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Nathanson Heights and the CSS Conjecture for Cayley Graphs

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Abstract

Let G be a finite directed graph, $\beta(G)$ the minimum size of a subset X of edges such that the graph $G' = (V, E \setminus X)$ is directed acyclic and $\gamma(G)$ the number of pairs of nonadjacent vertices in the undirected graph obtained from G by replacing each directed edge with an undirected edge. Chudnovsky, Seymour and Sullivan proved that if G is triangle-free, then $\beta(G) \leq \gamma(G)$. They conjectured a sharper bound (so called the "CSS conjecture") that $\beta(G) \leq \gamma(G)/2$. Nathanson and Sullivan verified this conjecture for the directed Cayley graph $\mathrm{Cay}(\mathbb{Z}/N\mathbb{Z}, E_A)$ whose vertex set is the additive group $\mathbb{Z}/N\mathbb{Z}$ and whose edge set E_A is determined by $E_A = \{(x, x+a) : x \in \mathbb{Z}/N\mathbb{Z}, a \in A\}$ when N is prime and $|A| \leq (N-1)/4$ by introducing "height". In this work, we extend the definition of height and apply to answer the CSS conjecture for $\mathrm{Cay}(\mathbb{Z}/N\mathbb{Z}, E_A)$ to any positive integer N and $|A| \leq (N-1)/4$.

Keywords: Cayley graphs, CSS conjecture, Nathanson heights

1. Introduction

A *finite directed graph* $G = (V, E)$ consists of two finite sets, the set $V = V(G)$ of *vertices* of G and the set $E = E(G) \subseteq V \times V$ of *edges* of G. Let v and v' be distinct vertices of the finite directed graph G. A *directed path of length* l in G from v to v' is a sequence of l edges $\{(v_{i-1}, v_i)\}_{i=1}^{l}$ such that $v = v_0$ and $v' = v_l$. A *directed cycle of length* l in G is a sequence of l edges $\{(v_{i-1}, v_i)\}_{i=1}^{l}$ such that $v_0 = v_l$. A *loop*, a *digon* and a *triangle* are directed cycle of length 1, 2 and 3, respectively. A *triangle free* graph is a graph with no loops, digons, or triangles. A directed graph is called *acyclic* if it has no directed cycles.

Let $\beta(G)$ be the minimum size of a subset X of edges such that the graph $G' = (V, E \setminus X)$ is directed acyclic, and let $\gamma(G)$ be the number of pairs of nonadjacent vertices in the undirected graph obtained from G by replacing each directed edge with an undirected edge. Chudnovsky, Seymour and Sullivan (Chudnovsky, M., 2007) proved that if G is a triangle-free digraph, then $\beta(G) \leq \gamma(G)$. They conjectured a sharper bound (so called the "CSS conjecture") that if G is a triangle-free digraph, then $\beta(G) \leq \gamma(G)/2$.

Let N be a positive integer and A a nonempty subset of $\mathbb{Z}/N\mathbb{Z} \setminus \{0\}$ of cardinality $d \leq N$. Consider the directed Cayley graph $G = \mathrm{Cay}(\mathbb{Z}/N\mathbb{Z}, E_A)$ whose vertex set is the additive group $\mathbb{Z}/N\mathbb{Z}$ and whose edge set E_A is determined by

$$E_A = \{(x, x+a) : x \in \mathbb{Z}/N\mathbb{Z}, a \in A\}.$$

Assume that G is triangle free. Then G has neither loops nor digons, so the number of pairs of adjacent vertices is the same as the number of directed edges, which is dN. Thus the number of pairs of nonadjacent vertices is

$$\gamma(G) = \binom{N}{2} - dN = \frac{N(N-1-2d)}{2}. \tag{1}$$

In this case, the inequality in the CSS conjecture becomes

$$\beta(G) \leq \frac{\gamma(G)}{2} = \frac{N(N-1-2d)}{4}.$$

By introducing the term "height in finite projective space", Nathanson and Sullivan verified this conjecture when N is prime in (Nathanson, M. B., 2007) and $d \leq (N-1)/4$. Later, the height on the finite projective line was studied extensively in (Batson, J., 2008).

Using the "height" idea together with some elementary number theory facts involving the unit group of $\mathbb{Z}/N\mathbb{Z}$ and its cardinality, we prove the CSS conjecture when N is any positive integer expanding Nathanson and Sullivan's results. The detail of our work is divided into two sections. Section 2 presents the definition and bound of the height defined for $\mathbb{Z}/N\mathbb{Z}$. The final section talks about the CSS conjecture and shows how to relate the height to it.

2. Heights

Let N and d be positive integers. We define an equivalence relation $\sim$ on the set of nonzero d-tuple $(\mathbb{Z}/N\mathbb{Z})^d \setminus (0,\ldots,0)$ by

$$(a_1,a_2,\ldots,a_d) \sim (b_1,b_2,\ldots,b_d) \Leftrightarrow (b_1,b_2,\ldots,b_d) = \lambda(a_1,a_2,\ldots,a_d)$$

for some $\lambda \in (\mathbb{Z}/N\mathbb{Z})^\times$. Here $(\mathbb{Z}/N\mathbb{Z})^\times$ stands for the unit group of $\mathbb{Z}/N\mathbb{Z}$ and we use $(\mathbb{Z}/N\mathbb{Z})^*$ for the set of nonzero element in $\mathbb{Z}/N\mathbb{Z}$. Observe that $(\mathbb{Z}/N\mathbb{Z})^\times = (\mathbb{Z}/N\mathbb{Z})^*$ if and only if N is a prime. Also, $|(\mathbb{Z}/N\mathbb{Z})^\times| = \phi(N)$, the *Euler ϕ-function*. Write $(a \mod N)$ for the least nonnegative integer in the congruence class $a \in \mathbb{Z}/N\mathbb{Z}$. We first compute

Lemma 1 *For $a \in (\mathbb{Z}/N\mathbb{Z})^*$,*

$$\sum_{k\in(\mathbb{Z}/N\mathbb{Z})^\times} (ka \mod N) = \frac{N\phi(N)}{2}.$$

Proof. Let $a \in (\mathbb{Z}/N\mathbb{Z})^*$. If $N = 2$, then $(a \mod 2) = 1 = 2\phi(2)/2$. Next we assume that $N > 2$. It is clear that $k \in (\mathbb{Z}/N\mathbb{Z})^\times \Leftrightarrow N-k \in (\mathbb{Z}/N\mathbb{Z})^\times$ for all $k \in (\mathbb{Z}/N\mathbb{Z})^*$. Since $N > 2$, $k \neq N-k$ for every $k \in (\mathbb{Z}/N\mathbb{Z})^\times$. Then

$$(\mathbb{Z}/N\mathbb{Z})^\times = \{k, N-k : k \in (\mathbb{Z}/N\mathbb{Z})^\times \text{ and } k < N/2\}$$

and so $\phi(N)$ is even. Note that

$$((N-k)a \mod N) = ((Na-ka) \mod N) = N - (ka \mod N)$$

for all $k \in (\mathbb{Z}/N\mathbb{Z})^\times$. Thus

$$\begin{aligned}\sum_{k\in(\mathbb{Z}/N\mathbb{Z})^\times} (ka \mod N) &= \sum_{\substack{k\in(\mathbb{Z}/N\mathbb{Z})^\times,\\ k<N/2}} [(ka \mod N) + ((N-k)a \mod N)] \\ &= \sum_{\substack{k\in(\mathbb{Z}/N\mathbb{Z})^\times,\\ k<N/2}} [(ka \mod N) + (N - (ka \mod N))] \\ &= \sum_{\substack{k\in(\mathbb{Z}/N\mathbb{Z})^\times,\\ k<N/2}} N = \frac{N\phi(N)}{2}.\end{aligned}$$

Hence we have the lemma. □

We denote the equivalence class of the point $(a_1, a_2, \ldots, a_d)$ by $\langle a_1, a_2, \ldots, a_d\rangle$ and the set of all equivalence classes by $\mathbf{P}^{d-1}(\mathbb{Z}/N\mathbb{Z})$. The *height* of the class $\mathbf{a} = \langle a_1, a_2, \ldots, a_d\rangle \in \mathbf{P}^{d-1}(\mathbb{Z}/N\mathbb{Z})$ is given by

$$h_N(\mathbf{a}) = \min\left\{\sum_{i=1}^{d}(ka_i \mod N) : k \in (\mathbb{Z}/N\mathbb{Z})^\times\right\}.$$

Since $\mathbf{a} \neq \mathbf{0}$, there exists $a_j \in (\mathbb{Z}/N\mathbb{Z})^*$ such that $(ka_j \mod N) > 0$ for every $k \in (\mathbb{Z}/N\mathbb{Z})^\times$, so $h_N : \mathbf{P}^{d-1}(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{Z}^+$. We use $d^*(\mathbf{a})$ to denote the number of nonzero components of $\mathbf{a} = \langle a_1, \ldots, a_d\rangle \in \mathbf{P}^{d-1}(\mathbb{Z}/N\mathbb{Z})$, that is, the number of $a_i \neq 0$, and we define

$$d^*(\mathcal{A}) = \max\{d^*(\mathbf{a}) : \mathbf{a} \in \mathcal{A}\}$$

for $\mathcal{A} \subseteq \mathbf{P}^{d-1}(\mathbb{Z}/N\mathbb{Z})$. Clearly, $h_N(\mathbf{a}) \leq d^*(\mathbf{a})(N-1)$ for all $\mathbf{a} \in \mathbf{P}^{d-1}(\mathbb{Z}/N\mathbb{Z})$. For any nonempty finite subset A of $\mathbb{Z}^+$ with $|A| = m$, we note that $\min A \leq (1/m)\sum_{a\in A} a$. By Lemma 1, we have

$$\begin{aligned} h_N(\mathbf{a}) &= \min\left\{\sum_{i=1}^{d}(ka_i \mod N) : k \in (\mathbb{Z}/N\mathbb{Z})^\times\right\} \\ &\leq \frac{1}{\phi(N)}\left(\sum_{k\in(\mathbb{Z}/N\mathbb{Z})^\times}\left(\sum_{i=1}^{d}(ka_i \mod N)\right)\right) \\ &= \frac{1}{\phi(N)}\left(\sum_{i=1}^{d}\left(\sum_{k\in(\mathbb{Z}/N\mathbb{Z})^\times}(ka_i \mod N)\right)\right) \\ &= \frac{1}{\phi(N)}\left(d^*(\mathbf{a})\frac{N\phi(N)}{2}\right) = \frac{d^*(\mathbf{a})N}{2}. \end{aligned}$$

Since heights are positive integers, $h_N(\mathbf{a}) \leq \lfloor d^*(\mathbf{a})N/2\rfloor$. Hence we get a better bound for $h_N(\mathbf{a})$. We summarize the above computation with its corollary as follows.

Lemma 2 *For $\mathbf{a} \in \mathbf{P}^{d-1}(\mathbb{Z}/N\mathbb{Z})$, $h_N(\mathbf{a}) \leq \lfloor d^*(\mathbf{a})N/2\rfloor$.*

Corollary 3 (i) *For $d \geq 1$ and $\mathbf{a} \in \mathbf{P}^{d-1}(\mathbb{Z}/2\mathbb{Z})$, $h_2(\mathbf{a}) = d^*(\mathbf{a})$.*
(ii) *For $N \geq 2$ and $\mathbf{a} = \langle a\rangle \in \mathbf{P}^0(\mathbb{Z}/N\mathbb{Z})$, $h_N(\mathbf{a}) \leq \lfloor N/2\rfloor$. In particular, if $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, then*

$$h_N(\mathbf{a}) = \min\{(ka \mod N) : k \in (\mathbb{Z}/N\mathbb{Z})^\times\} = \min\{(k \mod N) : k \in (\mathbb{Z}/N\mathbb{Z})^\times\} = 1.$$

3. The CSS Conjecture

In this section, we deal with the CSS conjecture for the Cayley graph $G = \mathrm{Cay}(\mathbb{Z}/N\mathbb{Z}, E_A)$. Notice that if the outdegree of every vertex in finite directed graph is at least one, then the graph contains a cycle. Thus every finite directed acyclic graph contain at least one vertex with outdegree 0. Nathanson and Sullivan used this to prove the following theorem and derived its consequence. Their proofs can be found in (Nathanson, M. B., 2007). We recall this work in

Theorem 4 *(Nathanson, M. B., 2007) Let $V = \{v_0, v_1, \ldots, v_{N-1}\}$ be the vertex set of the directed graph G. Then G is directed acyclic if and only if there is a permutation σ of $\{0, 1, \ldots, N-1\}$ such that $r < s$ for every edge $(v_{\sigma(r)}, v_{\sigma(s)})$ of the graph G.*

Corollary 5 *(Nathanson, M. B., 2007) Let $G = (V, E)$ be a directed graph with vertex set $\{v_0, v_1, \ldots, v_{N-1}\}$ and let $\Sigma \subseteq S_N$ be a set of permutations of $\{0, 1, \ldots, N-1\}$. For $\sigma \in \Sigma$, let B_σ be the set of edges $(v_{\sigma(r)}, v_{\sigma(s)}) \in E$ with $r \geq s$. Then $\beta(G) \leq \min\{|B_\sigma| : \sigma \in \Sigma\}$.*

This corollary yields an immediate result on our Cayley graph $\mathrm{Cay}(\mathbb{Z}/N\mathbb{Z}, E_A)$, namely,

Lemma 6 *Let $N \geq 2$, $d \geq 1$ and $A = \{a_1, \ldots, a_d\} \subseteq (\mathbb{Z}/N\mathbb{Z})^*$. Let $G = \mathrm{Cay}(\mathbb{Z}/N\mathbb{Z}, E_A)$ be the Cayley graph constructed from A. Let Σ be a set of permutations of $\mathbb{Z}/N\mathbb{Z}$ and $\sigma \in \Sigma$. For $i \in \mathbb{Z}/N\mathbb{Z}$ and $j \in \{1, \ldots, d\}$, define $t_{ij} \in \mathbb{Z}/N\mathbb{Z}$ by $\sigma(i) + a_j = \sigma(t_{ij})$. Then $E_A = \{(\sigma(i), \sigma(t_{ij})) : i \in \mathbb{Z}/N\mathbb{Z}$ and $j \in \{1, \ldots, d\}\}$. Let*

$$B_\sigma = \{(\sigma(i), \sigma(t_{ij})) : (i \mod N) > (t_{ij} \mod N) \text{ and } j \in \{1, \ldots, d\}\}.$$

Then the graph $G' = (\mathbb{Z}/N\mathbb{Z}, E_A \setminus B_\sigma)$ is directed acyclic for every permutation $\sigma \in \Sigma$ and $\beta(G) \leq \min\{|B_\sigma| : \sigma \in \Sigma\}$.

For $k \in (\mathbb{Z}/N\mathbb{Z})^\times$, define the permutation σ_k of $\mathbb{Z}/N\mathbb{Z}$ by $\sigma_k(i) = ki$ for all $i \in \mathbb{Z}/N\mathbb{Z}$. Let $\Sigma = \{\sigma_k : k \in (\mathbb{Z}/N\mathbb{Z})^\times\}$ be the set of $\phi(N)$ permutations of $\mathbb{Z}/N\mathbb{Z}$. Fix $k \in (\mathbb{Z}/N\mathbb{Z})^\times$. For $i \in \mathbb{Z}/N\mathbb{Z}$ and $j \in \{1, \ldots, d\}$, define $t_{ij} \in \mathbb{Z}/N\mathbb{Z} \setminus \{i\}$ by $\sigma_k(t_{ij}) = \sigma_k(i) + a_j$. Since $k \in (\mathbb{Z}/N\mathbb{Z})^\times$, there exists $u_k \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that $ku_k = 1$. Let $r_j = (u_k a_j \mod N)$. Then $1 \leq r_j \leq N - 1$ and $a_j = kr_j$. Thus

$$\sigma_k(t_{ij}) = \sigma_k(i) + a_j = ki + kr_j = k(i + r_j) = \sigma_k(i + r_j),$$

so $t_{ij} = i + r_j$. Since $1 \leq r_j \leq N - 1$, $(t_{ij} \mod N) = (i \mod N) + r_j - N < (i \mod N)$ if $(i \mod N) + r_j \geq N$. Moreover, if $(i \mod N) + r_j < N$, then $(t_{ij} \mod N) = (i \mod N) + r_j > (i \mod N)$. Hence $(i \mod N) > (t_{ij} \mod N) \Leftrightarrow N - r_j \leq (i \mod N)$.

Let $B_{\sigma_k} = \{(\sigma_k(i), \sigma_k(t_{ij})) : (i \mod N) > (t_{ij} \mod N)$ and $j \in \{1, \ldots, d\}\}$. Then

$$|B_{\sigma_k}| = |\{(\sigma_k(i), \sigma_k(t_{ij})) : N - r_j \leq (i \mod N) \leq N - 1\}| = \sum_{j=1}^{d} r_j = \sum_{j=1}^{d} (u_k a_j \mod N).$$

Applying Lemma 6 and the fact that $\{u_k : k \in (\mathbb{Z}/N\mathbb{Z})^\times\} = (\mathbb{Z}/N\mathbb{Z})^\times$, we get

$$\begin{aligned}
\beta(G) &\leq \min\{|B_{\sigma_k}| : k \in (\mathbb{Z}/N\mathbb{Z})^\times\} \\
&= \min\left\{\sum_{j=1}^{d} (u_k a_j \mod N) : k \in (\mathbb{Z}/N\mathbb{Z})^\times\right\} \\
&= \min\left\{\sum_{j=1}^{d} (k a_j \mod N) : k \in (\mathbb{Z}/N\mathbb{Z})^\times\right\} \\
&= h_N(\langle a_1, \ldots, a_d\rangle).
\end{aligned}$$

Thus $\beta(G) \leq h_N(\langle a_1, \ldots, a_d\rangle)$. Together with Lemma 2, we have

Lemma 7 *Let $N \geq 2$, $d \geq 1$ and $A = \{a_1, \ldots, a_d\} \subseteq (\mathbb{Z}/N\mathbb{Z})^*$. Let $G = \mathrm{Cay}(\mathbb{Z}/N\mathbb{Z}, E_A)$ be the Cayley graph constructed from A. Then*

$$\beta(G) \leq h_N(\langle a_1, \ldots, a_d\rangle) \leq \frac{dN}{2}.$$

This lemma gives

Theorem 8 *Let $N \geq 5, d \geq 1$ and $A = \{a_1, \ldots, a_d\} \subseteq (\mathbb{Z}/N\mathbb{Z})^*$. Let $G = \mathrm{Cay}(\mathbb{Z}/N\mathbb{Z}, E_A)$ be the Cayley graph constructed from A which has no digons. If $d \leq (N - 1)/4$, then $\beta(G) \leq \gamma(G)/2$.*

Proof. Assume that $d \leq (N - 1)/4$. Then

$$\frac{dN}{2} = dN - \frac{dN}{2} \leq \frac{N(N-1)}{4} - \frac{dN}{2} = \frac{N(N - 1 - 2d)}{4}.$$

By Lemma 7 and Eq. (1), we get

$$\beta(G) \leq \frac{dN}{2} \leq \frac{N(N - 1 - 2d)}{4} = \frac{\gamma(G)}{2}$$

as desired. □

Hamidoune proved the Caccetta-Häggkvist conjecture for Cayley graphs:

Theorem 9 *(Hamidoune, Y. 0., 1981, p.349-355 or Nathanson, M. B., 2006) Let* $A \subseteq (\mathbb{Z}/N\mathbb{Z})^*$ *and* $d = |A| \geq N/k$. *Then the Cayley graph* $G = \mathrm{Cay}(\mathbb{Z}/N\mathbb{Z}, E_A)$ *contains a cycle of length at most* k. *In particular, if* G *is triangle-free, then* $d < N/3$.

Back to the CSS conjecture. Since $dN/2 \leq N(N-1-2d)/4$ if and only if $d \leq (N-1)/4$, it follows that, for a fixed N, we only need to consider sets A of cardinality $d > N/4$. Combined with Theorem 9, in order to prove the CSS conjecture for the group $\mathbb{Z}/N\mathbb{Z}$, it remains to work only on the sets A of size d, where $N/4 < d < N/3$. The following example shows that sometimes the height is greater than $\gamma(G)/2$, so we cannot conclude the CSS conjecture without computing $\beta(G)$ explicitly.

Example 10 *Let* $N = 14$ *and* $A = \{1, 2, 8, 9\} \subset (\mathbb{Z}/14\mathbb{Z})^*$. *Then* $N/4 < d < N/3$. *Since* 0 *is not in* $A, 2A$ *and* $3A$, $G = \mathrm{Cay}(\mathbb{Z}/14\mathbb{Z}, E_A)$ *is a triangle-free digraph. We have* $h_{14}(\langle 1, 2, 8, 9\rangle) = 20$ *and* $\gamma(G) = 35$. *Thus* $h_{14}(\langle 1, 2, 8, 9\rangle) > \gamma(G)/2$.

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The Set R^* and Its Properties

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Abstract

In this paper, we shall give a set R^* and indicate its properties, and thus, some abnormal results, such as the limit number may be successor, the natural number may be transfinite, the infinite set can not be equipotent to its proper subset etc., will be obtained.

Keywords: Remainder matrix, Axis number, Double-direction induction

1. On $R^{(n)}$

Assume the ordered set formed by front (n+1) prime numbers on the number axis (We refer only to the non-negative integers on right of number axis) in accordance with natural order be

$$D_n = \{d_0, d_1, d_2, \cdots, d_n\}$$

make product

$$\mathfrak{Z}_n = \prod_{i=0}^{n} d_i$$

, and denote the directed and closed segment from 0 to $\mathfrak{Z}_n$ on the axis by

$$\overline{M}_n = [0 - \mathfrak{Z}_n]$$

The $(\mathfrak{Z}_n + 1)$ integers in the $\overline{M}_n$ form an ordered set, in symbols

$$Z_n = \{0, 1, 2, \cdots, \mathfrak{Z}_n\}$$

For any $m \in Z_n$, dividing the m successively by every prime in D_n, assume the remainders be respectively

$$r_{m0}, r_{m1}, \cdots, r_{mn}$$

and rewrite them as a single column matrix

$$R^{(n)}(m) = \begin{bmatrix} r_{m0} \\ r_{m1} \\ \vdots \\ r_{mn} \end{bmatrix}$$

Obviously, for any one of numbers in Z_n there is a definite matrix of single column corresponding with it. We arrange all these column matrices in natural order, such that they form a n-degree matrix of remainders

$$R^{(n)} = \begin{bmatrix} r_{00} & r_{10} & \cdots & r_{\mathfrak{Z}_n 0} \\ r_{01} & r_{11} & \cdots & r_{\mathfrak{Z}_n 1} \\ \vdots & \vdots & \ddots & \vdots \\ r_{0n} & r_{1n} & \cdots & r_{\mathfrak{Z}_n n} \end{bmatrix}$$

So the integral points in $\overline{M}_n$, the integers in Z_n and the columns in $R^{(n)}$ (respectively rejecting their last element) naturally form one-to-one correspondences (refer following 1.1, 1.2), each of them may therefore be replaced by the other.

The $R^{(n)}$ has following properties:

1.1 Any two different numbers in Z_n except $\mathfrak{Z}_n(R^{(n)}(\mathfrak{Z}_n) \equiv R^{(n)}(0))$ correspond to two different columns in $R^{(n)}$ (Two columns are called the same or equal when only when their corresponding elements at the same rows are all equal, in sign "≡").

Proof: Assume that there were integers $i,\ j \in [0,\ \mathfrak{Z}_n)$ and $i < j$, such that $R^{(n)}(i) \equiv R^{(n)}(j)$, then by congruence property (Hua, 1964, P. 3, P. 22.), the difference $(j - i)$ can be integrally divided by each of all primes in D_n, therefore by $\mathfrak{Z}_n$ also. This is contrary to known that $(j - i) < \mathfrak{Z}_n$.

1.2 The all different columns in $R^{(n)}$ (Both the same first and last columns termed 0-columns which are composed all by zeros may disregard the last one) include all possible combinations taking respectively one remainder of each primes in D_n.

