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Integral Oscillation Criteria for Second-Order Linear neutral Delay Dynamic Equations on Time Scales

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Abstract

In this paper we present several sufficient conditions for oscillation of the second-order linear neutral delay dynamic equation on a time scale $\mathbb{T}$. Our results as a special case when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$ improve some well-known oscillation results for second-order neutral delay differential and difference equations.

Keywords: Oscillation, Time scales, Dynamic equation

1. Introduction

In 1988, Stefan Hilger introduced the calculus of *measure chain* in order to unify continuous and discrete analysis. Berned Aulbach, who supervised Stefan Hilger's Ph.D. thesis (Hilger, S., 1990, p18-56), points out the three main purposes of this new calculus: Unification - Extension - Discretization.

For many purposes in analysis it is sufficient to consider a special case of a measure chain, a so-called *time scale*, which simply is a closed subset of the real numbers. We denote a time scale by the symbol $\mathbb{T}$. The two most popular examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ that represent the classical theories of differential and of difference equations. Since Stefan Hilger formed the definition of derivatives and integrals on time scales, several authors has expounded on various aspects of this new theory, see the paper by (Agarwal et al., 2002, p1-26) and the references cited therein. The books on the subject of time scales, i.e., measure chain, by Bohner and Peterson (2001, 2003) summarize and organize much of time scale calculus.

In this paper, we are concerned with the oscillation of the second-order linear dynamic equation

$$(y(t) + p(t)y(t-\tau))^{\Delta\Delta} + q(t)y(t-\delta) = 0 \tag{1}$$

on a time scale $\mathbb{T}$.

Since we are interested in asymptotic behavior of solutions, we will suppose that the time scales $\mathbb{T}$ under consideration is not bounded above; i.e., it is a time scale interval of the form $[t_0,\infty)_{\mathbb{T}} = [t_0,\infty) \cap \mathbb{T}$.

Throughout this paper we assume that: τ and δ are positive constants such that the delay functions $\tau(t) := t-\tau < t$ and $\delta(t) := t-\delta < t$ satisfy $\tau(t) : \mathbb{T} \to \mathbb{T}$ and $\delta(t) : \mathbb{T} \to \mathbb{T}$ for all $t \in \mathbb{T}$,
(H1) $p(t), q(t) \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ where $C_{rd}(\mathbb{T}, \mathbb{R}^+)$ denotes the set of all function $f : \mathbb{T} \to \mathbb{R}^+$ which are right-dense continuous on $\mathbb{T}$ and $0 \le p(t) < p < 1$;

(H2) $y \in C^2_{rd}(I, \mathbb{R})$ where $I = [t_*, \infty) \subset \mathbb{T}$ for some $t_* > 0$;

(H3) $\int_0^\infty \delta(s)q(s)(1 - p(\delta(s)))\Delta s = \infty$.

By a solution of equation (1), we mean a nontrivial real value function $y(t)$ which has the properties $(y(t) + p(t)y(t-\tau)) \in$

$C^2_{rd}[t_y, \infty)$, $t_y > t_0$ and satisfying equation (1) for all $t > t_y$. Our attention is restricted to those solutions of equation (1) which exist on some half line $[t_y, \infty)$ and satisfy $sup\{|y(t)| : t > t_1\} > 0$ for any $t_1 > t_y$. A solution $y(t)$ of equation (1) is said to be oscillatory if it neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

We note that when $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, $\mu(t) = 0$, $y^{\Delta}(t) = y'(t)$ and (1) becomes the second-order neutral delay differential equation

$$(y(t) + p(t)y(t-\tau))'' + q(t)y(t-\delta) = 0, \quad t \in \mathbb{T}. \tag{2}$$

If $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t + 1$, $\mu(t) = 1$, $y^{\Delta}(t) = \Delta y(t) = y(t+1) - y(t)$ and (1) becomes the second-order neutral delay difference equation

$$\Delta^2(y(t) + p(t)y(t-\tau)) + q(t)y(t-\delta) = 0, \quad t \in \mathbb{T}. \tag{3}$$

If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, we have $\sigma(t) = t + h$, $\mu(t) = h$,

$$y^{\Delta}(t) = \Delta_h y(t) = \frac{y(t+h) - y(t)}{h},$$

and (1) becomes the second-order neutral delay difference equation

$$\Delta_h^2(y(t) + p(t)y(t-\tau)) + q(t)y(t-\delta) = 0, \quad t \in \mathbb{T}.$$

If $\mathbb{T} = q^{\mathbb{N}} = \{t : t = q^n, n \in \mathbb{N}, q > 1\}$, we have $\sigma(t) = qt$, $\mu(t) = (q-1)t$,

$$y^{\Delta}(t) = \Delta_q y(t) = \frac{y(qt) - y(t)}{(q-1)t},$$

and (1) becomes the second-order q-neutral delay difference equation

$$\Delta_q^2(y(t) + p(t)y(t-\tau)) + q(t)y(t-\delta) = 0, \quad t \in \mathbb{T}.$$

Numerous oscillation criteria have been established for second-order neutral delay differential and difference equations (2), (3). See for examples [Grammatikopoulos et al., 1985, p267-274, Kubiaczyk et al., 2002, p185-212, Saker, 2003, p99-111, Sun et al., 2005, p909-918] and the references cited therein.

