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On a Formula of Mellin and Its Application to the Study of the Riemann Zeta-function

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Abstract

In this paper, we reconsider a formula of Mellin. We present a formula which relates the sum of two positive real numbers m, n to their product mn. We apply this formula to derivation of a relationship involving the Hurwitz zeta-function. Then we define a series function (stemming from the proved relationship) and discuss an analogy in regard to the Lindelöf hypothesis. Finally, a proof of the Lindelöf hypothesis of the Riemann zeta-function is deduced from this analogy.

Keywords: Lindelöf hypothesis, Mellin's formula, Hurwitz zeta-function, Riemann zeta-function

1. Introduction

In this paper, we present

Theorem 1 *Let* $Re(x_1), Re(x_2) > 0$ *and* $m, n > 0$. *We have*

$$\frac{2\pi\Gamma(x_1 + x_2)}{(m+n)^{x_1+x_2}} = \int_{-\infty}^{\infty} m^{-x_1-it} n^{-x_2+it} \Gamma(x_1 + it)\Gamma(x_2 - it)dt. \tag{1}$$

To prove the theorem, we recall a formula of Mellin (Montgomery & Vaughan, 2006): for $0 < c < \text{Re}(a)$,

$$\frac{\Gamma(a)}{(1+z)^a} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\Gamma(a-s)z^{-s}ds. \tag{2}$$

We perform the following manipulations on (2):

1) replace a by $x_1 + x_2$, with $\text{Re}(x_1), \text{Re}(x_2) > 0$;

2) replace c by x_1;

3) replace z by mn^{-1};

4) multiply both sides by $n^{-x_1-x_2}$.

By these, (2) is rewritten as

$$\frac{2\pi\Gamma(x_1 + x_2)}{(m+n)^{x_1+x_2}} = \int_{-\infty}^{\infty} m^{-x_1-it} n^{-x_2+it} \Gamma(x_1 + it)\Gamma(x_2 - it)dt.$$

This completes the proof of Theorem 1.

Regardless of the straightforward derivation of Theorem 1 from Mellin's formula, however, we decided to present it in a research paper because of its rare characteristic; that is, it relates the sum of two numbers m and n to their product mn.

Throughout the rest of the discussion, we choose $-\pi \le \arg z < \pi$ as the domain of complex logarithm $\log z$.

An application of Theorem 1 is given in a proof of the following formula.

Theorem 2 *Let* $Re(\delta) > -1$, $\epsilon > 0$, *and* α *be any real number. In addition, let* a *be any complex number such that* $\arg(a) > 0$.

Define

$$Z_\epsilon(1-\delta+\epsilon+i\alpha,a) := \sum_{m\geq 1}[-(m+a)]^{-(1-\delta+\epsilon)+i\alpha}\gamma(\epsilon,\delta,\alpha)$$
$$E(1-\delta+\epsilon+i\alpha,a) := \sum_{m\leq -1}[|m|-a]^{-(1-\delta+\epsilon)+i\alpha}\gamma(\epsilon,\delta,\alpha),$$

where

$$\gamma(\epsilon,\delta,\alpha) := \Gamma(\delta+i\alpha)\Gamma(1-\delta+\epsilon-i\alpha).$$

Then we have

$$Z_\epsilon(1-\delta+\epsilon-i\alpha,a)+E(1-\delta+\epsilon-i\alpha,a)+(-a)^{-(1-\delta+\epsilon)+i\alpha}\gamma(\epsilon,\delta,\alpha)$$
$$=\Gamma(1+\epsilon)\int_0^1 u^{\delta+i\alpha-1}\sum_{m\in\mathbb{Z}}\frac{1}{[u-(m+a)]^{1+\epsilon}}du+\Gamma(1+\epsilon)\int_0^1\zeta(1-\delta-i\alpha,r)\sum_{m\in\mathbb{Z}}\frac{1}{[r-(m+a)]^{1+\epsilon}}dr, \tag{3}$$

where $\zeta(s,a)$ is the Hurwitz zeta-function defined for $Re(s)>1$ by

$$\zeta(s,a) := \sum_{n\geq 1}\frac{1}{(n+a)^s}.$$

Notes:

1) The first integral on the right would be regarded as its analytic continuation to $\mathrm{Re}(\delta) > -1$ (see (15) for the existence of such an extension).

2) The left series Z_ϵ and E, converging for $\mathrm{Re}(\delta) < \epsilon$, should be considered as its analytic continuation whenever $\mathrm{Re}(\delta) \geq \epsilon$.

3) The conventional definition of $\zeta(s,a)$ takes the sum $\sum_{n\geq 0}$. Because we often consider the case $a = 0$, however, we separate the $n = 0$-term.

4) The subscript ϵ for Z_ϵ is to make clear the order of magnitude of the polynomially growing factor $|\alpha|^\epsilon$ in $\gamma(\epsilon,\delta,\alpha)$.

We note that the main term of the left side of Theorem 2 as $\alpha\to\infty$ is the function Z_ϵ. This is by the appearance of the number (-1) in its terms; the other function E vanishes exponentially because it lacks a factor which should cancel out the exponentially decaying factor γ.

As another objective of this paper other than presentation of Theorem 1 and 2, we shall relate the latter theorem to the study of the Riemann zeta-function.

