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Anisotropic Fractional Maximal Operator in Anisotropic Generalized Morrey Spaces

M. S. Dzhabrailov[1] & S. Z. Khaligova[1]

[1] Azerbaijan State Pedagogical University, Baku, Azerbaijan

Correspondence: S. Z. Khaligova, Azerbaijan State Pedagogical University, Baku, Azerbaijan. E-mail: seva.xaligova@hotmail.com

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Abstract

In this paper it is proved that anisotropic fractional maximal operator $M_{\alpha,\sigma}$, $0 \le \alpha < |\sigma|$ is bounded on anisotropic generalized Morrey spaces $M_{p,\varphi,\sigma}$, where $|\sigma| = \sum_{i=1}^{n} \sigma_i$ is the homogeneous dimension of $\mathbb{R}^n$. We find the conditions on the pair (φ_1, φ_2) which ensure the Spanne-Guliyev type boundedness of the operator $M_{\alpha,\sigma}$ from anisotropic generalized Morrey space $M_{p,\varphi_1,\sigma}$ to $M_{q,\varphi_2,\sigma}$, $1 < p \le q < \infty$, $1/p - 1/q = \alpha/|\sigma|$, and from the space $M_{1,\varphi_1,\sigma}$ to the weak space $WM_{q,\varphi_2,\sigma}$, $1 < q < \infty$, $1 - 1/q = \alpha/|\sigma|$. We also find conditions on the φ which ensure the Adams-Guliyev type boundedness of $M_{\alpha,\sigma}$ from $M_{p,\varphi^{\frac{1}{p}},\sigma}$ to $M_{q,\varphi^{\frac{1}{q}},\sigma}$ for $1 < p < q < \infty$ and from $M_{1,\varphi,\sigma}$ to $WM_{q,\varphi^{\frac{1}{q}},\sigma}$ for $1 < q < \infty$.

As applications, we establish the boundedness of some Schödinger type operators on anisotropic generalized Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class.

Keywords: anisotropic fractional maximal function, anisotropic generalized Morrey space

2000 Mathematics Subject Classification: 42B25; 42B35

1. Introduction

In the present paper we will prove the boundedness of the anisotropic fractional maximal operator in the anisotropic generalized Morrey spaces.

For $x \in \mathbb{R}^n$ and $t > 0$, let $B(x,t)$ denote the open ball centered at x of radius t and ${}^{\complement}B(x,t) = \mathbb{R}^n \setminus B(x,t)$. Let $0 \le b \le 1$, $\sigma = (\sigma_1, ..., \sigma_n)$ with $\sigma_i > 0$ for $i = 1, ..., n$, $|\sigma| = \sigma_1 + ... + \sigma_n$ and $t^{\sigma} x \equiv (t^{\sigma_1} x_1, ..., t^{\sigma_n} x_n)$ for $t > 0$. For $x \in \mathbb{R}^n$ and $t > 0$, let $E_\sigma(x,t) = \prod_{i=1}^{n} (x_i - t^{\sigma_i}, x_i + t^{\sigma_i})$ denote the open parallelepiped centered at x of side length t^{σ_i} for $i = 1, ..., n$.

By Besov, Il'in and Lizorkin (1966) and Fabes and Rivère (1966), the function $F(x,\rho) = \sum_{i=1}^{n} x_i^2 \rho^{-2\sigma_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x,\rho) = 1$ is uniquely solvable. This unique solution will be denoted by $\rho(x)$. Define $\rho(x) = \rho$ and $\rho(0) = 0$. It is a simple matter to check that $\rho(x - y)$ defines a distance between any two points $x, y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space (Besov, Il'in, & Lizorkin, 1966; Bramanti & Cerutti, 1996; Fabes & Rivère, 1966). Note that $\rho(x)$ is equivalent to $|x|_\sigma = \max_{1 \le i \le n} |x_i|^{\frac{1}{\sigma_i}}$.

One of the most important variants of the anisotropic maximal function is the so-called anisotropic fractional maximal function defined by the formula

$$M_{\alpha,\sigma} f(x) = \sup_{t>0} |E_\sigma(x,t)|^{-1+\alpha/|\sigma|} \int_{E_\sigma(x,t)} |f(y)| dy, \quad 0 \le \alpha < |\sigma|,$$

where $|E_\sigma(x,t)| = 2^n t^{|\sigma|}$ is the Lebesgue measure of the parallelepiped $E_\sigma(x,t)$.

It coincides with the anisotropic maximal function $M_\sigma f \equiv M_{0,\sigma} f$ and is intimately related to the anisotropic Riesz potential operator

$$I_{\alpha,\sigma} f(x) = \int_{\mathbb{R}^n} \frac{f(y) dy}{|x - y|_\sigma^{|\sigma| - \alpha}}, \qquad 0 < \alpha < |\sigma|.$$

If $\sigma = \mathbf{1}$, then $M_\alpha \equiv M_{\alpha,\mathbf{1}}$ and $I_\alpha \equiv I_{\alpha,\mathbf{1}}$ is the fractional maximal operator and Riesz potential, respectively. The operators M_α, $M_{\alpha,\sigma}$, I_α and $I_{\alpha,\sigma}$ play important role in real and harmonic analysis (see, for example Besov, Il'in, & Nikol'skii, 1996; Stein, 1993).

Definition 1.1 Let $0 \le b \le 1$ and $1 \le p < \infty$. We denote by $L_{p,b,\sigma} \equiv L_{p,b,\sigma}(\mathbb{R}^n)$ anisotropic Morrey space, the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norm

$$\|f\|_{L_{p,b,\sigma}} = \sup_{x\in\mathbb{R}^n,\, t>0} \left(t^{-b|\sigma|} \int_{E_\sigma(x,t)} |f(y)|^p dy \right)^{1/p}.$$

Remark 1.1 Note that $L_{p,0,\sigma} = L_p(\mathbb{R}^n)$ and $L_{p,1,\sigma} = L_\infty(\mathbb{R}^n)$. If $b < 0$ or $b > 1$, then $L_{p,b,\sigma} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. In the case $\sigma \equiv \mathbf{1} = (1, ..., 1)$ and $b = \frac{\lambda}{n}$ we get the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n) = L_{p,\frac{\lambda}{n},\mathbf{1}}(\mathbb{R}^n)$, $0 \le \lambda \le n$.

In the theory of partial differential equations, together with weighted $L_{p,w}(\mathbb{R}^n)$ spaces, Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ play an important role. Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see Morrey, 1938). Later, Morrey spaces found important applications to Navier-Stokes (Mazzucato, 2003; Taylor, 1992) and Schrödinger (Perez, 1990; Ruiz & Vega, 1991 & 1993; Shen, 2002 & 2003) equations, elliptic problems with discontinuous coefficients (Caffarelli, 1990; Fazio, Palagachev, & Ragusa, 1999; Softova, 2006 & 2008), and potential theory (Adams, 1975 & 1998). An exposition of the Morrey spaces can be found in the book Kufner, John and Fucik (1977).