In this paper we improve the sufficient conditions for oscillation of the special case of nonlinear neutral delay differential equation

$$(r(t)([y(t) + p(t)y(t-\tau)]^{\Delta})^{\gamma})^{\Delta} + f(t, y(t-\delta)) = 0 \tag{4}$$

in (Saker, 2006, p123-141), (Agarwal et al., 2004, p203-217) and (Hong-Wu Wu et al., 2006, p321-331), the special case of second-order nonlinear delay dynamic equation

$$(p(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + q(t)f(x(\tau(t))) = 0 \tag{5}$$

in (Zhenlai Han et al., 2007, p1-16) and the linear neutral delay differential equation

$$(y(t) + r(t)y(\tau(t)))^{\Delta\Delta} + p(t)y(\delta(t)) = 0 \tag{6}$$

in (Saker, 2007, p175-190).

Moreover, we intend to use the Riccati integral equations and the theory of integral inequalities (Kwong Man Kam, 2006, p1-18) for obtaining several oscillation criteria for (1). Hence the paper is organized as follows: In section 2, we present some preliminaries on time scales. In section 3, we establish some new sufficient conditions for oscillation of (1).

2. Some preliminaries on time scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale $\mathbb{T}$, we define the forward and backward jump operators by

$$\sigma(t) := inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := sup\{s \in \mathbb{T} : s < t\}.$$

A point $t \in \mathbb{T}$, $t > inf\mathbb{T}$ is said to be *left-dense* if $\rho(t) = t$, right-dense if $t < sup\mathbb{T}$ and $\sigma(t) = t$, *left-scattered* if $\rho(t) < t$ and *right-scattered* if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd*-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of *rd*-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is *rd*-continuous function is denoted by $C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$.

A function $p : \mathbb{T} \to \mathbb{R}$ is called positively regressive (we write $p \in \mathfrak{R}^+$) if it is *rd*-continuous function and satisfies $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by any Banach space) the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is right-dense then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists.

A function $f : [a, b] \to \mathbb{R}$ is said to be differentiable if its derivative exists, and a useful formula is

$$f^{\sigma} = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$) of two differentiable functions f and g

$$(f.g)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma},$$

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}.$$

For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta} \Delta t = f(b) - f(a),$$

and infinite integral is defined as

$$\int_{t_0}^{\infty} f(t) \Delta t = \lim_{b \to \infty} \int_{t_0}^{b} f(t) \Delta t.$$

An integration by parts formula reads

$$\int_a^b f(t) g^{\Delta}(t) \Delta t = [f(t)g(t)]_a^b - \int_a^b f^{\Delta}(t) g^{\sigma} \Delta t,$$

or

$$\int_a^b f^{\sigma} g^{\Delta}(t) \Delta t = [f(t)g(t)]_a^b - \int_a^b f^{\Delta}(t) g(t) \Delta t.$$

3. Main results

Before stating our main results in this paper, we start with the following lemmas.

Lemma 1 *Assume that (H3) hold and the inequality*

$$x^{\Delta\Delta}(t) + q(t)(1 - p(\delta(t))x(\delta(t)) \leq 0, \tag{7}$$

has a positive solution x on $[t_0, \infty)_{\mathbb{T}}$. *Then there exists a* $\mathrm{T} \in [t_0, \infty)_{\mathbb{T}}$, *sufficiently large, so that* $x^{\Delta}(t) \geq 0$ *and* $\frac{x^{\Delta}(t)}{x(t)} \leq \frac{1}{t}$ *for* $t \in [\mathrm{T}, \infty)_{\mathbb{T}}$.

proof. The proof is similar to the proof of Lemma 1 in (Erbe, L. ,2006, p65-78).

Lemma 2 *(Sahiner, 2005, pe1073-e1080) Suppose that the following conditions hold:*
(B1) $u \in C^2_{rd}(I, \mathbb{R})$ *where* $I = [t_*, \infty) \subset \mathbb{T}$ *for some* $t_* > 0$,
(B2) $u(t) > 0$, $u^\Delta > 0$ *and* $u^{\Delta\Delta} \leq 0$ *for* $t \geq t_*$.
Then, for each $k \in (0, 1)$*, there exists a constant* $t_k \in \mathbb{T}$, $t_k \geq t_*$*, such that*

$$u(\sigma(t)) \leq \frac{\sigma(t)}{k\delta(t)} u(\delta(t)) \quad for \quad t \geq t_k. \tag{8}$$

