On P-nipotence of Finite Groups

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Abstract

A subgroup H is said to be weakly c^*-normal in a group G if there exists a subnormal subgroup K of G such that $HK = G$ and $H \cap K$ is s-quasinormally embedded in G. We give some results which generalize some authors' results.

Keywords: Weakly c^*-normality, p-nilpotence, s-quasinormally embedded

1. Introduction

In this paper the word group is always finite. Ore (1937, p150) gives quasinormality of subgroups. A subgroup H is said to be quasinormal in G if for every subgroup K of G such that $HK = KH$. A subgroup H of a group G is said to be s-quasinormal in G if H permutes with every Sylow subgroup of G. This concept was introduced by Kegel (1962, p 205),and extensively studied (Deskins, 1963, p126-131). Ballester-Bolinches and Pedraza-Aguilera (1998, p114) introduce the conception of s-quasinormally embedded in G if for each prime divisor p of H, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-quasinormal subgroup of G. Wei and Wang (2007, p212)introduced the notion of c^*-normality, a subgroup H of G is said to be c^*-normal in G if there exists a subgroup $K \leq G$ such that $G = HK$ and $H \cap K$ is s-quasinormally embedded in G.

For some notions and notations,the reader is referred to Robinson (1995)and Huppert (1968).

2. Some definitions and preliminary results

A subgroup H is called weakly c-normal in a group G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the largest normal subgroup of G contained in H. The conception of weakly c-normality was introduced by Lu, Guo, and Shum (2002, p 5506).

Definition 2.1 A subgroup H is said to be weakly c^*-normal in G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_s G$, where $H_s G$ is s-quasinormally embedded subgroup of G contained in H.

Lemma 2.1 (Ballester-Bolinches and Pedraza-Aguilera, 1998, Lemma 1) Suppose that U is s-quasinormally embedded in a group G, and that $H \leq G$ and $K \lhd G$.

(1) If $U \leq H$, then U is s-quasinormally embedded in H.

(2) If UK is s-quasinormally embedded in G, then UK/K is s-quasiormally embedded in G/K.

(3) If $K \lhd H$ and H/K is s-quasinormally embedded in G/K, then H is s-quasinormally embedded in G.

Lemma 2.2 Let G be a group. Then the following statements hold.

(1) If H is weakly C^*-normal in G and $H \leq M \leq G$, then H is weakly c^*-normal in M.

(2) Let $N \lhd G$ and $N \leq H$. Then H is weakly c^*-normal in G if and only if H/N is weakly c^*-normal in G/N.

(3)Let π be a set of primes. H is a π-subgroup of G and N a normal π'-subgroup of G, if H is weakly c^*-normal in G, then HN/N is weakly c^*-normal in G/N.

(4)Let $L \leq G$ and $H \leq \Phi(L)$ If H is weakly c^*-normal in G, then H is s-quasinormally embedded in G.

(5)Let H is c^*-normal in G, then H is weakly c^*-normal in G.

Proof. (1) If H is weakly c^*-normal in G, that is, there exists a subnormal subgroup T of G such that $HT = G$ and $H \cap T$ is s-quasinormally embedded in G, then $M = M \cap G = (M \cap T)H$. Since T is subnormal in G, then $M \cap T$ is subnormal in M, and $H \cap (M \cap T)$ is s-quasinormally embedded in M. So we have H is weakly c^*-normal in M.

(2) If H is weakly c^*-normal in G, then there exists a subnormal T of G such that $G = HT$ and $H \cap T$ is s-quasinormally embedded in G. Then $G/N = (H/N)(TN/N)$, where TN/N is subnormal in G/N and $(H/N) \cap (TN/N)$ is s-quasinormally embedded in G/N. Then H/N is weakly c^*-normal in G/N. The converse part can be proved similarly.

(3) If H is weakly c^*-normal in G, then there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T$ is s-quasinormally embedded in G. Since $|G|_{\pi'} = |T|_{\pi'} = |TN|_{\pi'}$, then $N \leq T$. Clearly $(HN/N)(T/N) = G/N$ and $(HN/N) \cap (T/N) = (H \cap T)N/N$ is s-quasinormally embedded in G/N.

