$\frac{\partial f_i}{\partial a_{jj}} = -\lambda_i^{n-1} + tr(M_j)\lambda_i^{n-2} + \cdots + (-1)^n \det(M_j) \tag{8}$$

where M_j is the $(n-1) \times (n-1)$ matrix obtained by deleting the jth row and the jth column of A.

Theorem 2

$$\det J = -\prod_{i<j}(a_{ii} - a_{jj})(\lambda_i - \lambda_j) \tag{9}$$

Proof. From the expression

$$\frac{\partial f_i}{\partial a_{jj}} = -\lambda_i^{n-1} + tr(M_j)\lambda_i^{n-2} + \cdots + (-1)^n \det(M_j)$$

It is clear that the Jacobian matrix will have the ith and the kth rows equal if λ_i is the same as λ_k. Also the Jacobian matrix will have the jth and lth columns equal if the diagonal elements a_{jj} and a_{ll} are equal. The result follows from the vanishing of a determinant with repeated rows or repeated columns as well as (up to multiplication by a constant) from the factor theorem of elementary algebra. It is easily checked that by the definition of characteristic polynomial adopted, the constant multiple is -1. □

Theorem 3 *If the (anti-) Hermitian matrix A is of rank r, the characteristic polynomial functions reduce to the forms:*

$$f_i(a_{11}, a_{22}, \ldots, a_{nn}) = \sum_{k=0}^{r}(-1)^{2n-k} I_k \lambda_i^{n-k} \qquad i = 1, \cdots, n \tag{10}$$

Proof. If A is (anti-) Hermitian of rank r, then all the possible invariants of A for which $k > r$ vanish (from Equation (4)). □

Corollary 1 *If the (anti-) Hermitian matrix A is of rank one, then the characteristic polynomial functions reduce to the forms*

$$f_i(a_{11}, a_{22}, \ldots, a_{nn}) = \lambda_i^n - (tr\mathrm{A})\lambda_i^{n-1} \tag{11}$$

with A being (anti-) Hermitian and of rank 1, all the invariants of A vanish except the trace and the f_i become linear functions of the diagonal elements of A.

Theorem 4 (Newton's Algorithm) *Let $A^{(0)}$ be (anti-) Hermitian and $X^{(0)} = (a_{11}^{(0)}, a_{22}^{(0)}, \cdots, a_{nn}^{(0)})$. Then, provided the diagonal elements $a_{ii}^{(0)}$ of $A^{(0)}$ are distinct, the inverse eigenvalue problem stated above can be solved in a sufficiently small neighbourhood of $X^{(0)}$ by the Newton's method given by the scheme.*

$$X^{(N+1)} = X^{(N)} - J^{-1} f(X^{(N)}), \quad N = 0, 1, 2, \ldots \tag{12}$$

Here, the Jacobian matrix J must be invertible.

Theorem 5 (Singularity of J for $n > 2$ and distinct eigenvalues) *The Jacobian matrix corresponding to distinct target eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, $n > 2$ is singular.*