Lemma 3 *(Saker, 2006, p123-141) Let* $f(u) = Bu - Au^{\frac{\gamma+1}{\gamma}}$*, where* $A > 0$ *and* B *are constants,* γ *is a positive integer. Then* f *attains its maximum value on* $\mathbb{R}$ *at* $u^* = (\frac{B\gamma}{A(\gamma+1)})^\gamma$*, and*

$$\max_{u \in \mathbb{R}} f = f(u^*) = \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}.$$

Theorem 1 *Assume that (H1)-(H3) hold. Furthermore, assume that there exist positive rd-continuous* Δ*-differentiable functions* $\alpha(t)$ *and* $\beta(t)$ *with* $\beta(t) \geq t$ *such that* $\lim_{t\to\infty} \frac{\beta(t)}{\alpha(t)} = 0$, $\lim_{t\to\infty} \alpha(t) = \infty$*. If*

$$\limsup_{t\to\infty} \frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} \left(\alpha(s)q(s)(1 - p(s-\delta)) - \frac{(\alpha^\Delta(s))^2}{4\alpha(s)} \right) \Delta s \geq 0, \tag{9}$$

then every solution of Eq. (1) is oscillatory.

proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of equation (1). Without loss of generality, we may assume that $y(t)$ is an eventually positive solution of (1) with $y(t - N) > 0$ where $N = max\{\tau, \delta\}$ for all $t > t_0$ sufficiently large. We shall consider only this case, since the substitution $z(t) = -y(t)$ transform Eq. (1) into an equation of the same form. Set

$$x(t) = y(t) + p(t)y(t - \tau). \tag{10}$$

From (10) and (1) we have

$$x^{\Delta\Delta}(t) + q(t)y(t - \delta) = 0, \tag{11}$$

for all $t > t_0$, and so $x^\Delta(t)$ is an eventually decreasing function. We first show that $x^\Delta(t)$ is eventually nonnegative. Indeed, since $q(t)$ is a positive function, the deceasing function $x^\Delta(t)$ is either eventually positive or eventually negative. Suppose there exists an integer $t_1 \geq t_0$ such that $x^\Delta(t_1) = c < 0$, then $x^\Delta(t) < x^\Delta(t_1) = c$ for $t \geq t_1$, hence $x^\Delta(t) \leq c$, which implies that

$$x(t) \leq x(t_1) + c(t - t_1) \to -\infty \quad \text{as} \quad t \to \infty,$$

which contradicts the fact that $x(t) > 0$ for all $t > t_1$. Hence $x^\Delta(t)$ is eventually nonnegative. Therefore, we see that there is some t_1 such that

$$x(t) > 0, x^\Delta(t) \geq 0, x^{\Delta\Delta} < 0, t \geq t_1. \tag{12}$$

This implies that

$$\begin{aligned} y(t) = x(t) - p(t)y(t-\tau) &= x(t) - p(t)[x(t-\tau) - p(t-\tau)y(t-2\tau)] \\ &\geq x(t) - p(t)x(t-\tau) \geq x(t)(1 - p(t)). \end{aligned}$$

Then, for $t \geq t_1 = t_0 + \delta$ sufficiently large, we see that

$$y(t - \delta) \geq x(t - \delta)(1 - p(t - \delta)). \tag{13}$$

From (11) and (13) we obtain for $t \geq t_1$

$$x^{\Delta\Delta}(t) + q(t)(1 - p(t - \delta))x(t - \delta) \leq 0. \tag{14}$$

Then from (14), we have

$$\alpha(t)q(t)(1 - p(t - \delta)) \leq \frac{-x^{\Delta\Delta}(t)\alpha(t)}{x(t - \delta)}. \tag{15}$$

Integrating the above inequality from t_1 to t, we get

$$\int_{t_1}^{t} \alpha(s)q(s)(1-p(s-\delta))\Delta s \le -\int_{t_1}^{t} \frac{\alpha(s)x^{\Delta\Delta}(s)}{x(s-\delta)}\Delta s,$$

hence

$$\begin{aligned}
\int_{t_1}^{t} \alpha(s)q(s)(1-p(s-\delta))\Delta s &\le -\frac{\alpha(t)x^{\Delta}(t)}{x(t-\delta)} + \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} + \int_{t_1}^{t} x^{\Delta}(\sigma(s))(\frac{\alpha(s)}{x(s-\delta)})^{\Delta}\Delta s \\
&\le -\frac{\alpha(t)x^{\Delta}(t)}{x(t-\delta)} + \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} + \int_{t_1}^{t} \frac{x^{\Delta}(\sigma(s))\alpha^{\Delta}(s)}{x(\sigma(s)-\delta)} \\
&- \int_{t_1}^{t} \frac{x^{\Delta}(\sigma(s))\alpha(s)x^{\Delta}(s-\delta)}{x(s-\delta)x(\sigma(s)-\delta)}\Delta s. \qquad (16)
\end{aligned}$$

