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MTL Algebra of Fractions and Maximal MTL Algebra of Quotients

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Abstract

In this paper we introduce the notions of *MTL algebra of fractions* and *maximal MTL algebra of quotients for a MTL algebra* and prove constructively the existence of a maximal MTL algebra of quotients (see Buşneag & Piciu, 2005, for *BL* algebras).

Keywords: residuated lattice, Boolean algebra, MTL algebra, *BL* algebra, multiplier, MTL algebra of fractions, maximal MTL algebra of quotients

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1. Introduction

A *localization ring* $A_{\mathcal{F}}$ associated with a Gabriel topology $\mathcal{F}$ for a ring A is a very important construction in ring theory. For the therm *localization* we have in view Chapter IV: *Localization* in N. Popescu's book (1971). The notion of *complete ring of quotients* for a commutative ring is introduced in Lambek's book (1966). This localization is relative to the *dense ideals* and is a special case of localization ring. Schmid define in 1980, the concept of *maximal lattice of quotients* for a distributive lattice using partial morphisms introduced by Findlay and Lambek (1966). The *multipliers* (defined for a distributive lattice by W. H. Cornish in 1974 and 1980) plays an important role in this constructions.

Basic (Fuzzy) logic (*BL* from now on) is the many-valued residuated logic introduced by Hájek in 1998 to cope with the logic of continuous t-norms and their residua. *Monoidal logic* (*ML* from now on), introduced by Hőhle (1995), is a logic whose algebraic counterpart is the class of residuated lattices; MTL algebras (see Esteva & Godo, 2001) are algebraic structures for the Esteva-Godo monoidal t-norm based logic (MTL), a many-valued propositional calculus that formalizes the structure of the real unit interval $[0, 1]$, induced by a left-continuous t-norm. MTL algebras were independently introduced in Flondor, Georgescu, and Iorgulescu (2001) under the name *weak-BL algebras*. The results obtained in this paper for MTL algebras are analogously to the ones obtained for *BL* algebras in Buşneag and Piciu (2005). The main difference is that the equation $x \odot (x \to y) = x \wedge y$ is not valid for MTL algebras.

This paper is organized as follows: Section 2 is dedicated to basic definitions and rules of calculus in MTL algebras. In Section 3 we introduce the notion of *multiplier* for a MTL algebra. In the proof of Lemma 9 and Lemma 10 we have used mainly the rules c_{13} and c_{16} which are specific for MTL algebras (by Proposition 4 and Corollary 5). This explain why in this paper we have considered the particular case of MTL algebras and not the general case of residuated lattice.

In Section 4 we introduce the notions of *MTL algebra of fractions* and *maximal MTL algebra of quotients* for a MTL algebra. In Theorem 30 we prove the existence of the maximal MTL algebra of quotients for a MTL algebra.

This paper is a very important step in a future study of localization in the category of MTL algebras (and more general, in the category of residuated lattices).

For a survey relative to notion of fractions and localization in algebra of logic see Rudeanu (2010).

2. Definitions and First Properties

In this section we review the basic results relative to MTL algebras with more details and examples.

Definition 1 An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ equipped with an order $\leq$ is a residuated lattice (Blyth & Janovitz, 1972; Galatos, Jipsen, Kowalski, & Ono, 2007; Turunen, 1999), if it satisfies:

(LR_1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice relative to order $\leq$;

(LR_2) $(L, \odot, 1)$ is an ordered commutative monoid;

(LR_3) $(\odot, \rightarrow)$ is an adjoint pair ($z \leq x \rightarrow y$ iff $x \odot z \leq y$ for every $x, y, z \in L$).

For examples of residuated lattices see Buşneag and Piciu (2006), Galatos et al. (2007), and Turunen (1999).

In this section by L we denote the universe of a residuated lattice. We denote $x^* = x \rightarrow 0$ and $x^{**} = (x^*)^*$, for $x \in L$.

We review some rules of calculus for residuated lattices:

Theorem 1 (Buşneag & Piciu, 2006; Galatos et al., 2007) *Let* $x, y, z \in L$. *Then*:

(c_1) $x \rightarrow x = 1, 1 \rightarrow x = x, 0 \rightarrow x = 1, y \leq x \rightarrow y, x \odot (x \rightarrow y) \leq y, x \rightarrow 1 = 1, x \odot 0 = 0$;

(c_2) $x \leq y$ *iff* $x \rightarrow y = 1$;

(c_3) $x \leq y$ *implies* $x \odot z \leq y \odot z, z \rightarrow x \leq z \rightarrow y$ *and* $y \rightarrow z \leq x \rightarrow z$;

(c_4) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$, *so* $(x \odot y)^* = x \rightarrow y^* = y \rightarrow x^*$;

(c_5) $x \odot x^* = 0$ *and* $x \odot y = 0$ *iff* $x \leq y^*$;

(c_6) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;

(c_7) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

We shall denote $B(L) = \{x \in L$: x is a *complemented element* in $(L, \wedge, \vee, 0, 1)\}$, which is a Boolean algebra (called the *Boolean center* of L).

Theorem 2 (Buşneag & Piciu, 2006) *For* $a \in L$, $a \in B(L)$ *iff* $a \vee a^* = 1$.

Theorem 3 (Buşneag & Piciu, 2006; Galatos et al., 2007) *If* $a_1, a_2 \in B(L)$ *and* $x, y \in L$, *then:*

(c_8) $a_1 \odot x = a_1 \wedge x$;

(c_9) $x \odot (x \rightarrow a_1) = a_1 \wedge x, a_1 \odot (a_1 \rightarrow x) = a_1 \wedge x$;

(c_{10}) $a_1 \odot (x \rightarrow y) = a_1 \odot [(a_1 \odot x) \rightarrow (a_1 \odot y)]$;

(c_{11}) $x \odot (a_1 \rightarrow a_2) = x \odot [(x \odot a_1) \rightarrow (x \odot a_2)]$.

Definition 2 (Esteva & Godo, 2001) A MTL algebra is a residuated lattice satisfying the preliniarity equation:

(c_{12}) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

The variety of MTL algebras will be denoted by $\mathcal{MTL}$.

Example 1 (Iorgulescu, 2004) Let $L = \{0, a, b, c, d, 1\}$, with $0 < a, b < c < 1, 0 < b < d < 1$, but a, b and, respective c, d are incomparable. Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is an MTL algebra, where the operations $\odot$ and $\rightarrow$ are defined as follows:

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

Proposition 4 (Esteva & Godo, 2001) *Let L be a residuated lattice. The following conditions are equivalent*:

(i) $L \in \mathcal{MTL}$;

(ii) *L is a subdirect product of linearly ordered residuated lattices;*

(iii) (c_{13}) $x \to (y \vee z) = (x \to y) \vee (x \to z)$, *for any* $x, y, z \in L$;

(iv) (c_{14}) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$, for any $x, y, z \in L$.

Corollary 5 (Esteva & Godo, 2001; Flondor, Georgescu, & Iorgulescu, 2001) *Let* $L \in \mathcal{MTL}$. *For every* $x, y, z \in L$:

(c_{15}) $(x \wedge y)^* = x^* \vee y^*$;

(c_{16}) $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$;

(c_{17}) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

(c_{18}) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$.

