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Representation of Functions by Walsh's Series with Monotone Coefficients

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Abstract

There exists a series in the Walsh system $\{\varphi_n\}$ of the form

$$\sum_{i=1}^{\infty} a_i\varphi_i, \quad \text{with} \quad |a_i| \searrow 0,$$

that possess the following properties:

For any $\epsilon > 0$ and any function $f \in L^1(0,1)$ there exists set $E \subset [0,1]$ ($|E| > 1-\epsilon$) and a sequence $\{\delta_i\}_{i=0}^{\infty}$, $\delta_i = 0$ or 1, such that the series

$$\sum_{i=0}^{\infty} \delta_i a_i \varphi_i$$

converges to f on E in the $L^1(0,1)$-metric and on $[0,1]\diagdown E$ in the $L^r([0,1]\diagdown E)$ metric for all $r \in (0,1)$.

Keywords: orthonormal system, convergence, functional series

AMS 2000 Mathematics Subject Classification: 42C10,42C20

1. Introduction

The problem of representing a function f by a series in classical and general orthonormal systems has a long history.

A question posed by Lusin in 1915 asks whether it be possible to find for every measurable function $[0,2\pi]$ a trigonometric series, with coefficient sequence converging to function. For real-valued functions, this question was given an affirmative answer by Men'shov in 1941.

There are many other works (see Talalian, 1960; Men'shov, 1947, 1941; Grigorian, 1999, 2003, 2000; Ul'janov, 1972; Ivanov, 1989; Krotov, 1977; Kozlov, 1950) devoted to representations of functions by series in classical and general orthonormal systems and the existence of different types of universal series in the sense of convergence almost everywhere and by measure.

The papers by Men'shov (1947) and Kozlov (1950) were the first to construct some ordinary universal trigonometric series in the class of all measurable functions in the sense of a.e. convergence.

Grigoryan (2009) proved the following important result:

For any $\epsilon > 0$ and any function $f \in L[0,1]$ there exists a sequence $\{\delta_i\}_{i=0}^{\infty}$, $\delta_i = 0$ or 1, such that the series with monotone coefficients

$$\sum_{i=0}^{\infty} \delta_i a_i \varphi_i$$

converges to f in the $L^1(E)$-metric. In Grigoryan's paper properties of series outside the E remained open. In this paper we succeed to ensure the convergence of the series to f outside E in weaker metric. Note that convergence is impossible in the same metric.

Let r be the periodic function, of least period 1, defined on $[0, 1)$ by

$$r = \chi_{[0,1/2)} - \chi_{[1/2,1)}.$$

The Rademacher system, $R = r_n : n = 0, 1, \dots$, is defined by the conditions

$$r_n(x) = r(2^n x),\ \forall x \in R, n = 0, 1, \dots,$$

and, in the ordering employed by Payley (see Golubov, Efimov, & Skvartsov, 1987; Paley, 1932), the n-th element of the Walsh system $\{\varphi_n\}$ is given by

$$\varphi_n = \prod_{k=0}^{\infty} r_k^{n_k}, \tag{1}$$

where $\sum_{k=0}^{\infty} n_k 2^k$ is the unique binary expansion of n, with each n_k either 0 or 1.

In the present work we prove the following theorem:

Theorem *There exists a series in the Walsh system of the form*

$$\sum_{i=1}^{\infty} a_i \varphi_i, \quad \text{with} \quad |a_i| \searrow 0,$$

that possess the following properties:

For any $\epsilon > 0$ and any function $f \in L^1(0,1)$ there exists set $E \subset [0, 1]$ ($|E| > 1 - \epsilon$) and a sequence $\{\delta_i\}_{i=0}^{\infty}$, $\delta_i =$ 0 or 1, such that the series

$$\sum_{i=0}^{\infty} \delta_i a_i \varphi_i$$

converges to f on E in the $L^1(0, 1)$-metric and on $[0, 1]\diagdown E$ in the $L^r([0, 1]\diagdown E)$ metric for all $r \in (0, 1)$.

The following problem remains open: is the Theorem true for the trigonometric system?

