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Certain Flowers of Continued Fractions In the Garden of Generalized Lambert Series

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Abstract

In this paper, an attempt has been made to establish certain results involving Lambert series and Continued fractions.

Keywords: Lambert series, Continued fractions, Rogers-Ramanujan identity

1. Introduction

Lambert series is a well known class of series in Analytic function theory and number theory. This series has been elegantly used in a variety of context of Ramanujan's research work. The dimension provided by Ramanujan inspired Andrews, G. E. and Berndt, B. C. (2005) to prove a lot of identities given by S. Ramanujan (Andrews, G. E. & Berndt, B. C., 2005). Most of Ramanujan's unproved identities came in picture with their proofs nicely derived by Andrews, G. E. and Berndt, B. C. (2005) merely on the knowledge of Lambert series. The garden of Lambert series and generalized Lambert series enriched a lot due to the contribution of Agarwal, R. P. (1993), Denis, R. Y. (1985; 1984), Denis, R. Y., Singh, S. N., & Singh, S. P. (2009), Bhargava, S. & Somashekara, D. D. (1993), Naika, M. S. M. & Dharmendra, B. N. (2008), Tachiya, Y. (2004), Srivastava, B. (2011; 2001), Assche, W. V. (2001), Cassens, P. & Regan, F. (1970), Slater, L. J. (1952) and many others. We all are aware that continued fraction representations of basic hypergeometric series has given new direction for Ramanujan's work and mathematicians working in the field of continued fraction namely Agarwal, R. P. (1996), Denis, R. Y. (1985; 1983), Denis, R. Y., Singh, S. N., & Singh, S. P. (2009; 2010), Denis, R. Y. & Singh, S. N. (2000), Naika, M. S. M., Chandankumar, S., & Sushan Bairy, K. (2012), Srivastava, P. (2008; 2011) and others have established hundreds of results for basic and bilateral basic hypergeometric functions and their continued fraction representations. In the present work we have used Generalized Lambert series and developed new continued fractions.

2. Notations and Definitions

For a and q real or complex and $|q| < 1$, we shall have

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & n = 1,2,3\dots. \end{cases} \tag{1}$$

We also define

$$(a)_\infty = (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \text{ for } |q| < 1. \tag{2}$$

The infinite product diverges when $a \neq 0$.

Also

$$(a_1, a_2, \dots, a_r; q)_n = (a_1;q)_n(a_2;q)_n \dots (a_r;q)_n. \tag{3}$$

$$[a;q]_n = \frac{[a;q]_\infty}{[aq^n;q]_\infty}. \tag{4}$$

An expression of the form $\frac{a_0}{b_0+}\frac{a_1}{b_1+...}$ is said to be a continued fraction. The values of $a_1, a_2, a_3, ...$ and $b_1, b_2, b_3, ...$ can be either real or complex values. A finite simple continued fraction is a simple continued fraction with only a finite number of terms. An infinite simple continued fraction is a simple continued fraction with an infinite number of terms.

The series, $\sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}$, considered by Lambert in connection with the convergence of power series is called Lambert series.If the series $\sum_{n=1}^{\infty} a_n$ converges,then Lambert series,$\sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}$ converges for all values of x except for $x = \pm 1$, otherwise it converges for those values of x, for which the series $\sum_{n=1}^{\infty} a_n x^n$ converges. A series of the form $\sum_{n=-\infty}^{n=\infty} (-1)^{\epsilon n} q^{\lambda n^2} R(q^n)$, where $\epsilon = 0$ or 1, $\lambda > 0$, and $R(q^n)$ is rational function of q^n is called generalized Lambert series.

We use following continued fractions for analysis in the present paper.

