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On n-Paranormal Operators

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Abstract

A Hilbert space operator T is called n-paranormal and $*$-n-paranormal if $\|Tx\|^n \leq \|T^n x\| \cdot \|x\|^{n-1}$ and $\|T^* x\|^n \leq \|T^n x\| \cdot \|x\|^{n-1}$, respectively. Let $\mathfrak{P}(n)$ and $\mathfrak{S}(n)$ be the sets of all n-paranormal operators and $*$-n-paranormal operators, respectively. In this paper we study and discuss the relationship between these two sets of operators and especially show $\bigcap_{n=3}^{\infty} \mathfrak{P}(n) = \mathfrak{P}(3) \bigcap \mathfrak{P}(4)$. Finally we introduce $*$-n-paranormality for an operator on a Banach space and give some spectral properties.

Keywords: Hilbert space, n-paranormal operator, $*$-paranormal operator

1991 Mathematics Subject Classification: 47A10, 47B20

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is called n-paranormal if $\|Tx\|^n \leq \|T^n x\| \cdot \|x\|^{n-1}$ for all $x \in \mathcal{H}$. If an operator T satisfies the inequality for $n = 2$, then T is called paranormal. Paranormal operators are normaloid, i.e., $\|T\| = r(T)$ (the spectral radius of T) and if T is an invertible paranormal operator, then T^{-1} is also paranormal (cf. § 2.6 of Furuta, 2001). Ando in 1972 gave the useful characterization of a paranormal operator by some norm condition. Arun in (1976) introduced n-paranormal operators.

An operator $T \in B(\mathcal{H})$ is called $*$-paranormal if $\|T^* x\|^2 \leq \|T^2 x\| \cdot \|x\|$ for all $x \in \mathcal{H}$. Arora and Thukral in 1986 showed that $*$-paranormal operators are 3-paranormal and normaloid.

Uchiyama and Tanahashi in (preprint) studied spectral properties of $*$-paranormal operators and n-paranormal operators and presented an example of an invertible $*$-paranormal operator T such that T^{-1} is not normaloid. A $*$-paranormal operator is 3-paranormal by Arora and Thukral in 1986, which means that there exists an invertible 3-paranormal operator T such that T^{-1} is not 3-paranormal. Also Uchiyama and Tanahashi showed that Weyl's theorem holds for n-paranormal operators for every $n \geq 3$. Chō, Ôta, Tanahashi, and Uchiyama in 2012 showed $\|T^{-1}\| \leq \|T\| \cdot r(T^{-1})^2$ for an invertible $*$-paranormal operator T.

In this paper we study the relationship between two classes of n-paranormal operators and $*$-n-paranormal operators on a Hilbert space and discuss some spectral properties of $*$-n-paranormal operators. Finally we introduce a notion of $*$-n-paranormality for an operator on a Banach space and present some spectral properties.

2. Examples of n-Paranormal Operators

Definition 1 Let $n \geq 2$. An operator $T \in B(\mathcal{H})$ is said to be *n-paranormal* if

$$\|Tx\|^n \leq \|T^n x\| \cdot \|x\|^{n-1} \quad (\forall x \in \mathcal{H}).$$

We denote the set of all n-paranormal operators by $\mathfrak{P}(n)$.

Lemma. *Let $\mathcal{H} = \ell^2$ with the usual orthogonal base $\{e_k\}_{k\in\mathbb{Z}}$ and let $T \in B(\mathcal{H})$ be the bilateral weighted shift with bounded weights $\{w_k\}_{k\in\mathbb{Z}}$. Then $T \in \mathfrak{P}(n)$ if and only if $w_k^{n-1} \leq w_{k+1} \cdots w_{k+(n-1)}$ for all $k \in \mathbb{Z}$.*

Proof. Let $x \in \ell^2$ be $x = \sum x_k e_k$ such that $\|x\| = 1$. Suppose $w_k^{n-1} \le w_{k+1} \cdots w_{k+(n-1)}$ for all $k \in \mathbb{Z}$. Since $w_k > 0$,

$$w_k \le \Big(w_k \cdot w_{k+1} \cdots w_{k+(n-1)}\Big)^{\frac{1}{n}} \quad \text{for all } k \in \mathbb{Z}.$$