In view of (12), we obtain

$$\frac{x^{\Delta}(\sigma(t))}{x(\sigma(t)-\delta)} < \frac{x^{\Delta}(t)}{x(\sigma(t)-\delta)} < \frac{x^{\Delta}(t-\delta)}{x(\sigma(t)-\delta)} < \frac{x^{\Delta}(t-\delta)}{x(t-\delta)}. \qquad (17)$$

From (16) and (17) we get

$$\begin{aligned}
\int_{t_1}^{t} \alpha(s)q(s)(1-p(s-\delta))\Delta s &\le -\frac{\alpha(t)x^{\Delta}(t)}{x(t-\delta)} + \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} \\
&+ \int_{t_1}^{t} \frac{\alpha^{\Delta}(s)x^{\Delta}(\sigma(s))}{x(\sigma(s)-\delta)}\Delta s - \int_{t_1}^{t} \alpha(s)(\frac{x^{\Delta}(\sigma(s))}{x(\sigma(s)-\delta)})^2\Delta s \\
&\le \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} - \frac{\alpha(t)x^{\Delta}(t)}{x(t-\delta)} + \int_{t_1}^{t} \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\Delta s \\
&- \int_{t_1}^{t} (\frac{\alpha^{\Delta}(s)}{2\sqrt{\alpha(s)}} - \frac{\sqrt{\alpha(s)}x^{\Delta}(\sigma(s))}{x(\sigma(s)-\delta)})^2\Delta s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{t_1}^{t} \alpha(s)q(s)(1-p(s-\delta))\Delta s &\le \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} + \sqrt{\alpha(t)}(\frac{\alpha^{\Delta}(t)}{2\sqrt{\alpha(t)}} - \frac{\sqrt{\alpha(t)}x^{\Delta}(\sigma(t))}{x(\sigma(t)-\delta)}) - \frac{\alpha^{\Delta}(t)}{2} \\
&+ \int_{t_1}^{t} \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\Delta s - \int_{t_1}^{t} (\frac{\alpha^{\Delta}(s)}{2\sqrt{\alpha(s)}} - \frac{\sqrt{\alpha(s)}x^{\Delta}(\sigma(s))}{x(\sigma(s)-\delta)})^2\Delta s.
\end{aligned} \qquad (18)$$

Let

$$w(t) = \frac{\alpha^{\Delta}(s)}{2\sqrt{\alpha(s)}} - \frac{\sqrt{\alpha(t)}x^{\Delta}(\sigma(t))}{x(\sigma(t)-\delta)}. \qquad (19)$$

Then from (18), (19), it is easy to see that

$$\begin{aligned}
\int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s &\le \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} + \sqrt{\alpha(t)}w(t) - \frac{\alpha^{\Delta}(t)}{2} - \int_{t_1}^{t} w^2(s)\Delta s \\
&\le \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} - \frac{\alpha(t)x^{\Delta}(\sigma(t))}{x(\sigma(s)-\delta)}. \qquad (20)
\end{aligned}$$

Then, by lemma 1, for sufficiently large t, there exists $\beta(t) \ge t$ such that $\frac{1}{\beta(t)} \le \frac{x^{\Delta}(\sigma(t))}{x(\sigma(t))} \le \frac{1}{t}$. Hence

$$\frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s \le \frac{\beta(t)}{\alpha(t)} \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} - 1.$$

Since $\lim_{t\to\infty} \frac{\beta(t)}{\alpha(t)} = 0$ we have

$$\frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s < 0,$$

which contradicts the condition (9). The proof is complete.

Theorem 2 *Assume that (H1)-(H3) hold. Let* $\alpha(t)$, $\beta(t)$ *be as defined in Theorem 1 and*

$$\limsup_{t\to\infty} \frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} \left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)} \right) \Delta s \geq 0 \tag{21}$$

then every solution of Eq. (1) is oscillatory.

proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of Eq. (1) and let $t_1 \geq t_0$ be such that $y(t) \neq 0$ for all $t \geq t_1$, so without loss of generality, we may assume that $y(t)$ is an eventually positive solution of Eq. (1). From (15), we get

$$\frac{t-\delta}{t}\alpha(t)q(t)(1-p(t-\delta)) \leq \frac{-x^{\Delta\Delta}(t)\alpha(t)}{x(t-\delta)}. \tag{22}$$

We proceed as in the proof of Theorem 1 and it follows that

$$\frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} \left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)} \right) \Delta s \quad < \quad 0,$$

which contradicts the condition (21). The proof is complete.

Corollary 1 *Assume that (H1)-(H3) hold and. Let* $\alpha(t)$ *be as defined in Theorem 1 and*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)} \right) \Delta s = \infty, \tag{23}$$

then every solution of Eq. (1) is oscillatory.

proof. We proceed as in the proof of Theorem 1 to prove that there exists $t_1 \geq t_0$ such that (20) holds for $t \geq t_1$. From (20), it follows that

$$\int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)} \right) \Delta s \quad \leq \quad \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)},$$

which contradicts the condition (23).