Proof: Assume $d_i \in D_n$, the all possible remainders of d_i are

$$P(d_i) = \{0,\ 1,\ 2, \cdots,\ d_i - 1\}$$

Taking $r_i \in P(d_i)(i = 0,\ 1,\ 2, \cdots,\ n)$, we obtain a combination of remainders as $\begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_n \end{bmatrix}$. Evidently, the number of all possible such combinations is

$$C^1_{d_0} \bullet C^1_{d_1} \bullet \cdots \bullet C^1_{d_n} = d_0 d_1 \cdots d_n = \mathfrak{Z}_n$$

By 1.1, there exactly are $\mathfrak{Z}_n$ different columns in $R^{(n)}$, and thus the 1.2 has been proved.

We may separate all the columns in $R^{(n)}$ into two classes: A column which does not contain the element zero is called a column of first class, otherwise a column of second class. The integral points in $\overline{M}_n$ (or numbers in Z_n) corresponding with the columns of first class are termed residual points (or residual numbers) of degree n, easily known

1.3 The number (total) of all columns of first class in $R^{(n)}$ is

$$\mathfrak{Z}'_n = \prod_{i=0}^{n}(d_i - 1)$$

. By 1.2, we only except the element zero from $P(d_i)(i = 0,\ 1,\ 2, \cdots,\ n)$, then get the combinations so done. Denote the ordered set (in natural order) of all residual numbers in Z_n by

$$\triangle^{(n)} = \{\delta^{(n)}_0,\ \delta^{(n)}_1, \cdots,\ \delta^{(n)}_{\mathfrak{Z}'_n - 1}\}$$

easily known, $\delta^{(n)}_0 = 1$, whereas the $\delta^{(n)}_1$ must be the next prime $d_{(n+1)}$. Obviously, the primes are infinite.

Calling the column $R^{(n)}(\frac{1}{2}\mathfrak{Z}_n)$ (the element at the first row is 1, and the others all 0) the mid-column of $R^{(n)}$, we have

1.4 The two classes of columns in $R^{(n)}$ are all symmetrically distributive with reference to the mid-column. The mutually symmetric columns $R^{(n)}(m)$ and $R^{(n)}(\mathfrak{Z}_n - m)(m \in Z_n)$ satisfy

$$r_{mi} + r_{(\mathfrak{Z}_n - m)i} \equiv 0 \bmod(d_i) \quad (i = 0,\ 1,\ 2, \cdots,\ n)$$

We call $R^{(n)}(m)$ and $R^{(n)}(\mathfrak{Z}_n - m)$ the mutually conjugate columns, in symbols, $\overline{R^{(n)}(m)} = R^{(n)}(\mathfrak{Z}_n - m)$, and also so do for two relative points or numbers.

Two columns in $R^{(n)}$ are called mutually independent columns when only their corresponding elements of all the same rows are different, otherwise, mutually dependent columns. And so call the relative points or numbers.

1.5 For any integer $i \in Z_n$, when the i is an odd the column $R^{(n)}(i)$ does not have any independent column of first class; When the i is an even, then the $R^{(n)}(i)$ has such columns so many that the number q satisfies

$$\beth_n^{"} \le q \le \beth_n' \quad \text{where} \beth_n^{"} = \prod_{i=1}^{n}(d_i - 2)$$

1.6 For any even $h \in Z_n$, assume that its independent points of first class (i.e. independent residual points) laying within the interval $[h, \beth_n]$ have number q', then in $\overline{M}_n$ the number of all pairs of residual points being h apart (i.e. the distance between them is h length units) is necessarily the q', and the converse is also true.

Proof: Assume that $\delta \in [h, \beth_n]$ is a residual number independent of h, then the $R^{(n)}(\delta - h)$ must be a column of first class (i.e. the $(\delta - h)$ is also a residual number). If not, the element in $R^{(n)}(\delta)$ that corresponds with the element zero of $R^{(n)}(\delta - h)$ is necessarily equal to the relative element of $R^{(n)}(h)$. This contradicts that the h and δ are mutually independent. Conversely, assume δ_1 and δ_2 be two residual numbers in $\overline{M}_n$ and $\delta_2 - \delta_1 = h$, then $\delta_2 \in [h, \beth_n]$, above all, the $\beth_2$ must be independent of h. Because if not, for both $R^{(n)}(\delta_2)$ and $R^{(n)}(h)$, say, their elements of i-th row were equal, then the element of i-th row in $R^{(n)}(\delta_1)$ must be zero. This contradicts that the δ_1 is a residual number.

Particularly, in the $\overline{M}_n$, the number of all twin residual numbers is $(\beth_n^{"} - 1)$ pairs.

2. On R

According to congruence properties, the whole semi-axis of numbers forms infinitely many periodic segments: $[0, \beth_n], [\beth_n, 2\beth_n], [2\beth_n, 3\beth_n], \cdots$. Each of them corresponds to the same $R^{(n)}$.

Adding the next prime d_{n+1} into the D_n, we obtain relatively D_{n+1}, $\beth_{n+1}$, Z_{n+1}, $\overline{M}_{n+1}$ and $R^{(n+1)}$. Obviously, the $R^{(n+1)}$ involves d_{n+1} ones of the same $R^{(n)}$ ranging periodically, but because of increasing a row of new elements at last (ranging the $P(d_{n+1})$ periodically to $\beth_n$ times), among the old columns of first class in $R^{(n+1)}$ there now are $\beth_n'$ ones to have translated into second class. Evidently, the columns of first class decrease relatively. For the sake of convenience, the process of translating from $R^{(n)}$ to $R^{(n+1)}$ through increasing the next prime is called regular evolution. Clearly, under the regular evolution, the properties stated before remain unchanged always.

Let $R^{(n)}$ by the order of natural numbers regularly evolves on infinitely (i.e. $n \to \infty$), then

$$D_n \to D = \{d_0, d_1, \cdots, d_n, \cdots\}$$

$$\beth_n \to \beth = \prod_{i=0}^{\infty} d_i$$

$$\overline{M}_n \to \overline{M} = [0 - \beth]$$

$$Z_n \to Z = \{0, 1, 2, \cdots, \beth_n, \cdots, \beth\}$$

$$R^{(n)} \to R = \begin{bmatrix} r_{00} & r_{10} & \cdots & r_{\beth_n 0} & \cdots & r_{\beth 0} \\ r_{01} & r_{11} & \cdots & r_{\beth_n 1} & \cdots & r_{\beth 1} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ r_{0n} & r_{1n} & \cdots & r_{\beth_n n} & \cdots & r_{\beth n} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \end{bmatrix}$$

The R still preserves similar properties about mentioned before, above all

2.1 The R also possesses the last column $R(\beth)$ which is identical with the first column $R(0)$, both are composed by infinitely many zeros. Clearly, the proper factors of $\beth$ involve every prime in D, and thus $\beth \equiv 0 \bmod(d_i)(i = 0, 1, 2, \cdots)$. Again, the index of each prime factor of $\beth$ is degree 1, hence the $\beth$ is the minimal number by all primes as factors.

Similarly, the R possesses the mid-column $R(\frac{1}{2}\beth)$, its element at the first row is 1, and the others are all zeros.

2.2 In R, the first column $R(0)$ possesses the successor $R(1) = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix}$, the last column $R(\mathfrak{Z})$ possesses the predecessor $R(\mathfrak{Z}-1) = \begin{bmatrix} d_0-1 \\ d_1-1 \\ \vdots \\ d_n-1 \\ \vdots \end{bmatrix}$ (the $(\mathfrak{Z}-1)$ is therefore of definite meaning, and the $\mathfrak{Z}$ is a successor number), and the others possess both predecessor and successor. For this sake, we require only to subtract together 1 from every element of given column (if its some element, say one at mark i row, is zero, then do after replacing it by corresponding prime d_i), or to add together 1 to every element of the column (if the element at mark i row plus 1 is equal to d_i, then transform into zero).

Let $X = \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_n \\ \vdots \end{bmatrix} (r_i \in P(d_i),\ i = 0,\ 1,\ 2,\ \cdots)$, $X_k = \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_k \end{bmatrix}$, and R_k be the submatrix formed by the front (k+1) rows of R (the R_k is composed only by the $R^{(k)}$ and its periodic segments), then.

2.3 Any X is necessarily a column of R and any column of R must be some X.

Proof: Assume that there were an X being not a column of R, then there necessarily exists a certain $k \in \omega_0$ (the ω_0 is ordered set of natural numbers), such that the $X_k \notin R_k$, and therefore $X_k \notin R^{(k)}$ as well as any periodic segment of $R^{(k)}$. This obviously contradicts the 1.2. Similarly, any column of R can only be some X.

2.4 In R, any two columns but the $R(\mathfrak{Z})$ are mutually unlike.

Proof: (a) It may be alleged that, in R, there is no any other 0-column (its elements are all zeros) to be between the $R(0)$ and $R(\mathfrak{Z})$. Otherwise, if there were $\alpha(0 < \alpha < \mathfrak{Z})$ such that $R(\alpha)$ could be a 0-column, then $\alpha \equiv 0 \bmod(d_i)(i = 0,\ 1,\ 2, \cdots)$, thus every prime in D would be proper factor of the α, and so, $\alpha \geq \mathfrak{Z}$ (see 2.1), this is contrary to the assumption. (b) In R, there are no any two non-zero columns to be the same. Otherwise, if there were non-zero columns $R(\alpha)$ and $R(\beta)(\alpha, \beta \in Z, \text{and}\, \alpha < \beta)$, such that $R(\alpha) \equiv R(\beta)$, then by $R_{0-\alpha}$ denote the submatrix composed by all the columns from $R(0)$ to $R(\alpha)$ in R, and on the R parallely translate it to right, so that to coincide with $R_{0-\beta}$ up to their last column, thus the two submatrixes, because their last columns are identical, by the double-direction induction (see late 3.2) easily known, the column (denoted by $R(\beta - \alpha)$) coinciding with the first column of $R_{0-\alpha}$ must be a 0-column. By (a), this is impossible.

Denote the matrix obtained after to reject the last column $R(\mathfrak{Z})$ off the R by R^*, and conceive the R^* as an ordered set of its columns in natural order, then the R^* has minimal and maximal elements $R^*(0)$ and $R^*(\mathfrak{Z}-1)$ (called two extreme columns). Assume X be a non-extreme column of R^*, then clearly, the X has both immediate predecessor and successor. Therefore, the R^* is a discrete set without any "limit (or inaccessible) element".

As $R^{(n)}$, the R^* contains the columns of first class (e.g. in the column X let $r_n = n + 1(n \in \omega_0)$ we get one), and in the R^*, each column has its conjugate one (the $R^*(0)$ and $R^*(\frac{1}{2}\mathfrak{Z})$ are self-conjugate).

2.5 The potency (cardinal) of R^* is equal to the potency of the Continuum (Xie, 1979, P. 2.): $\overline{\overline{R^*}} = c$. Proof: Denote the Continuum [0, 1] by C, and let any infinite decimal $0.m_1m_2\cdots m_n\cdots(0 \leq m_i \leq 9,\ i = 1,\ 2, \cdots)$ in

C correspond to the element $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ m_1 \\ m_2 \\ \vdots \end{bmatrix}$ of R^*, we find that $\overline{\overline{C}} = c \leq \overline{\overline{R^*}}$. Next denote the set of all denumerable sequences consisting of non-negative integers by S, no doubt $\overline{\overline{S}} = c$, and $R^* \subset S$ when each column of R^* is naturally conceived as such a sequence, and so $\overline{\overline{R^*}} \leq \overline{\overline{S}} = c$. Thus, $\overline{\overline{R^*}} = c$ (here followed the old view of cardinals and only make reference).

Use the $R^{(n)*}$ to express the ordered set of all columns of $R^{(n)}$ but the last one $R^{(n)}(\mathfrak{Z}_n)$, in definition, always $\mathfrak{Z}_n = R^{(n)*}$ for any $n \in \omega_0$, again when $n \to \infty$,

$$\mathfrak{Z}_n \& R^{(n)*} \to \mathfrak{Z} \& R^*$$

then $\mathfrak{Z} = R^*$ and thus $\overline{\mathfrak{Z}} = c$.

3. On R^*

Since the columns in R^* and the integral points on number axis $\overline{M}$ (except the end point $\mathfrak{Z}$) form a one-to-one correspondence, thus we can directly use the columns of R^* to label the integral numbers on $\overline{M}$ in the relation of correspondence (the end number $\mathfrak{Z}$ is labeled by R^* itself). Obviously, these numbers not only include all natural numbers, but exceed them by far (e.g. the all residual numbers but the $R^*(1)$, the conjugate numbers of the natural numbers but zero etc. are all transfinite). Above all, in R^* except $R^*(0)$, the all other numbers are successor ones, and any two consecutive numbers correspond to two points spaced out 1 apart on $\overline{M}$, and thus the R^* (as well as the $\mathfrak{Z}$) may be regarded as extension of natural number set (such numbers as well as their points are still called integral ones).

Similarly to $R^{(n)}$, in R^* the amount of all columns of first class is $\mathfrak{Z}' = \prod_{i=0}^{\infty}(d_i - 1)$. These numbers expressed by such columns except the $R^*(1)$ may be called generalized primes (or transfinite primes). Easily known, in R^* the twin generalized primes are infinitely many pairs, their number is $(\mathfrak{Z}^{"} - 1)$ pairs, where $\mathfrak{Z}^{"} = \prod_{i=1}^{\infty}(d_i - 2)$. Of course, for any number in R^*, if its element of the first row is 0, it is called an even; if 1, an odd. In order to be clear about the orderity and connectivity of R^*, we have:

Axiom 1: The distribution of all integral points on $\overline{M}$ possesses: 1) equidistant property (all points are everywhere equispaced out by 1 length unit and, the extreme points have one-side consecutive points, the others have two-side ones); 2) increasing property (from left to right monotonically increasing); 3) completeness (single-linearly marked the all integers from 0 to $\mathfrak{Z}$). Such arrangement is termed the number axis order.

Axiom 2: Assume α, $\beta(\alpha < \beta)$ being two integral points on $\overline{M}$, by $R^*_{\alpha-\beta}$ denote the subset of a segment from $R^*(\alpha)$ to $R^*(\beta)$ in R^*, let U and V be two nonempty and mutually complementary subsets of $R^*_{\alpha-\beta}$(for any $R^*(x) \in R^*_{\alpha-\beta}$, either $R^*(x) \in U$ or $R^*(x) \in V$), then there exist $R^*(x_1) \in U$ and $R^*(x_2) \in V$, such that the $R^*(x_1)$ and $R^*(x_2)$ are consecutive.

3.1 No doubt, the R^* is an ordered set (with the number axis order).

Note that, taking $R^*(\alpha) \in R^*$, the subset formed by all front columns of the $R^*(\alpha)$ is denoted by $A = \{R^*(0), R^*(1), \cdots, R^*(\alpha - 1)\}$ (when $\alpha = 0$, $A = \emptyset$), particularly the A is called the pre-part of $R^*(\alpha)$ and signed as $R^*_{0-(\alpha-1)}$; the subset formed by all back columns of the $R^*(\alpha)$ by $B = \{R^*(\alpha + 1), R^*(\alpha + 2), \cdots, R^*(\mathfrak{Z} - 1)\}$ (when $\alpha = \mathfrak{Z} - 1$, $B = \emptyset$), then for any $R^*(x) \in R^*$, perhaps $R^*(x) \equiv R^*(\alpha)$, or $R^*(x) \in A$ (i.e. $R^*(x) < R^*(\alpha)$), or $R^*(x) \in B$ (i.e. $R^*(x) > R^*(\alpha)$). Obviously, it is impossible that $R^*(x) \in A$ and $R^*(x) \in B$ too (otherwise, there will be two identical columns in R^*).

3.2 Double-direction induction (DDI): By $R^*_{\alpha-\beta}$ denote the subset of a segment $(R^*(\alpha) < R^*(\beta))$ in R^*, there is $R^*(\xi) \in R^*_{\alpha-\beta}$ which possesses property Q, "for any column $R^*(x)$ of $R^*_{\alpha-\beta}$, assume the $R^*(x)$ possess the Q, then its consecutive columns ($R^*(x-1)$ and $R^*(x+1)$) also possess the Q", so every column of $R^*_{\alpha-\beta}$ possesses the

Q.

Proof: Assume some columns of $R^*_{\alpha-\beta}$ do not possess the Q, and all such columns form a set as U, its complementary set be V, then by axiom 2, in $R^*_{\alpha-\beta}$ there exist consecutive columns $R^*(x_1) \in U$ and $R^*(x_2) \in V$, because the $R^*(x_2)$ has possessed the Q, from assumption, the $R^*(x_1)$ possesses also the Q, this is a contradiction.

3.3 The R^* is not a well ordered set (see late 3.5). And thus, the $\mathfrak{Z}$ has been not an ordinal number (Thomas, 1978, P. 24-31.) in traditional meaning. Now define the numbers in R^* and the R^* itself (expressing the number $\mathfrak{Z}$) as axis numbers, then for each integral point on $\overline{M}$ there is a certain axis number corresponding with it. For consistence with $\mathfrak{Z} = R^*$ as ever, every axis number, as $R^*(\alpha) \in R^*$, may be defined as its pre-part, namely $R^*(\alpha) = R^*_{0-(\alpha-1)}$ (when $\alpha = 0$, its pre-part being Ø).

This paper doesn't stipulate operations on axis numbers, only affirms the determinacy of predecessor and successor for given axis number. The $\mathfrak{Z}$ is an end number no its successor, so the $(\mathfrak{Z}+1)$ has not been axis number. As to $\mathfrak{Z}'$ and $\mathfrak{Z}^{"}$, since all greater than 0 and smaller than $\mathfrak{Z}$, they must be axis numbers.

3.4 Each axis number (but zero) regarded as a set is not equipotent to its proper subset (PS).

Proof: Obviously, in R^* any nonzero natural number has possessed such property. For convenience, assume that nonempty-subset $R^*_{0-\alpha}$ of R^* is not equipotent to its any PS, we prove that $R^*_{0-\alpha_1}(\alpha_1 = \alpha - 1)$ and $R^*_{0-\alpha_2}(\alpha_2 = \alpha + 1)$ are also not equipotent to their PS. 1). Suppose that the $R^*_{0-\alpha_1}$ were equipotent to its PS W, then add the element $R^*(\alpha)$ respectively to $R^*_{0-\alpha_1}$ and W, and make it correspond to itself, so we find that the $R^*_{0-\alpha}$ and its PS $W \cup \{R^*(\alpha)\}$ are equipotent, this is contrary to assumption. 2) Suppose that the $R^*_{0-\alpha_2}$ were equipotent to its PS E, then a) If $R^*(\alpha_2) \notin E$, and $R^*(\alpha_2)$ corresponds to $R^*(x)$ of E, then reject both $R^*(\alpha_2)$ and $R^*(x)$ off $R^*_{0-\alpha_2}$ and E respectively, clearly $R^*(x) \in R^*_{0-\alpha}$, and so the $R^*_{0-\alpha}$ would be equipotent to its PS (E rejected the $R^*(x)$), being contrary to assumption. b) If $R^*(\alpha_2) \in E$, then in the corresponding relation of $R^*_{0-\alpha_2}$ and E, either $R^*(\alpha_2)$ is oneself correspondence, in this case, reject the $R^*(\alpha_2)$ off $R^*_{0-\alpha_2}$ and E, at once we find $R^*_{0-\alpha}$ being equipotent to its PS E^*(E rejected the $R^*(\alpha_2)$), this is similar contradiction; or each $R^*(\alpha_2)$ corresponds to other element, to say, $R^*(x_1) \in E$ and $R^*(x_2) \in R^*_{0-\alpha_2}$ are the maps of $R^*(\alpha_2)$ from its set onto the opposite respectively, then reject the $R^*(\alpha_2)$ off each set, and make $R^*(x_2)$ correspond to $R^*(x_1)$, so we still find the $R^*_{0-\alpha}$ being equipotent to its PS E^*, and also obtain a contradiction to assumption. By DDI, the 3.4 has been proved. So the axis number is also cardinal like natural number.

3.5 Traditionally, the ordinal number ω_0 defined by the ordered set of natural numbers is a "minimal transfinite number". In fact, the axis number system has shown that, between the finite and infinite there is no certain boundary and unbridgeable gulf. And so, so-called "minimal transfinite number" does not exist. Relatively, the "minimal transfinite cardinal number" does not exist too (old conclusion of infinite cardinals have been not enough to regard as criterion).

Usually, one stipulates the "natural numbers" being all finite. However, since each natural number itself also expresses the "number" of natural numbers (concretely, of all ones smaller than it), hence, if natural numbers are infinite many, then necessarily there will be some "infinite natural number" (generalized natural number). Since restricting of the decimal notation, when the order of units in a numeral changes into infinite, the traditional notation has had no method to express and to judge it, and so one counts it being no existing. However, when one has some new number scale and expressing means, the condition will be quite another.

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The Correlation between Variate-Values and Ranks in Samples from Complete Fourth Power Exponential Distribution

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Abstract

In this paper, we derive the correlation between variate-values and ranks in a sample from the Complete Fourth Power Exponential (CFPE) distribution. A sample from the CFPE distribution could be misclassified as if it is drawn from the normal distribution due to some similarities between the two distributions. In practice, ranks are used instead of real values (variate-values) when there is hardly any knowledge about the underlying distribution. This may lead to loss of some of the information contained in the actual values. In this paper we found that the amount of information loss, by using ranks instead of real data, is larger when the sample is from the CFPE distribution than if it is from the normal distribution. However, there is still a relatively high correlation between variate-values and the corresponding ranks. Comparisons between the correlation between variate-values and ranks for the CFPE distribution and some other distributions are provided.

Keywords: Complete Fourth Power Exponential Distribution, Variate-values, Normal distribution, Correlation coefficient, Ranks

1. Introduction

Statistical methods based on ranks have been heavily studied in the literature, specifically when there is hardly any knowledge of the underlying distribution of the data at hand. Such methods fall under the nonparametric techniques umbrella. However, ignoring the underlying distribution may lead to loss of some of the information contained in the data. And, in some circumstances, when there is a lack of information with regards to the underling distribution, nonparametric techniques – including the methods based on ranks – could be useful and lead to robust inferences. For more details on these methods, we refer the reader to Lehmann and D'Abrera (1976).

It can be difficult to discriminate between the Complete Fourth Power Exponential (CFPE) distribution and the normal distribution due to some similarities between these distributions, e.g. their shapes and other properties. Some people, especially non-statisticians, may mistakenly assume their dataset comes from a normal distribution, when in fact the data may actually come from the CFPE distribution, since it has properties similar to a normal distribution.

Amira and Mazloum (1993) studied in detail the CFPE distribution and focused on the geometric and statistical properties of this distribution and compared it to the normal distribution. Unfortunately, this distribution

has limited application at the moment with possible future works. We briefly give an overview of the CFPE distribution and some of its properties in the next section.

The misclassification of the dataset from the CFPE distribution as a normal distribution was studied in an unpublished MSc thesis entitled "Study on the Error in Random Samples Classification between the Normal Distribution and the Complete Fourth Power Exponential" by Baeshen (2000), King Abdulaziz University, Jeddah, Saudi Arabia.

In practice, the ranks of data are used instead of the original data to make an inference. Additionally, to find how much information we may lose by this action, Stuart (1954) derived the formula to calculate the relationship (correlation) between variate-values and their ranks. It showed that for some distributions we did not lose much information when the original data is replaced by their corresponding ranks. Moreover, Stuart (1955) considered the situation when the variance of a specific distribution does not exist and it showed that for a continuous distribution with no moments which is eventually monotone, the correlation between variate-values and their ranks is zero.

Later, O'Brien (1982) estimated via simulation the average correlation between variate-values and their ranks for small size samples from different distributions. The term "the degree of distortion or error" is used to indicate the loss of information by replacing the variate-values by their ranks. O'Brien found that these correlations are generally high, and when the sample size increase they reach the limiting values presented by Stuart (1954, 1955) . This finding supports the idea of using the ranks instead of variate-values with a small degree of distortion or error even for small samples.

In Section 2, we will overview the CFPE distribution and some of its properties. In Section 3, we use Stuart's formula to derive the correlation between variate-values and ranks in samples from the CFPE Distribution. Finally, we provide the exact and the approximation correlations between variate-values and ranks from some distributions in Section 4.

2. The Complete Fourth Power Exponential (CFPE) Distribution

In this section we overview briefly the CFPE distribution and some of its summaries that are needed in this paper. For more details we refer to Amira and Mazloum (1993).