Since $Tx = \sum x_k \cdot w_k e_{k+1}$,

$$\|Tx\|^{2n} = \Big(\sum |x_k|^2 \cdot w_k^2\Big)^n.$$

Since $T^n x = \sum x_k \cdot w_k \cdot w_{k+1} \cdots w_{k+(n-1)} e_{k+n}$,

$$\|T^n x\|^2 = \sum |x_k|^2 \cdot w_k^2 \cdot w_{k+1}^2 \cdots w_{k+(n-1)}^2.$$

Hence, by $w_k^2 \le \Big(w_k^2 \cdot w_{k+1}^2 \cdots w_{k+(n-1)}^2\Big)^{\frac{1}{n}}$

$$\begin{aligned}\|Tx\|^{2n} = \Big(\sum |x_k|^2 w_k^2\Big)^n &\le \Big(\sum |x_k|^2 (w_k^2 \cdot w_{k+1}^2 \cdots w_{k+(n-1)}^2)^{\frac{1}{n}}\Big)^n \\ &\le \sum |x_k|^2 w_k^2 \cdot w_{k+1}^2 \cdots w_{k+(n-1)}^2 = \|T^n x\|^2\end{aligned}$$

by Jensen's inequality for $f(x) = x^n$ on $x > 0$.

The converse is clear. □

Let T be the unilateral weighted shift with positive weights $\{w_k\}_{k=1}^{\infty}$. Then it is clear that, by the similar way to Lemma,

(1) $T \in \mathfrak{P}(3) \iff w_k^2 \le w_{k+1} \cdot w_{k+2}$ $(\forall k \in \mathbb{N})$,

(2) $T \in \mathfrak{P}(4) \iff w_k^3 \le w_{k+1} \cdot w_{k+2} \cdot w_{k+3}$ $(\forall k \in \mathbb{N})$.

Example 1 (*i*) Let T be the unilateral weighted shift with weights $\{w_k\}_{n=1}^{\infty}$ such that

$$w_k = \begin{cases} 1 & (k = 1) \\ 2 & (k = 2) \\ 1 & (k = 3) \\ 4 & (k = 4) \\ 1 & (k = 5) \\ 16 & (k \ge 6). \end{cases}$$

Then $T \in \mathfrak{P}(3)$ and $T \notin \mathfrak{P}(4)$.

(*ii*) Let T be the unilateral weighted shift with weights $\{w_k\}_{k=1}^{\infty}$ such that

$$w_k = \begin{cases} 1 & (k = 1) \\ 2 & (k = 2) \\ 3 & (k = 3, 4) \\ 2 & (k = 5) \\ 5 & (k \ge 6). \end{cases}$$

Then $T \in \mathfrak{P}(4)$ and $T \notin \mathfrak{P}(3)$.

Example 2 Let T be the unilateral weighted shift with weights $\{w_k\}_{k=1}^{\infty}$ such that

$$w_k = \begin{cases} 1 & (k = 1) \\ \sqrt{2} & (k = 2) \\ 4 & (k = 3) \\ \sqrt{2} & (k = 4) \\ 16 & (k \ge 5). \end{cases}$$

Then $T \in \mathfrak{P}(3) \bigcap \mathfrak{P}(4)$ and $T \notin \mathfrak{P}(2)$.

Let T be the bilateral weighted shift with positive bounded weights $\{w_k\}_{k\in\mathbb{Z}}$ such that $\inf w_k > 0$. Then T is invertible by Proposition II 6.8 of Conway (1985) and the following equivalent relations hold:

(3) $T, T^{-1} \in \mathfrak{P}(3) \iff w_k^2 \le w_{k+1}\cdot w_{k+2} \le w_{k+3}^2$ $(\forall k \in \mathbb{Z})$,

(4) $T, T^{-1} \in \mathfrak{P}(4) \iff w_k^3 \le w_{k+1}\cdot w_{k+2}\cdot w_{k+3} \le w_{k+4}^3$ $(\forall k \in \mathbb{Z})$.