Rogers - Ramanujan continued fraction, $c(q)$ is given by

$$\frac{1}{c(q)} = \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\frac{q^4}{1+\ldots} = \frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{[q;q]_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2}}{[q;q]_n}} = \frac{[q;q^5]_\infty[q^4;q^5]_\infty}{[q^2;q^5]_\infty[q^3;q^5]_\infty}, \qquad |q| < 1 \tag{5}$$

(Andrews, G. E. & Berndt, B. C., 2005, 10.4.1, p. 248).

$$\frac{[q;q^6]_\infty[q^5;q^6]_\infty}{[q^3;q^6]_\infty^2} = \frac{1}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+\ldots}, \tag{6}$$

(Andrews, G. E. & Berndt, B. C., 2005, 6.2.37, p. 154).

$$\frac{[q;q^8]_\infty[q^7;q^8]_\infty}{[q^3;q^8]_\infty[q^5;q^8]_\infty} = \frac{1}{1+}\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+\ldots}, \tag{7}$$

(Andrews, G. E. & Berndt, B. C., 2005, 6.2.38, p. 154).

$$\frac{[q;q^2]_\infty}{[q^2;q^4]_\infty^2} = \frac{1}{1+}\frac{q}{1+}\frac{q+q^2}{1+}\frac{q^3}{1+}\frac{q^2+q^4}{1+\ldots}, \tag{8}$$

(Andrews, G. E. & Berndt, B. C., 2005, 6.2.22, p. 150).

$$\frac{[q^3;q^4]_\infty}{[q;q^4]_\infty} = \frac{1}{1-}\frac{q}{1+q^2-}\frac{q^3}{1+q^4-}\frac{q^5}{1+q^6-\ldots}, \tag{9}$$

(Andrews, G. E. & Berndt, B. C., 2005, p. 156).

$$\frac{[q^2;q^3]_\infty}{[q;q^3]_\infty} = \frac{1}{1-}\frac{q}{1+q-}\frac{q^3}{1+q^2-}\frac{q^5}{1+q^3-\ldots}, \tag{10}$$

(Andrews, G. E. & Berndt, B. C., 2005, 8.1.1, p. 197).

The first remarkable result of Rogers - Ramanujan continued fraction involving Lambert series is given by

$$\frac{1}{c^3(q)} = \frac{\displaystyle\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{5n+2}}}{\displaystyle\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{5n+1}}} = \frac{\displaystyle\sum_{n=0}^{\infty} q^{5n^2+4n}\frac{1+q^{5n+2}}{1-q^{5n+2}} - \sum_{n=0}^{\infty} q^{5n^2+6n+1}\frac{1+q^{5n+3}}{1-q^{5n+3}}}{\displaystyle\sum_{n=0}^{\infty} q^{5n^2+2n}\frac{1+q^{5n+1}}{1-q^{5n+1}} - \sum_{n=0}^{\infty} q^{5n^2+8n+3}\frac{1+q^{5n+4}}{1-q^{5n+4}}}, \tag{11}$$

(Andrews, G. E. & Berndt, B. C., 2005, 4.4.1, 4.4.2, p. 117).

Here $C(q)$ is called Ramanujan continued fraction with its value equal to $G(q)/H(q)$, and $G(q), H(q)$ are called Rogers - Ramanujan identities, which are as follows

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{[q;q]_n} = \frac{1}{[q;q^5]_\infty [q^4;q^5]_\infty}, \tag{12}$$

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{[q;q]_n} = \frac{1}{[q^2;q^5]_\infty [q^3;q^5]_\infty}, \tag{13}$$

(Andrews, G. E. & Berndt, B. C., 2005, 4.3.3, 4.3.4, p. 114).

We also use the following identities for establishing our main results.

The Rogers-Fine identity is as follows

$$\sum_{n=0}^{\infty} \frac{[\alpha;q]_n}{[\beta:q]_n} z^n = \sum_{n=0}^{\infty} \frac{[\alpha;q]_n [\alpha z q/\beta;q]_n \beta^n z^n q^{n^2-n}(1-\alpha z q^{2n})}{[\beta;q]_n [z;q]_{n+1}}, \tag{14}$$

(Andrews, G. E. & Berndt, B. C., 2005, 9.1.1, p. 223).