Proof. Clearly, it is sufficient to prove this result for the case $n = 3$. □

$$\frac{\partial f_i}{\partial a_{jj}} = -\lambda_i^{n-1} + tr(M_j) \tag{13}$$

$$
\begin{aligned}
|J| &= \begin{vmatrix} -\lambda_1^2 + (a_{22}+a_{33})\lambda_1 & -\lambda_1^2 + (a_{11}+a_{33})\lambda_1 & -\lambda_1^2 + (a_{11}+a_{22})\lambda_1 \\ -\lambda_2^2 + (a_{22}+a_{33})\lambda_2 & -\lambda_2^2 + (a_{11}+a_{33})\lambda_2 & -\lambda_2^2 + (a_{11}+a_{22})\lambda_2 \\ -\lambda_3^2 + (a_{22}+a_{33})\lambda_3 & -\lambda_3^2 + (a_{11}+a_{33})\lambda_3 & -\lambda_3^2 + (a_{11}+a_{22})\lambda_3 \end{vmatrix} \\
&= \begin{vmatrix} -\lambda_1^2 & -\lambda_1^2 + (a_{11}+a_{33})\lambda_1 & -\lambda_1^2 + (a_{11}+a_{22})\lambda_1 \\ -\lambda_2^2 & -\lambda_2^2 + (a_{11}+a_{33})\lambda_2 & -\lambda_2^2 + (a_{11}+a_{22})\lambda_2 \\ -\lambda_3^2 & -\lambda_3^2 + (a_{11}+a_{33})\lambda_3 & -\lambda_3^2 + (a_{11}+a_{22})\lambda_3 \end{vmatrix} \\
&\quad +(a_{22}+a_{33}) \begin{vmatrix} \lambda_1^2 & -\lambda_1^2 + (a_{11}+a_{33})\lambda_1 & -\lambda_1^2 + (a_{11}+a_{22})\lambda_1 \\ \lambda_2^2 & -\lambda_2^2 + (a_{11}+a_{33})\lambda_2 & -\lambda_2^2 + (a_{11}+a_{22})\lambda_2 \\ \lambda_3^2 & -\lambda_3^2 + (a_{11}+a_{33})\lambda_3 & -\lambda_3^2 + (a_{11}+a_{22})\lambda_3 \end{vmatrix} \\
&= (a_{11}+a_{33}) \begin{vmatrix} -\lambda_1^2 & \lambda_1^2 & -\lambda_1^2 \\ -\lambda_2^2 & \lambda_2^2 & -\lambda_2^2 \\ -\lambda_3^2 & \lambda_3^2 & -\lambda_3^2 \end{vmatrix} + (a_{11}+a_{33})(a_{11}+a_{22}) \begin{vmatrix} -\lambda_1^2 & \lambda_1^2 & \lambda_1^2 \\ -\lambda_2^2 & \lambda_2^2 & \lambda_2^2 \\ -\lambda_3^2 & \lambda_3^2 & \lambda_3^2 \end{vmatrix} \\
&\quad +(a_{22}+a_{33}) \begin{vmatrix} \lambda_1^2 & -\lambda_1^2 & -\lambda_1^2 \\ \lambda_2^2 & -\lambda_2^2 & -\lambda_2^2 \\ \lambda_3^2 & -\lambda_3^2 & -\lambda_3^2 \end{vmatrix} + (a_{22}+a_{33})(a_{11}+a_{22}) \begin{vmatrix} \lambda_1^2 & -\lambda_1^2 & \lambda_1^2 \\ \lambda_2^2 & -\lambda_2^2 & \lambda_2^2 \\ \lambda_3^2 & -\lambda_3^2 & \lambda_3^2 \end{vmatrix}
\end{aligned}
$$

Therefore

$$|J| = 0 \tag{14}$$

Corollary 2 *If $A^{(0)}$ is (anti-) Hermitian, singular and of rank 1 and with distinct nonzero diagonal element, then f at $X^{(0)}$ is linear. Therefore, the solution of the IEP by Newton's method is exact and in one step only. This is only true for $n = 2$.*

Example 1 (The case $n = 2$) From Equation (11), for $n = 2$, the characteristic functions of an (anti-) Hermitian matrix A of rank one are given as,

$$f_1(a_{11}, a_{22}) = \lambda_1^2 - \lambda_1(a_{11} + a_{22})$$

$$f_2(a_{11}, a_{22}) = \lambda_2^2 - \lambda_2(a_{11} + a_{22})$$

From Equation (8), the corresponding Jacobian matrix is given as

$$J = \left[\frac{\partial f_i}{\partial a_{jj}}\right] = \begin{bmatrix} a_{22} - \lambda_1 & a_{11} - \lambda_1 \\ a_{22} - \lambda_2 & a_{11} - \lambda_2 \end{bmatrix} \tag{15}$$

The inverse of the Jacobian matrix is then computed as follows

$$J^{-1} = \frac{1}{(\lambda_2 - \lambda_1)(a_{11} - a_{22})} \begin{bmatrix} a_{11} - \lambda_2 & \lambda_1 - a_{11} \\ \lambda_2 - a_{22} & a_{22} - \lambda_1 \end{bmatrix} \tag{16}$$

Note that for J to be invertible, the target eigenvalues λ_1 and λ_2 must be distinct. So also must be the diagonal elements of A.

Numerical Example 1 Let $\lambda_1 = -1$ and $\lambda_2 = 3$. Now suppose, initial matrix is given by the singular symmetric matrix

$$A^{(0)} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

So that ${a_{11}}^{(0)} = 1,\ {a_{22}}^{(0)} = 4,\ f_1({a_{11}}^{(0)}, {a_{22}}^{(0)}) = 1 + 5 = 6$ and $f_2({a_{11}}^{(0)}, {a_{22}}^{(0)}) = 9 - 15 = -6$.