Moreover, implication (3) $\Longrightarrow$ (4) holds. In fact, statement (3) implies $w_k \le w_{k+3}$ for every $k \in \mathbb{Z}$. Hence

$$w_k^3 \le w_k\cdot w_{k+1}\cdot w_{k+2} \le w_{k+3}\cdot w_{k+1}\cdot w_{k+2} = w_{k+1}\cdot w_{k+2}\cdot w_{k+3} \le w_{k+4}\cdot w_{k+4}^2 = w_{k+4}^3.$$

Example 3 Let T be the bilateral weighted shift with weights $\{w_k\}_{k\in\mathbb{Z}}$ such that

$$w_k = \begin{cases} 1 & (k \le 1) \\ \sqrt{2} & (k = 2) \\ 2 & (k = 3) \\ \sqrt{3} & (k = 4) \\ 4 & (k \ge 5). \end{cases}$$

Then $T \in \mathfrak{P}(3)$ and by Proposition II 6.8 of Conway (1985) T is invertible. Since weights $\{w_k\}$ are not monotone increasing, $T \notin \mathfrak{P}(2)$.

3. n-Paranormal Operators

First we give the following.

Theorem 1 *Let T be in $B(\mathcal{H})$. If T belongs to $\mathfrak{P}(2)$, then T belongs to $\mathfrak{P}(n)$ for all $n \ge 3$.*

Proof. Suppose $T \in \mathfrak{P}(2)$. Since $\|Tx\|^4 \le \|T^2x\|^2\cdot\|x\|^2 \le \|T^3x\|\cdot\|Tx\|\cdot\|x\|^2$, it holds $T \in \mathfrak{P}(3)$.

We next assume $T \in \mathfrak{P}(n)$. Then since $\|Tx\|^n \le \|T^nx\|\cdot\|x\|^{n-1}$, it holds $\|T^2x\|^n \le \|T^{n+1}x\|\cdot\|Tx\|^{n-1}$. Therefore

$$\|Tx\|^{2n} \le \|T^2x\|^n\cdot\|x\|^n \le \|T^{n+1}x\|\cdot\|Tx\|^{n-1}\cdot\|x\|^n.$$

So we have $T \in \mathfrak{P}(n+1)$. Thus by induction, the proof is complete. □

Theorem 2 *Let T be in $B(\mathcal{H})$. If T belongs to $\mathfrak{P}(3)\bigcap\mathfrak{P}(4)$, then T belongs to $\mathfrak{P}(5)$.*

Proof. Let $T \in \mathfrak{P}(3)\cap\mathfrak{P}(4)$. Since $T \in \mathfrak{P}(3)$, it holds $\|T^3x\|^3 \le \|T^5x\|\cdot\|T^2x\|^2$. Hence $\|T^3x\|^6 \le \|T^5x\|^2\cdot\|T^2x\|^4$. Next since $T \in \mathfrak{P}(4)$, we have $\|T^2x\|^4 \le \|T^5x\|\cdot\|Tx\|^3$. Therefore

$$\|T^3x\|^6 \le \|T^5x\|^3\cdot\|Tx\|^3 \quad \text{that is} \quad \|T^3x\|^2 \le \|T^5x\|\cdot\|Tx\|.$$

Since $T \in \mathfrak{P}(3)$, it holds

$$\|Tx\|^6 \le \|T^3x\|^2\cdot\|x\|^4 \le \|T^5x\|\cdot\|Tx\|\cdot\|x\|^4,$$

that is, $T \in \mathfrak{P}(5)$. □

Theorem 3 *Let T be in $B(\mathcal{H})$. If T belongs to $\mathfrak{P}(3)\bigcap\mathfrak{P}(4)$, then T belongs to $\mathfrak{P}(n)$ for all $n \ge 5$.*

Proof. Let $T \in \mathfrak{P}(3)\bigcap\mathfrak{P}(4)$. We show the theorem by induction $T \in \mathfrak{P}(k)$ $(k = 3, 4, ..., n)$.

Since $T \in \mathfrak{P}(n-1)$, it holds $\|T^3x\|^{n-1} \le \|T^{n+1}x\|\cdot\|T^2x\|^{n-2}$.

By $T \in \mathfrak{P}(n)$, we have $\|T^2x\|^n \le \|T^{n+1}x\|\cdot\|Tx\|^{n-1}$.

By $T \in \mathfrak{P}(3)$, it holds $\|Tx\|^{3(n-1)} \le \|T^3x\|^{n-1}\cdot\|x\|^{2(n-1)}$.