$$\sum_{n=0}^{\infty} \frac{z^n}{(1-\alpha q^n)} = \sum_{n=0}^{\infty} \frac{(\alpha z)^n q^{n^2}(1-\alpha z q^{2n})}{(1-\alpha q^n)(1-zq^n)}, \tag{15}$$

$$\sum_{n=0}^{\infty} \frac{q^{nj}}{(1-q^{kn+i})} = \sum_{n=0}^{\infty} \frac{q^{kn^2+n(i+j)}(1-q^{2kn+i+j})}{(1-q^{kn+i})(1-q^{kn+j})}, \tag{16}$$

(Denis, R. Y., 1985, 2.2).

$$\sum_{n=0}^{\infty} \frac{q^{ni}}{(1-q^{kn+i})} = \sum_{n=0}^{\infty} \frac{q^{kn^2+2ni}(1+q^{kn+i})}{(1-q^{kn+i})}, \tag{17}$$

(Denis, R. Y., Singh, S. N., & Singh, S. P., 2009, 3.4).

$$\sum_{n=-\infty}^{\infty} \frac{q^{nj}}{(1-q^{kn+i})} = \frac{[q^k;q^k]_\infty^2 [q^{i+j};q^k]_\infty [q^{k-i-j};q^k]_\infty}{[q^i;q^k]_\infty [q^j;q^k]_\infty [q^{k-i};q^k]_\infty [q^{k-j};q^k]_\infty}, \tag{18}$$

(Denis, R. Y., Singh, S. N., & Singh, S. P., 2009, 3.1).

$$\sum_{n=0}^{\infty} \frac{(n+1)[q/\alpha;q]_n \alpha^n}{[\beta;q]_{n+1}} + \sum_{n=0}^{\infty} \frac{n[q/\beta;q]_n \beta^n}{[\alpha;q]_{n+1}} = \frac{[q;q]_\infty^3 [\alpha\beta;q]_\infty}{[\alpha;q]_\infty^2 [\beta;q]_\infty^2}, \tag{19}$$

(Bhargava, S. & Somashekara, D. D., 1993, 2.3, p. 556).

$$\sum_{n=0}^{\infty} \frac{(n+1)[q^{k-i};q^k]_n q^{in}}{[q^j;q^k]_{n+1}} + \sum_{n=0}^{\infty} \frac{n[q^{k-j};q^k]_n q^{jn}}{[q^i;q^k]_{n+1}} = \frac{[q^k;q^k]_\infty^3 [q^{i+j};q^k]_\infty}{[q^i;q^k]_\infty^2 [q^j;q^k]_\infty^2}, \tag{20}$$

(Denis, R. Y., Singh, S. N., & Singh, S. P., 2009, 3.7).

$$[q^5;q^5]_\infty^2 G(q) = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+2}}, \tag{21}$$

$$[q^5;q^5]_\infty^2 H(q) = \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{5n+1}}, \tag{22}$$

$$[q^5;q^5]_\infty^2 \frac{G^2(q)}{H(q)} = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+1}}, \tag{23}$$

$$[q^5;q^5]_\infty^2 \frac{H^2(q)}{G(q)} = \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+2}}, \tag{24}$$

$$[q^5;q^5]^2_\infty G(q) = \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+1}}, \tag{25}$$

$$[q^5;q^5]^2_\infty H(q) = \sum_{n=-\infty}^{\infty} \frac{q^{n}}{1-q^{5n+3}}, \tag{26}$$

$$[q^5;q^5]^2_\infty \frac{G^2(q)}{H(q)} = \sum_{n=-\infty}^{\infty} \frac{q^{5n^2+2n}(1+q^{5n+1})}{1-q^{5n+1}}, \tag{27}$$

$$[q^5;q^5]^2_\infty \frac{H^2(q)}{G(q)} = \sum_{n=-\infty}^{\infty} \frac{q^{5n^2+4n}(1+q^{5n+2})}{1-q^{5n+2}}, \tag{28}$$

$$[q^5;q^5]^2_\infty G(q) = \sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1-q^{10n+1}}, \tag{29}$$

$$[q^5;q^5]^2_\infty H(q) = \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{10n+3}}, \tag{30}$$

(Andrews, G. E. & Berndt, B. C., 2005, (21)...(30)).