(4) Since H is weakly c^*-normal in G, then there exists a subnormal subgroup T such that $G = HT$ and $H \cap T$ is s-quasinormally embedded in $G.L = L \cap (HT) = H(T \cap L)$. Since $H \leq \Phi(L)$, then $L = T \cap L$ and so $L \leq T$, then $T = G$ and $H = H \cap T$ is s-quasinormally embedded in G.

(5) The result is obvious.

Lemma 2.3 Let M be a maximal subgroup of G and P a normal Sylow p-subgroup of G such that $G = PM$, where p is a prime, then $P \cap M$ is a normal subgroup of G.

Lemma 2.4 (Wei and Wang, 2007, Lemma 2.5) Let G be a group, K an s-quasinormal subgroup of G, P a Sylow p-subgroup of K where p is a prime divisor of $|G|$. If either $P \leq O_P(G)$ or $K_G = 1$, then P is s-quasinormal in G.

Lemma 2.5 (Li, Wang and Wei, 2003, Lemma 2.2) Let G be a group and P is s-quasinormal p-subgroup of G where p is a prime, then $O_P(G) \leq N_G(P)$.

Lemma 2.6 (Wei and Wang, 2007, Lemma 2.8) Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.

(1) If N is normal in G of order p, then N is in $Z(G)$.

(2) If G has cyclic Sylow p-subgroups, then G is p-nilpotent.

(3)If $M \leq G$ and $|G : M| = p$, then $M \lhd G$.

Lemma 2.7 (Huppert, 1968, IV, 5.4) Suppose that G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Then G is a group which is not nilpotent but whose proper subgroups are all nilpotent.

Lemma 2.8 (Robinson, 1995, III, 5.2) Suppose that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then

(1) G has a normal Sylow p-subgroup for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.

(2) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

(3) If P is non-abelian and $p \neq 2$, then $exp(P) = p$.

(4) If P is non-abelian and $p = 2$, then $exp(P) = 4$.

(5)If P is abelian, then $exp(P) = p$.

Lemma 2.9 Let H be a subgroup of G. Then H is weakly c^*-normal in G if and only if there exists a subgroup K such that $G = HK$ and $H \cap K = H_{sG}$.

Proof. $\Leftarrow$ It is clear.

$\Rightarrow$ By definition 2.1, there exists a subnormal subgroup L of G such that $G = HL$ and $H \cap L \leq H_{sG}$ If $H \cap L < H_{sG}$, note that $K = LH_{sG}$, then $HK = HLH_{sG} = LHH_{SG} = LH = G$ and hence $H \cap K = H \cap LH_{sG} = (H \cap L)H_{sG} = H_{sG}$.

3. Main results

Theorem 3.1 Let G be a group, P a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If all maximal subgroups of P are weakly c^*-normal in G, then G is p-nilpotent.

Proof. Suppose that the result is false, then we chose a minimal order G as a counterexample. We will prove by the following steps:

Steps 1. For every proper subgroup of G is p-nilpotent, thus G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent.

Let M be a maximal subgroup of G, Then $P \cap M$ is a maximal p-subgroup of P. By hypothesis, $P \cap M$ is weakly c^*-normal in G and so $P \cap M$ is weakly c^*-normal in M by lemma 2.2(1). Thus M, $P \cap M$ satisfies the hypotheses of the theorem, the minimal choice of G implies that M is p-nilpotent. Then we have that G is not p-nilpotent but all proper subgroups are p-nilpotent. Then, by lemma 2.7 and lemma 2.8(1), G has a normal Sylow p-subgroup for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.

Steps 2. Let L be a minimal normal subgroup of G contained in P, then G/L is p-nilpotent, L is the unique minimal normal of G and $L \nleq \Phi(G)$.