Therefore

$$\begin{aligned} \|Tx\|^{3(n-1)} &\le \|T^3x\|^{n-1}\cdot\|x\|^{2(n-1)} \le \|T^{n+1}x\|\cdot\|T^2x\|^{n-2}\cdot\|x\|^{2(n-1)} \\ &\le \|T^{n+1}x\|\cdot(\|T^{n+1}x\|^{\frac{1}{n}}\cdot\|Tx\|^{\frac{n-1}{n}})^{n-2}\cdot\|x\|^{2(n-1)} \\ &= \|T^{n+1}x\|^{\frac{2(n-1)}{n}}\cdot\|Tx\|^{\frac{(n-1)(n-2)}{n}}\cdot\|x\|^{2(n-1)}. \end{aligned}$$

Hence we have

$$\|Tx\|^{n+1} \le \|T^{n+1}x\|\cdot\|x\|^n.$$

Thus $T \in \mathfrak{P}(n+1)$. Hence, by induction with Theorem 2 the proof is complete. □

The following corollary is the direct consequence.

Corollary 1 *In $B(\mathcal{H})$, it holds* $\mathfrak{P}(2) \subset \bigcap_{n=3}^{\infty} \mathfrak{P}(n) = \mathfrak{P}(3) \bigcap \mathfrak{P}(4)$.

It should be remarked that the above inclusion is proper by Example 2. Moreover, we have following

Theorem 4 *In $B(\mathcal{H})$, it holds* $\mathfrak{P}(3) \bigcap \mathfrak{P}(4) \subset \{T\colon T^2 \in \mathfrak{P}(2)\}$.

Proof. Let $T \in \mathfrak{P}(3) \cap \mathfrak{P}(4)$. Since $||Tx||^3 \leq ||T^3x|| \cdot ||x||^2,\ ||Tx||^4 \leq ||T^4x|| \cdot ||x||^3$, we have

$$||T^2x||^3 \leq ||T^4x|| \cdot ||Tx||^2,$$

hence

$$||T^2x||^6 \leq ||T^4x||^2 \cdot ||Tx||^4 \leq ||T^4x||^3 \cdot ||x||^3.$$

That is, $||T^2x||^2 \leq ||T^4x|| \cdot ||x||$ and $T^2 \in \mathfrak{P}(2)$. □

We proved following result for the weighted shift operator by Example 2.

Theorem 5 *Let T be in $B(\mathcal{H})$. If T and T^{-1} belong to $\mathfrak{P}(3)$, then T belongs to $\mathfrak{P}(4)$.*

Proof. Since $||T^{-1}x||^3 \leq ||T^{-3}x|| \cdot ||x||^2$ for every $x \in \mathcal{H}$, it holds

$$||T^3x||^3 \leq ||Tx|| \cdot ||T^4x||^2.$$

Since $T \in \mathfrak{P}(3)$, it holds

$$||Tx||^9 \leq ||T^3x||^3 \cdot ||x||^6.$$

Hence it holds

$$||Tx||^9 \leq ||Tx|| \cdot ||T^4x||^2 \cdot ||x||^6.$$

Therefore, we have

$$||Tx||^4 \leq ||T^4x|| \cdot ||x||^3.$$

□

Remark These results hold for Banach space operators.

4. $*$-n-Paranormal Operators

Next we study $*$-n-paranormal operators.

Definition 2 Let $n \geq 2$. An operator $T \in B(\mathcal{H})$ is said to be *$*$-n-paranormal* if

$$||T^*x||^n \leq ||T^nx|| \cdot ||x||^{n-1} \quad (\forall x \in \mathcal{H}).$$

In particular, in case that $n = 2$, T is called $*$-paranormal. We denote the set of all $*$-n-paranormal operators by $\mathfrak{S}(n)$.

It is well known the following result.

Theorem A (Arora and Thukral, 1986) *In $B(\mathcal{H})$, it holds* $\mathfrak{S}(2) \subset \mathfrak{P}(3)$.

Related to the above, we have

Theorem 6 *In $B(\mathcal{H})$, it holds $\mathfrak{S}(n) \subset \mathfrak{P}(n+1)$ for all $n \geq 2$.*

Proof. By the definition it holds $||T^*x||^n \leq ||T^nx|| \cdot ||x||^{n-1}$. Therefore

$$||T^*Tx||^n \leq ||T^{n+1}x|| \cdot ||Tx||^{n-1}$$

and

$$||Tx||^{2n} \leq ||T^*Tx||^n \cdot ||x||^n \leq ||T^{n+1}x|| \cdot ||Tx||^{n-1} \cdot ||x||^n.$$

Hence $T \in \mathfrak{P}(n+1)$. □

Hence we have

Theorem 7 *In $B(\mathcal{H})$, it holds* $\bigcap_{n=2}^{\infty} \mathfrak{S}(n) \subset \mathfrak{P}(3) \bigcap \mathfrak{P}(4)$.