Corollary 2 *Assume that (H1)-(H3) hold. Let* $\alpha(t)$ *be as defined in Theorem 1 and*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)} \right) \Delta s = \infty, \tag{24}$$

then every solution of Eq. (1) is oscillatory.

proof. We proceed as in the proof of Theorem 1 to prove that there exists $t_1 \geq t_0$ such that (20) holds for $t \geq t_1$. From (20), it follows that

$$\int_{t_1}^{t} \left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)} \right) \Delta s \quad \leq \quad \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)},$$

which contradicts the condition (24).

Theorem 3 *Assume that (H1)-(H3) hold. Let* $\alpha(t)$ *be as defined in Theorem 1 such that for some positive constant* $k \in (0, 1)$,

$$\limsup_{t\to\infty} \frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} \left(k(\frac{s-\delta}{\sigma(s)})\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)} \right) \Delta s \geq 0, \tag{25}$$

then every solution of Eq. (1) is oscillatory.

proof. Suppose that Eq. (1) has a nonoscillatory solution $y(t)$. We may assume without loss of generality that $y(t) > 0$ for all $t > t_0$. We will consider only this case, since the proof when $y(t)$ is eventually negative is similar. In view of Lemma 2, for each positive constant $k \in (0, 1)$, there exists a $t_1 = max\{t_k, t_0\}$ such that

$$x(t) \leq x(\sigma(t)) \leq \frac{\sigma(t)}{k(t-\delta)}x(t-\delta) \leq \frac{\sigma(t)}{k(t-\delta)}x(t) \quad \text{for} \quad t \geq t_1. \tag{26}$$

From (15) and from (26), we get

$$\frac{k(t-\delta)}{\sigma(t)}\alpha(t)q(t)(1-p(t-\delta)) \le \frac{-x^{\Delta\Delta}(t)\alpha(t)}{x(t-\delta)}. \tag{27}$$

We proceed as in the proof of Theorem 1, so we get

$$\begin{aligned}\int_{t_1}^{t}\left(\frac{k(t-\delta)}{\sigma(t)}\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s \quad &\le \quad \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} + \sqrt{\alpha(t)}w(t) - \frac{\alpha^{\Delta}(t)}{2} - \int_{t_1}^{t} w^2(s)\Delta s \\ &\le \quad \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} - \frac{\alpha(t)x^{\Delta}(\sigma(t))}{x(\sigma(s)-\delta)}.\end{aligned} \tag{28}$$

From lemma 1, for sufficiently large t, there exists $\beta(t) \ge t$ such that $\frac{1}{\beta(t)} \le \frac{x^{\Delta}(\sigma(t))}{x(\sigma(t))} \le \frac{1}{t}$. Hence

$$\frac{\beta(t)}{\alpha(t)}\int_{t_1}^{t}\left(\frac{k(t-\delta)}{\sigma(t)}\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s \quad \le \quad \frac{\beta(t)}{\alpha(t)}\frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} - 1.$$

Since $\lim_{t\to\infty}\frac{\beta(t)}{\alpha(t)} = 0$ we have

$$\frac{\beta(t)}{\alpha(t)}\int_{t_1}^{t}\left(\frac{k(t-\delta)}{\sigma(t)}\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s \quad < \quad 0,$$

which contradicts the condition (25). The proof is complete.

Corollary 3 *Assume that (H1)-(H3) hold. Let $\alpha(t)$ be as defined in Theorem 1 such that for some positive constant $k \in (0,1)$,*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(k(\frac{s-\delta}{\sigma(s)})\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s = \infty, \tag{29}$$

then every solution of Eq. (1) is oscillatory.

Assume that the condition (23) fails, and

$$R(t) = \int_{t}^{\infty}\left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s. \tag{30}$$

In this case we have the following result.

Theorem 4 *Assume that (H1)-(H3) hold and there exists a positive rd-continuous Δ-differentiable functions $\alpha(t)$ such that* $\lim_{t\to\infty}\alpha(t) = \infty$*,*

$$\limsup_{t\to\infty}\frac{t-\delta}{\alpha(t)}R(t) > 1, \tag{31}$$

then every solution of Eq. (1) *is oscillatory.*

proof. Assume that Eq. (1) has a positive solution $y(t)$ for all $t \ge t_1$. Then from condition (31) we have,

$$\frac{t_1-\delta}{\alpha(t_1)}R(t_1) > 1. \tag{32}$$

From lemma 1, for sufficiently large t, we have

$$\frac{tx^{\Delta}(t)}{x(t)} \le 1. \tag{33}$$

Then from (32) and (33) we get

$$\frac{t_1-\delta}{\alpha(t_1)}R(t_1) > \frac{(t_1-\delta)x^{\Delta}(t_1-\delta)}{x(t_1-\delta)} > \frac{(t_1-\delta)x^{\Delta}(t_1)}{x(t_1-\delta)}.$$

Let

$$N(t) = \frac{\alpha(t)x^{\Delta}(t)}{x(t-\delta)} > 0. \tag{34}$$

For that,

$$R(t_1) > N(t_1) \tag{35}$$

From (20), (34) and (35) we get

$$\frac{\alpha(t)N(\sigma(t))}{\alpha(\sigma(t))} < \int_{t_1}^{\infty}\left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s - \int_{t_1}^{t}\left(\alpha(s)q(s)(1-p(s-\delta)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)}\right)\Delta s. \tag{36}$$

From (36), for sufficiently large t, we have

$$\frac{\alpha(t)N(\sigma(t))}{\alpha(\sigma(t))} < 0.$$

which is a contradiction. This complete the proof.