3. Main Results

We established the following main results

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+1})(1-q^{5n+3})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1-q^{10n+6})}{(1-q^{5n+4})(1-q^{5n+2})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+3n}(1-q^{10n+3})}{(1-q^{5n+2})(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+7n+2}(1-q^{10n+7})}{(1-q^{5n+3})(1-q^{5n+4})}} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}. \tag{31}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+3n}(1-q^{10n+3})}{(1-q^{5n+2})(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+7n+2}(1-q^{10n+7})}{(1-q^{5n+3})(1-q^{5n+4})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1+q^{5n+1})}{(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+8n+3}(1+q^{5n+4})}{(1-q^{5n+4})}} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}. \tag{32}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+1})(1-q^{5n+3})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1-q^{10n+6})}{(1-q^{5n+2})(1-q^{5n+4})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1+q^{5n+1})}{(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+8n+3}(1+q^{5n+4})}{(1-q^{5n+4})}} = \left\{\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}\right\}^2. \tag{33}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1+q^{5n+2})}{(1-q^{5n+2})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1+q^{5n+3})}{(1-q^{5n+3})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+3n}(1-q^{10n+3})}{(1-q^{5n+1})(1-q^{5n+2})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+7n+2}(1-q^{10n+7})}{(1-q^{5n+4})(1-q^{5n+3})}} = \left\{\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}\right\}^2. \tag{34}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1+q^{5n+2})}{(1-q^{5n+2})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1+q^{5n+3})}{(1-q^{5n+3})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+3})(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1-q^{10n+6})}{(1-q^{5n+4})(1-q^{5n+2})}} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}. \tag{35}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+1})(1-q^{5n+3})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1-q^{10n+6})}{(1-q^{5n+4})(1-q^{5n+2})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+3n}(1-q^{10n+3})}{(1-q^{5n+2})(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+7n+2}(1-q^{10n+7})}{(1-q^{5n+4})(1-q^{5n+3})}} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}. \tag{36}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+1})(1-q^{5n+3})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1-q^{10n+6})}{(1-q^{5n+4})(1-q^{5n+2})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+3n}(1-q^{10n+3})}{(1-q^{5n+2})(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+7n+2}(1-q^{10n+7})}{(1-q^{5n+4})(1-q^{5n+3})}}=\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}. \tag{37}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+1})(1-q^{5n+3})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1-q^{10n+6})}{(1-q^{5n+4})(1-q^{5n+2})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+3n}(1-q^{10n+3})}{(1-q^{5n+2})(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+7n+2}(1-q^{10n+7})}{(1-q^{5n+4})(1-q^{5n+3})}}=\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}. \tag{38}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1+q^{5n+2})}{(1-q^{5n+2})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1+q^{5n+3})}{(1-q^{5n+3})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+3n}(1-q^{10n+3})}{(1-q^{5n+1})(1-q^{5n+2})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+7n+2}(1-q^{10n+7})}{(1-q^{5n+4})(1-q^{5n+3})}}=\left\{\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}\right\}^2. \tag{39}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+3})(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1-q^{10n+6})}{(1-q^{5n+2})(1-q^{5n+4})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1+q^{5n+1})}{(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+8n+3}(1+q^{5n+4})}{(1-q^{5n+4})}}=\left\{\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}\right\}^2. \tag{40}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1-q^{10n+4