Let X be a random variable from the CFPE distribution with the density function given by

$$f(x\,;\alpha,\,\beta) = \frac{2}{\beta\,\Gamma(\frac{1}{4})}\;\exp\{-\,(x-\alpha)^4/\beta^4\} \quad ,-\infty < x < \infty,\; -\infty < \alpha < \infty,\; \beta > 0 \tag{1}$$

where α is the location parameter and β is the scale parameter. The characteristic function of this distribution is

$$\phi_X(t) = \sum_{m=0}^{\infty}\sum_{j=0}^{\left[\frac{m}{2}\right]} C_{2j}^{m}\,\alpha^{(m-2j)}\beta^{2j}\,\frac{\Gamma\left(\frac{2j+1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\,\frac{(it)^m}{m!}$$

where $[m/2]$ is the greatest integer number less than $m/2$. Then, from this characteristic function, we can obtain the central and non-central moments as follows:

$$\mu_1' = \mu = E(X) = \alpha\;, \quad \mu_2' = E(X^2) = \alpha^2 + \beta^2\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\;, \quad \mu_3' = E(X^3) = \alpha^3 + 3\beta^2\alpha\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\;,$$

$$\mu_4' = E(X^4) = \alpha^4 + 6\beta^2\alpha^2\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} + \frac{1}{4}\beta^4 \quad , \quad \mu_2 = \sigma^2 = \beta^2\frac{\sqrt{2}\,\pi}{\left[\Gamma(\frac{1}{4})\right]^2} \quad , \quad \mu_3 = 0 \quad \text{and} \quad \mu_4 = \frac{1}{4}\beta^4$$

For simplicity and easy notation, we work on the special case of the CFPE distribution when $\alpha = 0$, since it is easy to centralize the data around the mean. Therefore

$$f(x\;;\,\beta) = \frac{2}{\beta\,\Gamma(\frac{1}{4})}\;\exp\{-\,(x/\beta)^4\} \quad ,-\infty < x < \infty,\; \beta > 0$$

and $\mu = \alpha = 0 \quad , \quad V(X) = \sigma^2 = \beta^2 \frac{\sqrt{2}\pi}{[\Gamma(\frac{1}{4})]^2}$.

3. The correlation between variate-values and ranks in samples from the CFPE Distribution

In this section, we use Stuart's formula (1954) to derive the correlation between variate-values and the ranks in sample from the CFPE distribution. From Stuart (1954), the correlation between variate-values X_i and their ranks R_i, $\rho(X_i, R_i)$, is given by

$$\rho(X_i, R_i) = \left[\int_{-\infty}^{\infty} xF(x)dF(x) - \frac{1}{2}\mu\right] \left[\frac{12(n-1)}{(n+1)\sigma^2}\right]^{\frac{1}{2}} \tag{2}$$

where n is the sample size. The second factor on the right side in (2) tends to $2\sqrt{3}/\sigma$ as n tends to infinity. In the following theorem, we use (2) to derive the correlation between variate-values and ranks in a sample from the CFPE distribution.

Theorem:

The correlation between variate-values and ranks in a sample from the CFPE distribution when $\alpha = 0$ is given by

$$\rho = \rho(X_i, R_i) = \frac{\sqrt{3}}{2\sqrt[4]{2}}\sqrt{\frac{n-1}{n+1}} \quad \text{and as} \quad n \to \infty \quad , \quad \rho \to \frac{\sqrt{3}}{2\sqrt[4]{2}} = 0.728 \tag{3}$$

Proof:

For ease notation, we consider the centered CFPE distribution, i.e. when $\alpha = 0$, therefore

$$f(x\,;\,\beta) = \frac{2}{\beta\,\Gamma(\frac{1}{4})}\,\exp\{-\,(x/\beta)^4\} \quad , -\infty < x < \infty,\ \beta > 0$$

and $\mu = \alpha = 0 \quad , \quad V(X) = \sigma^2 = \beta^2 \frac{\sqrt{2}\pi}{[\Gamma(\frac{1}{4})]^2}$ then

$$E[XF(X)] = \int_{-\infty}^{\infty} \{x\,F(x)\}\,f(x)\,dx = \frac{4}{\beta^2[\Gamma(\frac{1}{4})]^2}\int_{-\infty}^{\infty}\int_{-\infty}^{x} x\,e^{-\frac{1}{\beta^4}(t^4+x^4)}dt\,dx$$

Let $\frac{x^2}{\beta^2} = r\cos\theta \Rightarrow x = \beta\, r^{\frac{1}{2}}\cos^{\frac{1}{2}}\theta$ and $\frac{t^2}{\beta^2} = r\sin\theta \Rightarrow t = \beta\, r^{\frac{1}{2}}\sin^{\frac{1}{2}}\theta$.

The Jacobian of the transformation is

$$\begin{aligned} J &= \begin{vmatrix} \frac{1}{2}\beta\, r^{-\frac{1}{2}}\cos^{\frac{1}{2}}\theta & -\frac{1}{2}\beta\, r^{\frac{1}{2}}\cos^{-\frac{1}{2}}\theta\,\sin\theta \\ \frac{1}{2}\beta\, r^{-\frac{1}{2}}\sin^{\frac{1}{2}}\theta & \frac{1}{2}\beta\, r^{\frac{1}{2}}\sin^{-\frac{1}{2}}\theta\,\cos\theta \end{vmatrix} = \tfrac{1}{4}\beta^2\left\{\sin^{-\frac{1}{2}}\theta\cos^{\frac{3}{2}}\theta + \cos^{-\frac{1}{2}}\theta\sin^{\frac{3}{2}}\theta\right\} \\ &= \tfrac{1}{4}\beta^2\sin^{-\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta \end{aligned}$$

then

$$\begin{aligned} E[XF(X)] &= \frac{8}{\beta^2[\Gamma(\frac{1}{4})]^2}\int_0^{\infty}\int_0^{\frac{\pi}{4}} (\beta\, r^{\frac{1}{2}}\cos^{\frac{1}{2}}\theta)\; e^{-\{r^2\cos^2\theta + r^2\sin^2\theta\}}\left(\frac{1}{4}\beta^2\sin^{-\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta\right) dr\,d\theta \\ &= \frac{2\beta}{[\Gamma(\frac{1}{4})]^2}\int_0^{\infty}\int_0^{\frac{\pi}{4}} r^{\frac{1}{2}}\; e^{-r^2}\sin^{-\frac{1}{2}}\theta dr\,d\theta \end{aligned}$$

Let $z = r^2 \Rightarrow \frac{1}{2}z^{-\frac{1}{2}}dz = dr$, then

$$\begin{aligned} E[XF(X)] &= \frac{2\beta}{[\Gamma(\frac{1}{4})]^2}\left(\frac{1}{2}\int_0^{\infty} z^{-\frac{1}{4}}e^{-z}dz\right)\left(\int_0^{\frac{\pi}{4}}\sin^{-\frac{1}{2}}\theta\,d\theta\right) \\ &= \frac{2\beta}{[\Gamma(\frac{1}{4})]^2}\left\{\frac{1}{2}\Gamma(\tfrac{3}{4})\right\}\left\{\frac{1}{4}B(\tfrac{1}{4},\tfrac{1}{2})\right\} = \frac{\beta\,\Gamma(\frac{1}{2})}{4\Gamma(\frac{1}{4})} \end{aligned}$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Now by using (2) we have

$$\rho(X_i, R_i) = \left[\frac{12(n-1)[\Gamma(\frac{1}{4})]^2}{(n+1)\beta^2\sqrt{2}\,\pi}\right]^{\frac{1}{2}} \frac{\beta\,\Gamma(\frac{1}{2})}{4\Gamma(\frac{1}{4})} = \frac{\sqrt{3}}{2\sqrt[4]{2}}\sqrt{\frac{n-1}{n+1}}$$

and as $n \to \infty \quad , \quad \rho \to \frac{\sqrt{3}}{2\sqrt[4]{2}} = 0.728$.

4. Comparisons results

Stuart (1954) provided, using the formula in (2), the correlation between variate-values and the ranks in samples from Uniform (0, 1), Exponential ($\lambda = 1$), and Normal (0, 1) distributions. From (3), we have the correlation between variate-values and the ranks in the sample from the CFPE distribution (0, b), which is equal to 0.728 when n is significantly large, see Table 1.

Moreover, by using the simulation technique to estimate the quantity $E[XF(X)]$ in (2), we obtained the approximation results for the correlation between variate-values and the ranks in samples from the other distributions which did not have explicit formula. These approximation results are given in Table 2. We can point out that the corresponding correlation for the Log-normal distribution (0, 1) is closer to (less than) the CFPE distribution compared to other distributions.

5. Conclusion

In this paper, we derived the correlation between variate-values and ranks in a sample from the Complete Fourth Power Exponential (CFPE) distribution. We found that we lost more information when the data set came from the CFPE distribution than when the data came from normal distribution, when we used ranks instead of variate-values. This finding emphasizes the importance of distinguishing between these distributions. However, the correlation between variate-values and ranks in samples from the CFPE is still relatively high, which allowed us to use ranks instead of variate-values without losing a lot of information.

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Table 1. The Exact correlation between variate-values and ranks for some distributions

Distribution	μ	σ^2	$E\ [\,X\,F(x)\,]$	ρ	$n \to \infty$
Uniform (0,1)	1/2	1/12	1/3	$\sqrt{\frac{n-1}{n+1}}$	1
Exponential $\lambda = 1$	1	1	3/4	$\frac{\sqrt{3}}{2}\sqrt{\frac{n-1}{n+1}}$	$\frac{\sqrt{3}}{2} \approx .866$
Normal (0,1)	0	1	$\frac{1}{2\sqrt{\pi}}$	$\sqrt{\frac{3}{\pi}}\sqrt{\frac{n-1}{n+1}}$	$\sqrt{\frac{3}{\pi}} \approx .977$
Logistic (0,1)	0	$\frac{\pi^2}{3}$	1/2	$\frac{3}{\pi}\sqrt{\frac{n-1}{n+1}}$	$\frac{3}{\pi} \approx .955$
CFPE (0,b)	0	$\beta^2 \frac{\sqrt{2}\pi}{[\Gamma(\frac{1}{4})]^2}$	$\frac{\beta\Gamma(\frac{1}{2})}{4\Gamma(\frac{1}{4})}$	$\frac{\sqrt{3}}{2\sqrt[4]{2}}\sqrt{\frac{n-1}{n+1}}$	$\frac{\sqrt{3}}{2\sqrt[4]{2}} \approx .728$

The exact results of Uniform, Exponential and Normal are reported by Stuart (1954) while the logistic result presented in O'Brien (1982)

Table 2. The approximation correlation between variate-values and ranks for some distributions

Distribution	μ	σ^2	$E\ [\,X\,F(x)\,]$	ρ
Student's T (5 df)	0	1.667	0.346	0.928
Chi-square (5 df)	5	10	3.347	0.928
Gamma (5, 1)	5	5	3.117	0.956
F (4, 10)	1.250	1.563	0.909	0.787
Beta (1, 2)	0.333	0.056	0.233	0.973
Lognormal (0, 1)	1.649	4.671	1.253	0.687
Weibull (5, 1)	0.918	0.044	0.518	0.974

The approximation results obtained from 100 000 simulation samples of size 100 (n =100)

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Analyzing Plane-plate Bending with EFGM

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Abstract

The moving least-square technique is used to construct shape function in the Element Free Galerkin Method at present, but sometimes the algebra equations system obtained from the moving least-square approximation is ill-conditioned and the shape function needs large quantity of inverse operation. In this paper, the weighted orthogonal functions are used as basis ones, the application in the calculation of plate bending shows that the improved moving least-square approximation is effective and efficient.

Keywords: EFGM, Moving least-square approximation, The weighted orthogonal functions, Plate bending

1. Introduction

The finite element method (Long, 2001, pp. 135-167) has been the main calculation method in the area of mechanics recently, but when it deals with the simulation of large distortion, the grid cells become distorted, then we need to divide the grids again to make equilibriums. So it makes some difficulties. In views of this situation, mesh-free method comes out. This is a new numerical analysis method. Because it need only the information of nodes and thoroughly or partly cancelled the meshes, and also because it has the advantage of excellent accuracy and rapid convergence, the mesh-free method rapidly developed in recent years. Doe (Belytdchko, 1994, pp. 229-256. & Liu, 2005, 64-90) reported that Element Free Galerkin Method (EFGM) is one of this methods. It bases on Moving least-square approximate (MLS) (Belyschko, 1996, pp. 3-47. & Genki, 2000, pp. 1419-1433) approximation and its basic idea is to solve problems with some discrete nodes.

But in the practical calculations, to ever fixed node, the MLS degenerates the classical least-square approximate, and we have to face the ill-conditioned function. This paper aims at the characteristic of MLS, to modify it using the weighted orthogonal functions to be the basic functions. This method is applied to the plate bending which has been studied previously (Fu, 2004, pp. 232-307), and the result shows that this method is effective and efficient comparing with the former one.

2. Moving least-square approximation

EFGM is brought forward by Belytschko, the basic idea is in the domain Ω bounded by Γ, $u(x)$ is the field function, where x is a point in the field. The local approximation of $u(x)$ (Genki, Y., m., 2000) can be defined as

$$u^h(x) = \sum_{j=1}^{m} P_j(x)a_j(x) = p^T(x)a(x),\ \forall x \in \Omega \tag{1}$$

Where $p^T(x)$ is built utilizing Pascal's triangle with complete basis polynomial of m-dimensional, and the corresponding coefficient is $a(x)$, which is determinated by MLS

$$J = \sum_{I}^{n} w(x - x_I)[u^h(x) - u^*(x_I)]^2 = \sum_{I}^{n} w(x - x_I)[p^T(x_I)a(x) - u^*(x_I)]^2 \tag{2}$$

Where x_1 is the node around x, which is in the domain of influence. $w(x - x_1)$ is the weight function We use the weight function just as follows

$$\omega_i(r_i) = \begin{cases} \frac{r_{m_i}^2}{r_i^2+\varepsilon^2 r_{m_i}^2}(1 - \frac{r_i^2}{r_{m_i}^2}) & r_i \le r_{m_i} \\ 0 & r_i \ge r_{m_i} \end{cases}$$

(Nayroles, 1992, pp. 307-318), this function is simple in form than that brought forward by Belytschko. We can easily get the parameter, Otherwise the approximation solution with either high rand can be got theorily.

Eq. (4) can be expressed as

$$J(x) = (pa - u)^T \omega(x)(pa - u) \tag{5a}$$

Where

$$p = \begin{bmatrix} p_1(x) & p_2(x_1) & \cdots & p_m(x_1) \\ p_1(x_2) & p_2(x_2) & \cdots & p_m(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1(x_n) & p_2(x_n) & \cdots & p_m(x_n) \end{bmatrix} \tag{5b}$$

$$\omega(x) = \begin{bmatrix} \omega(x - x_1) & 0 & \cdots & 0 \\ 0 & \omega(x - x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega(x - x_1) \end{bmatrix} \tag{5c}$$

Taking the extreme value of Eq. (5), choosing functions $h(x)$, $g(x)$ at random, sign them as follows:

$$(h,\, g) = \sum_{I=1}^{n} \omega(x - x_1) h(x_I) g(x_I) \tag{6}$$

Then

$$a_1(x)(p_i,\, p_1) + a_2(x)(p_i,\, p_2) + \cdots + a_m(x)(p_i,\, p_m) = (p_i,\, u_I),\ i = 1,\, 2 \cdots,\, m \tag{7a}$$

forming it as matrix

$$\begin{bmatrix} (p_1,\, p_1) & (p_1,\, p_2) & \cdots & (p_1,\, p_m) \\ (p_2,\, p_1) & (p_2,\, p_2) & \cdots & (p_2,\, p_m) \\ \vdots & \vdots & \ddots & \vdots \\ (p_m,\, p_1) & (p_m,\, p_2) & \cdots & (p_m,\, p_m) \end{bmatrix} \begin{bmatrix} a_1(x) \\ a_2(x) \\ \vdots \\ a_m(x) \end{bmatrix} = \begin{bmatrix} (p_1,\, u_1) \\ (p_2,\, u_1) \\ \\ (p_m,\, u_1) \end{bmatrix} \tag{7b}$$

One obtains $a(x)$,

$$a = A^{-1} B u \tag{8a}$$

Where

$$A = P^T \omega p,\ B = p^T \omega \tag{8b}$$

Taking the Eq. (8) into Eq. (1),

$$u^h(x) = \sum_{I=1}^{n} \Phi_I(x) u_I \tag{9a}$$

where

$$\Phi(x) = \sum_{j=1}^{m} p_j(x)(A^{-1}B)_{jI} \tag{9b}$$

The virtue of Moving least-square approximation is obvious, but sometimes it may make the equation ill-condition. Otherwise, because of large quantity of inverse operations, the calculation time increases. To solve the problems, we bring forward the Moving least-square approximation.

3. The modified Moving least-square approximation

The modified moving least-square approximation employs the weighted orthogonal functions to be basis ones, and overcome the difficulties of MLS. Considering the Congregation of nodes $\{x_i\}$ and the weight $\{\omega_i\}(i = 1, 2, \cdots, m)$, if a group of function $\varphi_1(x)$, $\varphi_m(x), \cdots, \varphi_m(x)$ satisfies the situations just as follows:

$$(\varphi_k, \varphi_j) = \sum_{i=1}^{n} w_i \varphi_k(x_i)\varphi_j(x_i) = \begin{cases} 0 & k \neq j \\ A_k & k = j \end{cases} \quad (k, j = 1, 2, \cdots, m) \tag{10}$$

One calls $\varphi_1(x)$, $\varphi_2(x), \cdots, \varphi_m(x)$ the group of orthogonal functions which about Congregation of nodes $\{x_i\}$ with the weight $\{\omega_i\}$.

In the MLS, if the basic function $p_i(x)$, $(i = 1, 2, \cdots, m)$ is the orthogonal one that about the $\{x_i\}$ with the weight $\{\omega_i\}(i = 1, 2, \cdots, m)$, and the diagonal elements $(p_i, p_j) = 0$, $(i \neq j)$ in the Eq. (7) can be reduced as

$$\begin{bmatrix} (p_1, p_1) & 0 & \cdots & 0 \\ 0 & (p_2, p_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (p_m, p_m) \end{bmatrix} \begin{bmatrix} a_1(x) \\ a_2(x) \\ \vdots \\ a_m(x) \end{bmatrix} = \begin{bmatrix} (p_1, u_1) \\ (p_2, u_1) \\ \vdots \\ (p_m, u_1) \end{bmatrix} \tag{11}$$

Then $a_i(i = 1, 2, \cdots, m)$ are solved from Eq. (11).

It is easily to see that $a_i(x) = \frac{(p_i, u_I)}{(p_i, p_i)}$, take it to the Eq. (1),

$$u^h(x) = \sum_{i=1}^{m} p_i(x) \frac{\sum_{I=1}^{n} \omega(x - x_I) p_i(x_I) u_I}{(p_i, p_i)} = \sum_{I=1}^{n} \Phi_I(x) u_I \tag{12a}$$

The shape function is

$$\Phi_I(x) = \omega(x - x_I) \sum_{i=1}^{m} \frac{p_i(x) p(x_I)}{(p_i, p_i)}$$

So in the course of solving $a_i(i = 1, 2, \cdots, m)$, we avoid the inverse operations and the solution of ill-condition equations (Cheng, 2003, pp. 181-186), and the definition and efficiency are all improved. It can be seen from the above that once a group of basic function which composed by weighted orthogonal functions are obtained, then we can get $a_i(i = 1, 2, \cdots, m)$, usually we can construct the orthogonal polynomial base using Schmidt orthogonalization.

4. Basic formulas

According to the Mindlin plate theory (Wang, 1996, pp. 154-167), the displacement and corner of a point are the independent field functions, the approximation corner.deflection of this point can be imitate from the relational data in the domain, supposing that the nodes in the domain is n, and the array of the nodes of corner and deflection are

$$u^* = [u_1^* \quad u_2^* \cdots u_n^*]^T \tag{13a}$$

Where

$$u_i^* = [\omega_i^* \quad \theta_{x_i}^* \quad \theta_{y_i}^*]^T, (1, 2, \cdots, n) \tag{13b}$$

According to MLS, the field function

$$u = [\omega \quad \theta_x \quad \theta_y]^T = \Phi u^* \tag{14a}$$

Where

$$\Phi = [\phi_1 I \quad \phi_2 I \cdots \phi_n I] \tag{14b}$$

is the shape function, and I is $3 * 3$ unit matrix and n is the number of nodes in the field. The generalized strain of the plate is

$$k = \Big[-\frac{\partial \theta_x}{\partial x} - \frac{\partial \theta_y}{\partial y} - (\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}) \Big]^T = Bu^* \tag{15a}$$

Where θ_x, θ_y are the rotation whose direction agree with $\frac{\partial\omega}{\partial x}$, $\frac{\partial\omega}{\partial y}$ and B is the strain array.

$$B = \begin{bmatrix} -\phi_{1,x} & 0 & 0 & \cdots & -\phi_{n,x} & 0 & 0 \\ 0 & -\phi_{1,y} & 0 & \cdots & 0 & -\phi_{n,y} & 0 \\ -\phi_{1,y} & -\phi_{1,x} & 0 & \cdots & -\phi_{n,y} & -\phi_{n,x} & 0 \end{bmatrix} \tag{15b}$$

The generalized relation between stress and strain is

$$M = [M_x \quad M_y \quad M_{xy}]^T = DK \tag{16a}$$

Where D is the elastic matrix, to the isotropic material

$$D = \frac{Et^3}{12(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} = D_0 \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \tag{16b}$$

Where D_0 is the stiffness matrix of the plate and is the Poission ratio.

In order to easily solve the problem of the curving plate at any condition, penalty function is imposed on the essential boundary in the programming of this paper. To the elastic plate of isotropy, considering the functional of shearing deformed Mindlin plate theory,

$$\Pi = \frac{1}{2}\int_\Omega k^T D k d\Omega - \int_\Omega q\omega d\Omega + \alpha_1[\int_\Omega (\frac{\partial\omega}{\partial x} - \theta_x)^2 d\Omega + \int_\Omega (\frac{\partial\omega}{\partial y} - \theta_y)^2 d\Omega]$$

$$+\alpha_2 \int_{s_1} [(\theta_n - \overline{\theta}_n)^2 + (\theta_s - \overline{\theta}_s)^2] ds + \alpha_3 \int_{s_1+s_2} (\omega - \overline{\omega})^2 ds - \int_{s_3} \overline{\theta}_n \omega ds + \int_{s_1+s_2} (\overline{M}_n \theta_n + \overline{M_n^-} \theta_s) ds \tag{17}$$

In the above equation, α_1 is the penalty factor of the shearing deformation in the functional, $\alpha_1 = Gt/2k$, G is the shearing mold, t is the thickness of the plate, k is the regulated coefficient which considering the shearing deforming non-proportionate distributed along the orientation of thickness. Take $k = 6/5$, α_2, α_3 is the penalty function which fulfill the essential conditions (physical significance and expression of the other signs can be found in the document (Genki, 2000, pp. 1419-1443). The numberical result shows that, when one takes $\alpha_2 = \alpha_3 = (10^3 - 10^6)E$ (E is the elastic mold), the definition of the result is fairly good, it shows that the essential boundary conditions which caused by this method, make the coefficient matrix symmetry and positive, easily solved.

5. Numerical example

A rectangular plate sustaining even load, one of its subtense (x=0 and x=a)are freely-supported, the third is free, the forth is built-in (y=0)(as fig. 1). The load $q = 2.2 \times 10^4 N/m^2$, the length of side are $a = 10.0m$, $b = 5.5m$, the thickness $t = 0.1m$, Young's modulus $E = 2.0 \times 10^{11} pa$, Poission ratio $\mu = 0.3$. In this paper, we layout 16×9 even distribution nodes to get 15×8 integral sub-domains, and the Gaussian integration is 4×4. The deflection of the plate center on the cross-section is as Tab.1, it shows that the solution of this paper is near to the solution obtained by ANSYS. The deflection along the center Line being parallel to x and y axis of the Plate are as Fig.1 and Fig.2.

6. Conclusions

This paper, which aims at the shortcoming of the MLS being easily forming the ill-condition function; the weighted orthogonal functions are used as basis functions to revise the MLS. Compared with the original, it breaks away from the inverse operation, and the amount of calculation is small. The result of the calculation of plate bending shows that the improved MLS is effective and efficient comparing with the result which obtained by the ANSYS software.