2. Basic Concepts and Terminology

We put

$$I_k^{(j)}(x) = \begin{cases} 1, & \text{if } x \in [0, 1] \setminus \Delta_k^{(j)}, \\ 1 - 2^k, & \text{if } x \in \Delta_k^{(j)} = (\frac{j-1}{2^k}, \frac{j}{2^k}), \end{cases} \quad k = 1, 2, \dots,\ 1 \le j \le 2^k, \tag{2}$$

and periodically extend these functions on R^1 with period 1.

By $\chi_E(x)$ we denote the characteristic function of the set E, i.e.

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases} \tag{3}$$

Then, clearly

$$I_k^{(j)}(x) = \varphi_0(x) - 2^k \cdot \chi_{\Delta_k^{(j)}}(x), \tag{4}$$

and let for the natural numbers $k \ge 1$, and $j \in [1, 2^k]$

$$b_i(\chi_{\Delta_k^{(j)}}) = \int_0^1 \chi_{\Delta_k^{(j)}}(x)\varphi_i(x)dx = \pm\frac{1}{2^k},\ 0 \le i < 2^k \tag{5}$$

$$a_i(I_k^{(j)}) = \int_0^1 I_k^{(j)}(x)\varphi_i(x)dx = \begin{cases} 0, & \text{if } i = 0,\ \text{if } i \ge 2^k, \\ \pm 1, & \text{if } 1 \le i < 2^k. \end{cases} \tag{6}$$

Hence

$$\chi_{\Delta_k^{(j)}}(x) = \sum_{i=0}^{2^k-1} b_i(\chi_{\Delta_k^{(j)}})\varphi_i(x) \tag{7}$$

$$I_k^{(j)}(x) = \sum_{i=1}^{2^k-1} a_i(I_k^{(j)})\varphi_i(x) \tag{8}$$

Lemma 1 *Let dyadic interval* $\Delta = \Delta_m^{(k)} = ((k-1)/2^m; k/2^m),\ \ k \in [1, 2^m]$ *and numbers* $N_0 \in N,\ \gamma \neq 0,\ \epsilon \in (0,1), r_2 \in (0,1)$ *be given. Then there exists a measurable set* $E \subset [0,1]$ *and a polynomial* Q *in the Walsh system* $\{\varphi_k\}$ *of the following form*

$$Q = \sum_{k=N_0}^{N} a_k \varphi_k$$

which satisfy the following conditions:

1) the coefficients $\{a_k\}_{k=N_0}^N$ *are* 0 *or* $\pm\gamma \mid \Delta \mid$,

2) $|E| > (1-\varepsilon)|\Delta|$,

3) $Q(x) = \begin{cases} \gamma : & \text{if } x \in E \\ 0 : & \text{if } x \notin \ \Delta \end{cases}$,

4) $\int_\Delta |\gamma\chi_\Delta(x) - Q(x)|^r dx < \epsilon|\Delta||\gamma|^r\ ,\ \forall r \in (0, r_2)$,

5) $\max\limits_{N_0 \le m \le N} \int_0^1 \left| \sum_{k=N_0}^{m} a_k\varphi_k(x) \right| dx < \frac{|\gamma||\Delta|^{\frac{1}{2}}}{\epsilon^{\frac{1}{(1-r_2)}}}$.

Proof. Let

$$\nu_0 = \left[\frac{1}{1-r_2} \log_2 \frac{1}{\epsilon} \right]; s = [\log_2 N_0] + m. \tag{9}$$

We define the polynomial $Q(x)$ and the numbers c_n, a_i and b_j in the following form:

$$Q(x) = \gamma \cdot \chi_{\Delta_m^{(k)}}(x) \cdot I_{\nu_0}^{(1)}(2^s x), x \in [0;1]. \tag{10}$$

$$c_n = c_n(Q) = \int_0^1 Q(x)\varphi_n(x)dx, \forall n \ge 0, \tag{11}$$

$$b_i = b_i(\chi_{\Delta_m^{(k)}}),\ 0 \le i < 2^m,\ a_j = a_j(I_{\nu_0}^{(1)}),\ 0 < j < 2^{\nu_0}. \tag{12}$$