Thus

$$\begin{bmatrix} f_1({a_{11}}^{(0)}, {a_{22}}^{(0)}) \\ f_2({a_{11}}^{(0)}, {a_{22}}^{(0)}) \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

and, Newton's method then gives

$$\begin{bmatrix} a_{11}^{(1)} \\ a_{22}^{(1)} \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \frac{1}{12} \begin{bmatrix} -2 & -2 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence $A^{(1)} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. It is easily checked the eigenvalues of $A^{(1)}$ are -1 and 3.

Note also that the Hermitian matrices, $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 2i \\ 2i & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & -2i \\ -2i & 4 \end{bmatrix}$ also each yield the same solution of the IEP as $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Similarly, the (anti-) Hermitian matrices, $i\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$, $i\begin{bmatrix} 1 & 2i \\ 2i & 4 \end{bmatrix}$, $i\begin{bmatrix} 1 & -2i \\ -2i & 4 \end{bmatrix}$ also each yield the same solution of the IEP as $i\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, where $-i$ and $3i$ are the target eigenvalues in this case.

Theorem 6 *Let the target eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ have multiplicities p and q, where $p + q = n$ and $1 \le p, q < n$. And let the initial matrix A possess corresponding diagonal elements repeated p times and q times respectively with non diagonal elements as prescribed by equation 2. Then, the n characteristic polynomial functions may be reduced to only two:*

$$f_i(a_{11}, a_{22}, \ldots, a_{nn}) = f_i(\underbrace{a_{kk}, a_{kk}, \ldots, a_{kk}}_{p \text{ times}}, \underbrace{a_{mm}, a_{mm}, \ldots, a_{mm}}_{q \text{ times}}) = \phi_i(a_{kk}, a_{mm}) \quad i = 1, 2$$

where

$$\phi_1(a_{kk}, a_{mm}) = (p\lambda_k)^2 - p\lambda_k(pa_{kk} + qa_{mm})$$

$$\phi_2(a_{kk}, a_{mm}) = (q\lambda_m)^2 - q\lambda_m(pa_{kk} + qa_{mm})$$

with corresponding Jacobian matrix:

$$J_\phi = \left[\frac{\partial \phi_i}{\partial a_{jj}}\right] = \begin{bmatrix} qa_{mm} - p\lambda_k & pa_{kk} - p\lambda_k \\ qa_{mm} - q\lambda_m & pa_{kk} - q\lambda_m \end{bmatrix} \tag{17}$$

Which may be used to solve the for the diagonal element $a_{11}^{(1)}, a_{22}^{(1)}$ for the IEP by the Newtons method in only one step. Subsequently, at least $n-1$ non-diagonal elements may also be computed using Equation 4 as constraint.

Proof. The diagonal elements of the initial matrix $a_{11}^{(0)}, a_{22}^{(0)}$ are, without loss of generality replaced respectively by pa_{kk}, qa_{mm} in the Jacobian matrix in Equation (15), as well as in the corresponding characteristic functions. The target eigenvalues λ_k, λ_m are also respectively replaced by $p\lambda_k, q\lambda_m$ in the same expressions. Thus, the solution of the IEP by Newton's method is here also exact in only one step. Subsequently, at least $n-1$ non-diagonal elements may also be computed using Equation 4 as constraint. □

Example 2 (The case of $n = 3$) Consider a 3×3 (anti-) Hermitian initial matrix of rank 1 with non diagonal elements as precribed by Equation (2).

$$A = a_{11} \begin{bmatrix} 1 & \bar{k}_1 & \bar{k}_2 \\ k_1 & |k_1|^2 & \bar{k}_1 k_2 \\ k_2 & k_1\bar{k}_2 & |k_2|^2 \end{bmatrix}$$

In order to have repeated diagonal elements of the initial matrix (say $a_{22}^{(0)} = a_{33}^{(0)}$), we can put $k_1 = -k_2$ or $k_1 = \pm ik_2$. Where $A^{(0)}$ is an (anti-) Hermitian matrix of rank 1.