5. Spectral Properties of $*$-n-Paranormal Operators

Theorem 8 *Let T be in $B(\mathcal{H})$. If T belongs to $\mathfrak{S}(n)$ and $\mathcal{M}$ is an invariant subspace for T, then $T\big|_{\mathcal{M}}$ belongs to $\mathfrak{S}(n)$.*

Proof. Let P be the orthogonal projection onto $\mathcal{M}$. Then $TP = PTP$, so that

$$(T\big|_{\mathcal{M}})^* = PT^*P.$$

Hence, for $x \in \mathcal{M}$ we have

$$\|(T\big|_{\mathcal{M}})^* x\|^n = \|PT^*x\|^n \le \|T^*x\|^n \le \|T^n x\| \cdot \|x\|^{n-1} = \|(T\big|_{\mathcal{M}})^n x\| \cdot \|x\|^{n-1}.$$

Thus $T\big|_{\mathcal{M}} \in \mathfrak{S}(n)$. □

Theorem 9 *For $T \in B(\mathcal{H})$, let T belong to $\mathfrak{S}(n)$ and z be an eigen-value of T. If $(T-z)x = 0$, then $(T-z)^*x = 0$.*

Proof. We may assume $x \ne 0$ and $\|x\| = 1$. Then

$$\|T^*x\|^n \le \|T^n x\| \cdot \|x\|^{n-1} = |z|^n.$$

Hence $\|T^*x\| \le |z|$ and

$$0 \le \|(T-z)^*x\|^2 = \|T^*x\|^2 - 2\mathrm{Re}\,(T^*x, \bar{z}x) + |z|^2 \le 2|z|^2 - 2|z|^2 = 0.$$

Hence $(T-z)^*x = 0$. □

Therefore we have the following corollary.

Corollary 2 *For $T \in B(\mathcal{H})$, let T belong to $\mathfrak{S}(n)$ and z, w be distinct eigen-values of T. If x and y are corresponding eigen-vectors of z and w, respectively, then $(x, y) = 0$.*

We denote the approximate point spectrum of T by $\sigma_a(T)$.

Corollary 3 *For $T \in B(\mathcal{H})$, let T belong to $\mathfrak{S}(n)$.*

(1) If $z \in \sigma_a(T)$ and $\|(T-z)x_n\| \to 0$ for unit vectors x_n, then $\|(T-z)^*x_n\| \to 0$.

(2) Let z and w $(z \ne w)$ be in $\sigma_a(T)$. If $\|(T-z)x_n\| \to 0$ and $\|(T-w)y_n\| \to 0$ for unit vectors x_n, y_n, then $(x_n, y_n) \to 0$.

Proof is direct from above results.

If $T \in \mathfrak{P}(n)$, then T is normaloid and Weyl's Theorem holds for T. Hence if $T \in \mathfrak{S}(n)$ then T is normaloid and Weyl's Theorem holds for T, i.e., it holds $w(T) = \sigma(T) \setminus \pi_{00}(T)$, where $\sigma(T)$, $w(T)$ and $\pi_{00}(T)$ are the spectrum, Weyl spectrum and the set of all isolated eigen-values with finite multiplicity of T, respectively (see Conway, 1985, p. 49). If $T \in \mathfrak{S}(n)$, then $T \in \mathfrak{P}(n+1)$ by Theorem 6. Hence, results of operators of $\mathfrak{P}(n+1)$ hold for operators of $\mathfrak{S}(n)$. For example, let $T \in \mathfrak{S}(n)$, and if λ is an isolated point of $\sigma(T)$ and D is a domain of $\mathbb{C}$ such that $\lambda \in D^\circ$ (the interior of D) and $D \cap \sigma(T) = \{\lambda\}$, then $E = \dfrac{1}{2\pi i}\displaystyle\int_{\partial D} (T-z)^{-1} dz$ is an orthogonal projection and satisfies $E\mathcal{H} = \ker(T-\lambda)$ (cf. Uchiyama & Tanahashi, preprint).