Remark 1 A MTL algebra L is a BL algebra iff in L is verified the divisibility condition: $x \odot (x \to y) = x \wedge y$. So, BL algebras are examples of MTL algebras; for an example of MTL algebra which is not BL algebra see Turunen (1999, p. 16). Also, every linearly ordered residuated lattice is a MTL algebra.

3. Multipliers on a MTL Algebra

By L we denote the universe of a MTL algebra.

Let $\mathcal{I}_d(L) = \{I\colon\ I$ is an *ideal* in the lattice $(L, \wedge, \vee, 0, 1)\}$ (see Balbes & Dwinger, 1974) and $\mathcal{I}(L)$ the set of all decreasing subsets of L. We have that, $\mathcal{I}(L) \subseteq \mathcal{I}_d(L)$ and if $J_1, J_2 \in \mathcal{I}(L)$, then $J_1 \cap J_2 \in \mathcal{I}(L)$. Also, if $J \in \mathcal{I}(L)$, then $0 \in J$.

Definition 3 A map $p : J \to L$, with $J \in \mathcal{I}(L)$, is a partial multiplier on L if it verifies the axioms:

(M_1) $p(a \odot x) = a \odot p(x)$, $a \in B(L), x \in J$;

(M_2) $x \odot (x \to p(x)) = p(x)$, $x \in J$;

(M_3) *If* $a \in B(L) \cap J$, *then* $p(a) \in B(L)$;

(M_4) $x \wedge p(a) = a \wedge p(x)$, $a \in B(L) \cap J, x \in J$.

Remark 2 Since $x \odot (x \to p(x)) \leq x$, from (M_2) we conclude that $p(x) \leq x$, for $x \in J$.

Remark 3 We use *multiplier* instead *partial multiplier.*

By $d(p) \in \mathcal{I}(L)$ we denote the domain of p; we call p *total* if $d(p) = L$.

Example 2 Let $a \in B(L)$ and $J \in \mathcal{I}(L)$. Then the map p_a: $J \to L$, $p_a(x) = a \wedge x \overset{(c_8)}{=} a \odot x$, for every $x \in J$ is a multiplier on L. We called this multiplier *principal*.

The axioms $(M_1), (M_3)$ and (M_4) are verified as in the case of BL algebras (see Buşneag & Piciu, 2005). Also, for $x \in J$, $x \odot (x \to p_a(x)) = x \odot (x \to (a \wedge x)) \overset{(c_7)}{=} x \odot [(x \to a) \wedge (x \to x)] = x \odot (x \to a) \overset{(c_9)}{=} a \wedge x = p_a(x)$, hence (M_2) is verified.

We denote p_a by $\overline{p_a}$ if $d(p_a) = L$. In particular, for $a = 0, 1$ the maps $\overline{p_0} = \mathbf{0}$: $L \to L$, $\overline{p_0}(x) = \mathbf{0}(x) = 0$, for every $x \in L$ and $\overline{p_1} = \mathbf{1}$: $L \to L, \overline{p_1}(x) = \mathbf{1}(x) = x$, for every $x \in L$ are total multipliers on L.

Remark 4 From (M_4), if $J = L$, then for $a = 1$ we deduce that $x \wedge p(1) = p(x)$, so every total multiplier is principal.

For $a \in L$ and $J = (a] = \{x \in L\colon\ x \leq a\} \in \mathcal{I}(L)$ we consider the map g_a: $J \to L, g_a(x) = a \odot (a \to x)$ for every $x \in J$.

Lemma 6 *g_a verify $(M_1), (M_3)$ and (M_4).*

Proof. (M_1). For $x \in J$ and $e \in B(L) \cap J$ (hence $x \leq a, e \in B(L)$ and $e \leq a$) we have: $g_a(e \odot x) = a \odot (a \to (e \odot x)) \overset{(c_8)}{=} a \odot (a \to (e \wedge x)) \overset{(c_7)}{=} a \odot [(a \to e) \wedge (a \to x)] \overset{(c_{16})}{=} [a \odot (a \to e)] \wedge [a \odot (a \to x)] = (a \wedge e) \wedge g_a(x) = e \wedge g_a(x) = e \odot g_a(x)$.

(M_3). If $e \in B(L) \cap J$, then $e \in B(L)$ and $e \leq a$, hence $g_a(e) = a \odot (a \to e) = a \wedge e = e \in B(L)$.

(M_4). Consider $x \in J$ and $e \in B(L) \cap J$ (that is, $x, e \leq a$ and $e \in B(L)$). Thus, $e \wedge g_a(x) = e \wedge [a \odot (a \to x)] = e \odot a \odot (a \to x) = (e \wedge a) \odot (a \to x) = e \odot (a \to x)$ and $x \wedge g_a(e) = x \wedge [a \odot (a \to e)] = x \wedge (a \wedge e) = x \wedge e = e \odot x$. Since $x \leq a \to x$, then $e \odot x \leq e \odot (a \to x)$, hence $x \wedge g_a(e) \leq e \wedge g_a(x)$.

From $e \leq a$ we deduce that $a \to x \leq e \to x$ hence $e \odot (a \to x) \leq x$. Then $e \odot (a \to x) \leq e \wedge x = e \odot x$, hence $e \wedge$

$g_a(x) = x \wedge g_a(e)$. □

Following Lemma 6, we can obtain an example of multiplier which is not principal.

For this, we consider $L = [0, 1]$ (see Turunen, 1999, p. 16) and for all $x, y \in L$ we define

$$x \odot y = 0, \text{ if } x + y \leq \frac{1}{2} \text{ and } x \odot y = x \wedge y, \text{ if } x + y > \frac{1}{2}$$

$$x \to y = 1 \text{ if } x \leq y \text{ and } x \to y = \max\{\frac{1}{2} - x, y\} \text{ if } x > y.$$

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a $MTL-$algebra. Obviously, L is not a $BL-$algebra and $B(L) = \{0, 1\}$.

Lemma 7 *$g_{\frac{1}{3}}: (\frac{1}{3}] = [0, \frac{1}{3}] \to L = [0, 1], g_{\frac{1}{3}}(x) = \frac{1}{3} \odot (\frac{1}{3} \to x)$ for every $0 \leq x \leq \frac{1}{3}$ is a multiplier on $L = [0, 1]$ which is not principal.*

Proof. Following Lemma 6, it is suffice to prove that $g_{\frac{1}{3}}$ verify (M_2), that is, $x \odot (x \to g_{\frac{1}{3}}(x)) = g_{\frac{1}{3}}(x)$, for every $0 \leq x \leq \frac{1}{3}$.

For $0 \leq x \leq \frac{1}{3}$ we have $\frac{1}{3} \to x = \max\{\frac{1}{2} - \frac{1}{3}, x\} = \max\{\frac{1}{6}, x\} = \frac{1}{6}$ if $0 \leq x \leq \frac{1}{6}$ and $\frac{1}{3} \to x = x$ if $\frac{1}{6} < x \leq \frac{1}{3}$, so $g_{\frac{1}{3}}(x) = \frac{1}{3} \odot \frac{1}{6} = 0$ for $0 \leq x \leq \frac{1}{6}$ and $g_{\frac{1}{3}}(x) = \frac{1}{3} \odot x$ for $\frac{1}{6} < x \leq \frac{1}{3}$.