Definition 1.2 (Burenkov, Guliyev, H. V., & Guliyev, V. S., 2007) Let $1 \le p < \infty$ and $0 \le b \le 1$. We denote by $WL_{p,b,\sigma} \equiv WL_{p,b,\sigma}(\mathbb{R}^n)$ the weak anisotropic Morrey space as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$ with finite norm

$$\|f\|_{WL_{p,b,\sigma}} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, t>0} \left(t^{-b|\sigma|} \, |\{y \in E_\sigma(x,t) : \, |f(y)| > r\}| \right)^{1/p}.$$

Note that

$$WL_p(\mathbb{R}^n) = WL_{p,0,\sigma}(\mathbb{R}^n),$$
$$L_{p,b,\sigma}(\mathbb{R}^n) \subset WL_{p,b,\sigma}(\mathbb{R}^n) \text{ and } \|f\|_{WL_{p,b,\sigma}} \le \|f\|_{L_{p,b,\sigma}},$$

The anisotropic result by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then $I_{\alpha,\sigma}$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = |\sigma|\left(\frac{1}{p} - \frac{1}{q}\right)$ and for $p = 1 < q < \infty$, $I_{\alpha,\sigma}$ is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha = |\sigma|\left(1 - \frac{1}{q}\right)$. Spanne (see Spanne, 1966) and Adams (1975) studied boundedness of the Riesz potential I_α for $0 < \alpha < n$ in Morrey spaces $L_{p,\lambda}$. Later on Chiarenza and Frasca (1987) was reproved boundedness of the Riesz potential I_α in these spaces. By more general results of Guliyev (1994) (see also 1999 & 2009) one can obtain the following generalization of the results in Adams (1975), Chiarenza and Frasca (1987) and Spanne (1966) to the anisotropic case.

Theorem A *Let* $0 < \alpha < |\sigma|$ *and* $0 \le b < 1$, $1 \le p < \frac{(1-b)|\sigma|}{\alpha}$.

1) If $1 < p < \frac{(1-b)|\sigma|}{\alpha}$, *then condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{(1-b)|\sigma|}$ *is necessary and sufficient for the boundedness of the operators* $M_{\alpha,\sigma}$ *and* $I_{\alpha,\sigma}$ *from* $L_{p,b,\sigma}(\mathbb{R}^n)$ *to* $L_{q,b,\sigma}(\mathbb{R}^n)$.

2) If $p = 1$, *then condition* $1 - \frac{1}{q} = \frac{\alpha}{(1-b)|\sigma|}$ *is necessary and sufficient for the boundedness of the operators* $M_{\alpha,\sigma}$ *and* $I_{\alpha,\sigma}$ *from* $L_{1,b,\sigma}(\mathbb{R}^n)$ *to* $WL_{q,b,\sigma}(\mathbb{R}^n)$.

It is known that the anisotropic maximal operator M_σ is also bounded from $L_{p,b,\sigma}$ to $L_{p,b,\sigma}$ for all $1 < p < \infty$ and $0 < b < 1$ (see, for example Guliyev, 1994 & 1999), which isotropic case proved by Chiarenza and Frasca (1987).

In this work, we prove the boundedness of the fractional maximal operator $M_{\alpha,\sigma}$, $0 \le \alpha < |\sigma|$ from one generalized Morrey space $M_{p,\varphi_1,\sigma}$ to $M_{q,\varphi_2,\sigma}$, $1 < p \le q < \infty$, $1/p - 1/q = \alpha/|\sigma|$, and from $M_{1,\varphi_1,\sigma}$ to the weak space $WM_{q,\varphi_2,\sigma}$, $1 < q < \infty$, $1 - 1/q = \alpha/|\sigma|$. We also prove the Adams-Guliyev type boundedness of the operator $M_{\alpha,\sigma}$ from $M_{p,\varphi^{\frac{1}{p}},\sigma}$ to $M_{q,\varphi^{\frac{1}{q}},\sigma}$ for $1 < p < q < \infty$ and from $M_{1,\varphi,\sigma}$ to $WM_{q,\varphi^{\frac{1}{q}},\sigma}$ for $1 < q < \infty$. In all the cases the conditions for the boundedness are given it terms of supremal-type inequalities on (φ_1, φ_2) and φ, which do not assume any assumption on monotonicity of (φ_1, φ_2) and φ in r. By using the $(M_{p,\varphi_1,\sigma}, M_{q,\varphi_2,\sigma})$ boundedness of the fractional maximal operators we establish the boundedness of some Schödinger type operators on anisotropic generalized Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Notations

Everywhere in the sequel the functions $\varphi(x, r)$, $\varphi_1(x, r)$ and $\varphi_2(x, r)$ used in the body of the paper, are non-negative measurable function on $\mathbb{R}^n \times (0, \infty)$.

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 2.3 Let $1 \leq p < \infty$. The anisotropic generalized Morrey space $M_{p,\varphi,\sigma}$ is defined of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ by the finite norm

$$\|f\|_{M_{p,\varphi,\sigma}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x, r)^{-1} \, |E_\sigma(x, r)|^{-\frac{1}{p}} \, \|f\|_{L_p(E_\sigma(x,r))}.$$

According to this definition, when $\varphi(x, r) = r^{\frac{(b-1)|\sigma|}{p}}$, we can see that

$$M_{p,\varphi,\sigma}(\mathbb{R}^n) = L_{p,b,\sigma}(\mathbb{R}^n).$$

There are many papers discussed the conditions on φ to obtain the boundedness of integral operators on the generalized Morrey spaces (see Guliyev, 1994, 1999 & 2009; Mizuhara, 1991; Nakai, 1994 & 2006; Softova, 2006).

In Nakai (2006) the following statements were proved.

Theorem 2.1 *Let* $1 \leq p < \infty, 0 < \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$ *and* $\varphi(x, \tau)$ *satisfy conditions*

$$c^{-1}\varphi(x, r) \leq \varphi(x, \tau) \leq c\, \varphi(x, r), \tag{2.1}$$

whenever $r \leq \tau \leq 2r$, *where* c *does not depend on* r, τ *and* $x \in \mathbb{R}^n$,

$$\int_r^\infty \tau^{\alpha p} \varphi(x, \tau)^p \frac{d\tau}{\tau} \leq C\, r^{\alpha p} \varphi(x, r)^p. \tag{2.2}$$

Then for $p > 1$ *the operator* $M_{\alpha,\sigma}$ *is bounded from* $M_{p,\varphi,\sigma}$ *to* $M_{q,\varphi,\sigma}$ *and for* $p = 1$ $M_{\alpha,\sigma}$ *is bounded from* $M_{1,\varphi,\sigma}$ *to* $WM_{q,\varphi,\sigma}$.

The following statements, containing results obtained in Nakai (2006) was proved in Nakai (1994) (see also Guliyev, 1999 & 2009; Guliyev & Mustafayev, 1997 & 1998).

Theorem 2.2 *Let* $1 \leq p < \infty$, $0 < \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_\tau^\infty r^\alpha \varphi_1(x, r) \frac{dr}{r} \leq C\, \varphi_2(x, \tau), \tag{2.3}$$

where C *does not depend on* x *and* τ. *Then the operator* $M_{\alpha,\sigma}$ *is bounded from* $M_{p,\varphi_1,\sigma}$ *to* $M_{q,\varphi_2,\sigma}$ *for* $p > 1$ *and from* $M_{1,\varphi_1,\sigma}$ *to* $WM_{q,\varphi_2,\sigma}$ *for* $p = 1$.