Theorem 10 *For $T, S \in B(\mathcal{H})$, if T and S belong to $\mathfrak{P}(n)$, then $T \otimes S$ belongs to $\mathfrak{P}(n)$.*

Proof.

$$\begin{aligned}
\|(T\otimes S)(x\otimes y)\|^n &= \|Tx \otimes Sy\|^n = \|Tx\|^n \cdot \|Sy\|^n \\
&\le \|T^n x\| \cdot \|x\|^{n-1} \cdot \|S^n y\| \cdot \|y\|^{n-1} \\
&= \|(T^n \otimes S^n)(x\otimes y)\| \cdot \|x \otimes y\|^{n-1} \\
&= \|(T\otimes S)^n (x\otimes y)\| \cdot \|x\otimes y\|^{n-1}.
\end{aligned}$$

□

By Theorems 1 and 3 we have following corollaries.

Corollary 4 *For $T, S \in B(\mathcal{H})$, if T belongs to $\mathfrak{P}(2)$ and S belongs to $\mathfrak{P}(n)$, then $T \otimes S$ belongs to $\mathfrak{P}(n)$.*

Corollary 5 *For $T, S \in B(\mathcal{H})$, if T belongs to $\mathfrak{P}(3) \bigcap \mathfrak{P}(4)$ and S belongs to $\mathfrak{P}(n)$, then $T \otimes S$* belongs to $\mathfrak{P}(n)$ *for $n \geq 5$.*

Similarly, it holds

Theorem 11 *For $T, S \in B(\mathcal{H})$, if T and S belong to $\mathfrak{S}(n)$, then $T \otimes S$ belong to $\mathfrak{S}(n)$.*

6. Banach Space Operators

Finally, we introduce $*$-paranormal operators on Banach space. Let $\mathcal{X}$ be a complex Banach space and T be a bounded linear operator on $\mathcal{X}$. We define the subset $\Pi(\mathcal{X})$ of $\mathcal{X} \times \mathcal{X}^*$ by

$$\Pi(\mathcal{X}) = \{(x, f) \in \mathcal{X} \times \mathcal{X}^* : \|f\| = f(x) = \|x\| = 1\},$$

where $\mathcal{X}^*$ is the dual space of $\mathcal{X}$.

Definition 3 An operator $T \in B(\mathcal{X})$ is said to be *$*$-n-paranormal* if

$$\|T^* f\|^n \leq \|T^n x\| \quad (\forall (x, f) \in \Pi(\mathcal{X})),$$

where T^* is the dual operator of T.

We denote the same symbol $\mathfrak{S}(n)$ for the set of all $*$-n-paranormal operators on $\mathcal{X}$. Then we have following result.

Theorem 12 *Let T be a $*$-n-paranormal operator on $\mathcal{X}$. Then $\|Tx\|^{n+1} \leq \|T^{n+1}x\|$ for every unit vector x.*

Proof. For a unit vector x, let $(x, f) \in \Pi(\mathcal{X})$ and $Tx \neq 0$. Choose $g \in \mathcal{X}^*$ such that $\|g\| = g(\frac{Tx}{\|Tx\|}) = 1$. Hence since $(\frac{Tx}{\|Tx\|}, g) \in \Pi(\mathcal{X})$, it holds

$$\|Tx\|^n = (g(Tx))^n = \left((T^* g)(x)\right)^n \leq \|T^* g\|^n \cdot \|x\|^n \leq \|T^n(\frac{Tx}{\|Tx\|})\| \cdot \|x\|^n.$$

Therefore we have $\|Tx\|^{n+1} \leq \|T^{n+1}x\|$ for every unit vector x. □

By Theorem 12, for Banach space operators it holds $\mathfrak{S}(n) \subset \mathfrak{P}(n+1)$.

Definition 4 Let A and B be subspaces of $\mathcal{X}$. A is *orthogonal* to B (denoted $A \perp B$) if

$$\|a\| \leq \|a + b\| \quad (a \in A, \ b \in B).$$

Let $\ker(T)$ and $R(T)$ be the kernel and the range of $T \in B(\mathcal{X})$, respectively. We need following propositions:

Proposition 1 (Bonsall & Duncan, 1973, Lemma 20.2) *For an operator $T \in B(\mathcal{X})$, the implications $(iii) \Longrightarrow (ii) \Longleftrightarrow (i)$ hold between the statements*:

(i) $\ker(T^2) = \ker(T)$,

(ii) $\ker(T) \bigcap R(T) = \{0\}$,

(iii) $\ker(T) \perp R(T)$.