Since for $x > \frac{1}{6}, \frac{1}{3} + x > \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$ we deduce that $g_{\frac{1}{3}}(x) = \frac{1}{3} \odot x = \frac{1}{3} \wedge x = x$, so $g_{\frac{1}{3}}(x) = 0$, for $0 \leq x \leq \frac{1}{6}$ and $g_{\frac{1}{3}}(x) = x$, for $\frac{1}{6} < x \leq \frac{1}{3}$.

Then $x \to g_{\frac{1}{3}}(x) = x \to 0 = \max\{\frac{1}{2} - x, 0\} = \frac{1}{2} - x$ for $0 < x \leq \frac{1}{6}$ and $x \to g_{\frac{1}{3}}(x) = x \to x = 1$, for $\frac{1}{6} < x \leq \frac{1}{3}$. For $x = 0, 0 \to g_{\frac{1}{3}}(0) = 1$.

So $x \odot (x \to g_{\frac{1}{3}}(x)) = x \odot (\frac{1}{2} - x) = 0$, for $0 \leq x \leq \frac{1}{6}$ and $x \odot (x \to g_{\frac{1}{3}}(x)) = x \odot 1 = x$, for $\frac{1}{6} < x \leq \frac{1}{3}$. For $x = 0, 0 \odot (0 \to g_{\frac{1}{3}}(0)) = 0$.

We deduce that $x \odot (x \to g_{\frac{1}{3}}(x)) = g_{\frac{1}{3}}(x)$, for every $0 \leq x \leq \frac{1}{3}$, that is $g_{\frac{1}{3}}$ verify (M_2), hence $g_{\frac{1}{3}}$ is a multiplier on $L = [0, 1]$. It is easy to prove that $B(L) = \{0, 1\}$, so if suppose by contrary that $g_{\frac{1}{3}}$ is principal, then $g_{\frac{1}{3}} = p_0$ or $g_{\frac{1}{3}} = p_1$ (with p_0, p_1: $[0, \frac{1}{3}] \to [0, 1]$). Since $g_{\frac{1}{3}}(\frac{1}{3}) = \frac{1}{3} \odot (\frac{1}{3} \to \frac{1}{3}) = \frac{1}{3} \odot 1 = \frac{1}{3}$ and $p_0(\frac{1}{3}) = 0$ it follows that $g_{\frac{1}{3}} \neq p_0$.

Also, $g_{\frac{1}{3}}(\frac{1}{6}) = \frac{1}{3} \odot (\frac{1}{3} \to \frac{1}{6}) = \frac{1}{3} \odot 0 = 0$ and $p_1(\frac{1}{6}) = \frac{1}{6}$, so $g_{\frac{1}{3}} \neq p_1$. □

For $J \in \mathcal{I}(L)$, let $M(J, L) = \{p\colon J \to L \mid p$ is a multiplier on $L\}$, $M(L) = \bigcup\{M(J, L)\colon J \in \mathcal{I}(L)\}$ and $\overline{M}(L) = \{p\colon L \to L \mid p$ is a multiplier on $L\}$.

Proposition 8 *If $J_1, J_2 \in \mathcal{I}(L)$ and $p_i \in M(J_i, L), i = 1, 2,$ then*

(c_{19}) $p_1(t) \odot [t \to p_2(t)] = p_2(t) \odot [t \to p_1(t)]$, *for every $t \in J_1 \cap J_2$.*

Proof. For $t \in J_1 \cap J_2$ we have $p_1(t) \odot [t \to p_2(t)] \overset{(M_2)}{=} t \odot (t \to p_1(t)) \odot (t \to p_2(t)) = [t \odot (t \to p_2(t))] \odot (t \to p_1(t)) \overset{(M_2)}{=} p_2(t) \odot [t \to p_1(t)]$. □

Definition 4 For $J_1, J_2 \in \mathcal{I}(L)$ and $p_i \in M(J_i, L), i = 1, 2$, we define $p_1 \wedge p_2$, $p_1 \vee p_2$, $p_1 \otimes p_2$, $p_1 \rightsquigarrow p_2\colon J_1 \cap J_2 \to L$ by $(p_1 \wedge p_2)(t) = p_1(t) \wedge p_2(t), (p_1 \vee p_2)(t) = p_1(t) \vee p_2(t), (p_1 \otimes p_2)(t) = p_1(t) \odot [t \to p_2(t)] \overset{(c_{19})}{=} p_2(t) \odot [t \to p_1(t)], (p_1 \rightsquigarrow p_2)(t) = t \odot [p_1(t) \to p_2(t)]$, for every $t \in J_1 \cap J_2$.

Lemma 9 *$p_1 \wedge p_2 \in M(J_1 \cap J_2, L)$.*

Proof. It is sufficient to verify only (M_2) (for $(M_1), (M_3)$ and (M_4) see Buşneag & Piciu, 2005).

For any $t \in J_1 \cap J_2$ we have $t \odot [t \to (p_1 \wedge p_2)(t)] = t \odot [t \to (p_1(t) \wedge p_2(t))] \overset{(c_7)}{=} t \odot [(t \to p_1(t)) \wedge (t \to p_2(t))] \overset{(c_{16})}{=} [t \odot (t \to p_1(t))] \wedge [t \odot (t \to p_2(t))] \overset{(M_2)}{=} p_1(t) \wedge p_2(t) = (p_1 \wedge p_2)(t)$. □

Lemma 10 *$p_1 \vee p_2 \in M(J_1 \cap J_2, L)$.*

Proof. The axioms $(M_1), (M_3)$ and (M_4) are verified as in the case of BL algebras (see Buşneag & Piciu, 2005). To verify (M_2), let $t \in J_1 \cap J_2$. Then $t \odot [t \to (p_1 \vee p_2)(t)] = t \odot [t \to (p_1(t) \vee p_2(t))] \overset{(c_{13})}{=} t \odot [(t \to p_1(t)) \vee (t \to p_2(t))] \overset{(c_6)}{=} [t \odot (t \to p_1(t))] \vee [t \odot (t \to p_2(t))] \overset{(M_2)}{=} p_1(t) \vee p_2(t) = (p_1 \vee p_2)(t)$. □

Lemma 11 $p_1 \otimes p_2 \in M(J_1 \cap J_2, L)$.

Proof. (M_1) is verified as in the case of BL algebras (see Buşneag, & Piciu, 2005), using (c_{10}). To prove (M_2), let $t \in J_1 \cap J_2$ and denote $p = p_1 \otimes p_2$.

To prove the equality $t \odot (t \to p(t)) = p(t)$ it is sufficient to prove that $p(t) \leq t \odot (t \to p(t))$. We have $p(t) = p_1(t) \odot (t \to p_2(t)) = t \odot (t \to p_1(t)) \odot (t \to p_2(t))$ and $t \odot (t \to p(t)) = t \odot [t \to (p_1(t) \odot (t \to p_2(t)))] = t \odot [t \to (t \odot (t \to p_1(t)) \odot (t \to p_2(t)))]$. So, to prove that $p(t) \leq t \odot (t \to p(t))$ it is sufficient to prove that $t \odot (t \to p_1(t)) \odot (t \to p_2(t)) \leq t \odot [t \to (t \odot (t \to p_1(t)) \odot (t \to p_2(t)))]$, that is, $\varphi \leq t \to (t \odot \varphi)$ (with $\varphi \overset{not}{=} (t \to p_1(t)) \odot (t \to p_2(t))$), which is true, since $\varphi \to [t \to (t \odot \varphi)] \overset{(c_4)}{=} (\varphi \odot t) \to (t \odot \varphi) = 1$. (M_3) and (M_4) are verified as in the case of BL algebras (see Buşneag & Piciu, 2005), using (c_{10}) and (c_{11}). □

Lemma 12 $p_1 \rightsquigarrow p_2 \in M(J_1 \cap J_2, L)$.