In Guliyev, Aliyev, Karaman and Shukurov (2011) obtained sufficient conditions on the pair (φ_1, φ_2)

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x, s) s^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \leq C\, \varphi_2(x, r), \tag{2.4}$$

for the boundedness of M_α from $M_{p,\varphi,\mathbf{1}}$ to $M_{q,\varphi,\mathbf{1}}$, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

3. Boundedness of the Fractional Maximal Operator in the Spaces $M_{p,\varphi,\sigma}$

3.1 Spanne-Guliyev Type Result

Sufficient conditions on φ for the boundedness of M_σ and $M_{\alpha,\sigma}$ in generalized Morrey spaces $\mathcal{M}_{p,\varphi,\sigma}$ have been obtained in Akbulut, Guliyev and Mustafayev (2012), Burenkov, Gogatishvili, Guliyev and Mustafayev (2010), Guliyev (2009), Guliyev and Mustafayev (1997 & 1998) and Nakai (2006).

The following lemma is true.

Lemma 3.1 *Let* $1 \le p < \infty$, $0 \le \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$. *Then for* $p > 1$ *and any ball* $E_\sigma = E_\sigma(x, r)$ *the inequality*

$$\|M_{\alpha,\sigma} f\|_{L_q(E_\sigma(x,r))} \lesssim \|f\|_{L_p(E_\sigma(x,2r))} + r^{\frac{|\sigma|}{q}} \sup_{\tau>2r} \tau^{-|\sigma|+\alpha} \|f\|_{L_1(E_\sigma(x,\tau))} \tag{3.1}$$

holds for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover for $p = 1$ *the inequality*

$$\|M_{\alpha,\sigma} f\|_{WL_q(E_\sigma(x,r))} \lesssim \|f\|_{L_1(E_\sigma(x,2r))} + r^{\frac{|\sigma|}{q}} \sup_{\tau>2r} \tau^{-|\sigma|+\alpha} \|f\|_{L_1(E_\sigma(x,\tau))} \tag{3.2}$$

holds for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|\sigma|}$. For arbitrary ball $E_\sigma = E_\sigma(x, r)$ let $f = f_1 + f_2$, where $f_1 = f\chi_{2E_\sigma}$ and $f_2 = f\chi_{\complement(2E_\sigma)}$.

$$\|M_{\alpha,\sigma} f\|_{L_q(E_\sigma)} \le \|M_{\alpha,\sigma} f_1\|_{L_q(E_\sigma)} + \|M_{\alpha,\sigma} f_2\|_{L_q(E_\sigma)}.$$

By the continuity of the operator $M_{\alpha,\sigma} : L_p(\mathbb{R}^n) \to L_q(\mathbb{R}^n)$ we have

$$\|M_{\alpha,\sigma} f_1\|_{L_q(E_\sigma)} \lesssim \|f\|_{L_p(2E_\sigma)}.$$

Let y be an arbitrary point from E_σ. If $E_\sigma(y, t) \cap {}^{\complement}(2E_\sigma) \neq \emptyset$, then $t > r$. Indeed, if $z \in E_\sigma(y, t) \cap {}^{\complement}(2E_\sigma)$, then $t > |y - z|_\sigma \ge |x - z|_\sigma - |x - y|_\sigma > 2^{\sigma_{\min}} r - r = (2^{\sigma_{\min}} - 1)r$.

On the other hand, $E_\sigma(y, t) \cap {}^{\complement}(2E_\sigma) \subset E_\sigma(x, 2t)$. Indeed, $z \in E_\sigma(y, t) \cap {}^{\complement}(2E_\sigma)$, then we get $|x - z|_\sigma \le 2^{\sigma_{\min}}(|y - z|_\sigma + |x - y|_\sigma) < 2^{\sigma_{\min}} t + r < (2^{\sigma_{\min}} + 1)t$.

Hence

$$\begin{aligned} M_{\alpha,\sigma} f_2(y) &= \sup_{t>0} \frac{1}{|E_\sigma(y,t)|^{1-\alpha/|\sigma|}} \int_{E_\sigma(y,t) \cap {}^{\complement}(2E_\sigma)} |f(z)| dz \\ &\le 2^{n\left(1-\frac{\alpha}{|\sigma|}\right)} \sup_{t>r} \frac{1}{|E_\sigma(x,2t)|^{1-\alpha/|\sigma|}} \int_{E_\sigma(x,2t)} |f(z)| dz \\ &= 2^{n\left(1-\frac{\alpha}{|\sigma|}\right)} \sup_{t>2r} \frac{1}{|E_\sigma(x,t)|^{1-\alpha/|\sigma|}} \int_{E_\sigma(x,t)} |f(z)| dz. \end{aligned}$$

Therefore, for all $y \in E_\sigma$ we have

$$M_{\alpha,\sigma} f_2(y) \le 2^{n\left(1-\frac{\alpha}{|\sigma|}\right)} \sup_{t>2r} \frac{1}{|E_\sigma(x,t)|^{1-\alpha/|\sigma|}} \int_{E_\sigma(x,t)} |f(z)| dz. \tag{3.3}$$

Thus

$$\|M_{\alpha,\sigma} f\|_{L_q(E_\sigma)} \lesssim \|f\|_{L_p(2E_\sigma)} + |E_\sigma|^{\frac{1}{q}} \left(\sup_{t>2r} \frac{1}{|E_\sigma(x,t)|^{1-\alpha/|\sigma|}} \int_{E_\sigma(x,t)} |f(z)| dz \right).$$

Let $p = 1$. It is obvious that for any ball $E_\sigma = E_\sigma(x, r)$

$$\|M_{\alpha,\sigma} f\|_{WL_q(E_\sigma)} \le \|M_{\alpha,\sigma} f_1\|_{WL_q(E_\sigma)} + \|M_{\alpha,\sigma} f_2\|_{WL_q(E_\sigma)}.$$

By the continuity of the operator $M_{\alpha,\sigma} : L_1(\mathbb{R}^n) \to WL_q(\mathbb{R}^n)$ we have

$$\|M_{\alpha,\sigma} f_1\|_{WL_q(E_\sigma)} \lesssim \|f\|_{L_1(2E_\sigma)}.$$

Then by (3.3) we get the inequality (3.2).