Taking into consideration the following equation

$$\varphi_i(x) \cdot \varphi_j(2^s x) = \varphi_{j \cdot 2^s + i}(x),\ \text{if } 0 \le i,\ \ j < 2^s\ \ \text{(see (1))}\ ,$$

and having the following relations (5)-(8) and (10)-(12), we obtain that the polynomial $Q(x)$ has the following form:

$$Q(x) = \gamma \cdot \sum_{i=0}^{2^m-1} b_i\varphi_i(x) \cdot \sum_{j=1}^{2^{\nu_0}-1} a_j\varphi_j(2^s x) = \gamma \cdot \sum_{j=1}^{2^{\nu_0}-1} a_j \cdot \sum_{i=0}^{2^m-1} b_i\varphi_{j\cdot 2^s+i}(x) = \sum_{k=N_0}^{\bar{N}} c_k\varphi_k(x), \tag{13}$$

where

$$c_k = c_k(Q) = \begin{cases} \pm\frac{\gamma}{2^m} \text{ or } 0, & \text{if } k \in [N_0, \bar{N}], \\ 0, & \text{if } k \notin [N_0, \bar{N}], \end{cases} \quad \bar{N} = 2^{s+\nu_0} + 2^m - 2^s - 1. \tag{14}$$

Then let

$$E = \{x; Q(x) = \gamma\}.$$

Clearly that (see (2) and (10)),

$$|E| = 2^{-m}(1 - 2^{-\nu_0}) > (1-\epsilon)|\Delta|, \tag{15}$$

$$Q(x) = \begin{cases} \gamma, & \text{if } x \in E, \\ \gamma(1 - 2^{\nu_0}), & \text{if } x \in \Delta \setminus E, \\ 0, & \text{if } x \notin \Delta. \end{cases} \tag{16}$$

Hence and from (9) for all $r \in (0, r_2)$ we obtain

$$\int_\Delta |\gamma\chi_1(x) - Q(x)|^r dx = \int_{\Delta\setminus E} |2^{\nu_0}\gamma|^r dx \le |\gamma|^r|\Delta| \left(\frac{1}{2^{\nu_0}}\right)^{(1-r_2)} < |\gamma|^r|\Delta|\epsilon.$$

By Bessel's inequality and by (13)-(16) we have

$$\max_{N_0\le m\le N}\int_0^1\left|\sum_{k=N_0}^{m}a_k\varphi_k(x)\right|dx\le\max_{N_0\le m\le N}\left(\int_0^1\left|\sum_{k=N_0}^{m}a_k\varphi_k(x)\right|^2dx\right)^{\frac{1}{2}}\le\left(\int_0^1 Q^2(x)dx\right)^{\frac{1}{2}}\le\le 2^{\nu_0}|\gamma|\mid\Delta\mid^{\frac{1}{2}}\le\frac{2|\gamma|\mid\Delta\mid^{\frac{1}{2}}}{\epsilon^{\frac{1}{(1-r_2)}}}$$

Lemma 1 is proved.

Lemma 2 *Let given the numbers $\tilde{N}\in N$, $0<\epsilon<1$, $0<r_1<r_2<1$. Then for any function $f\in L^1(0,1)$, $\|f\|_{L_1}>0$, one can find a set $E\subset[0,1]$ and a polynomial in the Walsh system*

$$Q=\sum_{k=\tilde{N}+1}^{M}a_k\varphi_k\ ,$$

satisfying the following conditions:

1) $0\le a_k<\epsilon$ *and the non-zero coefficients in* $\{a_k\}_{k=\tilde{N}+1}^{M}$ *are in decreasing order,*

2) $\mid E\mid>1-\epsilon$,

3) $\int_0^1|Q(x)-f(x)|^r dx<\epsilon\ ,\ \forall r\in(r_1,r_2)$,

4) $\int_E|Q(x)-f(x)|dx<\epsilon$,

5) $\max\limits_{\tilde{N}+1\le m\le M}\int_0^1\left|\sum\limits_{k=\tilde{N}+1}^{m}a_k\varphi_k(x)\right|^r dx<\int_0^1|f|^r dx+\epsilon,\forall r\in(r_1,r_2)$,