})}{(1-q^{5n+3})(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1-q^{10n+6})}{(1-q^{5n+2})(1-q^{5n+4})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1+q^{5n+1})}{(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+8n+3}(1+q^{5n+4})}{(1-q^{5n+4})}}=\left\{\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}\right\}^2. \tag{41}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1+q^{5n+2})}{(1-q^{5n+2})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1+q^{5n+3})}{(1-q^{5n+3})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+3n}(1-q^{10n+3})}{(1-q^{5n+1})(1-q^{5n+2})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+7n+2}(1-q^{10n+7})}{(1-q^{5n+4})(1-q^{5n+3})}}=\left\{\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}\right\}^2. \tag{42}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1+q^{5n+2})}{(1-q^{5n+2})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n+1}(1+q^{5n+3})}{(1-q^{5n+3})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{10n^2+5n}(1-q^{20n+5})}{(1-q^{10n+1})(1-q^{10n+4})}-\sum_{n=0}^{\infty}\frac{q^{10n^2+15n+5}(1-q^{20n+15})}{(1-q^{10n+9})(1-q^{10n+6})}}=\left\{\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}\right\}^2. \tag{43}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{10n^2+5n}(1-q^{20n+5})}{(1-q^{10n+2})(1-q^{10n+3})}-\sum_{n=0}^{\infty}\frac{q^{10n^2+15n+5}(1-q^{20n+15})}{(1-q^{10n+7})(1-q^{10n+8})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1+q^{5n+1})}{(1-q^{5n+1})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+8n+3}(1+q^{5n+4})}{(1-q^{5n+4})}}=\left\{\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}\right\}^2. \tag{44}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+4n}(1+q^{5n+2})}{1-q^{5n+2}}-\sum_{n=0}^{\infty}\frac{q^{5n^2+6n}(1+q^{5n+3})}{1-q^{5n+3}}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+6n}(1-q^{10n+6})}{(1-q^{5n+2})(1-q^{5n+4})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+4n-1}(1-q^{10n+4})}{(1-q^{5n+1})(1-q^{5n+3})}}=-q\left\{\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}\right\}. \tag{45}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+6n}(1-q^{10n+6})}{(1-q^{5n+2})(1-q^{5n+4})}-\sum_{n=0}^{\infty}\frac{q^{5n^2+4n-1}(1-q^{10n+4})}{(1-q^{5n+1})(1-q^{5n+3})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{5n^2+2n}(1+q^{5n+1})}{1-q^{5n+1}}-\sum_{n=0}^{\infty}\frac{q^{5n^2+8n+3}(1+q^{5n+4})}{1-q^{5n+4}}}=-\frac{1}{q}\left\{\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+\dots}\right\}^2. \tag{46}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{6n^2+7n}(1-q^{12n+7})}{(1-q^{6n+4})(1-q^{6n+3})}-\sum_{n=0}^{\infty}\frac{q^{6n^2+5n-1}(1-q^{12n+5})}{(1-q^{6n+3})(1-q^{6n+2})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{6n^2+3n}(1-q^{12n+3})}{(1-q^{6n+1})(1-q^{6n+2})}-\sum_{n=0}^{\infty}\frac{q^{6n^2+9n}(1-q^{12n+9})}{(1-q^{6n+4})(1-q^{6n+5})}}=-\frac{1}{q}\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+\ldots}\right\}^2. \tag{47}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{6n^2+7n}(1-q^{12n+7})}{(1-q^{6n+4})(1-q^{6n+3})}-\sum_{n=0}^{\infty}\frac{q^{6n^2+5n}(1-q^{12n+5})}{(1-q^{6n+3})(1-q^{6n+2})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{6n^2+4n}(1+q^{6n+2})}{1-q^{6n+2}}-\sum_{n=0}^{\infty}\frac{q^{6n^2+8n-2}(1+q^{6n+4})}{1-q^{6n+4}}}=-\frac{1}{q}\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+\ldots}\right\}. \tag{48}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{8n^2+7n}(1-q^{16n+7})}{(1-q^{8n+5})(1-q^{8n+2})}-\sum_{n=0}^{\infty}\frac{q^{8n^2+9n+1}(1-q^{16n+9})}{(1-q^{8n+3})(1-q^{8n+6})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{8n^2+5n}(1-q^{16n+5})}{(1-q^{8n+2})(1-q^{8n+3})}-\sum_{n=0}^{\infty}\frac{q^{8n^2+11n+3}(1-q^{16n+11})}{(1-q^{8n+6})(1-q^{8n+5})}}=\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+\ldots}\right\}. \tag{49}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{8n^2+9n}(1-q^{16n+9})}{(1-q^{8n+7})(1-q^{8n+2})}-\sum_{n=0}^{\infty}\frac{q^{8n^2+7n-1}(1-q^{16n+7})}{(1-q^{8n+1})(1-q^{8n+6})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{8n^2+5n}(1-q^{16n+5})}{(1-q^{8n+4})(1-q^{8n+1})}-\sum_{n=0}^{\infty}\frac{q^{8n^2+11n+3}(1-q^{16n+11})}{(1-q^{8n+4})(1-q^{8n+7})}}$$