Lemma 3.2 *Let* $1 \le p < \infty$, $0 \le \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$. *Then for* $p > 1$ *and any ellipsoid* $E_\sigma = E_\sigma(x, r)$ *in* $\mathbb{R}^n$, *the inequality*

$$\|M_{\alpha,\sigma} f\|_{L_q(E_\sigma(x,r))} \lesssim r^{\frac{|\sigma|}{q}} \sup_{t>2r} t^{-\frac{|\sigma|}{q}} \|f\|_{L_p(E_\sigma(x,t))} \tag{3.4}$$

holds for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover for $p = 1$ *the inequality*

$$\|M_{\alpha,\sigma} f\|_{WL_q(E_\sigma(x,r))} \lesssim r^{\frac{|\sigma|}{q}} \sup_{t>2r} t^{-\frac{|\sigma|}{q}} \|f\|_{L_1(E_\sigma(x,t))} \tag{3.5}$$

holds for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$, $0 \le \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$. Denote

$$\mathcal{M}_1 := |E_\sigma|^{\frac{1}{q}} \left(\sup_{t>2r} \frac{1}{|E_\sigma(x,t)|^{1-\alpha/|\sigma|}} \int_{E_\sigma(x,t)} |f(z)| dz \right),$$
$$\mathcal{M}_2 := \|f\|_{L_p(2E_\sigma)}.$$

Applying Hölder's inequality, we get

$$\mathcal{M}_1 \lesssim |E_\sigma|^{\frac{1}{q}} \left(\sup_{t>2r} \frac{1}{|E_\sigma(x,t)|^{\frac{1}{q}}} \left(\int_{E_\sigma(x,t)} |f(z)|^p dz \right)^{\frac{1}{p}} \right).$$

On the other hand,

$$|E_\sigma|^{\frac{1}{q}} \left(\sup_{t>2r} \frac{1}{|E_\sigma(x,t)|^{\frac{1}{q}}} \left(\int_{E_\sigma(x,t)} |f(z)|^p dz \right)^{\frac{1}{p}} \right)$$
$$\gtrsim |E_\sigma|^{\frac{1}{q}} \left(\sup_{t>2r} \frac{1}{|E_\sigma(x,t)|^{\frac{1}{q}}} \right) \|f\|_{L_p(2E_\sigma)} \approx \mathcal{M}_2.$$

Since by Lemma 3.1

$$\|M_{\alpha,\sigma} f\|_{L_q(E_\sigma)} \le \mathcal{M}_1 + \mathcal{M}_2,$$

we arrive at (3.4).

Let $p = 1$. The inequality (3.5) directly follows from (3.2).

Theorem 3.3 *Let* $1 \le p < \infty$, $0 \le \alpha < \frac{|\sigma|}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{|\sigma|}$, *and* (φ_1, φ_2) *satisfies the condition*

$$\sup_{r<t<\infty} t^\alpha \varphi_1(x,t) \le C \varphi_2(x,r), \tag{3.6}$$

where C does not depend on x and r. Then for $p > 1$, $M_{\alpha,\sigma}$ *is bounded from* $M_{p,\varphi_1,\sigma}$ *to* $M_{q,\varphi_2,\sigma}$ *and for* $p = 1$, $M_{\alpha,\sigma}$ *is bounded from* $M_{1,\varphi_1,\sigma}$ *to* $WM_{q,\varphi_2,\sigma}$.

Proof. By Lemma we get

$$\|M_{\alpha,\sigma} f\|_{M_{q,\varphi_2,\sigma}} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \left(\sup_{t>r} t^{-\frac{|\sigma|}{q}} \|f\|_{L_p(E_\sigma(x,t))} \right)$$
$$\lesssim \|f\|_{M_{p,\varphi_1,\sigma}} \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \sup_{t>r} t^\alpha \varphi_1(x,t)$$
$$\lesssim \|f\|_{M_{p,\varphi_1,\sigma}}$$

if $p \in (1, \infty)$ and

$$\|M_{\alpha,\sigma} f\|_{WM_{q,\varphi_2,\sigma}} \lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \left(\sup_{t>r} t^{-\frac{|\sigma|}{q}} \|f\|_{L_1(E_\sigma(x,t))} \right)$$
$$\lesssim \|f\|_{M_{1,\varphi_1,\sigma}} \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \sup_{t>r} t^\alpha \varphi_1(x,t)$$
$$\lesssim \|f\|_{M_{1,\varphi_1,\sigma}}$$

if $p = 1$.

In the case $\alpha = 0$ and $p = q$ from Theorem 3.3 we get the following corollary, which proven in Akbulut, Guliyev and Mustafayev (2012) on $\mathbb{R}^n$.

Corollary 3.1 *Let $1 \le p < \infty$ and (φ_1, φ_2) satisfies the condition*

$$\sup_{r<t<\infty} \varphi_1(x,t) \le C\,\varphi_2(x,r), \tag{3.7}$$

where C does not depend on x and r. Then for $p > 1$, M_σ is bounded from $M_{p,\varphi_1,\sigma}$ to $M_{p,\varphi_2,\sigma}$ and for $p = 1$, M_σ is bounded from $M_{1,\varphi_1,\sigma}$ to $WM_{1,\varphi_2,\sigma}$.

Corollary 3.2 *Let $p \in [1,\infty)$ and let $\varphi : (0,\infty) \to (0,\infty)$ be an decreasing function. Assume that the mapping $r \mapsto \varphi(r)\, r^{\frac{|\sigma|}{p}}$ is almost increasing (there exists a constant c such that for $s < r$ we have $\varphi(s)\, s^{\frac{|\sigma|}{p}} \le c\varphi(r)\, r^{\frac{|\sigma|}{p}}$). Then there exists a constant $C > 0$ such that*

$$\|Mf\|_{\mathcal{M}_{p,\varphi,\sigma}} \le C\|f\|_{\mathcal{M}_{p,\varphi,\sigma}} \quad \text{if} \quad p > 1,$$

and

$$\|Mf\|_{W\mathcal{M}_{1,\varphi,\sigma}} \le C\|f\|_{\mathcal{M}_{1,\varphi,\sigma}}.$$

3.2 Adams-Guliyev Type Result

The following is a result of Adams-Guliyev type for the fractional maximal operator.

Theorem 3.4 *Let $1 \le p < q < \infty$, $0 < \alpha < \frac{|\sigma|}{p}$ and let $\varphi(x,t)$ satisfy the condition*

$$\sup_{r<t<\infty} \varphi(x,t) \le C\,\varphi(x,r) \tag{3.8}$$

and

$$\sup_{r<t<\infty} t^{\alpha}\varphi(x,t)^{\frac{1}{p}} \le C r^{-\frac{\alpha p}{q-p}}, \tag{3.9}$$

where C does not depend on $x \in \mathbb{R}^n$ and $r > 0$. Then the operator $M_{\alpha,\sigma}$ is bounded from $M_{p,\varphi^{\frac{1}{p}},\sigma}$ to $M_{q,\varphi^{\frac{1}{q}},\sigma}$ for $p > 1$ and from $M_{1,\varphi,\sigma}$ to $WM_{q,\varphi^{\frac{1}{q}},\sigma}$.

Proof. Let $1 \le p < q < \infty$, $0 < \alpha < \frac{|\sigma|}{p}$ and $f \in M_{p,\varphi^{\frac{1}{p}},\sigma}$. Write $f = f_1 + f_2$, where $E_\sigma = E_\sigma(x,r)$, $f_1 = f\chi_{2E_\sigma}$ and $f_2 = f\chi_{\complement_{(2E_\sigma)}}$.