Remark 1 *From Theorem 1 and Theorem 2 we can obtain different conditions for oscillation of Eq.* (1) *by choosing* $\alpha(t) = t$.

Corollary 4 *Assume that (H1)-(H3) hold. Let* $\beta(t)$ *as defined in Theorem 1 such that* $\lim_{t\to\infty}\frac{\beta(t)}{t} = 0$. *If*

$$\limsup_{t\to\infty}\frac{\beta(t)}{t}\int_{t_1}^{t}\left(sq(s)(1-p(s-\delta)) - \frac{1}{4s}\right)\Delta s > 0,$$

then every solution of Eq. (1) *is oscillatory.*

Corollary 5 *Assume that (H1)-(H3) hold. Let* $\beta(t)$ *as defined in Theorem 1 such that* $\lim_{t\to\infty}\frac{\beta(t)}{t} = 0$. *If*

$$\limsup_{t\to\infty}\frac{\beta(t)}{t}\int_{t_1}^{t}\left((s-\delta)q(s)(1-p(s-\delta)) - \frac{1}{4s}\right)\Delta s > 0,$$

then every solution of Eq. (1) *is oscillatory.*

Corollary 6 *Assume that (H1)-(H3) hold. If*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(sq(s)(1-p(s-\delta)) - \frac{1}{4s}\right)\Delta s = \infty,$$

then every solution of Eq. (1) *is oscillatory.*

Corollary 7 *Assume that (H1)-(H3) hold and*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left((s-\delta)q(s)(1-p(s-\delta)) - \frac{1}{4s}\right)\Delta s = \infty,$$

then every solution of Eq. (1) *is oscillatory.*

The following theorem gives Philos-type oscillation criteria for Eq. (1).
First, let us introduce now the class of functions $\mathfrak{R}$ which will be extensively used in the sequel.
Let $\mathbb{D}_0 \equiv \{(t,s) \in \mathbb{T}^2 : t > s \geq t_0\}$ and $\mathbb{D} \equiv \{(t,s) \in \mathbb{T}^2 : t \geq s \geq t_0\}$. The function $H \in C_{rd}(\mathbb{D},\mathbb{R})$ is said to belongs to the class $\mathfrak{R}$ if

(i) $H(t,t) = 0$, $t \geq t_0$, $H(t,s) > 0$ on $\mathbb{D}_0$,

(ii) H has a continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ on $\mathbb{D}_0$ with respect to the second variable. (H is *rd*-continuous function if H is *rd*-continuous function in t and s).

Theorem 5 *Assume that (H1)-(H3) hold. Furthermore, assume that there exist a positive rd-continuous* Δ*-differentiable functions* $\alpha(t)$ *and* $\beta(t)$ *with* $\beta(t) \geq t$ *such that* $\lim_{t\to\infty}\frac{\beta(t)}{\alpha(t)} = 0$, $\lim_{t\to\infty}\alpha(t) = \infty$ *and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\frac{\beta(t)}{\alpha(t)}\int_{t_1}^{t}\left(\alpha(s)q(s)(1-p(s-\delta))H(t,s) - \frac{B^2(t,s)\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s > 0,$$

where

$$B(t,s) = \frac{\alpha^{\Delta}(s)H(t,s)}{\alpha^{\sigma}} + H^{\Delta_s}(t,s), \tag{37}$$

then every solution of Eq. (1) *is oscillatory.*

proof. Suppose to the contrary that $y(t)$ is a nonoscillatory solution of Eq. (1) and let $t \geq t_1$ be such that $y(t) \neq 0$ for all $t \geq t_1$, so without loss of generality, we may assume that $y(t)$ is an eventually positive solution of Eq. (1) with $y(t-N) > 0$ where $N = \max\{\tau, \delta\}$ for all $t \geq t_1$ sufficiently large. We proceed as in the proof of Theorem 1. From (15) we get

$$\int_{t_1}^{t} \alpha(s)q(s)(1-p(s-\delta))H(t,s)\Delta s \leq -\int_{t_1}^{t} \frac{\alpha(s)H(t,s)x^{\Delta\Delta}(s)}{x(s-\delta)}\Delta s$$

and then

$$\begin{aligned}\int_{t_1}^{t} \alpha(s)q(s)(1-p(s-\delta))H(t,s)\Delta s \leq\ & \frac{\alpha(t_1)H(t,t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} - \frac{\alpha(t)H(t,t)x^{\Delta}(t)}{x(t-\delta)} + \int_{t_1}^{t} \frac{x^{\Delta}(\sigma(s))\alpha^{\Delta}(s)H(t,s)}{x(\sigma(s)-\delta)}\Delta s \\ +\ & \int_{t_1}^{t} \frac{x^{\Delta}(\sigma(s))\alpha^{\sigma}(s)H^{\Delta_s}(t,s)}{x(\sigma(s)-\delta)}\Delta s - \int_{t_1}^{t} \frac{\alpha(s)x^{\Delta}(\sigma(s))H(t,s)x^{\Delta}(s-\delta)}{x(s-\delta)x(\sigma(s)-\delta)}\Delta s.\end{aligned} \tag{38}$$