Numerical Example 2 In particular, for the matrix above with $a_{11} = 1, k_1 = 2, k_2 = -2$ or $\pm 2i$ we have the following possibilities.

$$A^{(0)} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix} \quad or \quad A^{(0)} = \begin{bmatrix} 1 & 2 & -2i \\ 2 & 4 & -4i \\ 2i & 4i & 4 \end{bmatrix} \quad or \quad A^{(0)} = \begin{bmatrix} 1 & 2 & 2i \\ 2 & 4 & 4i \\ -2i & -4i & 4 \end{bmatrix}$$

All these initial Hermitian matrices together with their anti Hermitian counterparts should lead to the same solution of the IEP given any target spectrum of multiplicity 2:

$$\lambda_1 \neq \lambda_2 = \lambda_3$$

The characteristic functions of the problem is given as ($p = 1$ and $q = 2$):

$$\phi_1(a_{11}, a_{22}) = \lambda_1^2 - \lambda_1(a_{11} + 2a_{22})$$

$$\phi_2(a_{11}, a_{22}) = (2\lambda_2)^2 - 2\lambda_2(a_{11} + 2a_{22})$$

The corresponding Jacobian matrix also is given as

$$J_\phi = \left[\frac{\partial \phi_i}{\partial a_{jj}}\right] = \begin{bmatrix} 2a_{22} - \lambda_1 & a_{11} - \lambda_1 \\ 2a_{22} - 2\lambda_2 & a_{11} - 2\lambda_2 \end{bmatrix}$$

The inverse of the Jacobian matrix is then computed as follows

$$J_\phi^{-1} = \frac{1}{(2\lambda_2 - \lambda_1)(a_{11} - 2a_{22})} \begin{bmatrix} a_{11} - 2\lambda_2 & \lambda_1 - a_{11} \\ 2\lambda_2 - 2a_{22} & 2a_{22} - \lambda_1 \end{bmatrix}$$

Thus

$$\begin{bmatrix} a_{11}^{(1)} \\ 2a_{22}^{(1)} \end{bmatrix} = \begin{bmatrix} a_{11}^{(0)} \\ 2a_{22}^{(0)} \end{bmatrix} - \frac{1}{(2\lambda_2 - \lambda_1)(a_{11} - 2a_{22})} \begin{bmatrix} a_{11} - 2\lambda_2 & \lambda_1 - a_{11} \\ 2\lambda_2 - 2a_{22} & 2a_{22} - \lambda_1 \end{bmatrix} \begin{bmatrix} \phi_1(a_{11}^{(0)}, a_{22}^{(0)}) \\ \phi_2(a_{11}^{(0)}, a_{22}^{(0)}) \end{bmatrix}$$

As a numerical example we now solve the inverse eigenvalue problem for $\lambda_1 = -1, \lambda_2 = \lambda_3 = 3/2$. $\phi_1(a_{11}, a_{22}) = 1 + (1 + 2(4)) = 10$, $\phi_2(a_{11}, a_{22}) = 9 - 3(1 + 2(4)) = -18$, so that $a_{11}^{(0)} = 1$ and $a_{22}^{(0)} = 4$ and

$$\begin{bmatrix} \phi_1(a_{11}^{(0)}, a_{22}^{(0)}) \\ \phi_2(a_{11}^{(0)}, a_{22}^{(0)}) \end{bmatrix} = \begin{bmatrix} 10 \\ -18 \end{bmatrix}$$

The Newton's method gives the diagonal elements:

$$\begin{bmatrix} a_{11}^{(1)} \\ 2a_{22}^{(1)} \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix} + \frac{1}{60} \begin{bmatrix} -2 & -2 \\ -13 & 17 \end{bmatrix} \begin{bmatrix} 10 \\ -18 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix} + \frac{1}{60} \begin{bmatrix} 16 \\ -436 \end{bmatrix} = \begin{bmatrix} 1.2667 \\ 0.7333 \end{bmatrix}$$

Further two non-diagonal elements a_{12} and a_{23} (putting $a_{13} = a_{31} = 0$) may be obtained by solving the constraints:

$$\frac{1}{2}\left[(\mathrm{tr}A)^2 - (\mathrm{tr}A^2)\right] = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$$

$$\det A = \lambda_1\lambda_2\lambda_3$$

Thus, a symmetric solution of the IEP is obtained as follows:

$$A^{(1)} = \begin{bmatrix} 1.2667 & 0.3700i & 0 \\ 0.3700i & 0.3667 & 1.3966 \\ 0 & 1.3966 & 0.3667 \end{bmatrix}$$

It is easily checked that the eigenvalues of $A^{(1)}$ are $(-1, 3/2, 3/2)$.