For $M_{\alpha,\sigma} f_2(y)$ for all $y \in E_\sigma$ from (3.3) we have

$$\begin{aligned} M_{\alpha,\sigma}(f_2)(y) &\le 2^{|\sigma|-\alpha} \sup_{t>2r} \frac{1}{|E_\sigma(x,t)|^{1-\alpha/|\sigma|}} \int_{E_\sigma(x,t)} |f(z)|dz \\ &\lesssim \sup_{t>2r} t^{-\frac{|\sigma|}{q}} \|f\|_{L_p(E_\sigma(x,t))} \end{aligned} \tag{3.10}$$

Then from conditions (3.9) and (3.10) for all $y \in E_\sigma$ we get

$$\begin{aligned} M_{\alpha,\sigma} f(y) &\lesssim r^{\alpha}\, Mf(y) + \sup_{t>2r} t^{\alpha-\frac{|\sigma|}{p}} \|f\|_{L_p(E_\sigma(x,t))} \\ &\le r^{\alpha}\, Mf(y) + \|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}} \sup_{t>2r} t^{\alpha}\varphi(x,t)^{\frac{1}{p}} \\ &\lesssim r^{\alpha}\, Mf(y) + r^{-\frac{\alpha p}{q-p}}\, \|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}. \end{aligned}$$

Hence choose $r = \left(\frac{\|f\|_{M_{p,\varphi^{1/p},\sigma}}}{Mf(y)}\right)^{\frac{q-p}{\alpha q}}$ for every $y \in E_\sigma$, we have

$$|M_{\alpha,\sigma} f(y)| \lesssim (Mf(y))^{\frac{p}{q}}\, \|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}^{1-\frac{p}{q}}.$$

Hence the statement of the theorem follows in view of the boundedness of the anisotropic maximal operator M_σ in $M_{p,\varphi^{\frac{1}{p}},\sigma}$ provided by Corollary 3.1 in virtue of condition (3.8).

$$\begin{aligned}
\|M_{\alpha,\sigma}f\|_{M_{q,\varphi^{\frac{1}{q}},\sigma}} &= \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{|\sigma|}{q}} \|M_{\alpha,\sigma}f\|_{L_q(E_\sigma(x,t))} \\
&\lesssim \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}},\sigma}} \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{|\sigma|}{q}} \|Mf\|^{\frac{p}{q}}_{L_p(E_\sigma(x,t))} \\
&= \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}},\sigma}} \left(\sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{p}} t^{-\frac{|\sigma|}{p}} \|Mf\|_{L_p(E_\sigma(x,t))} \right)^{\frac{p}{q}} \\
&= \|f\|^{1-\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}},\sigma}} \|Mf\|^{\frac{p}{q}}_{M_{p,\varphi^{\frac{1}{p}},\sigma}} \\
&\lesssim \|f\|_{M_{p,\varphi^{\frac{1}{p}},\sigma}}.
\end{aligned}$$

if $1 < p < q < \infty$ and

$$\begin{aligned}
\|M_{\alpha,\sigma}f\|_{WM_{q,\varphi^{\frac{1}{q}},\sigma}} &= \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{|\sigma|}{q}} \|M_{\alpha,\sigma}f\|_{WL_q(E_\sigma(x,t))} \\
&\lesssim \|f\|^{1-\frac{1}{q}}_{M_{1,\varphi,\sigma}} \sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-\frac{1}{q}} t^{-\frac{|\sigma|}{q}} \|Mf\|^{\frac{1}{q}}_{WL_1(E_\sigma(x,t))} \\
&= \|f\|^{1-\frac{1}{q}}_{M_{1,\varphi,\sigma}} \left(\sup_{x\in\mathbb{R}^n,\, t>0} \varphi(x,t)^{-1} t^{-|\sigma|} \|Mf\|_{WL_1(E_\sigma(x,t))} \right)^{\frac{1}{q}} \\
&= \|f\|^{1-\frac{1}{q}}_{M_{1,\varphi,\sigma}} \|Mf\|^{\frac{1}{q}}_{WM_{1,\varphi,\sigma}} \\
&\lesssim \|f\|_{M_{1,\varphi,\sigma}},
\end{aligned}$$

if $1 < q < \infty$.

In the case $\varphi(x,t) = t^{(b-1)\frac{|\sigma|}{p}}$, $0 < b < 1$ from Theorem we get the following Adams type result for the fractional maximal operator.

Corollary 3.3 *Let* $0 < \alpha < |\sigma|$, $1 \le p < \frac{|\sigma|}{\alpha}$, $0 < \lambda < |\sigma| - \alpha p$ *and* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{|\sigma|-\lambda}$. *Then for* $p > 1$, *the operator* $M_{\alpha,\sigma}$ *is bounded from* $L_{p,b,\sigma}$ *to* $L_{q,b,\sigma}$ *and for* $p = 1$, $M_{\alpha,\sigma}$ *is bounded from* $L_{1,b,\sigma}$ *to* $WL_{q,b,\sigma}$.

4. Parabolic Schrödinger Type Operators $V^\gamma(\frac{\partial}{\partial t} - \Delta + V)^{-\beta}$ and $V^\gamma\nabla^2(\frac{\partial}{\partial t} - \Delta + V)^{-\beta}$

In this section we consider the parabolic Schrödinger operator

$$\frac{\partial}{\partial t} - \Delta + V \quad \text{on } \mathbb{R}^{n+1},$$

where $V = V(x,t)$ is a nonnegative potential which belongs to the parabolic reverse Hölder class $B_q(\mathbb{R}^{n+1})$. Examples of such potentials are all positive polynomials but also singular functions like $\max\{|x|, t^{\frac{1}{2}}\}^\alpha$ for $\alpha > -\frac{n+2}{q}$. We prove the parabolic generalized Morrey $M_{p,\varphi_1,\sigma_0}(\mathbb{R}^{n+1}) \to M_{p,\varphi_2,\sigma_0}(\mathbb{R}^{n+1})$ estimates for the operators $V^\gamma(\frac{\partial}{\partial t} - \Delta + V)^{-\beta}$ and $V^\gamma\nabla^2(\frac{\partial}{\partial t} - \Delta + V)^{-\beta}$, where $\sigma_0 = (1, \dots, 1, 2)$.

The investigation of Schrödinger operators on the Euclidean space $\mathbb{R}^n$ with nonnegative potentials which belong to the reverse Hölder class has attracted attention of a number of authors (cf. Fefferman, 1983; Shen, 1995; Zhong, 1993). Shen (1995) studied the Schrödinger operator $-\Delta + V$, assuming the nonnegative potential V belongs to the reverse Hölder class $B_q(\mathbb{R}^n)$ for $q \ge n/2$ and he proved the L_p boundedness of the operators $(-\Delta + V)^{i\gamma}$, $\nabla^2(-\Delta + V)^{-1}$, $\nabla(-\Delta + V)^{-\frac{1}{2}}$ and $\nabla(-\Delta + V)^{-1}$. Kurata and Sugano generalized Shens results to uniformly elliptic operators in Kurata and Sugano (2000). Sugano (1998) also extended some results of Shen to the operator $V^\gamma(-\Delta + V)^{-\beta}$, $0 \le \gamma \le \beta \le 1$ and $V^\gamma\nabla(-\Delta + V)^{-\beta}$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ and $\beta - \gamma \ge \frac{1}{2}$. Following Shen's approach, Gao and Jiang extend the results to the parabolic case. In Gao and Jiang (2005), they consider the parabolic operator $\frac{\partial}{\partial t} - \Delta + V$ where $V \in B_q(\mathbb{R}^{n+1})$ is a nonnegative potential depending only on the space variables and, under the assumptions $n \ge 3$ and $p > (n+2)/2$, they proved the boundedness of $V(\frac{\partial}{\partial t} - \Delta + V)^{-1}$ in $L_p(\mathbb{R}^{n+1})$.