Firstly, using Theorem 2, we could think of an analogy between the function $Z_\epsilon(1-\delta+\epsilon-i\alpha,a)|\alpha|^{-\epsilon}$ and the Riemann zeta-function.

For instance, a property of the function Z analogous to the Riemann zeta-function is that when $-1 < \mathrm{Re}(\delta) < 0$, by (Whittaker & Watson, 2008)

$$\Gamma(s+it) \asymp |t|^{\mathrm{Re}(s)-1/2}e^{-\pi|t|/2} \quad \text{as } |t|\to\infty, \tag{4}$$

the Equation (15) below, and by Theorem 2, it is easy to see that as $\alpha\to\infty$,

$$Z_\epsilon(1-\delta+\epsilon-i\alpha,a)|\alpha|^{-\epsilon} \ll 1, \quad \text{for each } \epsilon>0 \text{ and } \delta\leq 0,$$

while for $\sigma > 1$,

$$\zeta(\sigma-it)\alpha^{-\epsilon} \ll 1.$$

Choosing $\epsilon = 1/2$ and $\delta = 1$ in Theorem 2 and using the well-known result (Whittaker & Watson, 2008)

$$\zeta(\sigma+it,a) = O(|t|^{1/2-\sigma}\log|t|), \quad \sigma\leq 0, \tag{5}$$

we obtain the following corollary to Theorem 2.

Corollary 1 *We have*

$$Z_{1/2}(1/2-i\alpha,a)\alpha^{-1/2} = O(\log\alpha), \quad \text{as } \alpha\to\infty.$$

Note: The corollary does not include the case $\alpha\to-\infty$.

The Lindelöf hypothesis states that for each $\epsilon > 0$, the Riemann zeta-function would satisfy

$$\zeta(1/2 + it) = O(|t|^{\epsilon}).$$

If we note that the function $Z_{\epsilon}(1 - \delta + \epsilon - i\alpha, a)|\alpha|^{-\epsilon}$ is essentially reduced to the Riemann zeta-function by putting $a = 0$, that is,

$$\sum_{m\geq 1}(-m)^{-(1-\delta+\epsilon)+i\alpha}\gamma(\epsilon,\delta,\alpha)|\alpha|^{-\epsilon} = \zeta(1-\delta+\epsilon+i\alpha)\gamma(\epsilon,\delta,\alpha)|\alpha|^{-\epsilon}(-1)^{-(1-\delta+\epsilon)+i\alpha}$$
$$\asymp \zeta(1-\delta+\epsilon+i\alpha), \quad \text{as } \alpha \to \infty,$$

(by our convention on complex logarithm, $(-1) = e^{-\pi i}$ and $(-1)^{i\alpha} = e^{\alpha\pi}$) we may conclude from the corollary that the Lindelöf hypothesis for the function $Z_{\epsilon}(1 - \delta + \epsilon - i\alpha, a)|\alpha|^{-\epsilon}$ is true.

Secondly, we will give a proof of the original Lindelöf hypothesis based on Theorem 2.

2. Method

All the techniques used in our analysis are in the realm of classics.

We mainly rely on the following well-established theoretical tools:

1) Complex analysis (especially analytic continuation);

2) Fourier transform;

3) Rigors by real analysis.

As for the rigors, we leave the reader for verifying any details, such as convergence in taking the summation $\sum_{m\geq 1}$, etc.

When we use the theory of Fourier integrals, we discuss in the Schwartz space of rapidly decreasing functions.

Two important tools in our discussion, together with analytic continuation, are Theorem 1 and the following function

$$F_s(z,a) := \sum_{m\in\mathbb{Z}} \frac{1}{[z - 2(m+a)\omega_1]^s}, \tag{6}$$

where a and ω_1 are some complex numbers. It is easy to show that the series $F_s(z,a)$ has the period $2\omega_1$ as a function of z. The extensive use of these tools are to be seen as follows.

2.1 Proof of Theorem 2

Throughout the argument, δ is a complex variable, while ϵ is a positive real number.

First, for any small δ, ϵ satisfying $0 < \mathrm{Re}(\delta) < \epsilon$, we put $x_1 = \delta$, $x_2 = 1 - \delta + \epsilon$ in Theorem 1 and get

$$\frac{2\pi\Gamma(1+\epsilon)}{(m+n)^{1+\epsilon}} = \int_{-\infty}^{\infty} m^{-\delta-it}n^{-(1-\delta+\epsilon)+it}\Gamma(\delta+it)\Gamma(1-\delta+\epsilon-it)dt. \tag{7}$$

Here, we note that if

$$|\arg[-2(m+a)\omega_1]| < \pi, \tag{8}$$

then with (4), it is easy to see that we could extend (7) in the variable n to a neighborhood of the number $-2(m+a)\omega_1$ by analytic continuation.