Proposition 2 (Bonsall & Duncan, 1973, Lemma 20.3) *For an operator $T \in B(\mathcal{X})$, the following statements are equivalent*:

(iii) $\ker(T) \perp R(T)$.

(iv) *If $Tx = 0$ for a unit vector x, then there exists $f \in \mathcal{X}^*$ such that $(x, f) \in \Pi(\mathcal{X})$ and $T^* f = 0$.*

If T is $*$-n-paranormal, then property (iv) holds for T. So we have following result.

Theorem 13 *Let T be a $*$-n-paranormal operator on $\mathcal{X}$. Then* $\ker(T) \perp R(T)$.

By Proposition 1 and Theorem 13 next theorem holds and shows that if T is $*$-n-paranormal, then $\mathrm{asc}(T) \leq 1$.

Theorem 14 *Let T be a $*$-n-paranormal operator on $\mathcal{X}$. If $T^2 x = 0$, then $Tx = 0$.*

Definition 5 (i) A normed space is said to be *strictly convex* if and only if x and y are linear dependent whenever

$$\|x + y\| = \|x\| + \|y\|.$$

(ii) A Banach space is said to be *uniformly convex* if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that if $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$, then

$$\|\frac{x + y}{2}\| \leq 1 - \delta.$$

It is well known that if $\mathcal{X}$ is uniformly convex, then $\mathcal{X}$ is strictly convex.

Theorem 15 *Let the dual space $\mathcal{X}^*$ of $\mathcal{X}$ be strictly convex and T be a $*$-n-paranormal operator on $\mathcal{X}$. If $Tx = zx$ for $z \in \mathbb{C}$ and $(x, f) \in \Pi(\mathcal{X})$, then $T^* f = zf$.*

Proof. Since T is $*$-n-paranormal, it holds

$$\|T^* f\|^n \leq \|T^n x\| = |z|^n \cdot \|x\| = |z|^n.$$

Hence we have $\|T^* f\| \leq |z|$. Therefore

$$2|z| \geq \|T^* f\| + \|zf\| \geq \|T^* f + zf\| \geq |(T^* f + zf)(x)| = 2|z|.$$

This shows that $\|T^* f + zf\| = \|T^* f\| + \|zf\|$, i.e., $T^* f$ and zf are linearly dependent. Since $\mathcal{X}^*$ is strictly convex, we have $T^* f = zf$. □

Theorem 16 *Let the dual space $\mathcal{X}^*$ of $\mathcal{X}$ be strictly convex and T be a $*$-n-paranormal operator on $\mathcal{X}$. If z is an eigen-value of T, then* $\ker(T - z) \perp R(T - z)$.

Proof. Let $x \in \ker(T - z)$. We may assume $\|x\| = 1$. Choose $f \in \mathcal{X}^*$ such that $(x, f) \in \Pi(\mathcal{X})$. Then by Theorem 15 it holds $(T - z)^* f = 0$. Hence by Proposition we have $\ker(T - z) \perp R(T - z)$. □

Theorem 17 *Let the dual space $\mathcal{X}^*$ of $\mathcal{X}$ be strictly convex and T be a $*$-n-paranormal operator on $\mathcal{X}$. If z and w are distinct eigen-values of T, then* $\ker(T - z) \perp \ker(T - w)$.

Proof. Let $(T - z)x = 0$ with $\|x\| = 1$ and $(T - w)y = 0$. Then by Theorem 16 it holds

$$1 \leq \|x + (w - z)^{-1}(T - z)y\| = \|x + y + (w - z)^{-1}(T - w)y\| = \|x + y\|.$$

Therefore we have $\ker(T - z) \perp \ker(T - w)$. □

Theorem 18 *Let $\mathcal{X}$ be uniformly convex and T^* be a $*$-n-paranormal operator on $\mathcal{X}$. If $T^* f = zf$ for $z \in \mathbb{C}$ and $(x, f) \in \Pi(\mathcal{X})$, then $Tx = zx$.*

Proof. Since $\mathcal{X}$ is uniformly convex, it holds $\mathcal{X}^{**} = \mathcal{X}$. Hence since $\mathcal{X}^{**}$ is strictly convex, $T^* f = zf$ and $(f, x) \in \Pi(\mathcal{X}^*)$, we have $T^{**}x (= Tx) = zx$ by Theorem 15. □

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