Then by using (17) we get

$$\begin{aligned}\int_{t_1}^{t} \alpha(s)q(s)(1-p(s-\delta))H(t,s)\Delta s \leq\ & \frac{\alpha(t_1)H(t,t_1)x^{\Delta}(t_1)}{x(t_1-\delta)} + \int_{t_1}^{t} (H^{\Delta_s}(t,s) + \frac{\alpha^{\Delta}(s)H(t,s)}{\alpha^{\sigma}})\frac{x^{\Delta}(\sigma(s))\alpha^{\sigma}}{x(\sigma(s)-\delta)}\Delta s \\ -\ & \int_{t_1}^{t} \alpha(s)H(t,s)(\frac{x^{\Delta}(\sigma(s))}{x(\sigma(s)-\delta)})^2\Delta s,\end{aligned} \tag{39}$$

where $H(t,t) = 0$. Therefore by using Lemma 3, with

$$\gamma = 1,\ B = (H^{\Delta_s}(t,s) + \frac{\alpha^{\Delta}(s)H(t,s)}{\alpha^{\sigma}}),\ A = \frac{\alpha(s)H(t,s)}{\alpha^{\sigma^2}} \quad \text{and} \quad u = \frac{x^{\Delta}(\sigma(s))\alpha^{\sigma}}{x(\sigma(s)-\delta)}, \tag{40}$$

we get that

$$\int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta))H(t,s) - \frac{B^2\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s \leq \frac{\alpha(t_1)H(t,t_1)x^{\Delta}(t_1)}{x(t_1-\delta)}. \tag{41}$$

So

$$\frac{\beta(t)}{\alpha(t)H(t,t_1)} \int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta))H(t,s) - \frac{B^2\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s \leq \frac{\beta(t)}{\alpha(t)} \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)}.$$

Since $\lim_{t\to\infty} \frac{\beta(t)}{\alpha(t)} = 0$ we have

$$\frac{\beta(t)}{\alpha(t)H(t,t_1)} \int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta))H(t,s) - \frac{B^2\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s \leq 0,$$

which contradicts the condition (37). Then every solution of Eq. (1) oscillates.

Corollary 8 *Assume that (H1)-(H3) hold. Let $\alpha(t)$, $B(t,s)$ be as defined in Theorem 5 and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta))H(t,s) - \frac{B^2(t,s)\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s = \infty, \tag{42}$$

then every solution of Eq. (1) is oscillatory.

proof. By proceeding as in the proof of Theorem 5 and from (41) we get

$$\frac{1}{H(t,t_1)} \int_{t_1}^{t} \left(\alpha(s)q(s)(1-p(s-\delta))H(t,s) - \frac{B^2\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s \leq \frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)},$$

which contradicts the condition (42). Then every solution of Eq. (1) oscillates.

Theorem 6 *Assume that (H1)-(H3) hold. Let $\alpha(t)$, $\beta(t)$, $B(t,s)$ be as defined in Theorem 5 and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} \left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta))H(t,s) - \frac{B^2(t,s)\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s > 0, \tag{43}$$

then every solution of Eq. (1) is oscillatory.

proof. By proceeding as in Theorem 5 we get

$$\int_{t_1}^{t}\left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta))H(t,s)-\frac{B^2\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s \leq \frac{\alpha(t_1)H(t,t_1)x^{\Delta}(t_1)}{x(t_1-\delta)}$$

and then,

$$\frac{\beta(t)}{\alpha(t)H(t,t_1)}\int_{t_1}^{t}\left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta))H(t,s)-\frac{B^2\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s \leq \frac{\beta(t)}{\alpha(t)}\frac{\alpha(t_1)x^{\Delta}(t_1)}{x(t_1-\delta)}.$$

Since $\lim_{t\to\infty}\frac{\beta(t)}{\alpha(t)}=0$ we have

$$\frac{\beta(t)}{\alpha(t)H(t,t_1)}\int_{t_1}^{t}\left(\frac{s-\delta}{s}\alpha(s)q(s)(1-p(s-\delta))H(t,s)-\frac{B^2\alpha^{\sigma^2}}{4\alpha(s)H(t,s)}\right)\Delta s \quad\leq\quad 0$$

which contradicts the condition (43). Then every solution of Eq. (1) oscillates.