Thus, varying $m \mapsto z$ (the value for $z > 0$ is kept fixed until a particular number is picked below), $n \mapsto -2(m+a)\omega_1$ in (7), we get

$$\frac{2\pi\Gamma(1+\epsilon)}{(z-2(m+a)\omega_1)^{1+\epsilon}} = \int_{-\infty}^{\infty} z^{-\delta-it}[-2(m+a)\omega_1]^{-(1-\delta+\epsilon)+it}\Gamma(\delta+it)\Gamma(1-\delta+\epsilon-it)dt. \tag{9}$$

At this point, we further assume that

$$\arg(\omega_1) > 0 \quad (\text{so that } -\pi < \arg(-\omega_1) < 0) \quad \text{and} \quad m \in \mathbb{Z}. \tag{10}$$

One example of ω_1 and a for which $\arg(a) > 0$, (8), and (10) are satisfied is $\omega_1 = 1 + i$ and $a = i/2$.

Then since the right integral of (9) is

$$\int_{-\infty}^{\infty} z^{-\delta-it} m^{-(1-\delta+\epsilon)+it}[-2(1+o(1))\omega_1]^{-(1-\delta+\epsilon)+it}\Gamma(\delta+it)\Gamma(1-\delta+\epsilon-it)dt, \quad \text{as } m\to\infty,$$

and

$$\int_{-\infty}^{\infty} z^{-\delta-it} |m|^{-(1-\delta+\epsilon)+it}[(2+o(1))\omega_1]^{-(1-\delta+\epsilon)+it}\Gamma(\delta+it)\Gamma(1-\delta+\epsilon-it)dt, \quad \text{as } m\to-\infty,$$

it is easy to show that the summation $\sum_{m\in\mathbb{Z}}$ in (9) gives

$$2\pi\Gamma(1+\epsilon)F_{1+\epsilon}(z,a) = \int_{-\infty}^{\infty} z^{-\delta-it}\sum_{m\in\mathbb{Z}}[-2(m+a)\omega_1]^{-(1-\delta+\epsilon)+it}\Gamma(\delta+it)\Gamma(1-\delta+\epsilon-it)dt. \tag{11}$$

Replacing z by ze^{α} in (11), we have

$$2\pi\Gamma(1+\epsilon)F_{1+\epsilon}(ze^{\alpha},a) = \int_{-\infty}^{\infty} (ze^{\alpha})^{-\delta-it}\sum_{m\in\mathbb{Z}}[-2(m+a)\omega_1]^{-(1-\delta+\epsilon)+it}\Gamma(\delta+it)\Gamma(1-\delta+\epsilon-it)dt, \tag{12}$$

or multiplying by $e^{\delta\alpha}$,

$$2\pi\Gamma(1+\epsilon)e^{\delta\alpha}F_{1+\epsilon}(ze^{\alpha},a) = \int_{-\infty}^{\infty} z^{-\delta-it}\sum_{m\in\mathbb{Z}}[-2(m+a)\omega_1]^{-(1-\delta+\epsilon)+it}\Gamma(\delta+it)\Gamma(1-\delta+\epsilon-it)e^{-i\alpha t}dt. \tag{13}$$

Applying the Fourier inversion theorem to (13), namely,

$$\mathcal{F}[f](\alpha) = \int_{-\infty}^{\infty} f(t)e^{-i\alpha t}dt \Longleftrightarrow f(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}[f](t)e^{i\alpha t}dt,$$

we obtain for $0<\mathrm{Re}(\delta)<\epsilon$,

$$z^{-\delta-i\alpha}\sum_{m\in\mathbb{Z}}[-2(m+a)\omega_1]^{-(1-\delta+\epsilon)+i\alpha}\Gamma(\delta+i\alpha)\Gamma(1-\delta+\epsilon-i\alpha) = \Gamma(1+\epsilon)\int_{-\infty}^{\infty} e^{\delta t+i\alpha t}F_{1+\epsilon}(ze^{t},a)dt. \tag{14}$$

Now, the right integral of (14) is a meaningful expression; in fact, in the same way as getting (12), an application of Theorem 1 gives for $\mathrm{Re}(\delta)<\mathrm{Re}(\delta')<\epsilon$,

$$2\pi\Gamma(1+\epsilon)F_{1+\epsilon}(ze^{\alpha},a) = \int_{-\infty}^{\infty} (ze^{\alpha})^{-\delta'-it}\sum_{m\in\mathbb{Z}}[-2(m+a)\omega_1]^{-(1-\delta'+\epsilon)+it}\Gamma(\delta'+it)\Gamma(1-\delta'+\epsilon-it)dt,$$

from which it follows readily that

$$F_{1+\epsilon}(ze^{\alpha},a) = O(e^{-\delta'\alpha}), \quad \text{as } \alpha\to\infty.$$

With the change of variable $u=e^{t}$, the right integral of (14) is rewritten as

$$\int_0^{\infty} u^{\delta+i\alpha-1}F_{1+\epsilon}(zu,a)du.$$

At this point, we break the right integral of (14) as

$$\begin{aligned}\Gamma(1+\epsilon)\int_0^{\infty} u^{\delta+i\alpha-1}F_{1+\epsilon}(zu,a)du &= \Gamma(1+\epsilon)\int_0^{1} u^{\delta+i\alpha-1}F_{1+\epsilon}(zu,a)du + \Gamma(1+\epsilon)\int_1^{\infty} u^{\delta+i\alpha-1}F_{1+\epsilon}(zu,a)du\\ &=: f_1(\delta+i\alpha,z)+f_2(\delta+i\alpha,z),\end{aligned}$$

and extend (14) in the variable δ to $-1<\mathrm{Re}(\delta)\le 0$ by analytic continuation. But in this process, it is plain that the left expression of (14) keeps analytic in δ and valid for $\mathrm{Re}(\delta)\le 0$, and that $f_2(\delta+i\alpha,z)$ is also meaningful and analytic in δ for $\mathrm{Re}(\delta)\le 0$.