Since P/L is a Sylow p-subgroup of G/L, we have M/L is a maximal subgroup in P/L, where M is a maximal subgroup of P. Since M is weakly c^*-normal in G, by lemma 2.2(2) M/L is weakly c^*-normal in G/L. Thus G/L, P/L satisfies the hypotheses of the theorem and so we have G/L is p-nilpotent by the minimal choice of G. If L_1 is an another minimal normal subgroup, then $G/1 \cong G/L \times G/L_1$ is p-nilpotent and so L is unique. If $L \leq \Phi(G)$, then $G/\Phi(G)$ is p-nilpotent, and so is G, a contradiction.

Steps 3. $\Phi(P) \neq 1$.

If $\Phi(P) = 1$, then P is abelian. By steps 1 and lemma 2.8(5), $exp(P) = p$. If $|P/\Phi(P)| = P^n$ and $P/\Phi(P) = < x_1\Phi(P), x_2\Phi(P), \cdots, x_n\Phi(P) >$, then $P = < x_1, x_2 \cdots, x_n >$. So we have $| < x_1 > | = p$, and $< x_i >$ char P, where i is nature number. And since P is normal in G, then $< x_i >$ are normal p-subgroup of G of order p. Thus by lemma 2.6(1), we have $< x_i > \leq Z(G)$ for all $i = 1, 2, \cdots, n$, then $P \leq Z(G)$, then G is p-nilpotent, a contradiction. Thus $\Phi(P) \neq 1$.

Steps 4. L is a Sylow p-subgroup of G.

By steps 3, $\Phi(P) \neq 1$, then $L \leq P$. If $L < P$, then for a maximal subgroup M of P, M is weakly c^*-normal in G and so there exists a subnormal subgroup K such that $G = MK$ and $M \cap K$ is s-quasinormally embedded in G. We consider the following cases.

1) $M \cap K = 1$.

Since $|K|_p = |G : M|_p = |PQ : MQ|_p = |P : M| = p$, then K has a normal p-complement Q_1 which is also a Sylow q-subgroup of G. By Sylow theorem, there exists an element $g \in G \setminus Q$ such that $Q_1^g = Q$. Since $M \lhd P$, then $G = MK = (MK)^g = MK^g$. Since $K^g \cong K$ and $Q = Q_1^g \leq K^g$, this implies $K^g \leq N_G(Q)$ in this case Q is not normal in G. So we have $G = MK = (MK)^g = MN_G(Q)$. So we have $M \cap N_G(Q) = 1$ and $N_G(Q) \leq K^g$. Thus $K^g = N_G(Q^g) = N_G(Q)$. If H be a sylow subgroup of $N_G(Q)$, then $K = HQ$ and $HP = PH = P$. This implies that H is s-quasinormal in G, then by lemma 2.5 we have $O_p(H) \leq N_G(H)$, and H is normal in G. Then $L = H \leq M$ since the minimality of L and L is unique. But $L \nleq M$, a contradiction.

2) $M \cap K = M = M_{sG}$.

Then we have $M \leq K$, then $G = K$ is p-nilpotent, a contradiction.

3) $1 < M_{sG} < M$.

Let $S = M \cap K$. Then S is s?quasinormally embedded in G. Thus there exists an s-quasinormal subgroup R such that S is a Sylow p-subgroup of R. Then by lemma 2.5, we have $O^p(G) \leq N_G(S)$ and so S is normal in G, then we have, $S = P$ or $S = L$. If $S = P$. On the other hand, $|M| < |P|$, a contradiction. Then $S = L$ is a minimal normal Sylow p-subgroup of some s-quasinormal subgroup R of G, then for any sylow q-subgroup Q, we have $RQ = QR$ is a subgroup of G and, if $QS < G$, $Q \lhd RQ$ by (1), and so $LQ = L \times Q$. By steps 1 and Burnside's theorem, we have G is solvable. Thus $Q \leq C_G(L) \leq L$, a contradiction. Then $QS = G$, then

$G = PQ = QS$ and so $P = S^g$ for some $g \in Q$, a contradiction.

Steps 5. Conclusions.