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Table.1 The deflection of the plate center on the cross-section ω ($\times 10^{-4} m$)

	x/a		x/b	
relative distance	ANSYS solution	this paper's solution	ANSYS solution	this paper's solution
0	0.000	0.000	0.000	0.000
0.1	1.432	1.432	2.256	2.236
0.2	3.154	3.179	4.221	4.325
0.3	4.788	4.812	5.675	5.690
0.4	5.979	6.032	6.487	6.571
0.5	7.056	7.078	7.065	7.078
0.6	7.450	7.524	6.487	6.571
0.7	8.243	8.267	5.675	5.690
0.8	8.807	8.821	4.221	4.325
0.9	9.454	9.546	2.256	2.236
1.0	10.60	10.72	0.000	0.000

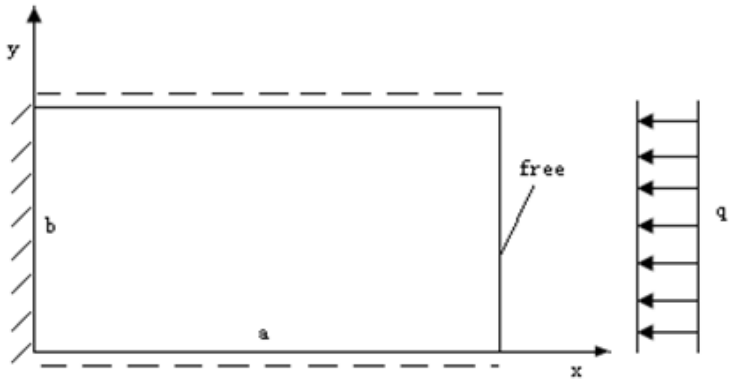

Figure 1. A loaded rectangular plate With complex edges conditions

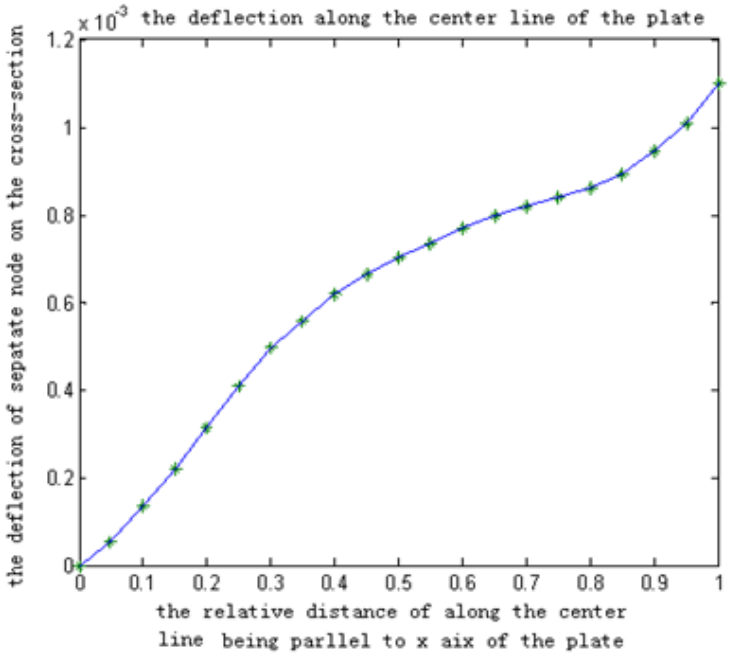

Figure 2. The deflection along the center Line being parallel to x axis of the Plate

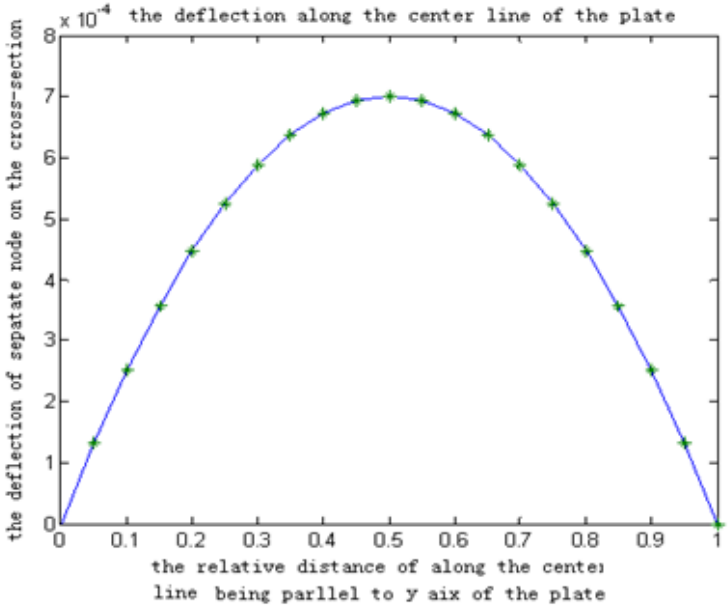

Figure 3. The deflection along the center Line being parallel to y axis of the Plate

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Perfect Glued Graphs at Complete Clones

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Abstract

A graph G is called perfect if the chromatic number and the clique number have the same value for every of its induced subgraph. A glued graph results from combining two vertex-disjoint graphs by identifying connected isomorphic subgraphs of both graphs. Such subgraphs are referred to as the clones.

We study the perfection of glued graphs whose clones are complete graphs. Our result generalizes the simplicial elimination ordering which is a characterization of chordal graphs.

Keywords: Perfect graphs, Glued graphs

1. Introduction

Let G_1 and G_2 be any two graphs with disjoint vertex sets. Let $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ be connected graphs such that $H_1 \cong H_2$ with an isomorphism f. *The glued graph of* G_1 *and* G_2 *at* H_1 *and* H_2 *with respect to* f, denoted by $G_1 \underset{H_1 \cong_f H_2}{\diamond} G_2$, is the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 with respect to the isomorphism f between H_1 and H_2. Let H be the copy of H_1 and H_2 in the glued graph. We refer to H, H_1 and H_2 as the *clones* of the glued graph, G_1 and G_2, respectively, and refer to G_1 and G_2 as the *original graphs*. The *glued graph of* G_1 *and* G_2 *at the clone* H, written $G_1 \underset{H}{\diamond} G_2$, means that there exist subgraph H_1 of G_1 and subgraph H_2 of G_2 and isomorphism f between H_1 and H_2 such that $G_1 \underset{H}{\diamond} G_2 = G_1 \underset{H_1 \cong_f H_2}{\diamond} G_2$ and H is the copy of H_1 and H_2 in the resulting graph. We use $G_1 \diamond G_2$ to denote an arbitrary graph resulting from gluing graphs G_1 and G_2 at any isomorphic subgraph $H_1 \cong H_2$ with respect to any of their isomorphism.

A *k-coloring* of a graph G is a coloring $f : V(G) \to S$, where $|S| = k$. A k-coloring is *proper* if adjacent vertices have different colors. The *chromatic number* of graph G, written $\chi(G)$, is the minimum number k such that G has a proper k-coloring. A *clique* of a graph G is a complete subgraph of G. The *clique number* of a graph G, written $\omega(G)$, is the order of the largest clique of G. For any graph G, it is always true that $\chi(G) \geq \omega(G)$. A graph G is called *perfect* if $\chi(F) = \omega(F)$ for every induced subgraph F of G, and a graph is called *imperfect* if it is not perfect. An *odd hole* of G is an induced subgraph of G which is an odd cycle of length at least 5. An *odd antihole* of G is an induced subgraph of G whose complement is an odd hole in $\overline{G}$. A graph having no odd hole and no odd antihole is called a *Berge graph*.

In 1972, Lovász proved the *Perfect Graph Theorem*, asserts that a graph is perfect if and only if its complement is perfect (Lovász , 1972). In 1961, Berge conjectured that a graph is perfect if and only if it is a Berge graph (Berge, 1961), affirmation well-known under the name of the *Strong Perfect Graph Conjecture*, and has just become a theorem since 2006 by Chudnovsky et al. (Chudnovsky et al., 2006). However, the proof was very

long (179 pages), recently, Chudnovsky and Seymour replaced the final 55 pages with a new much shorter proof (Chudnovsky & Seymour, 2009). This new theorem helps us to verify the perfection of our glued graphs.

Note that for vertex-disjoint graphs G_1 and G_2, $G_1 + G_2$ stands for the disjoint union of G_1 and G_2. The *join* of G_1 and G_2, written $G_1 \vee G_2$, is the graph obtained from the disjoint union $G_1 + G_2$ by adding the edge set $\{uv : u \in V(G_1), v \in V(G_2)\}$.

It is possible that a glued graph of imperfect graphs is perfect and it is also possible that a glued graph of perfect graphs is imperfect, see Examples A and B.

Example A *Perfect glued graphs of imperfect graphs*:

Let $G_1 = \overline{C_{2n+1}}$, $G_2 = K_1 \vee C_{2n-3}$ where $n \geq 4$ and $H = K_{1,2n-3}$. Both G_1 and G_2 are not Berge graphs, hence they are imperfect by the Strong Perfect Graph Theorem. Observe that $\overline{G_1 \underset{H}{\Phi} G_2} \cong P_6 + \overline{K_{2n-5}}$ which is perfect. By the Perfect Graph Theorem, we have that $G_1 \underset{H}{\Phi} G_2$ is perfect. If $n = 4$, then G_1 and G_2 are illustrated in Figure .

Note from Figure 1 that it is possible that a glued graph of simple graphs has multiple edges. However, multiple edges of a graph do not affect its the chromatic number and the clique number. Hence, we allow our glued graphs to have multiple edges.

Example B *Imperfect glued graphs of perfect graphs*:

Let $G_1 = C_{2n}$, $G_2 = K_1 \vee P_{2n-2}$ where $n \geq 3$ and $H = P_{2n-2}$. Both G_1 and G_2 are Berge graphs. Observe that $G_1 \underset{H}{\Phi} G_2$ contains C_5, so it is not a Berge graph. By the Strong Perfect Graph Theorem, G_1 and G_2 are perfect but $G_1 \underset{H}{\Phi} G_2$ is imperfect. If $n = 3$, then G_1 and G_2 are illustrated in Figure .

In this paper, we study the perfection of glued graphs of perfect graphs. Example B shows that a glued graph of perfect graphs may not be perfect. A condition is required to guarantee the perfection of a glued graph when the original graphs are perfect. Our main results reveal that the clone of the glued graph must be a complete graph in order to get the desired result.

Throughout the paper, G_1 and G_2 are graphs with disjoint vertex sets and the clone H is a connected graph. We use symbol $G(u_1, u_2, ..., u_n)$ for a graph G on the vertex set $\{u_1, u_2, ..., u_n\}$, and $\overline{G}(u_1, u_2, ..., u_n)$ for the complement of $G(u_1, u_2, ..., u_n)$. We use symbol $K_n(u_1, u_2, ..., u_n)$ for a complete graph on the vertex set $\{u_1, u_2, ..., u_n\}$, and $P_n(u_1, u_2, ..., u_n)$ and $C_n(u_1, u_2, ..., u_n)$ for a path and a cycle on the vertex set $\{u_1, u_2, ..., u_n\}$ and the edge set $\{u_1u_2, u_2u_3, ..., u_{n-1}u_n\}$ and $\{u_1u_2, u_2u_3, ..., u_{n-1}u_n, u_nu_1\}$, respectively. Other standard notations we follow West (West, 2001).

2. Main Results

When the clone H is an induced subgraph of both G_1 and G_2, it follows that G_1 and G_2 are induced subgraphs of the glued graph $G_1 \underset{H}{\Phi} G_2$. If $G_1 \underset{H}{\Phi} G_2$ is perfect, then both G_1 and G_2 must be perfect. This can be concluded here:

Proposition 2.1 Let G_1 and G_2 be graphs containing H as an induced subgraph. If $G_1 \underset{H}{\Phi} G_2$ is a perfect graph, then both G_1 and G_2 are perfect.

The inverse of Proposition 2.1 is not true. Namely, if H is not a complete graph, one can find perfect graphs G_1 and G_2 containing H as an induced subgraph while $G_1 \underset{H}{\Phi} G_2$ is not perfect.

Theorem 2.2 Let H be a connected incomplete graph. If H is a perfect graph, then there exist perfect graphs G_1 and G_2 containing H as an induced subgraph such that $G_1 \underset{H}{\Phi} G_2$ is not perfect.

Proof. Assume that H is a perfect graph. Let $|V(H)| = r$. Let $H_1(u_1, u_2, ..., u_r)$ and $H_2(v_1, v_2, ..., v_r)$ be the copies of H with an isomorphism $f : V(H_1) \to V(H_2)$ which is defined by $f(u_i) = v_i$ for all $i \in \{1, 2, ..., r\}$. Let $P_l(u_1, u_2, ..., u_l)$ and $P_l(v_1, v_2, ..., v_l)$ be the longest induced paths of H_1 and H_2, respectively. Since H_1 and H_2 are not complete graphs, $l \geq 3$. Choose $G_1 = H_1 \vee K_1(z)$; a join graph between H_1 and a new vertex z, and choose $G_2 = (H_2 \vee K_2(x, y)) - \{xv_l, yv_1\}$. Then G_1 and G_2 are perfect. Consider $G_1 \underset{H_1 \cong_f H_2}{\Phi} G_2$, we see that the corresponding vertices of v_1, x, y, v_l, z in $G_1 \underset{H_1 \cong_f H_2}{\Phi} G_2$ form C_5. By the Strong Perfect Graph Theorem, $G_1 \underset{H_1 \cong_f H_2}{\Phi} G_2$ is not perfect. □

The clone of a glued graph is called a *complete clone* if it is a complete graph.

The graph gluing at a complete clone preserves the perfection. Theorem 2.8 illustrates this fact and it is yielded by Lemmas 2.4 and 2.5.

Remark 2.3 For graphs G_1 and G_2, we have

1. $\chi(G_1 \Phi G_2) \geq \max\{\chi(G_1), \chi(G_2)\}$ and
2. $\omega(G_1 \Phi G_2) \geq \max\{\omega(G_1), \omega(G_2)\}$.

In general, $\chi(G_1 \Phi G_2) \leq \chi(G_1)\chi(G_2)$ (Promsakon & Uiyyasathian, 2006). When the clone is a complete graph, the chromatic numbers of glued graphs do not exceed the chromatic numbers of their original graphs, see Lemma 2.4.

For a positive integer r, a *glued graph at a complete clone*, $G_1 \overset{\Phi}{_{K_r}} G_2$, denotes an arbitrary glued graph between graphs G_1 and G_2 at any clone which is isomorphic to K_r. For convenience, K_r in our proofs always means the clone of the glued graph $G_1 \overset{\Phi}{_{K_r}} G_2$, not arbitrary subgraph K_r in the glued graph .

Lemma 2.4 For graphs G_1 and G_2, $\chi(G_1 \overset{\Phi}{_{K_r}} G_2) = \max\{\chi(G_1), \chi(G_2)\}$.

Proof. Let $\chi(G_1) = m$ and $\chi(G_2) = n$. Assume $m \geq n$. By Remark 2.3(1), it suffices to show that $\chi(G_1 \overset{\Phi}{_{K_r}} G_2) \leq m$. Let $a_1, a_2, ..., a_m$ be colors labeling vertices of G_1 by f and $b_1, b_2, ..., b_n$ colors labeling vertices of G_2 by g. Note that any pair of vertices in K_r must have different colors. Without loss of generality, for $i \in \{1, 2, ..., r\}$, let a_i and b_i be colors of the corresponding vertices of K_r in G_1 and G_2, respectively. Let $h : V(G_1 \overset{\Phi}{_{K_r}} G_2) \to \{a_1, a_2, ..., a_m\}$ defined by

$$h(v) = \begin{cases} f(v) & , \text{ if } v \in V(G_1); \\ a_i & , \text{ if } v \in V(G_2) - V(K_r) \text{ and } g(v) = b_i. \end{cases}$$

Since the clone is a complete graph, h is well-defined. To show that h is proper, let u and v be vertices in $G_1 \overset{\Phi}{_{K_r}} G_2$ such that u and v are adjacent. If $u, v \in V(G_1)$, then $h(u) = f(u) \neq f(v) = h(v)$. If $u, v \in V(G_2)$, then $g(u) = b_i$ and $g(v) = b_j$ for some $i \neq j$, so $h(u) = a_i \neq a_j = h(v)$. Besides, vertices in $V(G_1) - V(K_r)$ and $V(G_2) - V(K_r)$ are not adjacent. Hence, h is proper. That is, $G_1 \overset{\Phi}{_{K_r}} G_2$ has a proper m-coloring. Therefore, $\chi(G_1 \overset{\Phi}{_{K_r}} G_2) \leq m$. □

Unlike the chromatic number, we have not had an upper bound of the clique numbers of glued graphs in terms of the clique numbers of their original graphs. Promsakon conjectured that $\omega(G_1 \Phi G_2) \leq \omega(G_1)\omega(G_2)$ (Promsakon, 2006).

In general, the graph gluing can join two non-adjacent vertices in the clone of an original graph, Consider graphs G_1, G_2 and $G_1 \overset{\Phi}{_{H}} G_2$ whose clone H is shown as bold edges in Figure . We see that b and d are non-adjacent vertices in G_1 but the corresponding vertices of b and d in the glued graph are adjacent.

Since any pair of vertices in the complete clone are adjacent, there is no new edge created from the graph gluing. Thus all cliques in a glued graph at a complete clone are cliques in original graphs, so $\omega(G_1 \Phi G_2) \leq \omega(G_i)$ for $i = 1, 2$. The following lemma is concluded.

Lemma 2.5 For graphs G_1 and G_2, $\omega(G_1 \overset{\Phi}{_{K_r}} G_2) = \max\{\omega(G_1), \omega(G_2)\}$.

The condition in Lemmas 2.4 and 2.5 that the clone must be a complete graph is necessary. Theorem 2.7 confirms.

Theorem 2.6 (Brooks, 1941) If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$ where $\Delta(G)$ denotes the maximum degree of G.

Theorem 2.7 Let H be a connected graph. If H is not a complete graph, then there exist G_1 and G_2 such that $\chi(G_1 \overset{\Phi}{_{H}} G_2) > \max\{\chi(G_1), \chi(G_2)\}$ and $\omega(G_1 \overset{\Phi}{_{H}} G_2) > \max\{\omega(G_1), \omega(G_2)\}$.

Proof. Assume that H is not a complete graph. Let $|V(H)| = r$, so $r \geq 3$. Choose $G_1 \cong K_r$ and choose $G_2 \cong H \vee K_1$. Then $G_1 \overset{\Phi}{_{H}} G_2 \cong K_{r+1}$. If H is an odd cycle of length at least 5, $\chi(H) = 3 < r - 1$. Otherwise, H

is not an odd cycle of length at least 5, by Theorem 2.6, $\chi(H) \leq \Delta(H) \leq r-1$. Now, we have $\chi(H) \leq r-1$, so $\chi(G_2) = \chi(H \vee K_1) \leq (r-1)+1 = r$. Hence, $\chi(G_1 \underset{H}{\Phi} G_2) = r+1 > r = \max\{\chi(G_1), \chi(G_2)\}$. Since H is not a complete graph and $|V(H)| = r$, we get $\omega(H) \leq r-1$. So, $\omega(G_2) = \omega(H \vee K_1) \leq (r-1)+1 = r$. Therefore, $\omega(G_1 \underset{H}{\Phi} G_2) = r+1 > r = \max\{\omega(G_1), \omega(G_2)\}$. □

For graphs G_1 and G_2, $G_1 \cap G_2$ denotes the graph on the vertex set $V(G_1) \cap V(G_2)$ and the edge set $E(G_1) \cap E(G_2)$.

Theorem 2.8 For graphs G_1 and G_2,

$G_1 \underset{K_r}{\Phi} G_2$ is a perfect graph if and only if both G_1 and G_2 are perfect.

Furthermore, $\chi(G_1 \underset{K_r}{\Phi} G_2) = \omega(G_1 \underset{K_r}{\Phi} G_2) = \max\{\omega(G_1), \omega(G_2)\}$.

Proof. Necessity follows from Proposition 2.1. For sufficiency, assume that G_1 and G_2 are perfect graphs. We will show that $\chi(F) = \omega(F)$ for every induced subgraph F of $G_1 \underset{K_r}{\Phi} G_2$. Let F be an induced subgraph of $G_1 \underset{K_r}{\Phi} G_2$. If F is disconnected, we consider the perfection of each component of F. We may assume that F is connected. If F has no vertex in K_r, then F is an induced subgraph of either G_1 or G_2, so $\chi(F) = \omega(F)$. Assume that F has at least one vertex in K_r. Let $F_1 = F \cap G_1$ and $F_2 = F \cap G_2$. Since F_1 is an induced subgraph of a perfect graph G_1, we get $\chi(F_1) = \omega(F_1)$. Similarly, $\chi(F_2) = \omega(F_2)$. Now, let $F_r = F \cap K_r$. Then F_r is a complete graph. We have that $F = F_1 \underset{F_r}{\Phi} F_2$. By Lemmas 2.4 and 2.5, $\chi(F) = \max\{\chi(F_1), \chi(F_2)\}$ and $\omega(F) = \max\{\omega(F_1), \omega(F_2)\}$, respectively. Hence, $\chi(F) = \omega(F)$. Therefore, $G_1 \underset{K_r}{\Phi} G_2$ is perfect. Furthermore, $\chi(G_1 \underset{K_r}{\Phi} G_2) = \omega(G_1 \underset{K_r}{\Phi} G_2) = \max\{\omega(G_1), \omega(G_2)\}$. □

If the clone is not a complete graph, it fails to be concluded the perfection of glued graphs of perfect graphs. It is illustrated by Theorems 2.9 and 2.10.

Theorem 2.9 Let H be a connected graph. If H is not a complete graph, then there exist a perfect graph G_1 and an imperfect graph G_2 such that $G_1 \underset{H}{\Phi} G_2$ is perfect.

Proof. Assume that H is not a complete graph. Let $|V(H)| = r$, so $r \geq 3$. Let $H_1(u_1, u_2, ..., u_r)$ and $H_2(v_1, v_2, ..., v_r)$ be the copies of H with an isomorphism $f : V(H_1) \to V(H_2)$ which is defined by $f(u_i) = v_i$ for all $i \in \{1, 2, ..., r\}$. Since H_2 is not a complete graph, there are at least 2 non-adjacent vertices , say v_1 and v_r. Choose $G_1 = K_r(u_1, u_2, ..., u_r)$ and choose $G_2 = \overline{C_{2r-1}}(v_1, x_1, ..., v_{r-1}, x_{r-1}, v_r)$. Then G_1 is perfect but G_2 is not perfect. Since H_1 and H_2 are not complete graphs, $H_1 \subseteq K_r(u_1, u_2, ..., u_r)$ and $H_2 \subseteq K_r(v_1, v_2, ..., v_r) - v_1v_r \subseteq \overline{C_{2r-1}}(v_1, x_1, ..., v_{r-1}, x_{r-1}, v_r)$. We can verify that $G_1 \underset{H}{\Phi} G_2 \cong G_1 \underset{H_1 \cong_f H_2}{\Phi} G_2 \cong \overline{C_{2r-1}}(v_1, x_1, ..., v_{r-1}, x_{r-1}, v_r) + v_1v_r$, consequently, $\overline{G_1 \underset{H}{\Phi} G_2} \cong P_{2r-1}$. Since P_{2r-1} is perfect, by the Perfect Graph Theorem, $G_1 \underset{H}{\Phi} G_2$ is perfect. □

Theorem 2.10 Let H be a connected graph. If H is not a complete graph, then there exist perfect graphs G_1 and G_2 such that $G_1 \underset{H}{\Phi} G_2$ is not perfect.

Proof. Assume that H is not a complete graph. Let $|V(H)| = r$, so $r \geq 3$. Let $H_1(u_1, u_2, ..., u_r)$ and $H_2(v_1, v_2, ..., v_r)$ be the copies of H with an isomorphism $f : V(H_1) \to V(H_2)$ which is defined by $f(u_i) = v_i$ for all $i \in \{1, 2, ..., r\}$. Since H_2 is not a complete graph, there are at least 2 non-adjacent vertices, say v_1 and v_2. Choose $G_1 = K_r(u_1, u_2, ..., u_r)$ and choose $G_2 = \overline{C_{2r+1}}(v_1, x_1, ..., v_r, x_r, x_{r+1}) - v_1v_2$. Then G_1 and G_2 are perfect. Since H_1 and H_2 are not complete graphs, $H_1 \subseteq K_r(u_1, u_2, ..., u_r)$ and $H_2 \subseteq K_r(v_1, v_2, ..., v_r) - v_1v_2 \subseteq \overline{C_{2r+1}}(v_1, x_1, ..., v_r, x_r, x_{r+1}) - v_1v_2$. We can verify that $G_1 \underset{H}{\Phi} G_2 \cong G_1 \underset{H_1 \cong_f H_2}{\Phi} G_2 \cong \overline{C_{2r+1}}$. Thus, $G_1 \underset{H}{\Phi} G_2$ is not a Berge graph. By the Strong Perfect Graph Theorem, $G_1 \underset{H}{\Phi} G_2$ is not perfect. □

For a subset S of $V(G)$, a *neighborhood* of S in G, written $N_G(S)$ or $N(S)$, is the set of vertices in $V(G) - S$ which are adjacent to vertices in S. We use $G[S]$ and $G - S$ for the induced subgraph of G on the vertex set S and $V(G) - S$, respectively.