Furthermore, for $\mathrm{Re}(\delta) > 0$, we rewrite the integral for $f_1(\delta + i\alpha, z)$ with integration by parts as

$$\begin{aligned}\int_0^1 u^{\delta+i\alpha-1}F_{1+\epsilon}(zu,a)du &= \left[\frac{F_{1+\epsilon}(z,a)u^{\delta+i\alpha}}{\delta+i\alpha}\right]_0^1 - \frac{z}{\delta+i\alpha}\int_0^1 u^{\delta+i\alpha}F'_{1+\epsilon}(zu,a)du \\ &= \frac{F_{1+\epsilon}(z,a)}{\delta+i\alpha} - \frac{z}{\delta+i\alpha}\int_0^1 u^{\delta+i\alpha}F'_{1+\epsilon}(zu,a)du.\end{aligned} \tag{15}$$

Then the integral on the right converges also for $-1 < \mathrm{Re}(\delta) \leq 0$, and so this gives the extension of $f_1(\delta + i\alpha, z)$ into the the right half-plane $\mathrm{Re}(\delta) > -1$.

Thus, we have

$$\begin{aligned}&z^{-\delta-i\alpha}\sum_{m\in\mathbb{Z}}[-2(m+a)\omega_1]^{-(1-\delta+\epsilon)+i\alpha}\Gamma(\delta+i\alpha)\Gamma(1-\delta+\epsilon-i\alpha) \\ &= f_1(\delta+i\alpha,z) + f_2(\delta+i\alpha,z), \quad -1 < \mathrm{Re}(\delta) < \epsilon, \epsilon > 0.\end{aligned} \tag{16}$$

Next, we extend (16) in the variable z to some neighborhood of the point $z = 2\omega_1$ by analytic continuation. To validate this procedure, we first note that the left member of (16) and (by (15)) $f_1(\delta + i\alpha, z)$ readily have their desired extensions.

In addition, if we choose $\mathrm{Re}(\delta) < 0$, then with

$$F_{1+\epsilon}(2\omega_1 u, a) \ll 1 \quad \text{(as } u \text{ varies)},$$

the integral expression for $f_2(\delta + i\alpha, z)$ is also meaningful for $z = 2\omega_1$, and so has its desired analytic continuation for $-1 < \mathrm{Re}(\delta) < 0$.

Here we note that by (10) and $\arg(a) > 0$, we could vary z (= a positive real number) continuously upward from some point in the positive real line to the point $2\omega_1$, while any singularity is not brought out from $F_{1+\epsilon}(zu, a)$.

To give a new expression for $f_2(\delta + i\alpha, z)$ which is analytic in the variable $\delta + i\alpha$ in $\mathbb{C} - \{0\}$ and for $z = 2\omega_1$, we analyze as follows. (With this step, we could remove the restriction $\mathrm{Re}(\delta) < \epsilon$, and give a meaningful expression for the analytic continuation of the left members of (16) for $\mathrm{Re}(\delta) \geq \epsilon$.)

For $-1 < \mathrm{Re}(\delta) < 0$, if z is chosen to be $2\omega_1$, then by the $2\omega_1$-periodicity of $F_{1+\epsilon}(z, a)$, we have

$$\begin{aligned}\int_1^\infty u^{\delta+i\alpha-1}F_{1+\epsilon}(2\omega_1 u,a)du &= \sum_{n\geq1}\int_n^{n+1} u^{\delta+i\alpha-1}F_{1+\epsilon}(2\omega_1 u,a)du \\ &= \sum_{n\geq1}\int_0^1 (n+r)^{\delta+i\alpha-1}F_{1+\epsilon}(2\omega_1 n + 2\omega_1 r,a)dr \\ &= \int_0^1 \sum_{n\geq1}(n+r)^{\delta+i\alpha-1}F_{1+\epsilon}(2\omega_1 r,a)dr \\ &= \int_0^1 \zeta(1-\delta-i\alpha,r)F_{1+\epsilon}(2\omega_1 r,a)dr;\end{aligned} \tag{17}$$

here, we used the change of variable $u = n + r$, and the interchanging of summation and integration symbols is verified easily if we note $\mathrm{Re}(\delta) < 0$.

Factoring out $2\omega_1$ in the expression $F_{1+\epsilon}$ and multiplying by $(2\omega_1)^{1+\epsilon}$, with (16) and (17), we obtain (3) under the condition (8) on the number a.

But observing (3), it is easy to see that this restriction on a is improved (by analytic continuation in a) to $\arg(a) > 0$. (When a passes the real line, however, a singularity occurs.)

This completes the proof of Theorem 2.

2.2 Proof of the Lindelöf Hypothesis

The main difficulty is to treat several singularities which arise when we let $a \to 0$ in Theorem 2. Throughout this section, $a \neq 0$ is fixed, unless otherwise a is chosen to be some number.