By steps 4, $L = P$ is a Sylow p-subgroup of G, then, by hypothesis, maximal subgroup M of $L = P$ is weakly -normal in G. Then by lemma 2.10 there exists a subnormal subgroup K of G such that $G = MK$ and $M \cap K \le M_{sG}$. Since $M_{sG} < L = P < L = P$, then $M_{sG} = 1$ and LQ/L is p-nilpotent since G/L is p-nilpotent by steps 2, where Q is a Hall p'-subgroup of G, then $LQ/L \lhd G/L$ and so $LQ \lhd G$. It follows from Q char $LQ \lhd G$ that Q is normal in G. Therefore G is p-nilpotent.

Corollary 3.1 (Wei and Wang, 2007, Theorem 3.1) Let G be a group, P a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p − 1) = 1$. If all maximal subgroups of P are c^*-normal in G, then G is p-nilpotent.

Theorem 3.2 Let G be a group, P a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p−1) = 1$. If all cyclic subgroups of P of order p or 4 (if p = 2) are weakly c^*-normal in G, then G is p-nilpotent.

Proof. Suppose that the result is false, then we chose a minimal order G as a counterexample. We will prove by the following steps:

Steps 1. Let M be a proper subgroup of G, then M is p-nilpotent. So G is not p-nilpotent but all proper subgroups are p-nilpotent. Thus $G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non-normal cyclic Sylow q-subgroup of G. And so by Burnside's theorem G is solvable. Then $M \cap P$ is a Sylow p-subgroup of M. By hypothesis, for every cyclic subgroup of P of order p or 4 (if p = 2) is weakly -normal in G, then By lemma 2.2(1), for every cyclic subgroup of P M of order p or 4 (if p = 2) is weakly c^*-normal in M. Then M, $M \cap P$ satisfies the hypotheses of the theorem, M is p-nilpotent by the minimal choice of G, so we have: G is not p-nilpotent but all proper subgroups are p-nilpotent and so by lemma 2.7 and lemma 2.8(1), $G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non-normal cyclic Sylow q-subgroup of G.

Steps 2. Let L be a minimal normal subgroup of G contained in P, then L is unique minimal normal p-subgroup for some prime of $|G|$, G/L is p-nilpotent and $L \not\le \Phi(G)$. Furthermore, $L = F(G) = C_G(L)$.

Since all cyclic subgroups of P of order p or 4(if p = 2) is weakly c^*-normal in G, then by lemma 2.2(2) all cyclic subgroups of P/L with order p or 4 (if p = 2) is weakly c^*-normal in G/L, then the minimal choice of G implies that G/L is p-nilpotent. If $L \le \Phi(G)$, then $G/\Phi(G)$ is p-nilpotent and G is p-nilpotent, a contradiction. By lemma 2.6 (Li, etc, 2003), $F(G) = L$. By steps 1, solubility of G implies that $L \le C_G(F(G)) \le F(G)$ and so $C_G(L) = F(G) = L$ as L is abelian.

Steps 3. Conclusions.

By steps 2 $C_G(L) = F(G) = L$. But on the other hand, for $x \in P, < x >$ is weakly c^*-normal in G, then there exists a subnormal subgroup T of G such that $G =< x > T$ and $< x > \cap T$ is s-quasinormally embedded in G. By lemma 2.7 and lemma 2.8, we have if p is odd or P is abelian, then $exp(P) = p$ or if $p = 2 exp(P) = 4$. Since $F(G) =< x_1, x_2, \cdots, x_n >= L, | < x_i > | = p$ or 4 and $< x_i >$ char P since P is normal in G. Thus $F(G) = L =< x_i >$. So we have $LQ = QL = L \times Q$, Then $Q \le N_G(L)$, then $G = P \times Q$ is nilpotent, a contradiction.

Corollary 3.2 (Li and Wang, 2004, Theorem 4.1) Suppose G is a group, p is a fixed prime number. If every element of $P_p(G)$ is contained in $Z_\infty(G)$. If $p = 2$, every cyclic subgroup of order 4 of G is s-quasinormal in G, then G is p-nilpotent.

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