Remark 2 *With an appropriate choice of the functions $H \in C_{rd}(\mathbb{D},\mathbb{R})$ and $h \in C_{rd}(\mathbb{D}_0,\mathbb{R})$. We can take $H(t,s) = (t-s)^m, (t,s) \in \mathbb{D}$ with $m > 1$. It is clear that H belongs to the class $\mathfrak{R}$.*
Now, we claim that

$$((t-s)^m)^{\Delta_s} \leq -m(t-\sigma(s))^{m-1}. \tag{44}$$

proof.

We consider the following two case:
Case 1: If $\mu(t) = 0$ then $((t-s)^m)^{\Delta_s} = -m(t-\sigma(s))^{m-1}$.
Case 2: If $\mu(t) \neq 0$ then we have

$$\begin{aligned}((t-s)^m)^{\Delta_s} &= \frac{1}{\mu(s)}[(t-\sigma(s))^m-(t-s)^m] \\ &= \frac{-1}{\sigma(s)-s}[(t-s)^m-(t-\sigma(s))^m].\end{aligned} \tag{45}$$

Using Hardy et al. inequality (Hardy, 1952)

$$x^m - y^m \geq my^{m-1}(x-y) \qquad \text{for all} \quad x \geq y > 0 \quad \text{and} \quad m \geq 1 \tag{46}$$

we have

$$[(t-s)^m-(t-\sigma(s))^m] \geq m(t-\sigma(s))^{m-1}(\sigma(s)-s). \tag{47}$$

Then from (45) and (47), we have

$$((t-s)^m)^{\Delta_s} \leq -m(t-\sigma(s))^{m-1}. \tag{48}$$

and this proves (44).

From the above claim and Theorem 5, we have the following Kamenev-type oscillation criteria for Eq. (1).

Corollary 9 *Assume that (H1)-(H3) hold. Let $\alpha(t), \beta(t)$ be as defined in Theorem 5 and*

$$\limsup_{t\to\infty}\frac{1}{t^m}\frac{\beta(t)}{\alpha(t)}\int_{t_0}^{t}\left(\alpha(s)q(s)(1-p(s-\delta))(t-s)^m-\frac{C^2(t,s)\alpha^{\sigma^2}}{4\alpha(s)(t-s)^m}\right)\Delta s > 0,$$

where

$$C(t,s) = \frac{\alpha^{\Delta}(s)(t-s)^m}{\alpha^{\sigma}} - m(t-\sigma(s))^{m-1},$$

then every solution of Eq. (1) is oscillatory.

Corollary 10 *Assume that (H1)-(H3) hold. Let $\alpha(t)$ be as defined in Theorem 5 and $C(t,s)$ be as defined in Corollary 9*

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_0}^{t}\left(\alpha(s)q(s)(1-p(s-\delta))(t-s)^m-\frac{C^2(t,s)\alpha^{\sigma^2}}{4\alpha(s)(t-s)^m}\right)\Delta s = \infty,$$

then every solution of Eq. (1) is oscillatory.

Theorem 7 *Assume that (H1)-(H3) hold. Let $\alpha(t)$, $\beta(t)$ be as defined in Theorem 5 and*

$$\limsup_{t\to\infty} \frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} q(s)(1-p(s-\delta))\Delta s > 0, \tag{49}$$

then every solution of Eq. (1) *oscillates.*

proof. Suppose to the contrary that Eq. (1) has a nonoscillatory solution $y(t)$. We may assume that there exists $t_1 \geq t_0$ such that $y(t) > 0$ for all $t \geq t_1$.
We proceed in this theorem as in Theorem 5 and from (14) we get

$$\begin{aligned} x^{\Delta}(t_1) - x^{\Delta}(t) &\geq \int_{t_1}^{t} q(s)(1-p(s-\delta))x(s-\delta)\Delta s \\ x^{\Delta}(t_1) &\geq x(t_1-\delta)\int_{t_1}^{t} q(s)(1-p(s-\delta))\Delta s. \end{aligned}$$

For that,

$$\frac{x^{\Delta}(t_1)}{x(t_1-\delta)} \geq \int_{t_1}^{t} q(s)(1-p(s-\delta))\Delta s.$$

Since $\lim_{t\to\infty} \frac{\beta(t)}{\alpha(t)} = 0$ we have

$$\frac{\beta(t)}{\alpha(t)} \int_{t_1}^{t} q(s)(1-p(s-\delta))\Delta s \leq 0, \tag{50}$$

which is contradicts (49), and consequently, Eq. (1) has no eventually positive solution. Similarly, by using the same technique we can prove that Eq. (1) has no eventually negative solution. Thus Eq. (1) is oscillatory.

Corollary 11 *Assume that (H1) hold. If*

$$\limsup_{t\to\infty} \int_{t_1}^{t} q(s)(1-p(s-\delta))\Delta s = \infty,$$

then every solution of Eq. (1) *is oscillatory.*

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