To resolve the aforementioned problem, we first rewrite and sort out the $m = 0$-term of the left side of Theorem 2, namely,

$$(-a)^{-1+\delta+i\alpha}\Gamma(\delta + i\alpha)\Gamma(1 - \delta + \epsilon - i\alpha), \tag{18}$$

and the following terms of the right side,

$$\Gamma(1+\epsilon)\int_0^1 \zeta(1-\delta-i\alpha, r)\left[\frac{1}{(r-a)^{1+\epsilon}} + \frac{1}{(r-1-a)^{1+\epsilon}}\right]dr \tag{19}$$

and

$$\Gamma(1+\epsilon)\int_0^1 u^{\delta+i\alpha-1}\left[\frac{1}{(u-a)^{1+\epsilon}} + \frac{1}{(u-1-a)^{1+\epsilon}}\right]du. \tag{20}$$

Referring to (14), we have (considering only the $m = 0$-term in (14))

$$z^{-\delta-i\alpha}(-2a\omega_1)^{-(1-\delta+\epsilon)+i\alpha}\Gamma(\delta + i\alpha)\Gamma(1-\delta+\epsilon-i\alpha) = \Gamma(1+\epsilon)\int_{-\infty}^{\infty} e^{\delta t + i\alpha t}(ze^t - 2a\omega_1)^{-1-\epsilon}dt,$$

or with the change of variable $u = e^t$ on the right, multiplying by $(2\omega_1)^{1+\epsilon}$, and choosing $z = 2\omega_1$,

$$(-a)^{-(1-\delta+\epsilon)+i\alpha}\Gamma(\delta + i\alpha)\Gamma(1-\delta+\epsilon-i\alpha) = \Gamma(1+\epsilon)\int_0^{\infty} u^{\delta+i\alpha-1}(u-a)^{-1-\epsilon}du. \tag{21}$$

If we write

$$\Gamma(1+\epsilon)\int_0^{\infty} u^{\delta+i\alpha-1}(u-a)^{-1-\epsilon}du = \Gamma(1+\epsilon)\left(\int_0^1 + \int_1^{\infty}\right) =: \Gamma(1+\epsilon)\left(\int_0^1\right) + B_1,$$

then it is easy to see that (18) is equal to

$$(-a)^{-(1-\delta+\epsilon)+i\alpha}\Gamma(\delta + i\alpha)\Gamma(1-\delta+\epsilon-i\alpha) = \Gamma(1+\epsilon)\int_0^1 u^{\delta+i\alpha-1}(u-a)^{-1-\epsilon}du + B_1.$$

We note that B_1 is a meaningful expression for $a = 0$ and $\mathrm{Re}(\delta) \le 1$.

Next, we rewrite the integrals in (19) and (20) which contain the factor $(u-1-a)^{-1-\epsilon}$ by the change of variable $L = u - 1$ as

$$\int_0^1 \zeta(1-\delta-i\alpha, r)(r-1-a)^{-1-\epsilon}dr = \int_{-1}^0 \zeta(1-\delta-i\alpha, L+1)(L-a)^{-1-\epsilon}dL \tag{22}$$

and

$$\int_0^1 u^{\delta+i\alpha-1}(u-1-a)^{-1-\epsilon}du = \int_{-1}^0 (L+1)^{\delta+i\alpha-1}(L-a)^{-1-\epsilon}dL. \tag{23}$$

Summing up (22) and (23), if we recall the series definition of $\zeta(s, a)$ and note that the integral

$$\int_{-1}^0 (L+1)^{\delta+i\alpha-1}(L-a)^{-1-\epsilon}dL$$

has an analytic continuation to the region $-1 < \mathrm{Re}(\delta) \le 0$ (easily shown by integration by parts as we saw in (15)), we have

$$\int_{-1}^0 \zeta(1-\delta-i\alpha, L+1)(L-a)^{-1-\epsilon}dL + \int_{-1}^0 (L+1)^{\delta+i\alpha-1}(L-a)^{-1-\epsilon}dL = \int_{-1}^0 \zeta(1-\delta-i\alpha, L)(L-a)^{-1-\epsilon}dL.$$

Hence, the sum of two integrals in (19) and the second integral in (20) is

$$\int_{-1}^1 \zeta(1-\delta-i\alpha, L)(L-a)^{-1-\epsilon}dL. \tag{24}$$

Using all the results above, Theorem 2 is rewritten as, for $\mathrm{Re}(\delta) > 0$,

$$\begin{aligned} &Z_\epsilon(1-\delta+\epsilon-i\alpha, a) + E(1-\delta+\epsilon-i\alpha, a) \\ &= -B_1 + \Gamma(1+\epsilon)\int_{-1}^1 \zeta(1-\delta-i\alpha, L)(L-a)^{-1-\epsilon}dL + \Gamma(1+\epsilon)\int_0^1 [r^{\delta+i\alpha-1} + \zeta(1-\delta-i\alpha, r)]F^*_{1+\epsilon}(r, a)dr, \end{aligned} \tag{25}$$

where

$$F^*_{1+\epsilon}(r,a) := \sum_{m\in\mathbb{Z}-\{0,1\}} \frac{1}{[r-(m+a)]^{1+\epsilon}}.$$