Proof. (M_1) is verified as in the case of BL algebras (see Buşneag & Piciu, 2005) using (c_{10}). To prove (M_2), let $t \in J_1 \cap J_2$ and denote $p = p_1 \rightsquigarrow p_2 : J_1 \cap J_2 \to L$; then $p(t) = t \odot [p_1(t) \to p_2(t)]$. We have $p_1(t) \to p_2(t) \leq t \to [t \odot (p_1(t) \to p_2(t))]$, hence $t \odot [p_1(t) \to p_2(t)] \leq t \odot [t \to (t \odot (p_1(t) \to p_2(t)))] \Leftrightarrow p(t) \leq t \odot [t \to p(t)] \Leftrightarrow p(t) = t \odot [t \to p(t)]$.

(M_3) and (M_4) are verified as in the case of BL algebras (see Buşneag, & Piciu, 2005) using (c_{10}) and (c_{11}). □

Proposition 13

(i) For every $p \in M(L)$, $p \otimes \mathbf{1} = \mathbf{1} \otimes p = p$;

(ii) For every $p_1, p_2, p_3 \in M(L)$, $p_1 \otimes (p_2 \otimes p_3) = (p_1 \otimes p_2) \otimes p_3$ and for every $t \in d(p_1) \cap d(p_2) \cap d(p_3)$, $p_1(t) \leq (p_2 \rightsquigarrow p_3)(t)$ iff $(p_1 \otimes p_2)(t) \leq p_3(t)$;

(iii) For every $p_1, p_2 \in M(L)$ and $t \in d(p_1) \cap d(p_2)$, $(p_1 \rightsquigarrow p_2)(t) \vee (p_2 \rightsquigarrow p_1)(t) = \mathbf{1}(t)$.

Proof. (i) If $J = dom(p)$ and $t \in J$, then $(p \otimes \mathbf{1})(t) = p(t) \odot (t \to \mathbf{1}(t)) = p(t) \odot (t \to t) = p(t) \odot 1 = p(t)$ and $(\mathbf{1} \otimes p)(t) = t \odot (t \to p(t)) = p(t)$, that is, $p \otimes \mathbf{1} = \mathbf{1} \otimes p = p$.

(ii) Let $p_i \in M(J_i, L)$ where $J_i \in \mathcal{I}(L)$, $i = 1, 2, 3$. Thus, for $t \in J_1 \cap J_2 \cap J_3$ we have $[p_1 \otimes (p_2 \otimes p_3)](t) = ((p_2 \otimes p_3)(t)) \odot (t \to p_1(t)) = [p_2(t) \odot (t \to p_3(t))] \odot (t \to p_1(t)) = p_2(t) \odot [(t \to p_3(t)) \odot (t \to p_1(t))] = = p_2(t) \odot [(t \to p_1(t)) \odot (t \to p_3(t))] = [p_2(t) \odot (t \to p_1(t))] \odot (t \to p_3(t)) = ((p_1 \otimes p_2)(t)) \odot (t \to p_3(t)) = [(p_1 \otimes p_2) \otimes p_3](t)$, that is the operation $\otimes$ is associative.

For $t \in J_1 \cap J_2 \cap J_3$ we have $p_1(t) \leq (p_2 \rightsquigarrow p_3)(t) \Leftrightarrow p_1(t) \leq t \odot [p_2(t) \to p_3(t)]$. So, by (c_3), $p_1(t) \odot [t \to p_2(t)] \leq t \odot (t \to p_2(t)) \odot (p_2(t) \to p_3(t)) \overset{(M_2)}{\Leftrightarrow} p_1(t) \odot [t \to p_2(t)] \leq p_2(t) \odot (p_2(t) \to p_3(t)) \leq p_3(t) \Leftrightarrow (p_1 \otimes p_2)(t) \leq p_3(t)$, for any $t \in J_1 \cap J_2 \cap J_3$, that is, $p_1 \otimes p_2 \leq p_3$. Conversely, if $(p_1 \otimes p_2)(t) \leq p_3(t)$ we have $p_2(t) \odot [t \to p_1(t)] \leq p_3(t)$, for any $t \in J_1 \cap J_2 \cap J_3$. Obviously, $t \to p_1(t) \leq p_2(t) \to p_3(t) \overset{(c_3)}{\Leftrightarrow} t \odot (t \to p_1(t)) \leq t \odot (p_2(t) \to p_3(t)) \Leftrightarrow p_1(t) \leq (p_2 \rightsquigarrow p_3)(t)$.

(iii) We have $(p_1 \rightsquigarrow p_2)(t) \vee (p_2 \rightsquigarrow p_1)(t) = [t \odot (p_1(t) \to p_2(t))] \vee [t \odot (p_2(t) \to p_1(t))] = t \odot [(p_1(t) \to p_2(t)) \vee (p_2(t) \to p_1(t))] = t \odot 1 = t = \mathbf{1}(t)$. □

Corollary 14 $(\overline{M}(L), \wedge, \vee, \otimes, \rightsquigarrow, \mathbf{0}, \mathbf{1})$ *is a MTL algebra.*

Definition 5 (Esteva & Godo, 2001; Freytes, 2004) A MTL algebra L is called

(i) an $IMTL$ algebra (involutive algebra) if it satisfies the equation

(I) $x^{**} = x$;

(ii) a $SMTL$ algebra if it satisfies the equation

(S) $x \wedge x^* = 0$;

(iii) a WNM algebra (weak nilpotent minimum) if it satisfies the equation

(W) $(x \odot y)^* \vee [(x \wedge y) \to (x \odot y)] = 1$;

(iv) a $\Pi SMTL$ algebra if it is a $SMTL$ algebra satisfying the equation

(Π) $[z^{**} \odot ((x \odot z) \to (y \odot z))] \to (x \to y) = 1$.