6) $\max\limits_{\tilde{N}+1\le m\le M}\left|\int_E\sum\limits_{k=\tilde{N}+1}^{m}a_k\varphi_k(x)dx\right|<\int_E|f(x)|dx+\epsilon.$

Proof. Let

$$\delta=\frac{\epsilon}{2}\min\left\{\left(\frac{\epsilon}{2}\right)^{\frac{1-r_1}{r_1}};\left(\sup_{r\in(r_1,r_2)}\int_0^1|f(x)|^r dx+1\right)^{-1}\right\}. \tag{17}$$

We choose some non-overlapping binary intervals $\{\Delta_\nu\}_{\nu=1}^{\nu_0}$ and a step function

$$\varphi(x)=\sum_{\nu=1}^{\nu_0}\gamma_\nu\cdot\chi_{\Delta_\nu}(x) \tag{18}$$

satisfying the conditions

$$\sum_{\nu=1}^{\nu_0}|\Delta_\nu|=1,\quad\max_{1\le\nu\le\nu_0}|\gamma_\nu||\Delta_\nu|^{\frac{1}{2}}<\frac{\epsilon}{4}\cdot\delta^{\frac{1}{1-r_2}}, \tag{19}$$

$$\frac{\epsilon}{2}>|\gamma_1||\Delta_1|>...>|\gamma_\nu||\Delta_\nu|>...>|\gamma_{\nu_0}||\Delta_{\nu_0}|>0, \tag{20}$$

$$\left(\int_0^1|f-\varphi|dx\right)<\delta^2. \tag{21}$$

Now let

$$B=\{x\in[0,1]:\ |f(x)-\varphi(x)|<\delta\}. \tag{22}$$

Then by (17), (21) and (22)

$$|B|>1-\frac{\epsilon}{2}. \tag{23}$$

Successively applying Lemma 1, we determine some sets $E_\nu\subset[0,1]$ and polynomials

$$Q_\nu=\sum_{j=m_{\nu-1}}^{m_\nu-1}a_j\varphi_j,\ (m_0=\bar{N}+1),\ \nu=1,...,\nu_0, \tag{24}$$

where $a_j = 0$ or $\pm\gamma_j|\Delta_j|$, if $j \in [m_{\nu-1}, m_\nu)$,

$$|E_\nu| > (1 - \frac{\epsilon}{2}) \cdot |\Delta_\nu|, \tag{25}$$

$$Q_\nu = \begin{cases} \gamma_\nu : & \text{if } x \in E_\nu \\ 0 : & \text{if } x \notin \Delta_\nu \end{cases}, \tag{26}$$

$$\int_{\Delta_\nu} |\gamma_\nu \chi_{\Delta_\nu}(x) - Q_\nu(x)|^r dx < \delta |\Delta_\nu| |\gamma_\nu|^r, \ \forall r \in (0, r_2)\,, \tag{27}$$

$$\max_{m_{\nu-1} \le m \le m_\nu - 1} \left(\int_0^1 \left| \sum_{k=m_{\nu-1}}^{m} a_k \varphi_k(x) \right| dx \right) < \frac{2|\gamma_\nu| \mid \Delta_\nu \mid^{\frac{1}{2}}}{\delta^{\frac{1}{(1-r_2)}}}\,, \tag{28}$$

Then let

$$E = \bigcup_{\nu=1}^{\nu_0} E_\nu \bigcap B, \tag{29}$$

$$Q = \sum_{\nu=1}^{\nu_0} Q_\nu = \sum_{k=\tilde{N}+1}^{M} a_k \varphi_k, \tag{30}$$

From (20),(23), (24) (25) and (29) follows, that

$$|E| > 1 - \epsilon.$$

and $0 \le a_k < \epsilon$ and the non-zero coefficients in $\{a_k\}_{k=\tilde{N}+1}^{M}$ are in decreasing order.