Integrating (25) with respect to a over any path $[a_1, a_2]$ on the upper half-plane, we obtain (recall the series definition of Z_ϵ and E)

$$\begin{aligned}
&-(\delta-\epsilon+i\alpha)^{-1}\{[Z_\epsilon(-\delta+\epsilon-i\alpha,a)]_{a=a_1}^{a_2} + [E(-\delta+\epsilon-i\alpha,a)]_{a=a_1}^{a_2}\}\\
&= -\int_{a_1}^{a_2} B_1 da + \Gamma(1+\epsilon)(-\epsilon)^{-1}\int_{-1}^{1}\zeta(1-\delta-i\alpha,L)[(L-a)^{-\epsilon}]_{a=a_1}^{a_2}dL\\
&+\Gamma(1+\epsilon)\int_0^1 [r^{\delta+i\alpha-1}+\zeta(1-\delta-i\alpha,r)]\left[\int_{a_1}^{a_2} F^*_{1+\epsilon}(r,a)da\right]dr.
\end{aligned} \tag{26}$$

The final step of the proof is to estimate all the terms in (26) except for $Z_\epsilon(-\delta+\epsilon-i\alpha,a_1)$ one by one under the conditions

$$\epsilon = 1-\eta, \quad a_1\to 0, \quad |a_2|>2, \quad \delta=1/2, \quad \text{and} \quad \alpha\to\infty,$$

where $\eta>0$ is arbitrarily small.

Here, with Theorem 2 (choose $\epsilon = 1/2-\eta$ and $\delta = 1$) and (5), it is plain that

$$Z_{1/2-\eta}(1/2-\eta-i\alpha,a_2)\alpha^{1/2-\eta} \ll \alpha^{1/2}\log\alpha. \tag{27}$$

Besides, with integration by parts and (5), it is easy to show that

$$\begin{aligned}
\int_0^1 \zeta(1/2-i\alpha,r)G(r)dr &= [\zeta(-1/2-i\alpha,r)(1/2+i\alpha)^{-1}G(r)]_0^1 - \int_0^1 \zeta(-1/2-i\alpha,r)(1/2+i\alpha)^{-1}G'(r)dr\\
&\ll \log\alpha,
\end{aligned} \tag{28}$$

where

$$G(r) := \int_{a_1}^{a_2} F^*_{1+\epsilon}(r,a)da.$$

Thus, we are left with the integral

$$\int_{-1}^{1}\zeta(1/2-i\alpha,L)(L-a_1)^{-(1-\eta)}dL. \tag{29}$$

In order to rewrite (29) in a form to which estimates from Theorem 2 are applicable, we apply Cauchy's theorem in the variable L as follows.

We choose a_1 to be arbitrarily close to 0 (with $\arg(a_1)>0$), and consider

$$\int_{I_{g_1,g_2}}\zeta(1/2-i\alpha,L)(L-a_1)^{-(1-\eta)}dL + \int_{\Gamma_{g_1}+R_{g_1}}\zeta(1/2-i\alpha,z)(z-a_1)^{-(1-\eta)}dz = 0,$$

or

$$\int_{I_{g_1,g_2}}\zeta(1/2-i\alpha,L)(L-a_1)^{-(1-\eta)}dL = -\int_{\Gamma_{g_1}+R_{g_1}}\zeta(1/2-i\alpha,z)(z-a_1)^{-(1-\eta)}dz, \tag{30}$$

where

$$I_{g_1,g_2} := [-1+g_1,-g_2]\cup R_{g_2}\cup[g_2,1];$$

here, Γ_{g_1} is the incomplete semicircle

$$\Gamma_{g_1} := \{z : z = e^{it}, 0<t<\pi-g_1\}, \quad g_1>0 \text{ is arbitrarily small},$$

and R_{g_1}, R_{g_2} are any path from the point $z = e^{i(\pi-g_1)}$ to $z=-1+g_1$ and from $z=-g_2$ to $z=g_2$, respectively, such that the contour $I_{g_1,g_2}\cup\Gamma_{g_1}\cup R_{g_1}$ does not contain the point $z=-1$ and $z=a_1$. The analyticity of $\zeta(1-\delta-i\alpha,L)$

with respect to the variable L and with $\mathrm{Re}(\delta) < 1$ on the given contour (necessary in using Cauchy's theorem) follows from

$$\begin{aligned}\sum_{m\geq 1}(m+a)^{-s} &= s\int_1^\infty \lfloor t\rfloor (t+a)^{-s-1}dt \\ &= s\int_1^\infty (t+a)^{-s}dt + O(s\int_1^\infty (t+a)^{-s-1}dt) \\ &= \frac{s(1+a)^{-s+1}}{s-1} + O(s\int_1^\infty (t+a)^{-s-1}dt),\end{aligned}$$

which is shown with integration by parts in the sense of Riemann-Stieltjes integration.