The main purpose of this section is investigate the parabolic generalized Morrey $M_{p,\varphi_1,\sigma_0}(\mathbb{R}^{n+1}) \to M_{p,\varphi_2,\sigma_0}(\mathbb{R}^{n+1})$ boundedness of the operators

$$\mathcal{T}_1 = V^\gamma(\frac{\partial}{\partial t} - \Delta + V)^{-\beta}, \ \ 0 \le \gamma \le \beta \le 1,$$

$$\mathcal{T}_2 = V^\gamma \nabla^2(\frac{\partial}{\partial t} - \Delta + V)^{-\beta}, \ \ 0 \le \gamma \le \frac{1}{2} \le \beta \le 1, \ \beta - \gamma \ge \frac{1}{2}.$$

Note that the operator $\nabla^2(\frac{\partial}{\partial t} - \Delta + V)^{-1}$ in Gao and Jiang (2005) is the special case of $\mathcal{T}_2$.

It is worth pointing out that we need to establish pointwise estimates for $\mathcal{T}_1$, $\mathcal{T}_2$ and their adjoint operators by using the estimates of fundamental solution for the Schrödinger operator on $\mathbb{R}^{n+1}$ in Gao and Jiang (2005). And we prove the parabolic generalized Morrey estimates by using $M_{p,\varphi_1,\sigma_0}(\mathbb{R}^{n+1}) \to M_{p,\varphi_2,\sigma_0}(\mathbb{R}^{n+1})$ boundedness of the parabolic fractional maximal operators.

Definition 4.4 1) A nonnegative locally L_q integrable function V on $\mathbb{R}^{n+1}$ is said to belong to the parabolic reverse Hölder class $B_q(\mathbb{R}^{n+1})$ $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|K|}\int_K V(y,\tau)^q dy d\tau\right)^{\frac{1}{q}} \le \frac{C}{|K|}\int_K V(y,\tau) dy d\tau$$

holds for every parabolic cylinder

$$K = K((x,t),r) = \{(y,\tau) \in \mathbb{R}^{n+1} : |x_i - y_i| < r, |t-\tau| < r^2, i = 1, \dots, n\}$$

of center (x,t) and radius r in $\mathbb{R}^{n+1}$.

2) Let $V = V(x,t) \ge 0$. We say $V \in B_\infty(\mathbb{R}^{n+1})$, if there exists a constant $C > 0$ such that

$$\|V\|_{L_\infty(K)} \le \frac{C}{|K|}\int_K V(y,\tau) dy d\tau$$

holds for every parabolic cylinder $K = K((x,t),r)$ in $\mathbb{R}^{n+1}$.

Clearly, $B_\infty(\mathbb{R}^{n+1}) \subset B_q(\mathbb{R}^{n+1})$ for $1 < q < \infty$. But it is important that the $B_q(\mathbb{R}^{n+1})$ class has a property of "self-improvement"; that is, if $V \in B_q(\mathbb{R}^{n+1})$, then $V \in B_{q+\varepsilon}(\mathbb{R}^{n+1})$ for some $\varepsilon > 0$ (see Li, 1999).

By the functional calculus, we may write, for all $0 < \beta < 1$,

$$\left(\frac{\partial}{\partial t} - \Delta + V\right)^{-\beta} = \frac{1}{\pi}\int_0^\infty \lambda^{-\beta}\left(\frac{\partial}{\partial t} - \Delta + V + \lambda\right)^{-1} d\lambda.$$

Let $f \in C_0^\infty(\mathbb{R}^{n+1})$. From

$$\left(\frac{\partial}{\partial t} - \Delta + V + \lambda\right)^{-1} f(x,t) = \int_{\mathbb{R}^{n+1}} \Gamma(x,t;y,\tau;\lambda) f(y,\tau) dy d\tau,$$

it follows that

$$\mathcal{T}_1 f(x,t) = \int_{\mathbb{R}^{n+1}} K_1(x,t;y,\tau) V(y,\tau)^\gamma f(y,\tau) dy d\tau,$$

where

$$K_1(x,t;y,\tau) = \begin{cases} \frac{1}{\pi}\int_0^\infty \lambda^{-\beta}\,\Gamma(x,t;y,\tau;\lambda)\,d\lambda & \text{for } 0 < \beta < 1 \\ \Gamma(x,t;y,\tau;0) & \text{for } \beta = 1. \end{cases}$$

The following two pointwise estimates for $\mathcal{T}_1$ and $\mathcal{T}_2$ which proven in Zhong (1993), Lemma 3.2 with the potential $V \in B_\infty(\mathbb{R}^{n+1})$.

Theorem A *Suppose $V \in B_\infty(\mathbb{R}^{n+1})$ and $0 \le \gamma \le \beta \le 1$. Then, for any $f \in C_0^\infty(\mathbb{R}^{n+1})$*

$$|\mathcal{T}_1 f(x,t)| \lesssim M_{\alpha,\sigma_0} f(x,t),$$

where $\alpha = 2(\beta - \gamma)$.

Theorem B *Suppose* $V \in B_\infty(\mathbb{R}^{n+1})$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ *and* $\beta - \gamma \ge \frac{1}{2}$. *Then, for any* $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$|\mathcal{T}_2 f(x,t)| \lesssim M_{\alpha,\sigma_0} f(x,t),$$

where $\alpha = 2(\beta - \gamma) - 1$.

Note that the similar estimates for the adjoint operators T_1^* and T_2^* with the potential $V \in B_{q_1}$ for some $q_1 > \frac{n+2}{2}$ also valid (see Liu, 2009).

Theorem C *Suppose* $V \in B_{q_1}(\mathbb{R}^{n+1})$ *for some* $q_1 > \frac{n+2}{2}$, $0 \le \gamma \le \beta \le 1$ *and let* $\frac{1}{q_2} = 1 - \frac{\gamma}{q_1}$. *Then there exists a constant* $C > 0$ *such that*

$$|T_1^* f(x,t)| \le C \left(M_{\alpha q_2,\sigma_0}(|f|^{q_2})(x,t) \right)^{\frac{1}{q_2}}, \quad f \in C_0^\infty(\mathbb{R}^{n+1}),$$

where $\alpha = 2(\beta - \gamma)$.

Theorem D *Suppose* $V \in B_{q_1}(\mathbb{R}^{n+1})$ *for some* $q_1 > \frac{n+2}{2}$, $0 \le \gamma \le \frac{1}{2} < \beta \le 1$ *and* $\beta - \gamma \ge \frac{1}{2}$. *And let*

$$\frac{1}{q_2} = \begin{cases} 1 - \frac{\gamma}{q_1}, & \text{if } q_1 > n+2, \\ 1 - \frac{\gamma+1}{q_1} + \frac{1}{n+2}, & \text{if } \frac{n+2}{2} < q_1 < n+2. \end{cases}$$

Then there exists a constant $C > 0$ *such that*

$$|T_2^* f(x,t)| \le C \left(M_{\alpha q_2,\sigma_0}(|f|^{q_2})(x,t) \right)^{\frac{1}{q_2}}, \quad f \in C_0^\infty(\mathbb{R}^{n+1}),$$

where $\alpha = 2(\beta - \gamma) - 1$.

The above theorems will yield the parabolic generalized Morrey estimates for $\mathcal{T}_1$ and $\mathcal{T}_2$.