We now extend the *simplicial (perfect) elimination ordering* to a new definition as follows:

Definition 2.11 A subset V_i of $V(G)$ is **simplicial** if its *neighborhood* in G forms a clique. A **simplicial set elimination ordering** is an ordering $V_1, ..., V_k$ for deletion of nonempty vertex subsets so that each V_i is a simplicial vertex subset of the remaining graph induced by $\bigcup_{t=i}^{k} V_t$ with $|V_i| = 1$ or $|V_i| = 2$ for all $i \in \{1, 2, ..., k\}$, and $V_1, ..., V_k$ partitions $V(G)$.

Note that a simplicial elimination ordering of a graph G is a simplicial set elimination ordering of G with $|V_i| = 1$ for all $i \in \{1, 2, ..., n(G)\}$.

Remark 2.12 For any nontrivial graph G, let $V_1, ..., V_k$ be a partition of $V(G)$. Let $G_1 = G$, and for each $i \in \{2, 3, ..., k\}$, let $G_i = G - \bigcup_{t=1}^{i-1} V_t$. If $V_1, ..., V_k$ is a simplicial set elimination ordering of G, then for each $i \in \{1, 2, ..., k-1\}$, G_i is a glued graph between $G_i[V_i \cup N(V_i)]$ and G_{i+1} at a complete clone $G_i[N(V_i)]$.

Theorem 2.13 For a simple graph G, if G has a simplicial set elimination ordering, then G is a perfect graph.

Proof. Assume that G has a simplicial set elimination ordering $V_1, ..., V_k$. Let $G_1 = G$, and for each $i \in \{2, 3, ..., k\}$, let $G_i = G - \bigcup_{t=1}^{i-1} V_t$. Since $|V_k| = 1$ or 2, $G_k \cong K_1$ or K_2, so G_k is perfect. By Remark 2.12, G_i is a glued graph between $G_i[V_i \cup N(V_i)]$ and G_{i+1} at a complete clone $G_i[N(V_i)]$, it is enough to claim that $G_i[V_i \cup N(V_i)]$ is perfect for all $i \in \{1, 2, ..., k-1\}$. Let $i \in \{1, 2, ..., k-1\}$. Let C be an induced cycle in $G_i[V_i \cup N(V_i)]$. Since $N_{G_i}(V_i)$ forms a clique, at most 2 vertices in $N_{G_i}(V_i)$ can be in C. Together with vertices in V_i, C has length at most 4. Again, since $N_{G_i}(V_i)$ forms a clique, 2 vertices in $N_{G_i}(V_i)$ cannot be adjacent in the complement of $G_i[V_i \cup N(V_i)]$. Besides, each vertex in $N_{G_i}(V_i)$ must be adjacent to at least one vertex in V_i, so it can be adjacent to at most one vertex of V_i in the complement of $G_i[V_i \cup N(V_i)]$. Thus, there is no cycle in the complement of $G_i[V_i \cup N(V_i)]$. Hence, $G_i[V_i \cup N(V_i)]$ contains no odd hole and no odd antihole. By the Strong Perfect Graph Theorem, we get $G_i[V_i \cup N(V_i)]$ is perfect. By Theorem 2.8, G_i is perfect for all $i \in \{1, 2, ..., k-1\}$. □

The inverse of the theorem is not true, for instance, C_{2n}, where $n \geq 3$, is perfect while it has no simplicial set elimination ordering. It is well-known that the simplicial elimination ordering characterizes a subclass of perfect graphs, namely the chordal graphs. It would be interesting if one could characterize a subclass of perfect graphs by the simplicial set elimination ordering.

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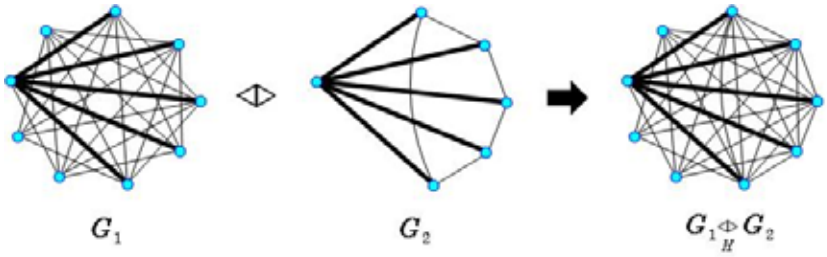

Figure 1. A perfect glued graph of imperfect graphs

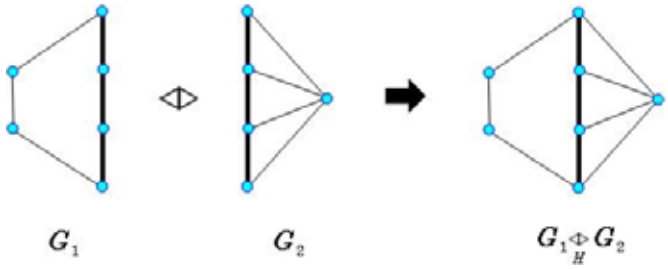

Figure 2. An imperfect glued graph of perfect graphs

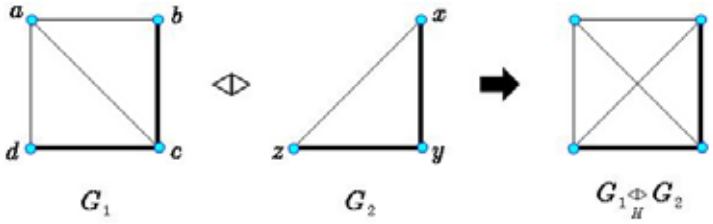

Figure 3. A glued graph containing a new edge

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An Improved Graph Method for Linear Goal Programming

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Abstract

The traditional Graph Method is a basic approach to solve Linear Goal Programming. The paper attained a modified method based on the traditional way. The modified method can attain the satisfaction solution, while the traditional Graph Method has its limitation in solving some Linear Goal Programming.

Keywords: Linear Goal Programming, Graph Method

1. Introduction

Goal programming is one of the most widely used methodologies in operations research and management science, and it encompasses most classes of multiple objective programming models. Goal programming is first brought up in 1961 by A. Charnes and W. W. Cooper (Editor Group of O.R., 1990), and then there were many scholars studying it and it developed rapidly in the past several decades. Goal programming allowed a decide-maker to put many requests within one decision. With goal programming the decide-maker should not look for an absolute optimal solution, instead, he should only look for a solution that can make himself more satisfactory than any other solutions. Because goal programming makes up some defects of linear programming, it is considered as a decision tool that is nearer to real decision process than linear programming (Hu Yunquan, 2003).

In 1972, Lee.Sang.M put forward the graph method for linear goal programming for the first time in his monograph 'Goal Programming for Decision Analysis (Editor Group of O.R., 1990). In the last 30 years, the graph method was included as an important content of linear goal programming in many textbooks, such as in (Hu Yunquan, 2003, Ignizio, 1976. Lee and Sang M., 1972). Analogous to the graph method for liner programming, the graph method for goal programming can only solve problems with no more than two decision variables. Since there are lots of problems with no more than two decision variables and graph method can give us some help on understanding the characteristics of the optimal solution of linear programming and the satisfaction solution of linear goal programming, graph method was an important part in almost all textbooks that included goal programming. Hence it's of theoretic sense to study graph methods for goal programming.

2. Goal programming

We describe linear goal programming in this section. Goal programming is a programming problem with multiple goals, in which there is a priority order among the goals. We first introduce some basic notations used commonly in goal programming as follows:

2.1 Deviation variable

For each decision goal, we introduce a positive deviation variable d^+ and a negative deviation variable d^-, while d^+ denotes how much the decision has exceeded the goal, and d^- denotes how far the decision is from the goal.

Here, we have $d^+ \geq 0$, $d^- \geq 0$, and $d^+ \bullet d^- = 0$.

2.2 Absolute constraint and goal constraint

If a constraint must be satisfied, we call it absolute constraint. Because it is a hard constraint, a solution is not a feasible one if it can not satisfy any one absolute constraint. Goal constraint is a special weak constraint that can only be seen in goal programming. And sometimes a feasible solution does not satisfy a goal constraint.

2.3 Priority factor and weight coefficient

Some goal is important, while some others are unimportant. If a goal is far more important than another one, we give it a priority factor P_l, and we give another goal a priority factor P_{l+1}. Here we have $P_l >> P_{l+1}$. If a goal is a little more important than another one, we can give it a bigger weight coefficient while we give them same priority factor.

2.4 Goal function

Goal function is composed of goal constraints' deviation variables, their priority factors and weight coefficient. Usually, goal programming try to minimize its deviation variables, such as $\min\{f(d^+ + d^-)\}$, $\min\{f(d^+)\}$ and $\min\{f(d^-)\}$.

2.5 Satisfaction solution

If a solution satisfies all the absolute constraints, and its cost value is no bigger than any other solution, we call it a satisfaction solution.

3 Traditional graph method for linear goal programming

While constraints and cost function are all linear, goal programming is linear goal programming.

In traditional graph method for linear goal programming, firstly the feasible solutions should satisfy all absolute constraints. Then we consider every goal constraint according to their priority factors. Generally, if R_j is the solution region for priority factor P_j, solution region R_{j+1} for priority factor P_{j+1} must be a subspace of R_j, i.e. $R_{j+1} \subseteq R_j$. If $R_j \neq \Phi$, and $R_{j+1} = \Phi$, there is a satisfaction solution in R_j. It satisfies goal $P_1, P_2, \cdots, P_j$, but it can not satisfy the other goals always.

Since there is no solution that can satisfy all the goals of P_{j+1}, we preferentially let deviation variable be zero whose cost coefficient is larger. For example, if the goal function is $2d_3^- + 3d_4^-$, we let d_4^- be zero preferentially as the coefficient 3 of d_4^- is larger than that 2 of d_3^-. So the satisfaction solution satisfies $d_4^- = 0$.

The above described method is just the Graph Method for Linear Goal Programming. But when we look for a satisfaction solution simply by whether the coefficient is larger or smaller, we ignore the difference among the influence that the constraint functions have upon different deviation variables, and so it is not reliable. Now let us look at an example of goal programming and solve it by Graph Method:

Example1: $\min\{P_1 d_1^-, P_2 d_2^+, P_3(2d_3^- + 3d_4^-), P_4 d_1^+\}$

$$st. \begin{cases} x_1 + 2x_2 + d_1^- - d_1^+ = 6 \\ x_1 + 2x_2 + d_2^- - d_2^+ = 9 \\ x_1 - 2x_2 + d_3^- - d_3^+ = 4 \\ x_1 - 2x_2 + d_4^- - d_4^+ = 2 \\ x_1, x_2, d_i^-, d_i^+ \geq 0, \ d_i^- * d_i^+ = 0, \quad i = 1, 2, 3, 4 \end{cases}$$

From Fig. 1, we know that the solution region R_2 for goal P_1 and P_2 is quadrangle **ABCD**. While considering the goal P_3, we minimize d_4^- in priority as its cost coefficient 3 is larger than that 2 of P_3. So the feasible solution region is reduced to quadrangle **CDEF**. Then we minimize d_3^-. But there is no point that satisfies $d_3^- = 0$ in quadrangle **CDEF**, so we have to try to look for a point that minimize d_3^-. The point is **F(5, 2)**. So the feasible solution for the Goal Programming is $x_1 = 5$, and $x_2 = 2$.

4. Defect of tradition graph method

In fact, the cost value of point **F(5, 2)** about $P_3(2d_3^- + 3d_4^-)$ is 3, and that of **G(6.5, 1.25)** is 2.25 which is smaller than 3. So point **G(6.5, 1.25)** is more satisfactory than point **F(5, 2)**, and the satisfaction solution is $x_1 = 6.5$, and $x_2 = 1.25$ indeed.

With the above analysis, we can get that the solution found by the traditional graph method is not very satisfactory sometimes. Now let us prove that point **G(6.5, 1.25)** is the satisfaction solution of the example.

Theorem1: G(6.5, 1.25) is the satisfaction solution of example1.

Proof: First, we will prove that any point I out of quadrangle **EFGH** must be not a satisfaction solution. Without loss of generality, we suppose that point **I** is in quadrangle **ABGH**. Firstly, we move point **I** to line $d_3^+ = d_3^- = 0$ in the vertical direction of line $d_4^+ = d_4^- = 0$, and the point of intersection is denoted by **I'**. Obviously, $d_3^-(I') = 0 < d_3^-(I)$, and $d_4^-(I') < d_4^-(I)$ so we have $(2d_3^- + 3d_4^-)(I') < (2d_3^- + 3d_4^-)(I)$.

But $d_3^-(G) + 3d_3^-(I') = 0$, and $d_4^-(G) < d_4^-(G')$, so

$$\begin{aligned}(2d_3^- + 3d_4^-)(G) &< (2d_3^- + 3d_4^-)(I') \\ &< (2d_3^- + 3d_4^-)(I)\end{aligned}$$

That is to say, point **G** is more satisfactory than point **I**. With same reason, point **F** is more satisfactory than all the points in quadrangle **CDEF**.

Now let us prove that the cost value of point **G** is smaller than that of all points in quadrangle **EFGH**. In quadrangle **EFGH**, all the points satisfy that $d_1^- = 0$, $d_2^+ = 0$, $d_3^+ = 0$ and $d_4^+ = 0$, so the original problem can be transferred into the following linear programming:

$$\min\{2d_3^- + 3d_4^-\}$$

$$s.t. \begin{cases} x_1 + 2x_2 - d_1^+ = 6 \\ x_1 + 2x_2 + d_2^- = 9 \\ x_1 - 2x_2 + d_3^- = 4 \\ x_1 - 2x_2 + d_4^- = 2 \\ x_1,\ x_2,\ d_i^-,\ d_i^+ \geq 0, \quad i = 1, 2, 3, 4 \end{cases}$$

The optimal solution of the above linear programming certainly can be found in the vertices of its feasible region, and they are **E, F, G** and **H**. In the four vertices, the cost value of point **F** is obviously smaller than that of point E, and that of point G is smaller than that of point H. And with the above calculation, we know that the cost value of point **G** is smaller than that of point **F**. So point **G** is the most satisfactory point, and point **G(6.5, 1.25)** is the satisfaction solution of Example1.

5. Improvement of graph method

With the proof of Theorem 1, we know that there must be a vertex which is the satisfaction solution of the goal programming. So we can improve the Traditional Graph Method. We put a concept algorithm of the improved graph method as follows:

Suppose that the feasible set of goal $P_1, P_2, \cdots, P_j$ is R_j, and there is no point that can satisfy all the goals in R_j. Then, for every deviation variable in P_{j+1}, we first draw a line whose deviation variables are all equal to zero rightly, then we calculate the intersection of each line and the boundary of R_j and the intersection of any two lines among them. Successively we calculate all the points' cost function of P_{j+1}. The point whose cost function is the smallest is just the satisfaction solution.

Remark 1: In Example 1, there is no point that can make all deviation variables in goal $P_3(2d_3^- + 3d_4^-)$ (d_3^- and d_4^-) equal to zero in R_2 (quadrangle **ABCD**). Then we draw the lines $d_3^+ = d_3^- = 0$ and $d_4^+ = d_4^- = 0$. Following the steps described in the above concept algorithm, we find the satisfaction solution G as prove in the previous section.

Remark 2: The concept algorithm posed above can find a satisfaction solution, because the satisfaction must be on the intersections among the lines of the deviation variables which are all zeros and the boundary of the feasible region of the problem.

6. Conclusion

The Improved Graph Method for Goal Programming can solve problems whose goals are to minimize their deviation variables with no more than two decision variables. While it can not solve the problems whose goals are to maximize their deviation variables, such as $\max 2d_2^+ + 3d_3^-$ and $\min 2d_2^+ - 3d_3^-$.

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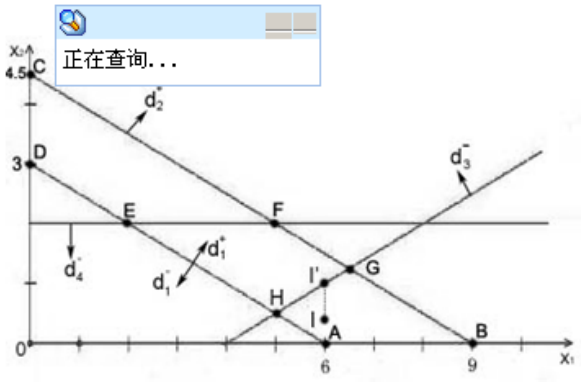

Figure 1. solution of Example 1

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Derivation of Augmented Arithmetic for Computing Gradient, Hessian and Jacobian through Forward Mode AD Using Dual Numbers

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Abstract

This paper presents a new approach to Automatic Differentiation (AD) for a scalar valued and twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$. A new arithmetic is obtained based on the chain rule and using augmented algebra of real numbers. The chain rule based differentiation arithmetic is used to find the Gradient and Hessian. Jacobian is evaluated using Gradient arithmetic by computing Gradient for components and is arranged in matrix form to give Jacobian value. The resulting derivative evaluation uses the operator overloading concept which uses computer programs written in C++.

Keywords: Automatic Differentiation, Augmented algebra, operator overloading, Forward mode

1. Introduction

Any efficient non-linear optimization routine needs good gradient approximations. Over the last decades, several research groups have developed the technique of Automatic Differentiation, which generates exact derivatives for a given code segment. A comprehensive introduction to this method can be found in (Griewank, A., 2000 & 1990; Naumann, U., 2008; Moore, R.E., 1962; Rall, L.B., 2007). Automatic Differentiation can be implemented in various ways, each of which is dependent on circumstances partially. Here, we use a new methodology to implement AD for computing Gradient, Hessian and Jacobian. A new arithmetic is obtained based on the chain rule and using augmented algebra of real numbers through forward mode of Automatic Differentiation.

2. The Differentiation Arithmetic for Evaluation of Gradient and Hessian

To obtain the arithmetic for Gradients and Hessians, an ordered triples of the form $U = (u_f, u_g, u_h)$ with $u_f \in \mathbb{R}, u_g \in \mathbb{R}^n, u_h \in \mathbb{R}^n \times \mathbb{R}^n$ where the scalar $u_f$denotes the function value $u(x)$of the twice differentiable function $u : \mathbb{R}^n \to \mathbb{R}$, the vector $u_g$and the matrix u_h denote the value of the gradient $\nabla u(x)$ and the Hessian $\nabla^2 u(x)$ respectively, each at a fixed point $x \in \mathbb{R}^n$ is considered. For the constant function $u(x) = c$, the triple is $U = (u_f, u_g, u_h) = (c, 0, 0)$.For the function $u(x) = x_k$ with $k \in \{1, 2, \cdots, n\}$, $(u_f, u_g, u_h) = (x_k, e^{(k)}, 0)$ is

defined, where $e^k \in \mathbb{R}^n$ denotes the k^{th} unit vector and 0 denotes the zero matrix, respectively. There are some rules for the differentiation arithmetic similar to one dimensional case. They are

$$W = U + V \Rightarrow \begin{pmatrix} w_f = u_f + v_f, \\ w_g = u_g + v_g, \\ w_h = u_h + v_h \end{pmatrix}$$

$$W = U - V \Rightarrow \begin{pmatrix} w_f = u_f - v_f, \\ w_g = u_g - v_g, \\ w_h = u_h - v_h \end{pmatrix}$$

$$W = U \cdot V \Rightarrow \begin{pmatrix} w_f = u_f \cdot v_f, \\ w_g = u_g v_f + u_f v_g, \\ w_h = v_f \cdot u_h + u_g \cdot v_g^T + v_g \cdot u_g^T + u_f \cdot v_h \end{pmatrix}$$

$$W = U/V \Rightarrow \begin{pmatrix} w_f = u_f/v_f, \\ w_g = (u_g - w_f \cdot v_g)/v_f, \\ w_h = (u_h - w_g \cdot v_g^T - v_g \cdot w_g^T - w_f \cdot v_h)/v_f \end{pmatrix}$$

where the familiar rules of calculus have been used in second and third components, and $v_f \neq 0$ is assumed in the case of division. The operations for w_f, w_g and w_h in these definitions are operations on real numbers, vectors and matrices. If the independent variables $x_i$of a formula for a function $f : \mathbb{R}^n \to \mathbb{R}$ and $x \to f(x)$ are replaced by $X_i = (x_i, e^{(i)}, 0)$, and if all constants c are replaced by their $(c, 0, 0)$ representation, then evaluation of f using the rules of differentiation arithmetic gives

$$f(X) = f\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \cdot \\ \cdot \\ \cdot \\ X_n \end{pmatrix} = f\begin{pmatrix} (x_1, e^{(1)}, 0) \\ (x_2, e^{(2)}, 0) \\ (x_3, e^{(3)}, 0) \\ \cdot \\ \cdot \\ \cdot \\ (x_n, e^{(n)}, 0) \end{pmatrix} = (f(x), \nabla u(x), \nabla^2 u(x))$$

For the elementary function $S : \mathbb{R} \to \mathbb{R}$ and $U = (u_f,\ u_g,\ u_h)$, the differentiation arithmetic is

$$W = S(U) \Rightarrow \begin{pmatrix} w_f = S(u_f), \\ w_g = S'(u_f) \cdot u_g, \\ w_h = S''(u_f) \cdot u_g \cdot u_g^T + S'(u_f) \cdot u_h \end{pmatrix}$$

2.1 Algorithmic Description

Here the algorithm for the elementary operators $\{+,\ -, \cdot,\ /\}$ and for elementary functions $S \in$ {power, exp, log, sin, cos, tan, cot, asin, acos, atan, acot, sinh, cosh, tanh, coth, asinh, acosh, atanh, acoth} of a Gradient and Hessian arithmetic for a twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$is discussed.

Algorithm for overloading the ± operator for U and V is given by

Step 1:	$\left[W_f\right] = \left[u_f\right] \pm \left[v_f\right];$	(Function value)
Step 2:	for $i = 1 : n$	
	$\left[W_g\right]_i = \left[u_g\right]_i \pm \left[v_g\right]_i;$	(Gradient component
	for $j = 1 : i$	
	$[W_h]_{ij} = [u_h]_{ij} \pm [v_h]_{ij};$	(Hessian component)
Step 3:	*return* $[W]$;	

The algorithmic description for the multiplication operator for (U, V) is given by

Step 1: $\left[W_f\right] = \left[u_f\right] \cdot \left[v_f\right]$; (Function value)
Step 2: for $i = 1 : n$
$\left[W_g\right]_i = \left[v_f\right] \cdot \left[u_g\right]_i + \left[u_f\right] \cdot \left[v_g\right]_i$; (Gradient component
for $j = 1 : i$
$[W_h]_{ij} = \left[v_f\right] \cdot [u_h]_{ij} + \left[u_g\right]_i \cdot \left[v_g\right]_j + \left[v_g\right]_i \cdot \left[u_g\right]_j + \left[u_f\right] \cdot [v_h]_{ij}$; (Hessian component)
Step 3: *return* $[W]$;

The algorithm for overloading the / operator for (U, V) is

Step 1: $\left[W_f\right] = \left[u_f\right] / \left[v_f\right]$ (Function value)
Step 2: for $i = 1 : n$
$\left[W_g\right]_i = (\left[u_g\right]_i - \left[W_f\right] \cdot \left[v_g\right]_i) / \left[v_f\right]$; (Gradient component)
for $j = 1 : i$
$[W_h]_{ij} = ([u_h]_{ij} - \left[W_g\right]_i \cdot \left[v_g\right]_j - \left[v_g\right]_i \cdot \left[W_g\right]_j - \left[W_f\right] \cdot [v_h]_{ij}) / \left[v_f\right]$ (Hessian component)
Step 3: *return* $[W]$;

The algorithm for overloading the elementary functions for (U, V) is

Step 1: $\left[W_f\right] = S\left[u_f\right]$; (Function value)
Step 2: for $i = 1 : n$
$\left[W_g\right]_i = S'(\left[u_f\right]) \cdot \left[u_g\right]_i$; (Gradient component)
for $j = 1 : i$
$[W_h]_{ij} = S''(\left[u_f\right]) \cdot \left[u_g\right]_i \cdot \left[u_g\right]_j + S'(\left[u_f\right])$; (Hessian component)
Step 3: *return* $[W]$;

Here, W returns three components of which the first one is the function value, the second one is the Gradient and the third one is the Hessian.