Note that as in the previous example, certain 3×3 initial Hermitian matrices with their anti-Hermitian counterparts should lead to the similar complex symmetric or (anti-) Hermitian solutions to the IEP, provided the non diagonal elements are appropriately prescribed using Equations (2) and (4).

Example 3 (The case of $n = 4$) For the case $n = 4$, when the target eigenvalues λ_1 and λ_2 each have multiplicities 2, the characteristic functions of the (anti-) Hermitian matrix of rank one, reduce to the following forms:

$$\phi_1(a_{11}, a_{22}) = (2\lambda_1)^2 - 2\lambda_1(2a_{11} + 2a_{22})$$

$$\phi_2(a_{11}, a_{22}) = (2\lambda_2)^2 - 2\lambda_2(2a_{11} + 2a_{22})$$

Then, the corresponding Jacobian matrix is also given as

$$J_\phi = \left[\frac{\partial \phi_i}{\partial a_{jj}}\right] = \begin{bmatrix} 2a_{22} - 2\lambda_1 & 2a_{11} - 2\lambda_1 \\ 2a_{22} - 2\lambda_2 & 2a_{11} - 2\lambda_2 \end{bmatrix}$$

The inverse of the Jacobian matrix is then computed as follows

$$J^{-1} = \frac{1}{(2\lambda_2 - 2\lambda_1)((2a_{11} - 2a_{22})} \begin{bmatrix} 2a_{11} - 2\lambda_2 & 2\lambda_1 - 2a_{11} \\ 2\lambda_2 - 2a_{22} & 2a_{22} - 2\lambda_1 \end{bmatrix}$$

Numerical Example 3 As a numerical example we now solve the inverse eigenvalue problem for $\lambda_1 = \lambda_3 = -1/2$ and $\lambda_2 = \lambda_4 = 3/2$. Then $\phi_1(a_{11}, a_{22}) = 1 + (2 + 2(4)) = 11,\ \phi_2(a_{11}, a_{22}) = 9 - 3(2 + 2(4)) = -21$, so that $a_{11}^{(0)} = 1$ and $a_{22}^{(0)} = 4$ and

$$\begin{bmatrix} \phi_1({a_{11}}^{(0)}, {a_{22}}^{(0)}) \\ \phi_2({a_{11}}^{(0)}, {a_{22}}^{(0)}) \end{bmatrix} = \begin{bmatrix} 11 \\ -21 \end{bmatrix}$$

and, Newton's method then gives the diagonal elements:

$$\begin{bmatrix} 2{a_{11}}^{(1)} \\ 2{a_{22}}^{(1)} \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} + \frac{1}{48} \begin{bmatrix} 1 & -5 \\ -13 & 17 \end{bmatrix} \begin{bmatrix} 11 \\ -21 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} + \frac{1}{48} \begin{bmatrix} 116 \\ -506 \end{bmatrix} = \begin{bmatrix} 4.41667 \\ -2.41667 \end{bmatrix}$$

Further two distinct non-diagonal elements a_{12}, a_{23} (putting $a_{12} = a_{34}$, and : $a_{14} = a_{41} = a_{13} = a_{31} = a_{24} = a_{42} = 0$) may be obtained by solving the constraints:

$$\frac{1}{2}\left[(\mathrm{tr}A)^2 - (\mathrm{tr}A^2)\right] = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4$$
$$\det A = \lambda_1\lambda_2\lambda_3\lambda_4$$

Thus, a symmetric solution of the IEP is obtained as follows:

$$A^{(1)} = \begin{bmatrix} 2.20833 & 1.3851 & 0 & 0 \\ 1.3851 & -1.20833 & 2.7701i & 0 \\ 0 & 2.7701i & 2.20833 & 1.3851 \\ 0 & 0 & 1.3851 & -1.20833 \end{bmatrix}$$

It is easily checked that the eigenvalues of $A^{(1)}$ are $(-1/2, 3/2, -1/2, 3/2)$.

Note that as in the previous examples, certain 4×4 initial Hermitian matrices with their anti-Hermitian counterparts should lead to the similar complex symmetric or (anti-) Hermitian solutions to the IEP, provided the non diagonal elements are appropriately prescribed using Equations (2) and (4).

4. Conclusion

We have obtained, in this study, a Newton's algorithm for solving the inverse eigenvalue problem for certain (anti-) Hermitian matrices and demonstrated that in the neighbourhood of certain singular (anti-) Hermitian matrices of rank 1, the solution is in one step.

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