Theorem 15

(i) *If L is a BL algebra, then for every $f_1, f_2 \in M(L)$, $(f_1 \otimes (f_1 \rightsquigarrow f_2))(x) = (f_1 \wedge f_2)(x)$, for every $x \in d(f_1) \cap d(f_2)$;*

(ii) *If L is an $IMTL$ algebra, then $f^{**} = f$, for every $f \in M(L)$;*

(iii) *If L is a $SMTL$ algebra, then for every $f \in \overline{M}(L)$, $f \wedge f^* = \mathbf{0}$;*

(iv) *If L is a WNM algebra, then for every $f_1, f_2 \in \overline{M}(L)$, $(f_1 \otimes f_2)^* \vee ((f_1 \wedge f_2) \rightsquigarrow (f_1 \otimes f_2)) = \mathbf{1}$;*

(v) *If L is a $\Pi SMTL$ algebra, then for every $f, g, h \in \overline{M}(L)$, $[h^{**} \otimes ((f \otimes h) \rightsquigarrow (g \otimes h))] \rightsquigarrow (f \rightsquigarrow g) = \mathbf{1}$.*

Proof. (i) Suppose L is a BL algebra (see Remark 1). Let $f_1, f_2 \in M(L)$, $f_1\colon J_1 \to L$, $f_2\colon J_2 \to L$, with $J_1, J_2 \in \mathcal{I}(L)$. For every $x \in J_1 \cap J_2$ we have $(f_1 \otimes (f_1 \rightsquigarrow f_2))(x) = (f_1 \wedge f_2)(x) \Leftrightarrow (f_1 \rightsquigarrow f_2)(x) \odot [x \to f_1(x)] = f_1(x) \wedge f_2(x)$ $\Leftrightarrow x \odot [f_1(x) \to f_2(x)] \odot [x \to f_1(x)] = f_1(x) \wedge f_2(x) \Leftrightarrow (x \odot [x \to f_1(x)]) \odot [f_1(x) \to f_2(x)] = f_1(x) \wedge f_2(x) \overset{(M_2)}{\Leftrightarrow}$ $f_1(x) \odot [f_1(x) \to f_2(x)] = f_1(x) \wedge f_2(x)$, which is true because L is supposed a BL algebra.

(ii) Suppose L is an $IMTL$ algebra. For $f \in M(L), f\colon J \to L$ and $x \in J$, we have $f^{**} = (f \rightsquigarrow \mathbf{0}) \rightsquigarrow \mathbf{0}$ and $f^{**}(x) = x \odot [x \odot f^*(x)]^* \overset{(c_4)}{=} x \odot [x \to (f(x))^{**}] \overset{(I)}{=} x \odot [x \to f(x)] \overset{(M_2)}{=} f(x)$, hence $f^{**} = f$.

(iii) Suppose L is a $SMTL$ algebra. If $f \in \overline{M}(L), f\colon L \to L$, then the equation $f \wedge f^* = \mathbf{0}$ is equivalent with $f \wedge (f \rightsquigarrow \mathbf{0}) = \mathbf{0} \Leftrightarrow f(x) \wedge [x \odot (f(x))^*] = 0$, for every $x \in L$, which is clearly (since $f(x) \wedge [x \odot (f(x))^*] \leq f(x) \wedge (f(x))^* = 0$), hence $f \wedge f^* = \mathbf{0}$.

(iv) Suppose L is a WNM algebra. Let $f_1, f_2 \in \overline{M}(L)$, $f_1, f_2\colon L \to L$ and $x \in L$. We denote $a = f_1(x), b = f_2(x)$. We have $((f_1 \otimes f_2)^* \vee ((f_1 \wedge f_2) \rightsquigarrow (f_1 \otimes f_2)))(x) = ((f_1 \otimes f_2)^*(x)) \vee (x \odot ((f_1 \wedge f_2)(x) \to (f_1 \otimes f_2)(x))) = (x \odot (a \odot (x \to b))^*) \vee (x \odot ((a \wedge b) \to (a \odot (x \to b)))) \overset{(c_6)}{=} x \odot ((a \odot (x \to b))^* \vee ((a \wedge b) \to (a \odot (x \to b))))$.

Since $b \leq x \to b$ we deduce that $a \wedge b \leq a \wedge (x \to b)$, hence (using (c_3)) $(a \wedge (x \to b)) \to (a \odot (x \to b)) \leq (a \wedge b) \to (a \odot (x \to b))$.

Since L is supposed a WNM−algebra we obtain $1 = (a \odot (x \to b))^* \vee ((a \wedge (x \to b)) \to (a \odot (x \to b))) \leq (a \odot (x \to b))^* \vee ((a \wedge b) \to (a \odot (x \to b)))$, hence $(a \odot (x \to b))^* \vee ((a \wedge b) \to (a \odot (x \to b))) = 1$. Then $((f_1 \otimes f_2)^* \vee ((f_1 \wedge f_2) \rightsquigarrow (f_1 \otimes f_2)))(x) = x \odot 1 = x = \mathbf{1}(x) \Leftrightarrow (f_1 \otimes f_2)^* \vee ((f_1 \wedge f_2) \rightsquigarrow (f_1 \otimes f_2)) = \mathbf{1}$.

(v) Suppose now L is a $\Pi SMTL$ algebra. From the condition $x \wedge x^* = 0$ $(x \in L)$, we deduce that $x^* \vee x^{**} \overset{(c_{15})}{=} (x \wedge x^*)^* = 0^* = 1$, that is, $x^* \in B(L)$. For $f, g, h\colon L \to L$, and $x \in L$ we denote $a_1 = f(x), a_2 = g(x)$ and $a_3 = h(x)$. Then $h^{**}(x) = x \odot (x \to a_3^{**}) \overset{(c_9)}{=} x \wedge a_3^{**} \overset{(c_8)}{=} x \odot a_3^{**}$, $[h^{**} \otimes ((f \otimes h) \rightsquigarrow (g \otimes h))](x) = [x \to h^{**}(x)] \odot [x \odot [(f \otimes h)(x) \to (g \otimes h)(x)]] = [x \to (x \odot a_3^{**})] \odot [x \odot [((x \to a_1) \odot a_3) \to ((x \to a_2) \odot a_3)]] =$ $[x \odot (x \to (x \odot a_3^{**}))] \odot [((x \to a_1) \odot a_3) \to ((x \to a_2) \odot a_3)] \overset{(c_1)}{\leq} (x \odot a_3^{**}) \odot [((x \to a_1) \odot a_3) \to ((x \to a_2) \odot a_3)] =$ $x \odot [a_3^{**} \odot [((x \to a_1) \odot a_3) \to ((x \to a_2) \odot a_3)]] \overset{(\Pi)}{\leq} x \odot [(x \to a_1) \to (x \to a_2)] = x \odot [(x \odot (x \to a_1)) \to a_2] \overset{(M_2)}{=}$ $x \odot (a_1 \to a_2) = (f \rightsquigarrow g)(x)$, hence $[h^{**} \otimes ((f \otimes h) \rightsquigarrow (g \otimes h))] \rightsquigarrow (f \rightsquigarrow g) = \mathbf{1}$. □

Corollary 16 *If L is a BL algebra (resp. an $IMTL$ algebra, a $SMTL$ algebra, a WNM algebra, a $\Pi SMTL$ algebra) then $\overline{M}(L)$ is a BL algebra (resp. an $IMTL$ algebra, a $SMTL$ algebra, a WNM algebra, a $\Pi SMTL$ algebra).*

Using the rules $(c_6), (c_{10})$ and (c_{11}) we obtain:

Lemma 17 *$v_L : B(L) \to \overline{M}(L)$, $v_L(a) = \overline{p_a}$ for every $a \in B(L)$, is a monomorphism of MTL algebras.*

Definition 6 A subset $J \subseteq L$ is called regular if for every $x, y \in L$ such that $x \wedge f = y \wedge f$ for every $f \in B(L) \cap J$, then $x = y$.

Denote $\mathrm{Re}\, g(L)$ the set of all regular subset of L.