By the uniform convergence of the integral

$$\int_{-1}^{1}\zeta(1/2-i\alpha,z)(z-a_1)^{-(1-\eta)}dz$$

throughout a_1 in any neighborhood of $a_1 = 0$ and

$$\lim_{g_2,a_1\to 0}\int_{R_{g_2}}\zeta(1/2-\alpha,z)(z-a_1)^{-(1-\eta)}dz = 0, \tag{31}$$

we have

$$\lim_{g_1,g_2,a_1\to 0}\int_{[-1+g_1,-g_2]\cup[g_2,1]}\zeta(1/2-\alpha,z)(z-a_1)^{-(1-\eta)}dz = \lim_{a_1\to 0}\int_{-1}^{1}\zeta(1/2-\alpha,z)(z-a_1)^{-(1-\eta)}dz. \tag{32}$$

Now, if we let $g_1 \to 0^+$ in (30), then integrals with diverging integrands on the right side are bounded by, using $\arg(1+L) > 0$,

$$\int_{\Gamma_{g_1}}(1+L)^{-1/2+i\alpha}(L-a_1)^{-(1-\eta)}dL \ll \int_{\Gamma_0}|1+L|^{-1/2}|L-a_1|^{-(1-\eta)}dL, \quad \text{as } \alpha\to\infty \tag{33}$$

and

$$\int_{R_{g_1}}(1+L)^{-1/2+i\alpha}(L-a_1)^{-(1-\eta)}dL \ll \int_{R_{g_1}}|1+L|^{-1/2}|L-a_1|^{-(1-\eta)}dL \to 0. \tag{34}$$

Next, we shall estimate the remaining integrals with analytic integrands on the right side of (30), namely,

$$\int_S \zeta(1/2-i\alpha,L+1)(L-a_1)^{-(1-\eta)}dL, \quad S = \Gamma_{g_1}, R_{g_1}$$

as follows. A difficulty at this step is to give an estimate for

$$\sum_{m\geq 2}\frac{1}{(m+e^{it})^{1-\delta-i\alpha}}, \quad 0\leq t\leq \pi.$$

That is, one might like to obtain a bound for this function which is valid for any closed segment $t\in[\pi-h,\pi]$ (or $t\in[0,h]$) with $h>0$ arbitrarily small and fixed, while (25) alone is not sufficient for this purpose. (For instance, when $a\to -1$ in (25), we need to handle the singularity $(L-a)^{-1-\epsilon}$ inside the integral $\int_{-1}^{1}$.)

In order to resolve the issue, we consider another contour K_h such that

$$K_h = [-1,-1+h]\cup(s_h = \{e^{it} : \pi-h\leq t\leq\pi\})\cup l_h,$$

where l_h is any small segment which goes from $z=-1+h$ to $z=e^{i(\pi-h)}$, and directly calculate the part

$$\lim_{a_1\to 0}\int_{s_h}\zeta(1/2-i\alpha,z+1)(z-a_1)^{-(1-\eta)}dz.$$

With the contour K_h, we have

$$\int_{s_h} \zeta(1/2 - i\alpha, z+1)(z-a_1)^{-(1-\eta)}dz = -\left(\int_{[-1,-1+h]} + \int_{l_h}\right)\zeta(1/2 - i\alpha, z+1)(z-a_1)^{-(1-\eta)}dz. \tag{35}$$

Inserting each $l \in l_h$ (so that $l = -1 + \mu$ with $\mu \neq 0$ and $\mu > 0$ as $l \to -1 + h$) for a in (25), and choosing $\epsilon = 1/2$, $\delta = 1$, we get, as $\alpha \to \infty$,

$$\zeta(1/2 - i\alpha, l)\alpha^{1/2} \ll \int_{-1}^{1} |\zeta(-i\alpha, L)| dL + \int_0^1 |\zeta(-i\alpha, r)| dr, \tag{36}$$

where we used

$$(L - l)^{-3/2}, F^*_{3/2}(u, l) \ll 1.$$

Applying

$$\zeta(s, a) - (1 + a)^{-s} = \zeta(s, a + 1)$$

and (5) to the first and second integrals on the right side of (36), respectively, we get for all $l \in l_h$,

$$\zeta(1/2 - i\alpha, l) \ll \log\alpha. \tag{37}$$

Besides, with integration by parts and (5), we have

$$\begin{aligned}\int_{-1}^{-1+h} \zeta(1/2 - i\alpha, z+1)(z-a_1)^{-(1-\eta)}dz &= [\zeta(-1/2 + i\alpha, z+1)(1/2 + i\alpha)^{-1}(z-a_1)^{-(1-\eta)}]_{-1}^{-1+h} \\ &\quad + (1-\eta)\int_{-1}^{-1+h} \zeta(-1/2 + i\alpha, z+1)(1/2 + i\alpha)^{-1}(z-a_1)^{-2+\eta}dz \\ &\ll \log\alpha.\end{aligned} \tag{38}$$

Thus, substituting (37) and (38) in (35), we obtain as $a_1 \to 0$ and $\alpha \to \infty$,

$$\lim_{a_1 \to 0} \int_{s_h} \zeta(1/2 - i\alpha, z+1)(z-a_1)^{-(1-\eta)}dz \ll \log\alpha. \tag{39}$$

Similar arguments are applied to estimation of another integral

$$\lim_{a_1 \to 0} \int_{w_h} \zeta(1/2 - i\alpha, z+1)(z-a)^{-(1-\eta)}dz, \quad w_h := \{e^{it} : 0 \le t \le h\},$$

which is bounded by

$$\lim_{a_1 \to 0} \int_{w_h} \zeta(1/2 - i\alpha, z+1)(z-a)^{-(1-\eta)}dz \ll \log\alpha. \tag{40}$$