$$=-\frac{1}{q}\left\{\frac{1}{1+}\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+\ldots}\right\}\left\{1+\frac{q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^6}{1+}\frac{q^4+q^8}{1+\ldots}\right\}. \tag{50}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{4n^2+5n}(1-q^{8n+5})}{(1-q^{4n+3})(1-q^{4n+2})}-\sum_{n=0}^{\infty}\frac{q^{4n^2+3n}(1-q^{8n+3})}{(1-q^{4n+1})(1-q^{4n+2})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{4n^2+2n}(1+q^{4n+1})}{1-q^{4n+1}}-\sum_{n=0}^{\infty}\frac{q^{4n^2+6n+2}(1+q^{4n+3})}{1-q^{4n+3}}}=-\frac{1}{q}\left\{\frac{1}{1+}\frac{q}{1+}\frac{q+q^2}{1+}\frac{q^3}{1+\ldots}\right\}^2. \tag{51}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{4n^2+3n}(1-q^{8n+3})}{(1-q^{4n+2})(1-q^{4n+1})}-\sum_{n=0}^{\infty}\frac{q^{4n^2+5n+1}(1-q^{8n+5})}{(1-q^{4n+2})(1-q^{4n+3})}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{4n^2+2n}(1+q^{4n+1})}{1-q^{4n+1}}-\sum_{n=0}^{\infty}\frac{q^{4n^2+6n+2}(1+q^{4n+3})}{1-q^{4n+3}}}=\left\{\frac{1}{1+}\frac{q}{1+}\frac{q+q^2}{1+}\frac{q^3}{1+}\frac{q^2+q^4}{1+\ldots}\right\}^2. \tag{52}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(2n+1)[q^4;q^6]_n q^{2n}}{[q^2;q^6]_{n+1}}}{\displaystyle\sum_{n=0}^{\infty}\frac{(2n+1)[q^2;q^6]_n q^{4n}}{[q^4;q^6]_{n+1}}}=(1-q^2)\left\{\frac{1}{1-}\frac{q^2}{1+q^2-}\frac{q^6}{1+q^4-}\frac{q^{10}}{1+q^6-\ldots}\right\}^5. \tag{53}$$

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(2n+1)[q^6;q^8]_n q^{2n}}{[q^2;q^8]_{n+1}}}{\displaystyle\sum_{n=0}^{\infty}\frac{(2n+1)[q^2;q^8]_n q^{6n}}{[q^6;q^8]_{n+1}}}=(1-q^4)\left\{\frac{1}{1-}\frac{q^2}{1+q^4-}\frac{q^6}{1+q^8-}\frac{q^{10}}{1+q^{12}-\ldots}\right\}^4. \tag{54}$$

4. Proof of Main Results

Proof (31)-(44):

As an illustration, we shall prove (31).

From (21) and (22) we have

$$\frac{H(q)}{G(q)} = \frac{\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{5n+1}}}{\sum_{n=-\infty}^{\infty} \frac{q^{n}}{1-q^{5n+2}}} = \frac{\sum_{n=0}^{\infty} \frac{q^{3n}}{1-q^{5n+1}} - \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1-q^{5n+4}}}{\sum_{n=0}^{\infty} \frac{q^{n}}{1-q^{5n+2}} - \sum_{n=0}^{\infty} \frac{q^{4n+2}}{1-q^{5n+3}}}. \tag{55}$$

Making use of (5) and (16) in (55) and after simplification we obtain (31).

Proceeding in the same way and using the results (5), (16), (17) and (21)-(30), one can establish the results (32)-(44).

Proof of (45)-(54):

In order to prove results (45) to (54), we proceed as follows:

As an illustration, we shall prove (47) and (50).