Example 3 We give an example of non-trivial regular subset in a MTL algebra. Consider $L = \{0, a, b, c, d, 1\}$ the MTL-algebra from Example 1. We have that $B(L) = \{0, a, d, 1\}$ and if consider $J = \{0, b, d, a\}$, then $J \in \mathcal{I}(L)$. It is easy to prove that for any $x, y \in L$ with $x \neq y$, there is $f \in J \cap B(L) = \{0, a, d\}$ such that $x \wedge f \neq y \wedge f$, that is, $J \in Reg(L)$.

Remark 5 The condition $J \in Reg(L)$ is equivalent with: if $x, y \in L$ and $p_{x|J \cap B(L)} = p_{y|J \cap B(L)}$, then $x = y$.

Lemma 18 *If $J_1, J_2 \in \mathcal{I}(L) \cap Reg(L)$, then $J_1 \cap J_2 \in \mathcal{I}(L) \cap Reg(L)$.*

Denote $M_{reg}(L) = \{p \in M(L)\colon d(p) \in \mathcal{I}(L) \cap Reg(L)\}$.

Remark 6 By Lemmas 9-12 and 18 we deduce that if $p_1, p_2 \in M_{reg}(L)$, then $p_1 \otimes p_2, p_1 \rightsquigarrow p_2 \in M_{reg}(L)$.

Proposition 19 *Let $p\colon J \to L$ be a multiplier on L with $J \in \mathcal{I}(L) \cap Reg(L)$. Then $(p \vee p^*)(x) = x$, for every $x \in J$.*

Proof. Let $a \in B(L) \cap J$ and $x \in J$. Then

$$\begin{aligned}
a \wedge [p \vee p^*](x) &= a \wedge [p(x) \vee (x \odot (p(x))^*)] = [a \wedge p(x)] \vee [a \wedge (x \odot (p(x))^*)] \\
&= [x \odot p(a)] \vee [x \odot a \odot (p(x))^*] \overset{c_{10}}{=} [x \odot p(a)] \vee [x \odot a \odot (a \odot p(x))^*] \\
&\overset{M_4}{=} [x \odot p(a)] \vee [x \odot a \odot (x \odot p(a))^*] = [x \odot p(a)] \vee [x \odot a \odot (x \wedge p(a))^*] \\
&\overset{c_{15}}{=} [x \odot p(a)] \vee [x \odot a \odot (x^* \vee (p(a))^*)] \overset{c_6}{=} [x \odot p(a)] \vee [a \odot ((x \odot x^*) \vee (x \odot (p(a))^*))] \\
&= [x \odot p(a)] \vee [a \odot (0 \vee (x \odot (p(a))^*))] = [x \odot p(a)] \vee [a \odot x \odot (p(a))^*] \\
&= [x \odot p(a)] \vee [x \odot (a \odot (p(a))^*)] \overset{c_6}{=} x \odot [p(a) \vee (a \odot (p(a))^*)] \\
&= x \odot [p(a) \vee (a \wedge (p(a))^*)] = x \odot [(p(a) \vee a) \wedge (p(a) \vee (p(a))^*)] \\
&= x \odot (a \wedge 1) = x \odot a = x \wedge a,
\end{aligned}$$

so $(p \vee p^*)(x) = x$, since $J \in Reg(L)$. □

Definition 7 Let two multipliers p_1, p_2 on L. We say that p_2 extends p_1 if $d(p_1) \subseteq d(p_2)$ and $p_{2|d(p_1)} = p_1$; if p_2 extends p_1, we write $p_1 \sqsubseteq p_2$. If we can not be extended a multiplier p to a strictly larger domain, we called p maximal.

Lemma 20

(i) If $p_1, p_2 \in M(L)$, $p \in M_{reg}(L)$ and $p \sqsubseteq p_1, p \sqsubseteq p_2$, then p_1 and p_2 coincide on the $d(p_1) \cap d(p_2)$;

(ii) any $p \in M_r(L)$ can be extended to a maximal multiplier. For any principal multiplier p_a, $a \in B(L)$, $d(p_a) \in \mathcal{I}(L) \cap Reg(L)$ there is an uniquely total multiplier $\overline{p_a}$ such that $p_a \sqsubseteq \overline{p_a}$ and for any non-principal multiplier p there is a maximal non-principal multiplier r such that $p \sqsubseteq r$.

On $M_{reg}(L)$ we consider the relation ρ_L defined by $(p_1, p_2) \in \rho_L$ iff $p_{1|d(p_1)\cap d(p_2)} = p_{2|d(p_1)\cap d(p_2)}$.

Lemma 21 ρ_L is an equivalence relation on $M_{reg}(L)$ compatible with $\wedge, \vee, \otimes$ and $\rightsquigarrow$.

Proof. Obviously, ρ_L is an equivalence relation on $M_{reg}(L)$ compatible with $\wedge$ and $\vee$.

For the compatibility of ρ_L with $\otimes$ and $\rightsquigarrow$ on $M_{reg}(L)$, let $(p_1, p_2), (r_1, r_2) \in \rho_L$.

Let $t \in d(p_1) \cap d(p_2) \cap d(r_1) \cap d(r_2)$. We have $p_1(t) = p_2(t)$ and $r_1(t) = r_2(t)$, so

$$(p_1 \otimes r_1)(t) = p_1(t) \odot (t \to r_1(t)) = p_2(t) \odot (t \to r_2(t)) = (p_2 \otimes r_2)(t),$$

$$(p_1 \rightsquigarrow r_1)(t) = t \odot [p_1(t) \to r_1(t)] = t \odot [p_2(t) \to r_2(t)] = (p_2 \rightsquigarrow r_2)(t),$$

that is, $(p_1 \otimes r_1, p_2 \otimes r_2), (p_1 \rightsquigarrow r_1, p_2 \rightsquigarrow r_2) \in \rho_L$. □

For $p \in M_{reg}(L)$ with $J = d(p) \in \mathcal{I}(L) \cap Reg(L)$, we denote by $[p, J]$ the congruence class of p modulo ρ_L and $L'' = M_{reg}(L)/\rho_L$.

On L'' we define the order relation $[p_1, J_1] \leq [p_2, J_2]$ iff $p_1(x) \leq p_2(x)$, for every $x \in J_1 \cap J_2$.

It is a routine to prove the following result:

Lemma 22 $(L'', \leq)$ *is a bounded lattice, where for* $[p_1, J_1], [p_2, J_2] \in L''$, $[p_1, J_1] \wedge [p_2, J_2] = [p_1 \wedge p_2, J_1 \cap J_2]$ *and* $[p_1, J_1] \vee [p_2, J_2] = [p_1 \vee p_2, J_1 \cap J_2]$, $\mathbf{0} = [\mathbf{0}, L]$, $\mathbf{1} = [\mathbf{1}, L]$.

For $[p_1, J_1], [p_2, J_2] \in L''$, we define $[p_1, J_1] \otimes [p_2, J_2] = [p_1 \otimes p_2, J_1 \cap J_2]$ and $[p_1, J_1] \rightsquigarrow [p_2, J_2] = [p_1 \rightsquigarrow p_2, J_1 \cap J_2]$ (where $[p_1 \otimes p_2$ and $p_1 \rightsquigarrow p_2$ are defined in Definition 4).