Corollary 4.4 *Assume that* $V \in B_\infty(\mathbb{R}^{n+1})$, *and* $0 \le \gamma \le \beta \le 1$. *Let* $1 \le p \le q < \infty$, $2(\beta - \gamma) = (n+2)\left(\frac{1}{p} - \frac{1}{q}\right)$ *and the condition (3.6) be satisfied for* $\alpha = 2(\beta - \gamma)$.

Then, for any $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$\|\mathcal{T}_1 f\|_{M_{q,\varphi_2,\sigma_0}} \lesssim \|f\|_{M_{p,\varphi_1,\sigma_0}}, \quad \text{for } p > 1$$

and

$$\|\mathcal{T}_1 f\|_{WM_{q,\varphi_2,\sigma_0}} \lesssim \|f\|_{M_{1,\varphi_1,\sigma_0}} \quad \text{for } p = 1$$

Corollary 4.5 *Assume that* $V \in B_\infty(\mathbb{R}^{n+1})$, $0 \le \gamma \le \frac{1}{2} \le \beta \le 1$ *and* $\beta - \gamma \ge \frac{1}{2}$. *Let* $1 \le p \le q < \infty$, $2(\beta - \gamma) - 1 = (n+2)\left(\frac{1}{p} - \frac{1}{q}\right)$ *and the condition (3.6) be satisfied for* $\alpha = 2(\beta - \gamma) - 1$.

Then, for any $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$\|\mathcal{T}_2 f\|_{M_{q,\varphi_2,\sigma_0}} \lesssim \|f\|_{M_{p,\varphi_1,\sigma_0}}, \quad \text{for } p > 1$$

and

$$\|\mathcal{T}_2 f\|_{WM_{q,\varphi_2,\sigma_0}} \lesssim \|f\|_{M_{1,\varphi_1,\sigma_0}} \quad \text{for } p = 1$$

Corollary 4.6 *Assume that* $V \in B_{q_1}(\mathbb{R}^{n+1})$ *for* $q_1 > \frac{n+2}{2}$, *and* $0 \le \gamma \le \beta \le 1$. *Let* $1 \le p < \frac{1}{\frac{\gamma}{q_1} + \frac{\alpha}{n+2}}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+2}$, $\frac{1}{q_2} = 1 - \frac{\gamma}{q_1}$ *and the condition (3.6) be satisfied for* $\alpha = 2(\beta - \gamma)$.

Then, for any $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$\|\mathcal{T}_1 f\|_{M_{q,\varphi_2,\sigma_0}} \lesssim \|f\|_{M_{p,\varphi_1,\sigma_0}}, \quad \text{for } p > 1$$

and

$$\|\mathcal{T}_1 f\|_{WM_{q,\varphi_2,\sigma_0}} \lesssim \|f\|_{M_{1,\varphi_1,\sigma_0}} \quad \text{for } p = 1$$

Corollary 4.7 *Assume that* $V \in B_{q_1}(\mathbb{R}^{n+1})$ *for* $q_1 > \frac{n+2}{2}$, *and*

$$\begin{cases} 0 \le \gamma \le \frac{1}{2} \le \beta \le 1, & \text{if } q_1 > n+2, \\ 0 \le \gamma \le \frac{1}{2} < \beta \le 1, & \text{if } \frac{n+2}{2} < q_1 < n+2. \end{cases}$$

Let $\beta - \gamma \geq \frac{1}{2}$, $1 \leq p < \frac{1}{\frac{\gamma}{q_1} + \frac{\alpha}{n+2}}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+2}$, $\frac{1}{q_2} = 1 - \frac{\gamma}{q_1}$ *and the condition (3.6) be satisfied for* $\alpha = 2(\beta - \gamma) - 1$, *where*

$$\frac{1}{p_1} = \begin{cases} \frac{\gamma}{q_1}, & \text{if } q_1 > n+2, \\ \frac{\gamma+1}{q_1} - \frac{1}{n+2}, & \text{if } \frac{n+2}{2} < q_1 < n+2. \end{cases}$$

Then, for any $f \in C_0^\infty(\mathbb{R}^{n+1})$

$$\|\mathcal{T}_2 f\|_{M_{q,\varphi_2,\sigma_0}} \lesssim \|f\|_{M_{p,\varphi_1,\sigma_0}}, \quad \text{for } p > 1$$

and

$$\|\mathcal{T}_2 f\|_{WM_{q,\varphi_2,\sigma_0}} \lesssim \|f\|_{M_{1,\varphi_1,\sigma_0}} \quad \text{for } p = 1$$

References

Adams, D. R. (1975). A note on Riesz potentials. *Duke Math., 42*, 765-778. http://dx.doi.org/10.1215/S0012-7094-75-04265-9

Adams, D. R. (1998). Choquet integrals in potential theory. *Publ. Mat., 42*, 3-66.

Akbulut, A., Guliyev, V. S., & Mustafayev, R. (2012). On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces. *Math. Bohemica, 137*(1) , 27-43.

Besov, O. V., Il'in, V. P., & Lizorkin, P. I. (1966). The L_p-estimates of a certain class of non-isotropically singular integrals (Russian). *Dokl. Akad. Nauk SSSR, 169*, 1250-1253.

Besov, O. V., Il'in, V. P., & Nikol'skii, S. M. (1996). *Integral representations of functions, and embedding theorems* (Russian, Second edition). Fizmatlit "Nauka", Moscow.

Bramanti, M., & Cerutti, M. C. (1996). Commutators of singular integrals on homogeneous spaces. *Boll. Un. Mat. Ital., B, 10*(7), 843-883.

Burenkov, V., Gogatishvili, A., Guliyev, V. S., & Mustafayev, R. (2010). Boundedness of the fractional maximal operator in local Morrey-type spaces. *Complex Var. Elliptic Equ., 55*(8-10), 739-758. http://dx.doi.org/10.1080/17476930903394697

Burenkov, V. I., Guliyev, H. V., & Guliyev, V. S. (2007). Necessary and sufficient conditions for boundedness of the fractional maximal operators in the local Morrey-type spaces. *J. Comput. Appl. Math., 208*(1), 280-301. http://dx.doi.org/10.1016/j.cam.2006.10.085

Caffarelli, L. (1990). Elliptic second order equations. *Rend. Sem. Mat. Fis. Milano, 58*, 253-284. http://dx.doi.org/10.1007/BF02925245

Chiarenza, V., & Frasca, M. (1987). Morrey spaces and Hardy-Littlewood maximal function. *Rend. Math., 7*, 273-279.

Fabes, E. B., & Rivère, N. (1966). Singular integrals with mixed homogeneity. *Studia Math., 27*, 19-38.

Fazio, G. Di, Palagachev, D. K., & Ragusa, M. A. (1999). Global Morrey regularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients. *J. Funct. Anal., 166*, 179-196. http://dx.doi.org/10.1006/jfan.1999.3425

Fefferman, C. (1983). The uncertainty principle. *Bull. Amer. Math. Soc., 9*, 129206. http://dx.doi.org/10.1090/S0273-0979-1983-15154-6

Gao, W., & Jiang, Y. (2005). L_p estimates for parabolic Schrödinger operator with certain potentials. *J. Math. Anal. Appl., 310*, 128-143. http://dx.doi.org/10.1016/j.jmaa.2005.01.049

Giaquinta, M. (1983). *Multiple integrals in the calculus of variations and nonlinear elliptic systems*. Princeton, NJ: Princeton Univ. Press.