By (17),(18),(21),(30) for all $r \in (r_1, r_2)$ we have

$$\int_0^1 |Q(x) - f(x)|^r dx \le \left(\int_0^1 |f(x) - \varphi(x)| dx \right)^r + \sum_{\nu=1}^{\nu_0} \int_0^1 |\gamma_\nu \chi_{\Delta_\nu}(x) - Q_\nu(x)|^r dx < \delta^{r_1} + \delta \cdot \int_0^1 |\varphi(x)|^r < \epsilon,$$

i.e. the statements 1), 2), 3) of Lemma 2 are valid.

To verify the statements 5) and 6), for any $\tilde{N} < m \le M$ determine ν from the condition $m_{\nu-1} \le m < m_\nu$. Then by (23) and (30)

$$\sum_{k=\tilde{N}+1}^{m} a_k \varphi_k = \sum_{n=1}^{\nu-1} Q_n + \sum_{k=m_\nu-1}^{m} a_k \varphi_k. \tag{31}$$

Hence by (17)-(19), (21), (26), (27) and (28) we obtain that for all $r \in (r_1, r_2)$

$$\int_0^1 | \sum_{k=\tilde{N}+1}^{m} a_k \varphi_k(x)|^r dx \le \sum_{n=1}^{\nu-1} \int_0^1 |Q_n(x) - \gamma_n \chi_{\Delta_n}(x) \mid^r dx + \sum_{n=1}^{\nu_0} \int_0^1 |\gamma_n \chi_{\Delta_n}(x)|^r dx +$$

$$+ \int_0^1 | \sum_{k=m_\nu-1}^{m} a_k \varphi_k(x)|^r \le \delta \cdot \int_0^1 |\varphi(x)|^r dx + \int_0^1 |f(x)|^r dx + \delta^{r_1} + \frac{\epsilon}{2} \le \int_0^1 |f(x)|^r dx + \epsilon.$$

Since for any point $x \in E$, $Q(x) = \varphi(x)$ (see (26),(29) and (30)), then from the conditions (17),(21), (28), (29) and (31), we have

$$\int_E |Q(x) - f(x)| dx = \int_E |\varphi(x) - f(x)| dx < \epsilon.$$

$$\left| \int_E | \sum_{k=\tilde{N}+1}^{m} a_k \varphi_k(x) dx \right| \le \int_E \left| \sum_{n=1}^{\nu-1} \gamma_n \chi_{\Delta_n}(x) dx \right| + \int_E \left| \sum_{n=m_\nu-1}^{m} a_n \varphi_n(x) dx \right| dx \le \int_E |\varphi(x)| dx + \frac{\epsilon}{2} \le \int_E |f(x)| dx + \epsilon.$$

Lemma 2 is proved.

3. Proof of the Theorem

Let

$$\{f_k(x)\}_{k=1}^{\infty}, \quad x \in [0, 1], \tag{32}$$

be the sequence of all algebraic polynomials with rational coefficients. Applying repeatedly Lemma 2, we obtain sequences of $\{E_k\}_{k=1}^{\infty}$ sets and polynomials in the Walsh systems $\{\varphi_n(x)\}$

$$Q_k(x) = \sum_{i=N_k}^{M_k} a_{n_i}\varphi_{n_i}(x), \tag{33}$$

where

$$N_1 = 1;\ N_k = M_{k-1} + 1,\ k \geq 2,$$

which satisfy the following conditions:

$$2^{-j} > |a_{n_i}| \geq |a_{n_{i+1}}| > 0,\ \forall i \in [N_k, M_k],\ k = 1, 2, \ldots, \tag{34}$$

$$\int_0^1 |Q_k(x) - f_k(x)|^r dx < 2^{-4(k+1)},\ \forall r \in [2^{-k}, 1 - 2^{-k}], \tag{35}$$

$$\int_{E_k} |Q_k(x) - f_k(x)| dx < 2^{-4k}, \tag{36}$$

$$\max_{N_k \leq m \leq M_k} \int_0^1 \left| \sum_{i=N_k}^{m} a_{n_i}\varphi_{n_i}(x) \right|^r dx < \int_0^1 |f_j(x)|^r dx + 2^{-k-1},\ k = 1, 2, \ldots, \tag{37}$$