Proposition 23 $(L'', \wedge, \vee, \otimes, \rightsquigarrow, \mathbf{0}, \mathbf{1})$ *is a MTL-algebra.*

Proof. We verify the axioms of MTL-algebras.

(LR_1) Follows from Lemma 22;

(LR_2) Follows from Proposition 13, (i), (ii);

(LR_3) Follows from Proposition 13, (ii);

The preliniarity equation (c_{12}) follows from Proposition 13, (iii). □

Remark 7 From Theorem 2, Propositions 19 and 23 we deduce that L'' is a Boolean algebra.

Remark 8 If consider $\mathcal{F} = \mathcal{I}(L) \cap \operatorname{Re} g(L)$ and the partially ordered systems $\{\delta_{I,J}\}_{I,J\in\mathcal{F},I\subseteq J}$ (for $I, J \in \mathcal{F}$, $I \subseteq J, \delta_{I,J}$: $M(J, L) \to M(I, L)$ is defined by $\delta_{I,J}(f) = f_{|I}$), then $L'' = \lim_{\overrightarrow{I\in\mathcal{F}}} M(I, L)$.

Lemma 24 *If consider $\overline{v_L} : B(L) \to L''$ defined by $\overline{v_L}(e) = [\overline{p_e}, L]$ for any $e \in B(L)$, then:*

(i) $\overline{v_L}$ is a monomorphism of Boolean algebras;

(ii) $\overline{v_L}(B(L)) \in \operatorname{Re} g(L'')$.

Proof. (i) See Lemma 17.

(ii) To prove $\overline{v_L}(B(L)) \in \operatorname{Re} g(L'')$, we suppose by contrary that there exist $p_1, p_2 \in M_{reg}(L)$ such that $[p_1, d(p_1)] \neq [p_2, d(p_2)]$ (hence we have $a \in d(p_1)\cap d(p_2)$ such that $p_1(a) \neq p_2(a)$) and $[p_1, d(p_1)]\wedge[\overline{p_e}, L] = [p_2, d(p_2)]\wedge[\overline{p_e}, L]$ for any $[\overline{p_e}, L] \in \overline{v_A}(B(L)) \cap B(L'') \Rightarrow p_1(x) \wedge e \wedge x = p_2(x) \wedge e \wedge x$ for any $x \in d(p_1) \cap d(p_2)$ and any $e \in B(L)$. For $e = 1$ and $x = a$ we deduce that $p_1(a) \wedge a = p_2(a) \wedge a \Leftrightarrow p_1(a) = p_2(a)$, a contradiction. □

Remark 9 Following Lemma 20 we can identify $[\overline{p_e}, L]$ with $\overline{p_e}$, for every $e \in B(L)$. So, the boolean elements can be identified with the elements of $\{\overline{p_e}$: $e \in B(L)\}$.

Following the above consideration we deduce, as in the case of BL-algebras (see Buşneag & Piciu, 2005), that:

Lemma 25 *If $[p, d(p)] \in L''$ (with $p \in M_{reg}(L)$ and $J = d(p) \in \mathcal{I}(L) \cap \operatorname{Re} g(L)$), then $J \cap B(L) \subseteq \{e \in B(L)$: $\overline{p_e} \wedge [p, d(p)] \in B(L)\}$.*

4. Maximal MTL Algebra of Quotients

In this section by L we denote a MTL-algebra.

Definition 8 A MTL algebra G is called MTL algebra of fractions of L if:

(Fr_1) $B(L)$ is a MTL subalgebra of G;

(Fr_2) For every $f, g, h \in G, f \neq g$, there is $e \in B(L)$ such that $e \wedge f \neq e \wedge g$ and $e \wedge h \in B(L)$.

We write $L \sqsubseteq G$ if G is a MTL algebra of fractions of L.

Definition 9 $Q(L)$ is the maximal MTL algebra of quotients of L if $L \sqsubseteq Q(L)$ and for any MTL algebra G with $L \sqsubseteq G$ there is a injective morphism of MTL algebras j: $G \to Q(L)$.

Proposition 26 *If L is a $MTL-$ algebra and $L \sqsubseteq G$, then G is a Boolean algebra.*

Proof. Indeed, if suppose that G is not a Boolean algebra, by Theorem 2, there is $f \in G$ such that $f \vee f^* \neq 1$. Since $L \sqsubseteq G$, then there is a boolean element g such that $g \wedge f$ is boolean and $g \wedge (f \vee f^*) \neq g$. Since $g \wedge f \in B(L)$, then $(g \wedge f) \vee (g \wedge f)^* = 1 \Rightarrow (g \wedge f) \vee (g^* \vee f^*) = 1 \Rightarrow [(g \wedge f) \vee g^*] \vee f^* = 1 \Rightarrow [(g \vee g^*) \wedge (f \vee g^*)] \vee f^* = 1 \Rightarrow [1 \wedge (f \vee g^*)] \vee f^* = 1 \Rightarrow (f \vee f^*) \vee g^* = 1$. By the unicity of the complement of g, we deduce that $f \vee f^* = g$. Then from $g \wedge (f \vee f^*) \neq g$ we obtain $g \wedge g \neq g \Rightarrow g \neq g$, a contradiction. Hence G is a Boolean algebra. □

Corollary 27 *$Q(L)$ is a Boolean algebra.*

As in Buşneag and Piciu (2005), we have:

Remark 10 If L is a Boolean algebra, obviously, $B(L) = L$. By Proposition 26, $Q(L)$ is a Boolean algebra; the axioms M_1, M_2, M_3 are equivalent with M_4 and $Q(L)$ is just Dedekind-MacNeille completion of L (Schmid, 1980).

Lemma 28 *Let $L \sqsubseteq G$; then for every $f, g \in G, f \neq g, h_1, ..., h_n \in G$, there exists a boolean element e such that $e \wedge f \neq e \wedge g$ and $e \wedge h_i \in B(L)$ for $i = 1, 2, ..., n$ $(n \geq 2)$.*

Lemma 29 *Let $L \sqsubseteq G$ and $g \in G$. Then $I_g = \{e \in B(L)$: $e \wedge g \in B(L)\} \in \mathcal{I}(B(L)) \cap M_{reg}(L)$.*

Theorem 30 *L'' (defined in Section 3) is the maximal (boolean) MTL algebra $Q(L)$ of quotients of L.*

Proof. From Lemma 24, (i), $B(L)$ is a MTL subalgebra of L''. Consider $[f_1, d(f_1)], [f_2, d(f_2)], [f_3, d(f_3)] \in L''$, $f_1, f_2, f_3 \in M_{reg}(L)$ such that $[f_2, d(f_2)] \neq [f_3, d(f_3)]$. Then we have $x' \in d(f_2) \cap d(f_3)$ with $f_2(x') \neq f_3(x')$.