Guliyev, V. S. (1994). *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$. Doctor of Sciencies, Moscow, Mat. Inst. Steklova (Russian), 1-329.

Guliyev, V. S. (1999). *Function spaces, integral operators and two weighted inequalities on homogeneous groups.* Some applications. Baku (Russian), 1-332.

Guliyev, V. S. (2009). Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces. *J. Inequal. Appl.*, Art. ID 503948, 20.

Guliyev, V. S., Aliyev, S. S., Karaman, T., & Shukurov, P. S. (2011). Boundedness of sublinear operators and commutators on generalized Morrey Space. *Int. Eq. Op. Theory., 71*, 327-355. http://dx.doi.org/10.1007/s00020-011-1904-1

Guliyev, V. S., & Mustafayev, R. Ch. (1997). Integral operators of potential type in spaces of homogeneous type (Russian). *Doklady Ross. Akad. Nauk, 354*(6), 730-732.

Guliyev, V. S., & Mustafayev, R. Ch. (1998). Fractional integrals in spaces of functions defined on spaces of homogeneous type (Russian). *Anal. Math., 24*(3), 181-200.

Kufner, A., & John, O., & Fucik, S. (1977). Function Spaces. Noordhoff, Leyden, and Academia, Prague.

Kurata, K., & Sugano, S. (2000). A remark on estimates for uniformly elliptic operators on weighted L_p spaces and Morrey spaces. *Math. Nachr., 209*, 137-150. http://dx.doi.org/10.1002/(SICI)1522-2616(200001)209:1<137::AID-MANA137>3.0.CO;2-3

Li, H. Q. (1999). Estimations L_p des operateurs de Schrödinger sur les groupes nilpotents. *J. Funct. Anal., 161*, 152-218. http://dx.doi.org/10.1006/jfan.1998.3347

Liu, Yu. (2009). The weighted estimates for the operators $V^{\alpha}(-\Delta_G + V)^{-\beta}$ and $V^{\alpha}\nabla_G(-\Delta_G + V)^{-\beta}$ on the stratified Lie group $\mathbb{G}$. *J. Math. Anal. Appl., 349*, 235-244. http://dx.doi.org/10.1016/j.jmaa.2008.08.043

Mazzucato, A. L. (2003). Besov-Morrey spaces: function space theory and applications to non-linear PDE. *Trans. Amer. Math. Soc., 355*, 1297-1364. http://dx.doi.org/10.1016/j.jmaa.2008.08.043

Mizuhara, T. (1991). Boundedness of some classical operators on generalized Morrey spaces. *Harmonic Analysis* (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo, 183-189.

Morrey, C. B. (1938). On the solutions of quasi-linear elliptic partial differential equations. *Trans. Amer. Math. Soc., 43*, 126-166. http://dx.doi.org/10.1090/S0002-9947-1938-1501936-8

Nakai, E. (1994). Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces. *Math. Nachr., 166*, 95-103. http://dx.doi.org/10.1002/mana.19941660108

Nakai, E. (2006). The Campanato, Morrey and Hölder spaces on spaces of homogeneous type. *Studia Math., 176*(1), 1-19. http://dx.doi.org/10.4064/sm176-1-1

Perez, C. (1990). Two weighted norm inequalities for Riesz potentials and uniform L_p-weighted Sobolev inequalities. *Indiana Univ. Math. J., 39*, 3144. http://dx.doi.org/10.1512/iumj.1990.39.39004

Ruiz, A., & Vega, L. (1991). Unique continuation for Schrödinger operators with potential in Morrey spaces. *Publ. Mat. 35*, 291-298.

Ruiz, A., & Vega, L. (1993). On local regularity of Schrödinger equations. *Int. Math. Res. Notices, 1993*(1), 13-27. http://dx.doi.org/10.1155/S1073792893000029

Sawano, Y. (2008). Wavelet characterizations of Besov Morrey and Triebel Lizorkin Morrey spaces. *Funct. Approx. Comment. Math., 38*, 93-107.

Sawano, Y. (2009). A note on Besov Morrey spaces and Triebel Lizorkin Morrey spaces. *Acta Mathematica Sinica, English Series, 25*(8), 1223-1242. http://dx.doi.org/10.1007/s10114-009-8247-8

Sawano, Y., & Tanaka, H. (2007). Decompositions of Besov Morrey spaces and Triebel Lizorkin Morrey spaces. *Math. Z., 257*(4), 871-905. http://dx.doi.org/10.1007/s00209-007-0150-3

Sawano, Y., & Tanaka, H. (2009). Besov Morrey spaces and Triebel Lizorkin Morrey spaces for non-doubling measures. *Math. Nachr., 282*(12), 1788-1810. http://dx.doi.org/10.1002/mana.200610818

Sawano, Y., Sugano, Y., & Tanaka, H. (2009). Identification of the image of Morrey spaces by the fractional integral operators. *Proc. A. Razmadze Math. Inst., 149*(2009), 87-93.

Shen, Z. W. (1995). L_p estimates for Schrödinger operators with certain potentials. *Ann. Inst. Fourier (Grenoble), 45*, 513-546. http://dx.doi.org/10.5802/aif.1463

Shen, Z. (2002). The periodic Schrödinger operators with potentials in the Morrey class. *J. Funct. Anal., 193*, 314-345. http://dx.doi.org/10.1006/jfan.2001.3933

Shen, Z. (2003). Boundary value problems in Morrey spaces for elliptic systems on Lipschitz domains. *Amer. J. Math., 125*, 1079-1115.

Softova, L. (2006). Singular integrals and commutators in generalized Morrey spaces. *Acta Mathematic Sinica, 22*(3), 757-766. http://dx.doi.org/10.1007/s10114-005-0628-z

Softova, L. (2008). Singular integral operators in Morrey spaces and interior regularity of solutions to systems of linear PDEs. *J. Glob. Optim., 40*, 427-442.

Spanne, S. (1966). Sur l'interpolation entre les espaces $\mathcal{L}_k^{p,\Phi}$. *Ann. Schola Norm. Sup. Pisa, 20*, 625-648.

Stein, E. M. (1993). Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals. Princeton, NJ: Princeton University Press.

Sugano, S. (1998). Estimates for the operators $V^\alpha(-\Delta+V)^{-\beta}$ and $V^\alpha\nabla(-\Delta+V)^{-\beta}$ with certain nonnegative potentials V. *Tokyo J. Math., 21*, 441-452.

Tang, L., & Xu, J. (2005). Some properties of Morrey type Besov Mriebel spaces. *Math. Nachr., 278*(7-8), 904-917. http://dx.doi.org/10.1002/mana.200310281

Taylor, M. E. (1992). Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations. *Comm. Partial Differential Equations, 17*, 1407-1456.

Yuan, W., Sickel, W., & Yang, D. (2010). *Morrey and Campanato meet Besov, Lizorkin and Triebel. Lecture Notes in Math., 2005*. Berlin: Springer. http://dx.doi.org/10.1007/978-3-642-14606-0

Zhong, J. P. (1993). *Harmonic analysis for some Schrödinger type operators*. PhD thesis, Princeton University.