INPUT: (i) Multi-variate functions of dimension 3.
(ii) Values of the components x, y, z
OUTPUT: (i) Value of the function
(ii) Value of the Gradient
(iii) Value of the Hessian

2.2 Numerical Results

The input function is $f(x) = x\sin(x) + \cos(y^2) + z^2$. The function is evaluated at the point $(2 \cdot 5, 3 \cdot 5, 4 \cdot 5)$

Value of the function = $523 \cdot 6875$

$$\text{Value of the Gradient} = \begin{pmatrix} 210 \cdot 875 \\ 220 \cdot 5 \\ 149 \cdot 625 \end{pmatrix}$$

$$\text{Value of the Hessian} = \begin{pmatrix} 42 & 91 \cdot 45 & 49 \\ 91 \cdot 75 & 40 \cdot 5 & 42 \cdot 75 \\ 49 & 42 \cdot 75 & 31 \cdot 5 \end{pmatrix}$$

3. Evaluation of Jacobians

Jacobian can be evaluated using Gradient arithmetic by computing Gradient for components f_i with $i = 1, 2, \cdots, n$. The same differentiation arithmetic, which is used for finding the functional value along with the Gradient, is used. Then, the Gradients for each $f_i, i = 1, 2, \cdots, n$ are computed and are arranged in the matrix form to give the Jacobian value. Using this new technique the Jacobian is computed exactly with minimum human effort.

4. Conclusion

Here a new technique is used for implementing AD for a scalar valued and twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$. Automatic Differentiation is a useful tool as it facilitates the automatic generation of

Gradient and Hessian which enhances the robustness of an optimization algorithm that requires derivatives. The chain rule based differentiation arithmetic is used to find the Gradient and Hessian. Jacobian is also evaluated using Gradient arithmetic by computing Gradient for components and is arranged in the matrix form to give the Jacobian value.

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Solution of a Class of Minimal Surface Problem with Obstacle

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Abstract

Plateau's problem is to determine the surface with minimal area that lies above an obstacle with given boundary conditions. In this paper, a special example of this class of the problem is given and solved with the linear finite element method. First, we triangulate the domain of definition, and transform the linear finite element approximation into a constrained nonlinear optimization problem. Then we introduce a simple and efficient method, named sequential quadratic programming, for solving the constrained nonlinear optimization problem. The sequential quadratic programming is implemented by the fmincon function in the optimization toolbox of MATLAB. Also, we discuss the relations between the number of grids and the computing time as well as the precision of the result.

Keywords: Minimal surface problem with obstacle, Finite element approximation, Constrained nonlinear optimization, Sequential quadratic programming

1. Introduction

Plateau's problem is to determine the surface of minimal area with a given closed curve in R^3 as boundary (Elizabeth, etc., 2004, pp.39-40). Suppose that the surface can be represented in nonparametric form $z : R^2 \to R$, and the requirement is $z \geq z_L$ for some obstacle z_L. The solution of this obstacle problem minimizes the function $f : K \to R$

$$f(z) = \int_D \sqrt{1 + \|\nabla z(x)\|^2} dx \tag{1}$$

over the convex set

$$K = \{z \in H^1(D) | z(x) = z_D(x) \text{ for } x \in \partial D,\ z(x) \geq z_L(x) \text{ for } x \in D\} \tag{2}$$

where $\| \bullet \|$ represents the Euclidean norm, $H^1(D)$ is the space of functions with gradients in $L^2(D)$. The function $z_D : \partial D \to R$ defines the boundary data, and $z_L : D \to R$ is the obstacle. We assume that $z_L \leq z_D$ on the boundary ∂D.

The linear finite element approximation to the minimal surface with obstacle, defined by (1) and (2), can be obtained by triangulating D and minimizing f over the space of piecewise linear functions. The linear finite element approximation for the minimal surface with obstacle is analyzed, the existence and uniqueness of the solution for the discrete problem are shown, and the error estimate of the finite element approximation is obtained (Shen, etc., 1992, pp. 42-51).

In this paper, we intend to solve an example of the problem with the linear finite element approximation. In the example, we set $D = [0, 1] \times [0, 1]$, and use the boundary data

$$z_D(x, y) = \begin{cases} 1 - (2x - 1)^2 & y = 0, 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and the obstacle

$$z_L(x, y) = \begin{cases} 1 & \text{if} \quad \sqrt{(x - x_0)^2 + (y - y_0)^2} \leq 0.25 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where (x_0, y_0) is the center of the projection circle on X-Y plane of the obstacle. The question is: when $(x_0, y_0) = (0.5, 0.5)$, what is the minimal area? And when (x_0, y_0) can move freely within D, what is the maximum minimal area?

2. Mathematic Model

To obtain the linear finite element approximation to the minimal surface, we use a triangulation with, respectively, n_x and n_y internal grid points in the coordinate directions, as shown in Figure 1, where (m, n) represents the index of the coordinate data.

We set $n_x = n_y = n$ for simplicity, and let z_{ij} denote $z(x_{ij}, y_{ij})$. Thus the surface area is approximately $2n^2$ small triangle area, and the linear finite element approximation of the problem (1)(3)(4) may be transformed into the following constrained nonlinear optimization problem

$$\min_{z_{ij}} \quad S = \sum_{k=1}^{2n^2} S_k(\{z_{ij}, i, j = 1, \cdots, n + 1\}) \tag{5}$$

subject to

$$\begin{cases} z_{i1} = 0 \\ z_{1i} = 1 - (2x_{1i} - 1)^2 \qquad i = 1, 2, \cdots, n + 1 \\ z_{i,n+1} = 0 \\ z_{n+1,i} = 1 - (2x_{n+1,i} - 1)^2 \end{cases} \tag{6}$$

and

$$z_{i,j} \geq \begin{cases} 1 & \text{if} \quad \sqrt{(x_{ij} - x_0)^2 + (y_{ij} - y_0)^2} \leq 0.25 \quad i, j = 1, 2, \cdots, n + 1 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

The minimum of this class of constrained nonlinear multivariable function can be solved by the fmincon function of MATLAB.

3. Solving Algorithm

The fmincon function uses the sequential quadratic programming (SQP) method to solve Medium-Scale Optimization problem of the following form

$$\begin{array}{lrcl} \min\limits_{x} & f(x) & & \\ s.t. & g_u(x) & \leq & 0 \quad (u = 1, 2, \cdots, p) \\ & h_v(x) & = & 0 \quad (v = 1, 2, \cdots, m) \end{array} \tag{8}$$

The recursive procedure of the SQP method is as follows (Li, 2006, pp. 117-119; Zhang, etc., 2007, pp. 117-122):

Give the initial point x_0, the initial Hessian matrix $H_0 = I$ (identity matrix), and set k=0.

(1) A QP problem of the following form is solved to get d_k.

$$\begin{array}{ll} \min\limits_{d\in R^n} & \nabla f(x_k)^T d + \frac{1}{2} d^T H_k d \\ s.t. & g_u(x_k) + \nabla g_u(x_k)^T d \le 0 \quad (u = 1, 2, \cdots, p) \\ & h_v(x_k) + \nabla h_v(x_k)^T d = 0 \quad (v = 1, 2, \cdots, m) \end{array} \tag{9}$$

(2) Use linear search to form a new iterate

$$x_{k+1} = x_k + \alpha_k d_k \tag{10}$$

where the step length α_k is determined according to certain rules.

(3) Update the Hessian matrix using the BFGS formula of the Quasi-Newton method.

$$H_{k+1} = H_k + \frac{q_k q_k^T}{q_k^T s_k} - \frac{H_k H_k^T}{s_K^T H_k s_k} \tag{11}$$

where

$$s_k = x_{k+1} - x_k$$

$$q_k = \nabla f(x_{k+1}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x_{k+1}) - [\nabla f(x_k) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x_k)]$$

and $\lambda_i (i = 1, \cdots, m)$ is an estimate of the Lagrange multipliers.

4. Results and Analysis

4.1 The minimal area when $(x_0, y_0) = (0.5, 0.5)$

Write MATLAB program using the fmincon function (choose Medium-Scale Algorithm), and run it to get the results. The minimal surface is shown in Figure 2. The minimal surface area is minS=2.4075. It can be seen from Figure 2 b) that the surface is sunken in the y direction.

The minimal surface area depends on the number of the grid points. Figure 3 shows the relationship between the minimal area and the number of grid points.

As can be seen from Figure 3, with the increase of the number of grid points, the minimal area is increasing on the whole. This is because the area of a triangular element unit is smaller than that of the curved surface, and the approximate error is reduced with more grid points, as shown in Figure 4. And the volatility is weakening, i.e. it has the trend to a constant. This is in line with the theoretical result of (Shen, Shumin, 1992), which concludes that the finite element approximate solution converges to the true solution when the number of grid points tends to infinity.

In addition, with the increase of the number of grid points, the number of the variable is increasing, so the computation becomes slower. For example, it takes only 37.48s for 10 grid points to finish computation, an hour for 35 grid points, while about 4 hours for 40 grid points. The computed result also depends on the parameters of the fmincon function. In the experiment, TolFun(termination tolerance on the function value), TolCon(termination tolerance on the constraint violation) and TolX (termination tolerance on x) are all set to be 10^{-5}.

4.2 The maximum minimal area when moves freely within D

In this case, x_0, y_0 can be treated as extra parameters to be optimized. Due to symmetry of the boundary data, it only need to consider the case when $(x_0, y_0) \in [0, 0.5] \times [0, 0.5]$.

Write MATLAB program using the fmincon function (choose Medium-Scale Algorithm), and run it to get the results. As shown in Figure 5 (30 grid points), the maximum minimal area is obtained when $(x_0, y_0) = (0.25, 0.5)$. The corresponding maximum minimal area is maxminS=2.5158.

From the above analysis, we have known that the minimal surface is sunken in the y direction. Obviously the surface with minimal area without obstacle is as Figure 6 shows. With the center of the plate obstacle approaching the y axis, the surface is bulging more, and the area of the surface will increase accordingly. So it is reasonable that the maximum minimal area is obtained when the plate obstacle is tangent to the y axis.

The maximum minimal area can also be obtained with $(x_0, y_0) = (0.75, 0.5)$ for symmetry. Computed result that the minimal area is also 2.5158 confirms the conclusion. The corresponding surface (30 grid points) is shown in Figure 7.

5. Conclusion

This paper solves the minimal surface problem with obstacle with linear finite element approximation. We transform the minimal surface problem with obstacle into a constrained nonlinear optimization problem. Our method is based on the sequential quadratic programming method, and we use the fmincon function in the optimization toolbox of MATLAB. Based on the results, the relation between the number of grid points and precision of the results as well as computation time is discussed.

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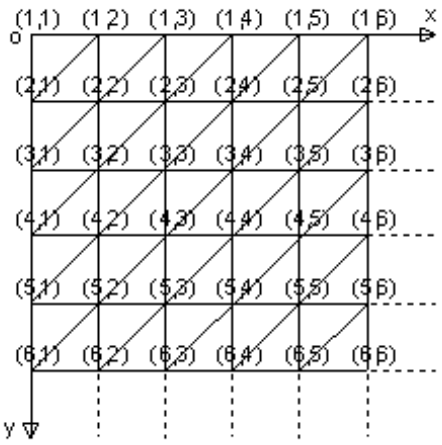
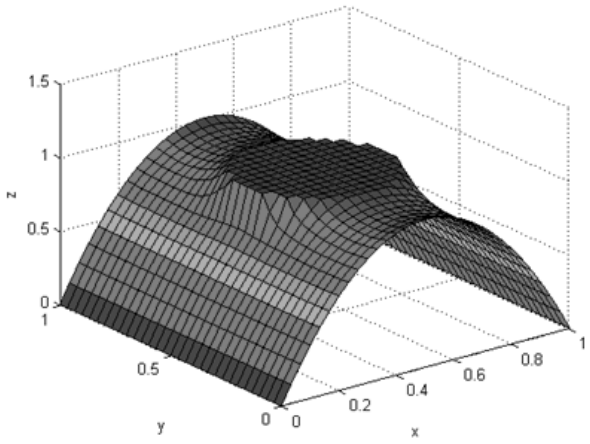
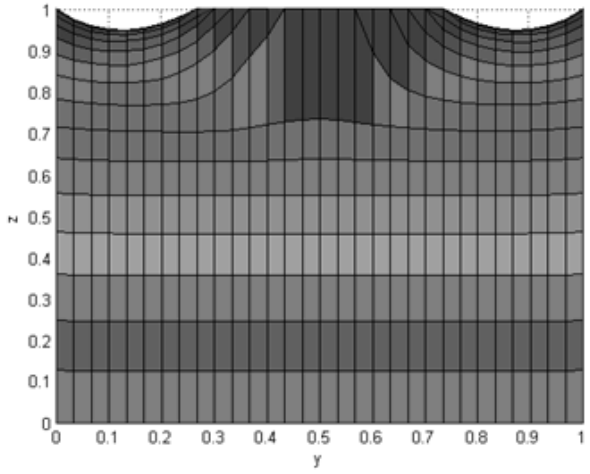


Figure 1. Triangulation of the region $D = [0, 1][0, 1]$



a) Three- dimensional graph



b) View from x direction

Figure 2. Minimal surface with obstacle (30 grid points)

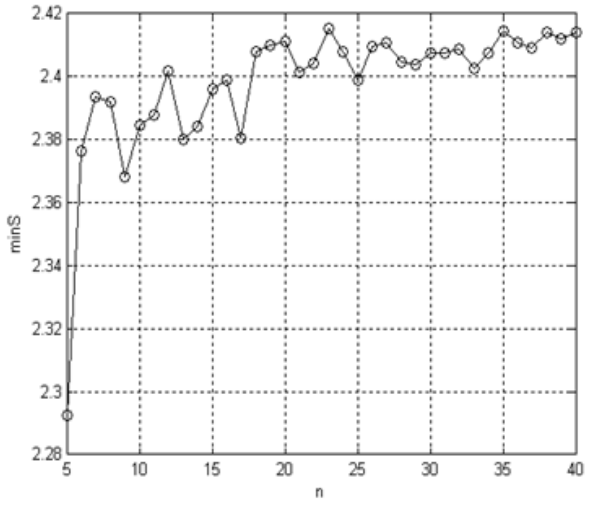
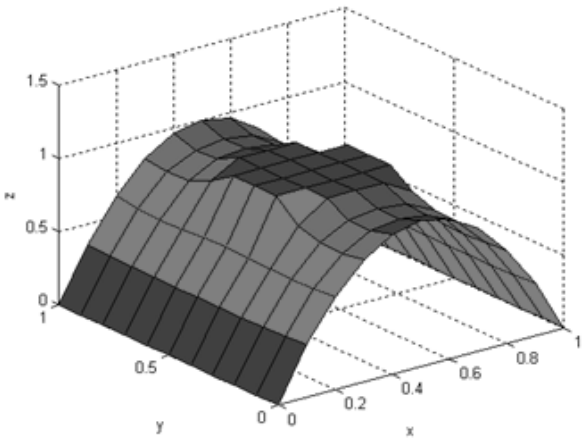
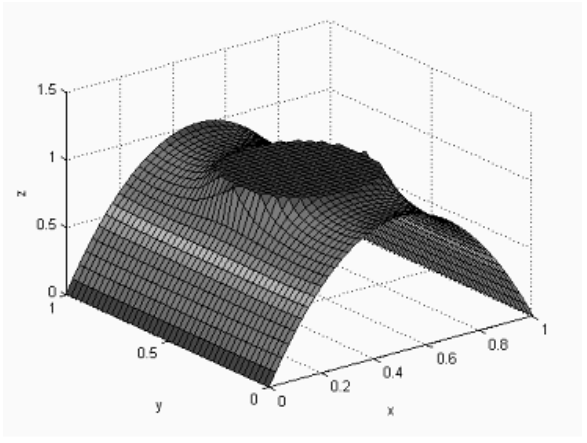


Figure 3. The relationship between the minimal area and the number of grid points



a) 10 grid points



b) 40 grid points

Figure 4. Comparison of the minimal surface with 10 grid points and 40 grid points

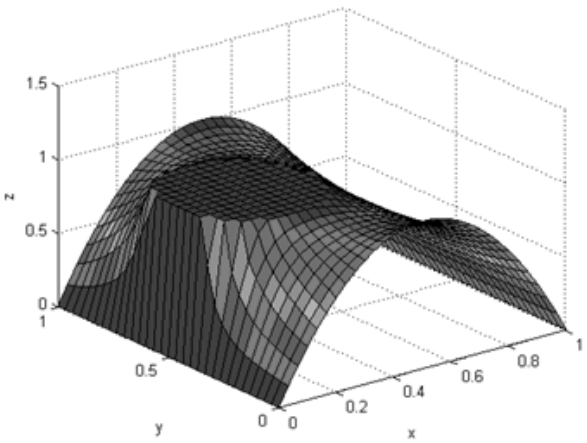


Figure 5. Surface with maximum minimal area

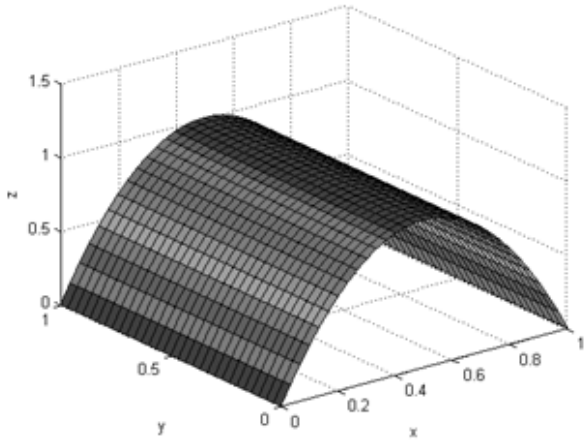


Figure 6. Minimal surface without obstacle

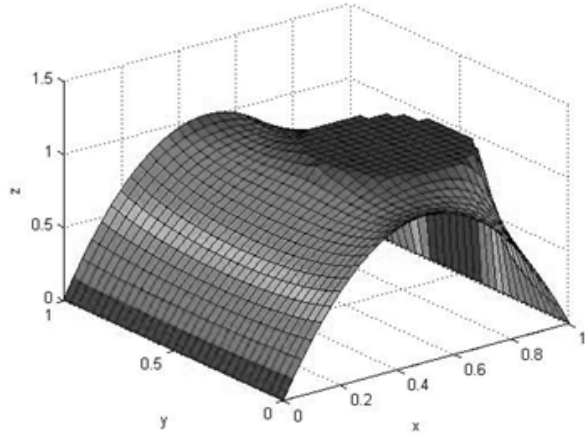


Figure 7. Surface with maximum minimal area

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Analysis of a Multigrid Algorithm for Mortar Element Method

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Abstract

In this paper, a multigrid algorithm is studied for mortar element method for rotated Q_1 element, the mortar condition is only dependent on the degrees of the freedom on subdomains interfaces. We prove the convergence of W-cycle multigrid and construct a variable V-cycle multigrid preconditioner which is available.

Keywords: Multigrid, Mortar element method, Rotated Q_1 element

1. Introduction

The mortar element method is a nonconforming domain decomposition method with non-overlapping subdomains. The meshes on different subdomains need not align across subdomains interfaces, and the matching of discretizations on adjacent subdomains is only enforced weakly. This method offers the advantages of freely choosing highly varying mesh sizes on different subdomains. The rotat Q_1 element is an important nonconforming element. It was first proposed and analysised for numerically solving the Stokes problem, the rotated Q_1 element provides the simplest example of discretely divergence-free nonconforming element on quadrilaterals.

Let $\Omega \in R^2$ be a rectangular or L-shape bounded domain with boundary $\partial\Omega$. Partition Ω into geometrically conforming rectangular substructures, i.e..

$\overline{\Omega} = \bigcup_{k=1}^{N} \overline{\Omega}_k$ and $\Omega_k \bigcap \Omega_l = \phi$, $k \neq l$, $\overline{\Omega}_k \bigcap \overline{\Omega}_l$ is empty set or a vertex or an edge for $k \neq l$.

Let $T_1^i = T_1^i(\Omega_i)$ be a coarsest quasi-uniform triangulation of the subdomain Ω_i, which made of elements that are rectangles whose edges are parallel to X-axis or Y-axis. Let $T_1 = \bigcup_{i=1}^{N} T_1^i$. The mesh parameter h_1 is the diameter of the largest element in T_1 the global triangulation of Ω. We refine the triangulation T_1 to produce T_2 by joining the midpoints of the edges of the rectangles in T_1. Obviously, the mesh size h_2 in T_2 satisfies $h_2 = \frac{1}{2}h_1$. Repeating this process, we get a sequel of triangulations $T_1(l = 1, 2, \cdots, L)$. Let $\Omega_{i,l}$ and $\partial\Omega_{i,l}$ be the set of vertices of the triangulation T_1^i that are in $\overline{\Omega}_i$ and $\partial\Omega_i$ respectively.

We construct the rotated Q_l element for each triangulation $T_l(\Omega_i)$ as follows.

$X_l(\Omega_i) = \{v \in L^2(\Omega_i)|v|_E = \alpha_E^1 + \alpha_E^2 x + \alpha_E^3 y + \alpha_E^4(x^2 - y^2)$, $\alpha_E^2 \in R$, $\int_{\partial E|\partial\Omega} v|_{\partial\Omega} ds = 0$, $\forall E \in T_l(\Omega_i)$; for $E_1, E_2 \in T_l(\Omega_i)$, if $\partial E_1|\partial E_2 = e$, then $\int_e v|_{\partial E_1} ds = \int_e v|_{\partial E_2} ds\}$

The global discrete space is defined by

$$X_l(\Omega) = \prod_{i=1}^{N} X_l(\Omega_i)$$

The interface $\Gamma = \bigcup_{i=1}^{N} \partial\Omega_i \backslash \partial\Omega$ is broken into a set of disjoint open straight segments $\gamma_m (1 \leq m \leq M)$, i.e., $\Gamma = \bigcup_{m=1}^{M} \overline{\gamma}_m$, $\gamma_m \cap \gamma = \phi$, if $m \neq n$.

By $\gamma_{m(i)}$ we denote an edge of Ω_i called mortar and by $\delta_{m(j)}$ an edge of Ω_i that geometrically occupies the same place called nonmortar, then $\gamma_{m(i)=\delta_{m(j)}=\gamma_m}$. Since γ_m inherits two different triangulations, by $T_l(\gamma_{m(i)})$ and $T_l(\delta_{m(j)})$ denote the different triangulations across γ_m (Assume the fine side is chosen as mortar). Define $S_l(\delta_{m(j)})$ to be a subspace of $L^2(\gamma_m)$, such that its functions are piecewise constants on $T_l(\delta_{m(j)})$. The dimension of $S_l(\delta_{m(j)})$ is equal to the number of elements on the $\delta_{m(j)}$. For each nonmortar edge $\delta_{m(j)}$, define an L^2-projection operator $Q_{l,\delta} : L^2(\gamma_m) \to S_l(\delta_{m(j)})$ by

$$(Q_{l,\delta}\nu, \psi)_{L^2(\delta_{m(j)})} = (\nu, \psi)_{L^2(\delta_{m(j)})}, \ \forall \psi \in S_l(\delta_{m(j)}) \tag{1}$$

The purpose of this paper is to study the multigrid method for mortar element for the rotated Q_1 element. An intergrid transfer operator is presented for nonnested mortar element spaces. On the basis of this operator, we give a multigrid algorithm. Using the theory developed by Bramble, Pasciak, Xu, we prove the W-cycle multigrid is optimal, i.e., the convergence rate is independent of mesh size and mesh level. Furthermore, a variable V-cycle multigrid preconditioner is developed, which results in a preconditioned system with uniformly bounded condition number.

The remainder of this paper is organized as follows. In section two we introduce Multigrid algorithm. Section three presents some lemmas. Last section gives our results.