Consider $J = d(f_1) \in \mathcal{I}(L) \cap Reg(L)$ and $J_{[f_1,d(f_1)]} = \{a \in B(L) : \overline{p_a} \wedge [f_1, d(f_1)] \in B(L)\}$. From Lemma 25, $J \cap B(L) \subseteq I_{[f_1,d(f_1)]}$. If suppose that for any $a \in J \cap B(L)$, $\overline{p_a} \wedge [f_2, d(f_2)] = \overline{p_a} \wedge [f_3, d(f_3)]$, then $[\overline{p_a} \wedge f_2, d(f_2)] =$

$[\overline{p_a} \wedge f_3, d(f_3)]$, so for any $x \in d(f_2) \cap d(f_3)$ we have $(\overline{p_a} \wedge f_2)(x) = (\overline{p_a} \wedge f_3)(x)$ i.e. $a \wedge f_2(x) = a \wedge f_3(x)$. Because $J \in \mathrm{Re}\, g(L)$, $f_2(x) = f_3(x)$ for any $x \in d(f_2) \cap d(f_3)$ so $[f_2, d(f_2)] = [f_3, d(f_3)]$, a contradiction.

If $[f_2, d(f_2)] \neq [f_3, d(f_3)]$, then there is $a \in J \cap B(L)$, such that $\overline{p_a} \wedge [f_2, d(f_2)] \neq \overline{p_a} \wedge [f_3, d(f_3)]$.

Since by Lemma 25, $J \cap B(L) \subseteq J_{[f_1,d(f_1)]}$for this $a \in J \cap B(L)$ we have $\overline{p_a} \wedge [f_1, d(f_1)] \in B(L)$.

Now, consider G a MTL algebra such that $L \sqsubseteq G$; obviously, $B(L) \subseteq B(G)$

$$\begin{array}{ccc} L & \sqsubseteq & G \\ & & \swarrow j \\ & L'' & \end{array}$$

By Lemma 29, For $a' \in G, J_{a'} = \{e \in B(L)\colon e \wedge a' \in B(L)\} \in \mathcal{I}(B(L)) \cap Reg(L)$.

$p_{a'}\colon J_{a'} \to L$, $p_{a'}(x) = x \wedge a'$ is a multiplier. Indeed, (M_1) and (M_2) are verified, because if $e \in B(L)$ and $x \in J_{a'}$, then $p_{a'}(e \odot x) = (e \odot x) \wedge a' = (e \wedge x) \wedge a' = e \wedge (x \wedge a') = e \odot (x \wedge a') = e \odot p_{a'}(x)$, and $x \odot (x \to p_{a'}(x)) = x \odot [x \to (x \wedge a')] \overset{(c_9)}{=} x \wedge (x \wedge a') = x \wedge a' = p_{a'}(x)$. To verify (M_3), let $e \in J_{a'} \cap B(L) = J_{a'}$. Thus, $p_{a'}(e) = e \wedge a' \in B(L)$ (since $e \in J_{a'}$). The condition (M_4) is obviously verified, hence $[p_{a'}, J_{a'}] \in L''$.

We define $j\colon G \to L''$, by $j(a') = [p_{a'}, J_{a'}]$, for every $a' \in G$. Obviously, $j(0) = \mathbf{0}$. For $a', b' \in G$ and $x \in J_{a'} \cap J_{b'}$, $(j(a') \otimes j(b'))(x) = (a' \wedge x) \odot [x \to (b' \wedge x)] = (a' \odot x) \odot [x \to (b' \wedge x)] = a' \odot [x \odot (x \to (b' \wedge x))] = a' \odot [x \wedge (b' \wedge x)] = a' \odot (b' \wedge x) = a' \odot (b' \odot x) = (a' \odot b') \odot x = (a' \odot b') \wedge x = j(a' \odot b')(x)$, hence $j(a') \otimes j(b') = j(a' \odot b')$ and $(j(a') \rightsquigarrow j(b'))(x) = x \odot [j(a')(x) \to j(b')(x)] = x \odot [(a' \wedge x) \to (b' \wedge x)] = x \odot [(x \odot a') \to (x \odot b')] \overset{(c_{10})}{=} x \odot (a' \to b') = x \wedge (a' \to b') = j(a' \to b')(x)$, hence $j(a') \rightsquigarrow j(b') = j(a' \to b')$.

Now, let $a', b' \in G$ such that $j(a') = j(b')$. It follows that $[p_{a'}, J_{a'}] = [p_{b'}, J_{b'}]$, so $p_{a'}(x) = p_{b'}(x)$ for any $x \in J_{a'} \cap J_{b'}$. So $a' \wedge x = b' \wedge x$ for any $x \in J_{a'} \cap J_{b'}$. By Lemma 28, if $a' \neq b'$, since $L \sqsubseteq G$, there is a boolean element e such that $e \wedge a', e \wedge b' \in B(A)$ and $e \wedge a' \neq e \wedge b'$ which is contradictory (since $e \wedge a', e \wedge b' \in B(L)$ implies $e \in J_{a'} \cap J_{b'}$). □

Proposition 31 *Let L be a MTL-algebra. The following are equivalent:*

(i) Every maximal multiplier on L has domain L;

(ii) For every multiplier $p \in M(J, L)$ there is $e \in B(L)$ such that $p = p_e$, (i.e., $p(x) = e \wedge x$ for any $x \in J$);

(iii) $Q(L) \approx B(L)$.

Proof. $(i) \Rightarrow (ii)$ Assume (i) and for $p \in M(J, L)$ let p' its the maximal extension (by Lemma 20). By (i), we have $p'\colon L \to L$. Put $e = p'(1) \in B(L)$ (by M_3), then for every $x \in J$, $p(x) = p(x) \wedge 1 \overset{M_4}{=} x \wedge p(1) = x \wedge e = p_e(x)$, that is $p = p_e$.

$(ii) \Rightarrow (iii)$ Follow from Lemma 24.

$(iii) \Rightarrow (i)$ Follow from Lemma 20 and Lemma 24. □

Remark 11

1) If L is a MTL algebra with $B(L) = L_2$ and $L \sqsubseteq G$ then $G = \{0, 1\}$, hence $Q(L) = L'' \approx L_2$. Indeed, if $a_1, a_2, a_3 \in G, a_1 \neq a_2$, then there exists $e \in B(L)$ (by (Fr_2)) such that $e \wedge a_1 \neq e \wedge a_2$ (hence $e \neq 0$) and $e \wedge a_3 \in B(L)$. Clearly, $e = 1$, hence $a_3 \in B(L)$, that is, $G = B(L)$.

2) More general, if L is a $MTL-$algebra such that $B(L)$ is finite and $L \sqsubseteq G$ then $G = B(L)$, hence $Q(L) = B(L)$. Indeed, since $L \sqsubseteq G$, we have $B(L) \subseteq G$. Let $a \in G$. Then there is $e \in B(L)$ such that $e \wedge a \in B(L)$. $Q(L)$ is finite, so, there is a largest element $e_a \in Q(L)$ with $e_a \wedge a \in B(L)$. Suppose $e_a \vee a \neq e_a$. Then there is $e \in B(L)$ such that $e \wedge (e_a \vee a) \neq e \wedge e_a$ and $e \wedge a \in B(L)$. Because $e \wedge a \in B(L)$ we deduce $e \leq e_a$ so $e = e \wedge (e_a \vee a) \neq e \wedge e_a = e$, a contradiction. Hence $e_a \vee a = e_a$, so $a \leq e_a$, consequently $a = a \wedge e_a \in Q(L)$, that is $G \subseteq Q(L)$. Then $G = Q(L)$, hence $Q(L) \approx B(L)$.

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