Proof of (47) is as follows Taking $i = 3,\ j = 4,\ k = 6$ in (18) we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1-q^{6n+3}} = \frac{[q^6;q^6]_\infty^2 [q^7;q^6]_\infty [q^{-1};q^6]_\infty}{[q^3;q^6]_\infty^2 [q^4;q^6]_\infty [q^2;q^6]_\infty}. \tag{56}$$

Again setting $i = 2,\ j = 1,\ k = 6$ in (18) we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{n}}{1-q^{6n+2}} = \frac{[q^6;q^6]_\infty^2 [q^3;q^6]_\infty^2}{[q^2;q^6]_\infty [q;q^6]_\infty [q^4;q^6]_\infty [q^5;q^6]_\infty}. \tag{57}$$

Now taking the ratio of (56) and (57) and making use of (16) for the assigned values of $i,\ j$ and k, we get

$$\frac{\sum_{n=0}^{\infty} \frac{q^{6n^2+7n}(1-q^{12n+7})}{(1-q^{6n+4})(1-q^{6n+3})} - \sum_{n=0}^{\infty} \frac{q^{6n^2+5n-1}(1-q^{12n+5})}{(1-q^{6n+3})(1-q^{6n+2})}}{\sum_{n=0}^{\infty} \frac{q^{6n^2+3n}(1-q^{12n+3})}{(1-q^{6n+1})(1-q^{6n+2})} - \sum_{n=0}^{\infty} \frac{q^{6n^2+9n}(1-q^{12n+9})}{(1-q^{6n+4})(1-q^{6n+5})}} = \frac{[q^7;q^6]_\infty [q^{-1};q^6]_\infty}{[q;q^6]_\infty [q^5;q^6]_\infty} \frac{[q;q^6]_\infty^2 [q^5;q^6]_\infty^2}{[q^3;q^6]_\infty^4}. \tag{58}$$

Finally making use of (6) in (58) and after some calculation, we get (47).

In order to prove (50) we proceed as follows consider $i = 2,\ j = 7,\ k = 8$ in (18) we get

$$\sum_{n=-\infty}^{\infty} \frac{q^{7n}}{1-q^{8n+2}} = \frac{[q^8;q^8]_\infty^2 [q^9;q^8]_\infty [q^{-1};q^8]_\infty}{[q^2;q^8]_\infty [q^7;q^8]_\infty [q^6;q^8]_\infty [q;q^8]_\infty}. \tag{59}$$

Further taking $i = 1,\ j = 4,\ k = 8$ in (18) we obtain

$$\sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1-q^{8n+1}} = \frac{[q^8;q^8]_\infty^2 [q^5;q^8]_\infty [q^3;q^8]_\infty}{[q;q^8]_\infty [q^4;q^8]_\infty [q^7;q^8]_\infty [q^4;q^8]_\infty}. \tag{60}$$

Now taking the ratio of (59) and (60) and making use of (16) for the assigned values of $i,\ j$ and k, we get

$$\frac{\sum_{n=0}^{\infty} \frac{q^{8n^2+9n}(1-q^{16n+9})}{(1-q^{8n+7})(1-q^{8n+2})} - \sum_{n=0}^{\infty} \frac{q^{8n^2+7n-1}(1-q^{16n+7})}{(1-q^{8n+1})(1-q^{8n+6})}}{\sum_{n=0}^{\infty} \frac{q^{8n^2+5n}(1-q^{16n+5})}{(1-q^{8n+4})(1-q^{8n+1})} - \sum_{n=0}^{\infty} \frac{q^{8n^2+11n+3}(1-q^{16n+11})}{(1-q^{8n+4})(1-q^{8n+7})}} = \frac{[1-q^{-1}][q;q^8]_\infty [q^7;q^8]_\infty}{[1-q][q^5;q^8]_\infty [q^3;q^8]_\infty} \frac{[q^4;q^8]_\infty^2}{[q^2;q^4]_\infty}. \tag{61}$$

Now making use of (7) and (8) in (61), and after some calculation we obtain (50).

For suitable selection of values for i, j and k and using the same process for the results (5) to (10) and (16) to (20), one can establish the results (45), (46), (48), (49) and (51) to (54).

5. Conclusion

The object of this article has been to show the effective applications of generalized Lambert series in obtaining a diverse variety of continued fractions.

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