2. Multigrid algorithm

We must define a suitable intergrid transfer operator for nonnested mesh space V_l. First introduce a local intergrid operator J_l^i from $X_{l-1}(\Omega_i)$ to $X_l(\Omega_i)$ by

$$\frac{1}{|\,e\,|}\int_e J_l^i \nu ds \begin{cases} 0 & e \subset \partial\Omega_i \cap \partial\Omega \\ \frac{1}{|e|}\int_e \nu ds & e \subset \partial\Omega_i \setminus \partial\Omega \\ \frac{1}{|e|}\int_e \nu ds & e \not\subset \partial E \qquad E \in T_{l-1}^i \\ \frac{1}{2|e|}\int_e (\nu\,|_{E_1} + \nu\,|_{E_2}) ds & e \subset \partial E_1 \cap \partial E_2 \quad E_1,\, E_2 \in T_{l-1}^i \end{cases}$$

Where $e \in \partial E$, $E \in T_l^i$.

Based on the operator J_l^i, a global intergrid transfer operator $J_l : X_{l-1}(\Omega) \to X_l(\Omega)$ introduced as follows.

$$J_l\nu = (J_l^1\nu,\, J_l^2\nu^2, \cdots, J_l^N\nu^N), \quad \forall \nu = (\nu^1,\, \nu^2, \cdots, \nu^N) \in X_{l-1}(\Omega)$$

To construct an intergrid operator in mortar element spaces we define an operator $\varepsilon_{l,\,\delta_{m(j)}}$:

$$X_l(\Omega) \to X_l(\Omega) \quad by \quad \int_e \varepsilon_{l,\delta_{m(j)}}(\nu) ds = \begin{cases} \int_e Q_{l,\delta}\left(I_l^\gamma Q_{l,\gamma}\nu\,|_{\gamma_{m(i)}} - \nu\,|_{\delta_{m(j)}}\right) ds & e \in T_l(\delta_{m(j)}) \\ 0 & otherwise \end{cases}$$

Then for any $\nu \in X_l(\Omega)$, let

$$\nu^* = \nu + \sum_{m=1}^{M} \varepsilon_{l,\delta_{m(j)}}(\nu) \tag{2}$$

It is easy to check that $\nu^* \in V_l$, since for any $\psi \in S_l(\delta_{m(j)})$, we can derive

$$\begin{aligned} \int_{\delta_{l,m(j)}} \nu^*|_{\delta_{m(j)}}\psi ds &= \int_{\delta_{m(j)}} \nu|_{\delta_{m(j)}}\psi ds + \int_{\delta_{m(j)}} \varepsilon_{l,\delta_{m(j)}}(\nu)|_{\delta_{m(j)}}\psi ds \\ &= \int_{\delta_{m(j)}} \nu|_{\delta_{m(j)}}\psi ds + \int_{\delta_{m(j)}} Q_{l,\delta}(I_l^\gamma Q_{l,\gamma}\nu|_{\gamma_{m(i)}} - \nu|_{\delta_{m(j)}})\psi ds \\ &= \int_{\delta_{m(j)}} \nu\,|_{\delta_{m(j)}}\,\psi ds + \int_{\delta_{m(j)}} (I_l^\gamma Q_{l,\gamma}\nu|_{\gamma_{m(i)}} - \nu|_{\delta_{m(j)}})\psi ds \\ &= \int_{\delta_{m(j)}} I_l^\gamma Q_{l,\gamma}\nu|_{\gamma_{m(i)}}\psi ds \\ &= \int_{\delta_{m(j)}} I_l^\gamma Q_{l,\gamma}\nu^*|_{\gamma_{m(i)}}\psi ds \end{aligned}$$

After above preparation, we can construct an intergrid transfer operator I_l in mortar element spaces.

$$I_l : X_{l-1}(\Omega) \to V_l \quad \text{by} \quad I_l v = J_l v + \sum_{m=1}^{M} \varepsilon_{l,\delta_{m(j)}}(J_l v),\ \forall v \in X_{l-1}(\Omega) \tag{3}$$

To present our multigrid algorithm, we describe some auxiliary operators. For $l = 1, 2, \cdots, L$, define $A_l : V_l \to V_l$, $P_{l-1} : V_l \to V_{l-1}$, and $P^0_{l-1} : V_l \to V_{l-1}$ respectively by $(A_l u, v) = \alpha_l(u, v)$, $\forall u, v \in V_l$, $(P^0_{l-1}u, v) = (u, I_l v)$, $\forall u \in V_l$, $v \in V_{l-1}$, $a_{l-1}(P_{l-1}u, v) = a_l(u, I_l v)$, $\forall u \in V_l$, $v \in V_{l-1}$,

Furthermore we must find smoothing operator R_l, including Gauss-Seidel, conjugate gradient iterations and so on, which satisfy the following condition.

(R). There exists a constant $C_R \geq 1$ independent of l such that

$$\frac{\|u\|_0^2}{\lambda_1} \leq C_R(\overline{R}_l u, v),\ \forall u \in V_l \tag{4}$$

For both $\overline{R}_l = (I - K_l^* K_l)A_l^{-1}$ or $\overline{R}_l = (I - K_l K_l^*)A_l^{-1}$, where $K_l = I - R_l A_l$, $K_l^* = I - R_l^T A_l$, R_l^T is the adjoint of R_l with respect to $(\cdot,\cdot)$ and λ_l is the maximum eigenvalue of A_l.

$$\text{Define } R_l^{(k)} = \begin{cases} R_l & k \text{ is odd} \\ R_l^T & k \text{ is even} \end{cases}$$

A general multigrid operator $B_l : V_l \to V_l$ can be defined recursively as follows.

Multigrid Algorithm. Set $B_1 = A_1^{-1}$. Let $2 \leq l \leq L$ and p be a positive integer, assume that B_{l-1} has been defined and define B_{1g} for $g \in V_l$ by

(1) Set initial value X^0 and let $q^0 = 0$.

(2) Define x^k for $k = 1, 2, \cdots, m(l)$ by $x^k = x^{k-1} + R_l^{(k+m(l))}(g - A_l x^{k-1})$.

(3) Define $y^{m(l)} = x^{m(l)} + I_l q^p y$, where q^i for $i = 1, \cdots, p$ are determined by $q^i = q^{i-1} + B_{l-1}(P^0_{l-1}(g - A_l x^{m(l)}) - A_{l-1}q^{i-1})$

(4) Define y^k for $k = m(l) + 1, \cdots, 2m(l)$ by $y^k = y^{k-1} + R_l^{(k+m(l))(g - A_l y^{k-1})}$.

(5) Set $B_{1g} = y^{2m(l)}$.

Remark. In the Multigrid Algorithm, $m(l)$ gives the number of presmoothing and postsmoothing steps, it can vary as a function of l. If $p = 1$, we have a V-cycle method, and $p = 2$ denotes a W-cycle method. A variable V-cycle algorithm is one in which the number of smoothing $m(l)$ increase exponentially as l decreases, i.e., the number of smoothing $m(l)$ satisfies $\beta_0 m(l) \leq m(l-1) \leq \beta_1 m(l)$, with $1 < \beta_0 < \beta_1$.

3. Some lemmas

To reach our conclusion, we present some auxiliary technical lemmas and prove an approximation assumption.

Define an operator $M_{l,i} : X_l(\Omega_i) \to V_l^{\frac{1}{2}}(\Omega_i)$ as follows.

Definition 1. Given $v \in X_l(\Omega_i)$, let $M_{l,i}v \in V_l^{\frac{1}{2}}(\Omega_i)$ by the values of $M_{l,i}v$ at the vertices of the partition $T_l^{\frac{1}{2}}(\Omega_i)$.

(1) If P is a central point of E, $E \in T_l(\Omega_i)$, then $(M_{l,i}v)(P) = \frac{1}{4}\sum_{e_i \in \partial E} \frac{1}{|e_i|}\int_{e_i} v ds$.

(2) If P is a midpoint of one edge $e \in \partial E$, $E \in T_l(\Omega_i)$, then $(M_{l,i}v)(P) = \frac{1}{|e_i|}\int_e v ds$.

(3) If $P \in \Omega_{i,l} \setminus \partial\Omega_{i,l}$, then $(M_{l,i}v)(P) = \frac{1}{4}\sum_{e_i} \frac{1}{|e_i|}\int_{e_i} v ds$. Where the sum is taken over all edges e_i with the common vertex P, $e_i \in \partial E_i$, $E_i \in T_l(\Omega_i)$.

(4) If $P \in \partial\Omega_{i,l} \setminus \{c_1, \cdots, c_n\}$, then $(M_{l,i}v)(P) = \frac{1}{2}(\frac{1}{|e_l|}\int_{e_l} v ds + \frac{1}{|e_\gamma|}\int_{e_\gamma} v ds)$, where $e_l \in \partial E_1 \cap \partial\Omega_i$ and $e_\gamma \in \partial E_2 \cap \partial\Omega_i$ are the left and right neighbor edges of P, E_1, $E_2 \in T_l(\Omega_i)$, $c_1, \cdots, c_n$ are the vertices of subdomain Ω_i.

(5) If $P \in \{c_1, \cdots, c_n\}$, then

$$(M_{l,i}v)(P) = \frac{|e_l|}{|e_l| + |e_\gamma|}(\frac{1}{|e_l|}\int_{e_l} v ds) + \frac{|e_\gamma|}{|e_l| + |e_\gamma|}(\frac{1}{|e_\gamma|}\int_{e_\gamma} v ds)$$

For the above operator $M_{l,j}$, we have the following result.

Lemma 1. For any $v \in X_l(\Omega_i)$, we have $|M_{l,i}v|_{H^1(\Omega_i)} \approx \|v\|_{l,i}$.

Lemma 2. $\|v - Q_{l,\delta}v\|_{L^2(\gamma m)} \leq h_l^{\frac{1}{2}}|v|_{H^{\frac{1}{2}}(\gamma m)}$ $\forall v \in H^{\frac{1}{2}}(\gamma m)$.

Lemma 3. For any $v \in X_l(\Omega_i)$, then $\|Q_{l,\delta}I_l^\gamma Q_{l,\gamma}v|_{\gamma m(i)} - I_l^\gamma Q_{l,\gamma}v|_{\gamma m(i)}\|_{L^2(\gamma m(i))} \leq h_l^{\frac{1}{2}}\|v\|_{l,i}$.

$\|I_l^r Q_{l,r}v|_{\gamma m(i)} - Q_{l,r}v|_{\gamma m(i)}\|_{L^2(\gamma m(i))} \leq h_l^{\frac{1}{2}}\|v\|_{l,i}$

Lemma 4. For any $v^i \in V_{l-1}(\Omega_i)$, we have $\|J_l^i v^i\|_{l,i} \leq \|v^i\|_{l-1,i}$, $\|v^i - J_l^i v^i\|_{0,i} \leq h_l\|v^i\|_{l-1,i}$.

Lemma 5. For any $v \in V_{l-1}$, it holds that $\|I_l v\|_l \leq \|v\|_{l-1}$, $\|v - I_l v\|_0 \leq h_l\|v\|_{l-1}$.

Lemma 6. For the operator Π_l, we have $\|\xi - \Pi_l\xi\|_0 + h_l\|\xi - \Pi_l\xi\|_l \leq h_l^2|\xi|_2$, $\forall \xi \in H_0^1(\Omega) \cap H^2(\Omega)$.

Lemma 7. For any $\xi \in H_0^1(\Omega) \cap H^2(\Omega)$, we have $\|\xi - I_L\Pi_{l-1}\xi\|_l \leq h_l|\xi|_2$.

The proofs of the above all lemmas can be found in relevant references. Let's come to see the last two lemmas.

Lemma 8. The operator P_{l-1} has following property $\|v - P_{l-1}v\|_0 \leq h_l\|v\|_l$, $\forall v \in V_l$.

Proof. Consider the auxiliary problem as follows

$$\begin{cases} -\triangle\xi = v - P_{l-1}v & in\ \Omega \\ \xi = 0 & on\ \partial\Omega \end{cases}$$

then $\|v - P_{l-1}v\|_0^2 = (-\triangle\xi, v - P_{l-1}v) = (\alpha_l(\xi, v) - \alpha_{l-1}(\xi, P_{l-1}v)) - \sum_{K\in T_l}\oint_{\partial K}\frac{\partial\xi}{\partial n}v ds + \sum_{K\in T_{l-1}}\oint_{\partial K}\frac{\partial\xi}{\partial n}P_{l-1}v ds$
$:= F_1 + F_2 + F_3$

Lemma 4 and Lemma 2 reveal $|F_2| \leq h_l|\xi|_2\|v\|_l = h_l\|v - P_{l-1}v\|_0\|v\|_l$. Using Lemma 5, we can see $\|P_{l-1}v\|_{l-1}^2 = \alpha_{l-1}(P_{l-1}v, P_{l-1}v) = \alpha_l(v, I_l P_{l-1}v) \leq \|v\|_l\|P_{l-1}v\|_l$. So $\|P_{l-1}v\|_{l-1} \leq \|v\|_l$.

By Lemma 2 and above inequality, we have $|F_3| \leq h_l|\xi|_2\|P_{l-1}v\|_{l-1} = h_l\|v - P_{l-1}v\|_0\|v\|_l$. Now we estimate F_1.

$$\begin{aligned} |F_1| &= |\alpha_l(\xi, v) - \alpha_{l-1}(\Pi_{l-1}\xi, P_{l-1}v) + \alpha_{l-1}(\Pi_{l-1}\xi, P_{l-1}v) - \alpha_{l-1}(\xi, P_{l-1}v)| \\ &\leq |\alpha_l(\xi - I_l\Pi_{l-1}\xi, v)| + |\alpha_{l-1}(\xi - \Pi_{l-1}\xi, P_{l-1}v)| \\ &\leq h_l|\xi|_2(\|v\|_l + \|P_{l-1}v\|_{l-1}) \leq h_l|\xi|_2\|v\|_l \leq |v - P_{l-1}v\|_0\|v\|_l \end{aligned}$$

All the above inequalities give the proof. Now, the approximation assumption theory is given as follows.

Lemma 9. $|\alpha_l((I - I_l P_{l-1})v, v)| \leq \left(\frac{\|A_l v\|_0^2}{\gamma_l}\right)^{\frac{1}{2}}\alpha_l(v, v)^{\frac{1}{2}}$, $\forall v \in V_l$. Proof. By triangular inequality, Lemma 5 and Lemma 8, we derive $\|v - I_l P_{l-1}v\|_0 \leq \|v - P_{l-1}v\|_0 + \|(I - I_l)P_{l-1}v\|_0 \leq h_l(\|v\|_l + \|P_{l-1}v\|_{l-1}) \leq h_l\|v\|_l$ On the other hand

$$\begin{aligned} \|v - I_l P_{l-1}v\|_l &= \sup_{\omega\in V_l, \|\omega\|_l=1} \alpha_l(v - I_l P_{l-1}v, \omega) \\ &= \sup_{\omega\in V_l, \|\omega\|_l=1} \alpha_l(v, \omega - I_l P_{l-1}\omega) \\ &\leq \sup_{\omega\in V_l, \|\omega\|_l=1} \|A_l v\|_0\|\omega - I_l P_{l-1}\omega\|_0 \\ &\leq h_l\|A_l v\|_0 \end{aligned}$$

Then, we can obtain

$$|\alpha_l((I - I_l P_{l-1})v, v)| \leq \|(I - I_l P_{l-1})v\|_l\|v\|_l \leq h_l\|A_l v\|_0\|v\|_l \leq \left(\frac{\|A_l v\|_0^2}{\lambda_l}\right)^{\frac{1}{2}}\alpha_l(v, v)^{\frac{1}{2}}.$$

4. Main result

We now state the convergence results for the multigrid algorithm. The convergence rate for the multigrid algorithm on the l th level is measured by a convergence factor

$$\delta_l \text{ satisfying } |\alpha_l((I - B_l A_l)v, v)| \leq \delta_l \alpha_l(v, v), \ \forall v \in V_l \tag{5}$$

Following the above analysis, we propose two propositions:

Proposition 1. (W-cycle). Under Lemma 9, if $p = 2$ and $m(l) = m$ is large enough, then the convergence factor in (5) is $\delta_l = \frac{C}{C+m^{\frac{1}{2}}}$

Proposition 2. (variable V-cycle preconditioner) Under Lemma 9, and the number of smoothing $m(l)$ increases as decreases in such a way that $\beta_0 m(l) \leq m(l-1) \leq m(l)$, hold with $1 \leq \beta_0 \leq \beta_1$. then there exists $M > 0$ independent of L such that $C_0^{-1}\alpha_l(v, v) \leq \alpha_l(B_l A_l v, v) \leq C_0 \alpha_l(v, v)$, $\forall v \in V_l$, with $C_0 = \frac{M+m(l)^{\frac{1}{2}}}{m(l)^{\frac{1}{2}}}$.

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Study on the Oscillation of a Class of Nonlinear Delay Functional Differential Equations

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Abstract

In this paper, a class of nonlinear delay functional differential equations with variable coefficients is linearized, and through analogizing the oscillation theory of linear functional differential equation, we obtain many oscillation criteria of this class of equation by using the Schauder fixed point theorem.

Keywords: Variable coefficient, Nonlinear, Functional differential equation, Oscillation

1. Introduction

There are many researchers about the oscillation of the linear delay functional differential equation with constant coefficients and the linear delay functional differential equation with variable coefficients, and a series of conclusions has been acquired. However, the literatures about the nonlinear delay functional differential equation with variable coefficients are very few. In the following study, we suppose the functional differential equation accords with the whole existence of solution, and we will use the Schauder fixed point theorem when proving the existence of positive solution.

Consider the nonlinear delay functional differential equation with variable coefficients

$$x'(t) + \sum_{i=1}^{n} Q_i(t) f(x(t-\tau_i)) = 0 \tag{1}$$

and the linear delay functional differential equation with constant coefficients

$$x'(t) + \sum_{i=1}^{n} q_i x(t-\tau_i) = 0 \tag{2}$$

where, $f \in C[R,R]$, $q_i \in [0,+\infty)$, $\tau_i \in [0,+\infty)$, $Q_i \in C\Big[[t_0,+\infty),R^+\Big](i=1,2\cdots n)$.
Replace the variable coefficients in the equation (1) by the constant q_i, we can obtain the equation

$$x'(t) + \sum_{i=1}^{n} q_i f(x(t-\tau_i)) = 0 \tag{3}$$

Gyori's article (Gyori, 1991) studied the oscillation of equation (3) and proved that if the following conditions

(H_1) $\lim\limits_{u\to 0}\frac{f(u)}{u} = 1$

(H_2) When $u \neq 0, uf(u) > 0$

(H_3) $\sigma > 0$ exists and makes when $u \in [0,\sigma)$, $f(u) \leqslant u$, and when $u \in (-\sigma,0]$, $f(u) \geqslant u$ comes into existence, so the sufficient and necessary condition of the oscillation of differential equation (3) is the equation (2) is oscillatory.

In the article, we will discuss the oscillation of the equation (1) which is more common than the equation (3), and the result will extend the conclusion in Gyori's article. To prove the main result, we first introduce the following lemma.

Lemma 1.1: For the delay differential inequation $x'(t) + qx(t-\tau) \leqslant 0$, where, $q \in R^+$, and x(t) is its final positive solution, so the inequation $x(t-\tau) \leqslant \left(\frac{2}{q\tau}\right)^2 x(t)$ comes into existence finally.

Prove: Suppose when $t \geqslant t_0 - \tau, x(t) > 0$, x(t) fulfills the delay differential inequation $x'(t) + qx(t-\tau) \leqslant 0$.

Make integral to the above inequation from s to $s + \frac{\tau}{2}$, we can obtain

$$x(s+\frac{\tau}{2}) - x(s) + \int_s^{s+\frac{\tau}{2}} qx(s-\tau)ds \leqslant 0, s > t_0 + \tau \tag{4}$$

Because $x'(t) \leqslant -qx(t-\tau)$, so $x(t)$ doesn't increase monotonically, so

$$\frac{q\tau}{2} x(s-\frac{\tau}{2}) \leqslant x(s) \tag{5}$$

Take $t = s + \frac{\tau}{2}$, from (5), we can obtain

$$\frac{q\tau}{2} x(t-\tau) \leqslant x(t-\frac{\tau}{2}),\ t \geqslant t_0 + \frac{3\tau}{2} \tag{6}$$

Change s in (5) by t, and from (6), we can obtain $x(t-\tau) \leqslant \left(\frac{2}{q\tau}\right)^2 x(t)$.

Lemma 1.2: Suppose $u(t) \in C^1[[t_0, \infty), R^+]$, and when t is enough big, the following inequation comes into existence.

$$u'(t) \leqslant 0,\ u(t-\alpha) < Au(t) \tag{7}$$

Where, $\alpha, A \in R^+$, suppose $\Omega = \{\lambda \geqslant 0 : u'(t) + \lambda u(t) \leqslant 0$ comes into existence finally$\}$, so when $A > 1, \lambda_0 = \frac{lnA}{\alpha} \notin \Omega$ exists.

Prove: Suppose $\lambda_0 = \frac{lnA}{\alpha} \in \Omega$, so $u'(t) + \lambda_0 u(t) \leqslant 0$, i.e.$\frac{d}{dt}[e^{\lambda_0 t}u(t)] \leqslant 0$, that indicates $e^{\lambda_0 t}u(t)$ is final unincreasing, so for the enough big t,

$$e^{\lambda_0(t-\alpha)}u(t-a) \geqslant e^{\lambda_0 t}u(t) \tag{8}$$

$$u(t-\alpha) \geqslant e^{\lambda_0 \alpha}u(t) = Au(t)$$

So, (7) is contrary with (8), which indicates the suppose doesn't come into existence, and the theorem is proved.

Lemma 1.3 (Gyori, 1991): The sufficient and necessary condition of the oscillation of the differential equation (2) is the characteristic equation $\lambda + \sum_{i=1}^{n} q_i e^{-\tau_i \lambda} = 0$ has no real root.

Lemma 1.4 (Zhang, 1987) (Schauder fixed point theory): Suppose M is the closed convex subset in the Banach space X, $T : M \to M$ is continuous, and is the relative compact subset of X, so T must have a fixed point $x \in M$ to make $Tx = x$.

2. Main results and proofs

For the need of following proofs, we give following conditions after $(H_1), (H_2)$ and (H_3).

(H_4) $\lim_{t\to\infty} Q_i(t) = q_i\,(i = 1, 2 \cdots n)$

(H_5) $Q_i(t) \leqslant q_i\,(i = 1, 2 \cdots n)$

(H_6) $\sum_{i=1}^{n} q_i > 0$

Theorem 2.1: Suppose conditions (H_2) and (H_6) come into existence, and if $x(t)$ is the non-oscillatory solution of the equation (1), so $x(t)$ is finally monotonically, and $\lim_{x\to\infty} x(t) = 0$.

Prove: Suppose $x(t)$ is the non-oscillatory solution and the finally positive solution of the equation (1), and for the situation of finally negative solution, we can prove it analogously. From the equation (1), we can obtain

$$x'(t) = -\sum_{i=1}^{n} Q_i(t) f(x(t-\tau_i)) < 0 \tag{9}$$

So $x(t)$ is finally monotonically decreasing function, and suppose $\lim_{t\to\infty} x(t) = l$, so $l = 0$, or else, $l > 0$, from the equation (1), we can obtain

$$\lim_{x\to\infty} x'(t) = -\sum_{i=1}^{n} q_i f(l) < 0 \tag{10}$$

The above equation indicates $\lim_{t\to\infty} x(t) = -\infty$, that is contrary with the condition that $x(t)$ is the finally positive solution. So the theorem is proved.

Theorem 2.2: Under the condition of (H_6), if the equation (2) is oscillatory, so one j_0 exists at least and makes $q_{j_0} > 0$ and $\tau_{j_0} > 0$.

Prove: Because the equation (2) is oscillatory, from Lemma 1.3 (Gyori, 1991), we know the characteristic equation

$$F(\lambda) = \lambda + \sum_{i=1}^{n} q_i e^{-\tau_i \lambda} = 0 \tag{11}$$

has no real root. And because $F(\infty) > 0$, $F(0) = \sum_{i=1}^{n} q_i > 0$, so one j_0 exists at least to make $q_{j_0} > 0$ and $\tau_{j_0} > 0$, or else, $\tau_i = 0\,(i = 1, 2\cdots n)$, $\lambda = -\sum_{i=1}^{n} q_i < 0$ is one negative real root of the characteristic equation $\lambda + \sum_{i=1}^{n} q_i e^{-\tau_i \lambda} = 0$, but that is impossible. The theorem is proved.