$$\max_{N_k \leq m \leq M_k} \int_{E_k} \left| \sum_{i=N_k}^{m} a_{n_i}\varphi_{n_i}(x) \right| dx < \int_0^1 |f_k(x)| dx + 2^{-k-1}, \tag{38}$$

$$|E_k| > 1 - 2^{-k-1}. \tag{39}$$

Consider a series

$$\sum_{s=1}^{\infty} a_s\varphi_s(x), \text{where } a_s = a_{n_i} \text{ if } s \in [n_i, n_{i+1}) \qquad \text{(see (33))}, \tag{40}$$

and a set

$$E = \bigcap_{k=k_0}^{\infty} E_k,\ \text{where } k_0 = [\log_2 \frac{1}{\epsilon}] + 1. \tag{41}$$

Clearly that (see (33), (34) and (39)-(41))

$$|a_k| \searrow 0 \qquad \text{and} \qquad |E| > 1 - \epsilon.$$

Let $r \in (0, 1)$, then for some $j_0 > 1$ we have $r \in \left(\frac{1}{j_0}, 1 - \frac{1}{j_0}\right)$, and let $f(x) \in L^r_{(0,1)}$.

We choose some $f_{\nu_1}(x),\ \nu_1 > j_0$, from sequence (32), to have

$$\int_0^1 |f(x) - f_{\nu_1}(x)|^r dx < 2^{-4}.$$

Denote that the numbers $j_0 < \nu_1 < ... < \nu_{q-1}$ and polynomials $Q_{\nu_1}(x), \ldots, Q_{\nu_{q-1}}(x)$ are already determined satisfying to the following conditions:

$$\int_0^1 |f(x) - \sum_{n=1}^{s} Q_{\nu_n}(x)|^r dx < 2^{-4s},\ s \in [2, q-1], \tag{42}$$

$$\max_{N_{\nu_n} \leq m \leq M_{\nu_n}} \int_0^1 \left| \sum_{i=N_{\nu_n}}^{m} a_{n_i}\varphi_{n_i}(x) \right|^r dx < 2^{-n},\ n \in [2, q-1] \tag{43}$$

Let a function $f_{\nu_q}(x)$, $\nu_q > \nu_{q-1}$ is from the sequence (32) such that

$$\int_0^1 \left| \left[f(x) - \sum_{j=1}^{q-1} Q_j^{(\nu_j)}(x) \right] - f_{\nu_q}(x) \right|^r dx < 2^{-4(q+1)}. \tag{44}$$

Hence by (42) we obtain

$$\int_0^1 |f_{\nu_q}|^r dx < 2^{-q-1}. \tag{45}$$

From the conditions (35), (37), (44) and (45) follows that

$$\int_0^1 |f(x) - \sum_{j=1}^{q} Q_{\nu_j}(x)|^r dx < 2^{-4q}, \tag{46}$$

$$\max_{N_{\nu_q} \le m \le M_{\nu_q}} \int_0^1 \left| \sum_{i=N_{\nu_q}}^{m} a_{n_i} \varphi_{n_i}(x) \right|^r dx < 2^{-q}, \tag{47}$$

Then we obtain that the series

$$\sum_{k=1}^{\infty} \delta_k a_k \varphi_k(x) \quad \text{(see (40))},$$

where

$$\delta_k = \begin{cases} 1, & \text{if } k = n_i, \text{ where } i = \bigcup_{q=1}^{\infty} [N_{\nu_q}, M_{\nu_q}], \\ 0, & \text{otherwise .} \end{cases}$$

converges to $f(x)$ in the $L^r_{(0,1)}$.

From the conditions (36), (38), (44) and (41) follows that

$$\sum_{k=1}^{\infty} \delta_k a_k \varphi_k(x) \quad \text{(see (40))},$$

series converges to $f(x)$ in the $L^1(E)$ metric.

Theorem is proved.

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