The other part of the integral $\int_{\Gamma_0}$, namely $\int_{\Gamma_0 - s_h - w_h}$, is estimated more easily, since in this case, we could obtain bounds for $\zeta(1/2 - i\alpha, 1 + e^{it})$ almost directly from (25); we get

$$\zeta(1/2 - i\alpha, 1 + e^{it}) \ll \log\alpha, \quad h \le t \le \pi - h.$$

Hence, it is easy to see that

$$\lim_{a_1 \to 0} \int_{\Gamma_0 - s_h - w_h} \zeta(1/2 - i\alpha, z+1)(z-a_1)^{-(1-\eta)}dz \ll \log\alpha. \tag{41}$$

In total, by (30), (32), (33), (34), (39), (40), and (41), we have

$$\begin{aligned}\lim_{a_1 \to 0} \int_{-1}^{1} \zeta(1/2 - i\alpha, z)(z-a_1)^{-(1-\eta)}dz &= -\lim_{g_1, a_1 \to 0} \int_{\Gamma_{g_1} + R_{g_1}} \zeta(1/2 - i\alpha, z)(z-a_1)^{-(1-\eta)}dz \\ &= -\lim_{a_1 \to 0}\left(\int_{\Gamma_0 - s_h - w_h} + \int_{s_h + w_h}\right)\zeta(1/2 - i\alpha, z+1)(z-a_1)^{-(1-\eta)}dz + O(1) \\ &\ll \log\alpha.\end{aligned} \tag{42}$$

With (27), (28), and (42), (26) is reduced to

$$Z_{1-\eta}(1/2 - \eta - i\alpha, 0)(-1/2 + \eta + i\alpha)^{-1} \asymp \zeta(1/2 - \eta - i\alpha)\alpha^{-\eta} \ll \log\alpha,$$

or

$$\zeta(1/2 - \eta - i\alpha) \ll \alpha^{\eta}\log\alpha, \quad \eta > 0. \tag{43}$$

Finally, by the Phragmen-Lindelöf principle on the growth rate of $\zeta(\sigma + it)$ (Ivic, 1985), (43) implies the Lindelöf hypothesis.

This completes the proof of the Lindelöf hypothesis.

3. Results

We presented Theorem 1, which may be useful for other fields in mathematics, not restricted to the theory of the Riemann zeta-function.

To summarize our arguments for Theorem 2 and the Lindelöf hypothesis given above, using Theorem 1, we first broke the series $F_s(z, a)$ into the product of $z^{-\delta-it}$, the sum $\sum_{m\in\mathbb{Z}}[-2(m + a)\omega_1]^{-(1-\delta+\epsilon)+it}$, and the function γ. Then by Fourier transform, we reproduced the latter product in terms of an integral involving the series $F_s(z, a)$. With the $2\omega_1$-periodicity of $F_s(z, a)$, we could relate this integral to the Hurwitz zeta-function, and Theorem 2 was established.

In order to prove the Lindelöf hypothesis, we needed to handle several singularities arising as a approaches 0 in Theorem 2. In resolving this difficulty, we rewrote Theorem 2 in a suitable form, and used Cauchy's theorem in the theory of residues.

References

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The Erratum to "On a Formula of Mellin and Its Application to the Study of the Riemann zeta-function"

Throughout this discussion, the *original article* refers to one as in the title of this note. It appeared on Vol. 4, No. 6 of this journal (Shinya, 2012).

First, as a minor mistypo, the definition of $Z_\epsilon(1-\delta+\epsilon+i\alpha, z)$ should have been written as

$$Z_\epsilon(1-\delta+\epsilon+i\alpha, a) := \sum_{m\geq 1}[-(m+a)]^{-(1-\delta+\epsilon)-i\alpha}\gamma(\epsilon,\delta,\alpha);$$

the sign attached to $i\alpha$ on the right is $-$ (the same with $E(1-\delta+\epsilon+i\alpha, a)$).

Next, a more important correction is that the whole of the subsection 2.2 should be forgotten (or needs a significant revision). Although we could find many mistakes there, one of the most fatal is described as follows.

From the equation (30) of the original article onwards, we tried to replace the integral

$$\int_{-1}^{1}\zeta(1/2-i\alpha, z)(z-a_1)^{-(1-\eta)}dz$$

by

$$\lim_{g_1\to 0}\int_{\Gamma_{g_1}}\zeta(1/2-i\alpha, z)(z-a_1)^{-(1-\eta)}dz$$

using Cauchy's residue theorem. However, the expression $z-a_1$ actually passes the negative real axis as z moves over Γ_{g_1} for small g_1. If we recall how the complex logarithm is handled, then it is easy to see that the function $(z-a_1)^{-(1-\eta)}$ is not even continuous on Γ_{g_1}.

This is fatal in our argument, because in this case, Cauchy's theorem is not applicable.

References

Shinya, H. (2012). On a Formula of Mellin and Its Application to the Study of the Riemann zeta-function. *Journal of Mathematics Research*, *4*(6), 12-21. doi:http://dx.doi.org/10.5539/jmr.v4n6p12