Theorem 2.3: Suppose (H_1) and (H_4) are fulfilled, and if the equation (1) has finally positive solution $x(t)$, for the enough big $T_0 \geqslant t_0$, make the set $\Lambda = \{\lambda \geqslant 0 : x'(t) + \lambda x(t-\tau_{j_0}) \leqslant 0, t \geqslant T_0\}$, so the set $\wedge$ is nonempty and bounded.

Prove: Because $x(t)$ is the finally positive solution, according to the conditions of (H_1), (H_4) and Theorem 2.1, we can obtain

$$\lim_{t\to\infty} Q_i(t) \frac{f(x(t-\tau_i))}{x(t-\tau_i)} = q_i\,(i = 1, 2, \cdots n) \tag{12}$$

So, to any appointed positive number $\varepsilon \in (0, 1)$, enough big $T_0 \geqslant t_0$ exists, and when $t \geqslant T_0$, the following inequation exists.

$$Q_i(t) \frac{f(x(t-\tau_i))}{x(t-\tau_i)} \geqslant q_i - \varepsilon\,(i = 1, 2, \cdots n) \tag{13}$$

From the equation (1) and (13), for j_0, the following differential inequation exists.

$$x'(t) + \frac{1}{\theta}(q_{j_0} - \varepsilon) x(t-\tau_{j_0}) \leqslant 0 \tag{14}$$

For the set $\Lambda = \{\lambda \geqslant 0 : x'(t) + \lambda x(t-\tau_{j_0}) \leqslant 0, t \geqslant T_0\}$, from (6) and Lemma 1.1 and Lemma 1.2, we can obtain $A = \frac{4\theta^2}{(q_{j_0}-\varepsilon)^2 \tau_{j_0}^2} > 1$, $\lambda_0 = \frac{\ln A}{\tau_{j_0}} \notin \Lambda$ (where $\theta \geqslant 1$ is certain number appointed). So the set Λ is nonempty and bounded.

Theorem 2.4: Suppose (H_1), (H_2), (H_4) and (H_6) are fulfilled, and if the equation (2) is oscillatory, so the equation (1) is oscillatory.

Prove: Otherwise, the equation(1) has the non-oscillatory solution $x(t)$. Suppose $x(t)$ is the finally positive solution, we can analogously prove the situation of finally negative solution. From the theorem 2.3, the set $\Lambda \equiv \{\lambda \geqslant 0 : x'(t) + \lambda x(t-\tau_{j_0}) \leqslant 0, t \geqslant T_0\}$ is nonempty and bounded.

Because the equation (2) is oscillatory, from Lemma 1.3, we can obtain the characteristic equation

$$F(\lambda) = \lambda + \sum_{i=1}^{n} q_i e^{-\tau_i \lambda} = 0 \tag{15}$$

has not real root. Suppose $K = \min_{\lambda \in R} F(\lambda)$, so the inequation exists.

$$\lambda + \sum_{i=1}^{n} q_i e^{-\tau_i \lambda} \geqslant K \tag{16}$$

Because the set Λ is nonempty, and suppose $\lambda_0 \in \Lambda$ and $\phi(t) = e^{\lambda_0 t} x(t)$, we can obtain $\frac{d\phi(t)}{dt} \leqslant 0$. Same to the deduction in the proof of Theorem 2.3, we can prove (13) and (16), and from (14) and (16), we can obtain

$$\begin{aligned} x'(t) + (\lambda_0 + \frac{k}{2})x(t) &= -\sum_{i=1}^{n} Q(t)_i f(x(t-\tau_i)) + (\lambda_0 + \frac{k}{2})x(t) \\ &\leqslant -\sum_{i=1}^{n} (q_i - \varepsilon)x(t-\tau_i) + (\lambda_0 + \frac{k}{2})x(t) \\ &\leqslant \phi(t)e^{-\lambda_0 t}\big[-\sum_{i=1}^{n}(q_i - \varepsilon)e^{\lambda_0 \tau_i} + (\lambda_0 + \frac{k}{2})\big] \\ &\leqslant \phi(t)e^{-\lambda_0 t}\big[-\lambda_0 - k + \varepsilon \sum_{i=1}^{n} e^{\lambda_0 \tau_i} + \lambda_0 + \frac{k}{2}\big] \\ &\leqslant \phi(t)e^{-\lambda_0 t}\big[\varepsilon \sum_{i=0}^{n} e^{\lambda_0 \tau_i} - \frac{k}{2}\big] \end{aligned} \tag{17}$$

When any positive number $\varepsilon \leqslant \frac{k}{2}(\sum_{i=1}^{n} e^{\lambda_0 \tau_i})^{-1}$, $x'(t) + (\lambda_0 + \frac{k}{2}) \leqslant 0$ exists. So, $\lambda_0 + \frac{k}{2} \in \Lambda$, and from the induction, we can deduce that when n is the enough big positive number, $\lambda_0 + \frac{K}{2} n \in \Lambda$ exists, so the set Λ is the unbounded set, which is contrary the the condition that the set Λ is bounded. So the theorem is proved.

Theorem 2.5: Suppose (H_1), (H_2), (H_3), (H_4), (H_5) and (H_6) are fulfilled, and if the equation (1) is oscillatory, so the equation (2) is oscillatory.

Prove: Otherwise, the equation (2) is non-oscillatory. From Lemma 1.3, we know the characteristic equation $F(\lambda) \equiv \lambda + \sum_{i=1}^{n} q_i e^{-\tau_i \lambda} = 0$ has real root u, and $u < 0$. If $\tau = \max_{1 \leqslant i \leqslant n} \{\tau_i\}$, X is the Banach space which is composed by the collectivity of bounded continuous function with supremum norm in $[t_0 - \tau, \infty]$, M in X is the set composed by the function $x(t)$ which could fulfill following characters.

(1) When $t \geqslant t_0$, $x(t)$ is non-increasing, and when $t \in [t_0 - \tau, t_0]$, $x(t) = x_0 \exp(u(t - t_0))$.

(2) When $t \geqslant t_0$, $x_0 \exp(u(t - t_0)) \leqslant x(t) \leqslant x_0 \leqslant \sigma \exp(u\tau)$.

(3) When $t \geqslant t_o$, $x(t - \tau_j) \leqslant x(t) \exp(-u\tau_j)\,(j = 1, 2 \cdots n)$.

Define the mapping $(Tx)(t)$ in M as follows.

$$(Tx)(t) = \begin{cases} x_0 \exp(u(t - t_0)), & t \in [t_0 - \tau, t_0] \\ x_0 \exp(-\sum_{i=1}^{n} \int_{t_o}^{t} \frac{Q_i(s) f(x(s-\tau_i))}{x(s)} ds), & t \in [t_0, \infty). \end{cases}$$

Next, we will use Lemma 1.4 (Schauder fixed point theorem) to prove that the fixed point exists in T on M. Obviously, $(Tx)(t)$is the continuously monotonically decreasing function, and $(Tx)(t) \leqslant x_0$.

When $t \geqslant t_0$, we can obtain the following inequations.

$$\begin{aligned}
(Tx)(t) &= x_0 \exp(-\sum_{i=1}^{n} \int_{t_0}^{t} \frac{Q_i(s) f(x(s-\tau_i))}{x(s)} ds) \\
&\geqslant x_0 \exp\big(-\sum_{i=1}^{n} q_i \int_{t_0}^{t} \frac{f(x(s-\tau_i))}{x(s-\tau_i)} \frac{x(s-\tau_i)}{x(s)} ds\big) \\
&\geqslant x_0 \exp\big(-\sum_{i=1}^{n} q_i \int_{t_0}^{t} \frac{x(s-\tau_i)}{x(s)} ds\big) \\
&\geqslant x_0 \exp\big(-\sum_{i=1}^{n} q_i \exp(-u\tau_i) \int_{t_0}^{t} ds\big) \\
&\geqslant x_0 \exp(-(t-t_0) \sum_{i=1}^{n} q_i \exp(-u\tau_i)) \\
&= x_0 \exp(u(t-t_0)) \qquad (18)
\end{aligned}$$

$$\begin{aligned}
(Tx)(t-\tau_j) &= x_0 \exp(-\sum_{i=1}^{n} \int_{t_0}^{t-\tau_j} \frac{Q_i(s) f(x(s-\tau_i))}{x(s)} ds) \\
&= (Tx)(t) \exp\big(\sum_{i=1}^{n} q_i \int_{t-\tau_j}^{t} \frac{f(x(s-\tau_i))}{x(s-\tau_i)} \frac{x(s-\tau_i)}{x(s)} ds\big) \\
&\leqslant (Tx)(t) \exp\big(\sum_{i=1}^{n} q_i \int_{t-\tau_j}^{t} \frac{x(s-\tau_i)}{x(s)} ds\big) \\
&\leqslant (Tx)(t) \exp\big(\sum_{i=1}^{n} q_i \int_{t-\tau_j}^{t} \exp(-u\tau_i) ds\big) \\
&\leqslant (Tx)(t) \exp\big(\tau_j \sum_{i=1}^{n} q_i \exp(-u\tau_i)\big) \\
&= (Tx)(t) \exp(-u\tau_j) \qquad (19)
\end{aligned}$$

From (18) and (19), we can obtain $(Tx)(t) \in M$, and the set M is the closed convex nonempty set. Next, we prove the M is relatively compact subset of X, and we only need to prove $(Tx)(t)$ is equicontinuous, i.e. $\frac{d(Tx)(t)}{dt}$ is uniformly bounded. In fact,

$$\Big|\frac{d(Tx)(t)}{dt}\Big| \leqslant x_0 \sum_{i=1}^{n} \frac{Q_i(t) f(x(t-\tau_i))}{x(t)} \leqslant x_0 \sum_{i=1}^{n} q_i \frac{x(t-\tau_i)}{x(t)} \leqslant x_0 \sum_{i=1}^{n} q_i \exp(-u\tau_i) = -x_0 u$$

So, $\frac{d(Tx)(t)}{dt}$ is uniformly bounded.

From above proofs, we can see that the mapping $(Tx)(t)$ from M to M fulfills the condition of Schauder fixed point theorem, so the fixed point $x(t)$ exists and $(Tx)(t) = x(t)$, and $x(t) > 0$ fulfills the equation (1), i.e. the equation (1) has finally positive solution, which is contrary with the condition that the equation (1) is oscillatory. The theorem is proved. From Theorem 2.4 and Theorem 2.5, we can obtain following deductions.

Deduction 2.1: Under the conditions of (H_1), (H_2), (H_3),(H_4), (H_5) and (H_6), the sufficient and necessary condition that the differential equation (1) is oscillatory is the differential equation (2) is oscillatory.

Example: We know the nonlinear functional differential equation

$$x'(t) + Q_1(t) f(x(t - \frac{\pi}{4})) + Q_2(t) f(x(t - \frac{3\pi}{4})) = 0 \qquad (20)$$

Where, $Q_1(t) = \frac{t^2}{\sqrt{2}t^2+1} e^{-\frac{\pi}{4}}$, $Q_2 = \frac{t^2+\sqrt{2}t}{\sqrt{2}t^2+t+1} e^{-\frac{3\pi}{4}}$, $f(u) = \arctan u$, so the equation (20) is oscillatory.

Prove: It is easily to prove the function $f(u)$ fulfills the conditions of (H_1), (H_2) and (H_3),

$$q_1 = \lim_{t\to\infty} Q_1(t) = \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}, q_2 = \lim_{t\to\infty} Q_2(t) = \frac{1}{\sqrt{2}}e^{-\frac{3\pi}{4}}$$

$$Q_1(t) \leqslant \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}, Q_2(t) \leqslant \frac{1}{\sqrt{2}}e^{-\frac{3\pi}{4}}$$

i.e. (H_4), (H_5) and (H_6) are fulfilled, and the corresponding linear delay functional differential equation with constant coefficient is

$$x'(t) + \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}x(t-\frac{\pi}{4}) + \frac{1}{\sqrt{2}}e^{-\frac{3\pi}{4}}x(t-\frac{3\pi}{4}) = 0 \tag{21}$$

Through computation, we can obtain

$$\sum_{i=1}^{2} q_i\tau_i = \frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}} \times \frac{\pi}{4} + \frac{1}{\sqrt{2}}e^{-\frac{3\pi}{4}} \times \frac{3\pi}{4} = 0.411 > \frac{1}{e} \tag{22}$$

So the equation (21) is oscillatory, and from the deduction 2.1, we can obtain the equation (20) is oscillatory.

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Two-Person Non Zero-Sum Bimatrix Game with Fuzzy Payoffs and Its Equilibrium Strategy

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Abstract

In this paper, we consider fuzzy bimatrix games with fuzzy payoffs. Based on fuzzy max order, for such games, we define three kinds of concepts of minim ax equilibrium strategies. Some basic results obtained.

Keywords: Fuzzy bimatrix games, Fuzzy payoffs, Minimax equilibrium strategy

1. Introduction

Since seminal works by Neumann-Morgenstern(J.VonNeunann, 1994) and Nash(J.F.Nash., 1950 & J.F.Nash., 1951), Game theory has played an important role in the fields of decision making theory such as economics, management, and operations research, etc. When we apply the game theory to model some practical problems which we encounter in real situations, we have to know the values of payoffs exactly. However, it is difficult to know the exact values of payoffs and we could only know the values of payoffs approximately. In such situations, it is useful to model the problems as games with fuzzy payoffs. In this case, since the expected payoffs of the game should be fuzzy-valued, there are no concepts of equilibrium strategies to be accepted widely. So, it is an important task to define the concept of equilibrium strategies and investigate their properties. In this paper, we consider fuzzy bimatrix games, namely, the games where the number of players are two and fuzzy payoffs. For such a game, we shall define three kinds of concepts of minim ax equilibrium strategies.

2. Preliminary

Let R^n be n-dimensional Euclidean space, and $x = (x_1, x_2, \cdots, x_n)^T \in R^n$ be any vector, where $x_i \in R$, $i = 1, 2, \cdots, n$ and T denotes the transpose of the vector. For any two vectors $x, y \in R^n$, we write $x \geq y$ if $x_i \geq y_i$, $i = 1, 2, \cdots, n$, $x > y$ and $x \neq y$, respectively.

Definition 2.1 Let m be any real number and let h be any positive number. A fuzzy number $\tilde{a}$ whose membership function is given by the following formula

$$\mu_{\tilde{a}}(x) = \begin{cases} 1 - |\frac{x-m}{h}| & x \in [m-h, m+h] \\ 0 & x \notin [m-h, m+h] \end{cases}$$

is called a symmetric triangular fuzzy number, and we denote the set of all symmetric triangular fuzzy numbers by F_T.

Real numbers m and h are called the center and the deviation parameter of $\tilde{a}$, respectively. Since any symmetric triangular fuzzy number $\tilde{a}$ is characterized by the center m and the deviation parameter h of $\tilde{a}$ we denote the symmetric triangular fuzzy number $\tilde{a}$ by $\tilde{a} \equiv (m, h)_T$.

Let $\tilde{a}$ be any fuzzy number and let $\alpha \in [0, 1]$ be any real number. The set $[\tilde{a}]^\alpha \equiv x \in R|\mu_{\tilde{a}}(x) \geq \alpha$ is called the α-level set of $\tilde{a}$. For $\alpha = 0$, we set $[\tilde{a}]^0 = cl\{x \in R|\mu_{\tilde{a}}(x) > 0\}$. where cal denotes the closure of sets.

Since the set $[\tilde{a}]^{\alpha}$ is a closed interval for each $\alpha \in [0, 1]$, we denote the α-level set of a $\tilde{a}$ by $[\alpha_{\alpha}^{L}, \alpha_{\alpha}^{R}]$, where $a_{\alpha}^{L} \equiv \inf[\tilde{a}]^{\alpha}$, $a_{\alpha}^{R} = \sup[\tilde{a}]^{\alpha}$.

For any two fuzzy numbers $\tilde{a}, \tilde{b} \in F_T$, we introduce three kinds of binary relations.

Definition 2.2 For any symmetric triangular fuzzy numbers $\tilde{a}$, $\tilde{b} \in F_T$, we write

$$\begin{aligned} \tilde{a} = \tilde{b}, \quad \text{if } (a_{\alpha}^{L}, a_{\alpha}^{R})^T &= (b_{\alpha}^{L}, b_{\alpha}^{R})^T \quad \forall \alpha \in [0, 1] \\ \tilde{a} \succeq \tilde{b}, \quad \text{if } (a_{\alpha}^{L}, a_{\alpha}^{R})^T &\geq (b_{\alpha}^{L}, b_{\alpha}^{R})^T \quad \forall \alpha \in [0, 1] \\ \tilde{a} \succ \tilde{b}, \quad \text{if } (a_{\alpha}^{L}, a_{\alpha}^{R})^T &> (b_{\alpha}^{L}, b_{\alpha}^{R})^T \quad \forall \alpha \in [0, 1] \end{aligned}$$

We call binary relations $\succeq$, and $\succ$ a strict fuzzy max order and a strong fuzzy max order, respectively.

Theorem 2.1 let $\tilde{a} = (a, \alpha)_T$, and $\tilde{b} = (b, \beta)_T$ be any symmetric triangular fuzzy numbers. Then, it holds that

$$\begin{aligned} \tilde{a} \succeq \tilde{b} \quad If \quad |\alpha - \beta| \leq a - b \\ \tilde{a} \succ \tilde{b} \quad If \quad |\alpha - \beta| < a - b \end{aligned}$$

3. Two-person zero-sum game with fuzzy payoffs and its equilibrium strategy

Let I, J denote players and let $M = \{1, 2, \cdots, m\}$, $N = \{1, 2, \cdots, n\}$ be the sets of all pure strategies available for player I, and J, respectively. We denote the sets of all mixed strategies available for player I, and J by

$$X = \{(x_1, x_2, \cdots, x_m) \in R_{+}^{m} | x_i \geq 0, \ i = 1, 2, \cdots, m, \ \sum_{i=1}^{m} x_i = 1\}$$

$$Y = \{(y_1, y_2, \cdots, y_n) \in R_{+}^{n} | y_j \geq 0, \ j = 1, 2, \cdots, n, \ \sum_{j=1}^{n} y_j = 1\}$$

By $\tilde{a}_{ij} = (a_{ij}, h_{ij})_T$, and $\tilde{b}_{ij} = (b_{ij}, h'_{ij})_T$, we denote the payoff that player I receives and J loses when player I plays the pure strategy i and player J plays the pure strategy i. Then we have the fuzzy payoff matrix $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$, $\tilde{B} = (\tilde{b}_{ij})_{m \times n}$, $A = (a_{ij})_{m \times n}$, $H = (h_{ij})_{m \times n}$, $B = (b_{ij})_{m \times n}$, $H' = (h'_{ij})_{m \times n}$, we call this game two-person fuzzy zero bimatrix game, and we denote it by

$$\tilde{\Gamma} = < \{I, J\}, X, Y, \tilde{A}, \tilde{B} >$$

Definition 3.1 A point $(x^*, y^*) \in X \times Y$ is said to be a minim ax equilibrium strategies to Game $\tilde{\Gamma}$ if it holds that

(1) $x^T \tilde{A} y^* \preceq x^{*T} \tilde{A} y^* \quad \forall x \in X$

(2) $x^{*T} \tilde{B} y \preceq x^{*T} \tilde{B} y^* \quad \forall y \in Y$

Definition 3.2 A point $(x^*, y^*) \in X \times Y$ us said to be a non-dominated minim ax equilibrium strategy to Game $\tilde{\Gamma}$ if it holds that

(1) There exist no $x \in X$ such that $x^{*T} \tilde{A} y^* \preceq x^T \tilde{A} y^*$

(2) There exist no $y \in Y$ such that $x^{*T} \tilde{B} y^* \preceq x^{*T} \tilde{B} y$

Definition 3.3 A point $(x^*, y^*) \in X \times Y$ is said to be a weak non-dominated minim ax equilibrium strategy to Game $\tilde{\Gamma}$ if it holds that

(1) There exist no $x \in X$ such that $x^{*T} \tilde{A} y^* \prec x^T \tilde{A} y^*$

(2) There exist no $y \in Y$ such that $x^{*T} \tilde{B} y^* \prec x^{*T} \tilde{B} y$

By definition, it is obvious that the following relationship holds among these definitions.

(1) If a strategy (x^*, y^*) is a minim ax equilibrium strategy to Game $\tilde{\Gamma}$, it is a non-dominated minim ax equilibrium strategy.

(2)If a strategy (x^*, y^*) is a non-dominated minim ax equilibrium strategy to Game $\tilde{\Gamma}$, it is a weak non-dominated minim ax strategy.

Theorem 3.1 In order that a strategy $(x^*, y^*) \in X \times Y$ be a minim ax equilibrium strategy to Game $\tilde{\Gamma}$, it is necessary and sufficient that, for all $x \in X$, $y \in Y$

$$x^T Ay^* + x^T Hy^* \leq x^{*T} Ay^* + x^{*T} Hy^* \tag{1}$$

$$x^T Ay^* - x^T Hy^* \leq x^{*T} Ay^* - x^{*T} Hy^* \tag{2}$$

$$x^{*T} By + x^{*T} H'y \leq x^{*T} By^* + x^{*T} H'y^* \tag{3}$$

$$x^{*T} By + x^{*T} H'y \leq x^{*T} By^* + x^{*T} H'y^* \tag{4}$$

Hold.

Proof. Let $(x^*, y^*) \in X \times Y$ be any minim ax equilibrium strategy to Game $\tilde{\Gamma}$. Then from Theorem 2.1, for all $x \in X$, $y \in Y$ we have

$$|x^{*T} Hy^* - x^T Hy^*| \leq x^{*T} Ay^* - x^T Ay^* \tag{5}$$

$$|x^{*T} H'y^* - x^{*T} H'y| \leq x^{*T} By^* - x^{*T} By \tag{6}$$

By expanding and rearranging (5), we have

$$x^T Ay^* + x^T Hy^* \leq x^{*T} Ay^* + x^{*T} Hy^* \tag{7}$$

$$x^T Ay^* - x^T Hy^* \leq x^{*T} Ay^* - x^{*T} Hy^* \tag{8}$$

On the other hand, by expanding and rearranging (6), we have

$$x^{*T} By + x^{*T} H'y \leq x^{*T} By^* + x^{*T} H'y^* \tag{9}$$

$$x^{*T} By - x^{*T} H'y \leq x^{*T} By^* - x^{*T} H'y^* \tag{10}$$

From (7) to (10), we have (1), (2), (3), and (4).

Since

$$x^{*T} A_0^R y^* = x^{*T}(A + H)y^* = x^{*T} Ay^* + x^{*T} Hy^*$$

$$x^{*T} A_0^L y^* = x^{*T}(A - H)y^* = x^{*T} Ay^* - x^{*T} Hy^*$$

$$x^{*T} B_0^R y^* = x^{*T}(B + H')y^* = x^{*T} By^* + x^{*T} H'y^*$$

$$x^{*T} B_0^L y^* = x^{*T}(B - H')y^* = x^{*T} By^* - x^{*T} H'y^*$$

Theorem 3.1 shows that players I, J faces a pair of two-person zero-sum games with crisp payoffs $\Gamma_1 =< \{I, J\}, X, Y, A_0^L, B_0^L >$, $\Gamma_2 =< \{I, J\}, X, Y, A_0^R, B_0^R >$. By setting $x^T A\, y = (x^T A_0^L y, x^T A_0^R y)^T$, $x^T B\, y = (x^T B_0^L y, x^T B_0^R y)^T$, for each $x \in X$, $y \in Y$, form (1), (2), (3),(4), we have $x^T A\, y^* \leq x^{*T} A\, y^*$, and $x^{*T} By \leq x^{*T} By^*$.

Theorem 3.2 In order that a strategy $(x^*, y^*) \in X \times Y$ be a non-dominated minim ax equilibrium strategy to Game $\tilde{\Gamma}$, it is necessary and sufficient that the following conditions holds:

(1) There is no $x \in Y$ such that $x^{*T} Ay^* \leq x^T Ay^*$ holds.

(2) There is no $y \in Y$ such that $x^{*T} By^* \leq x^{*T} By$ holds.

Proof. Let $(x^*, y^*) \in X \times Y$ be a non-dominated minim ax equilibrium strategy to Game $\tilde{\Gamma}$. First, we suppose that there exists a strategy $\overline{x} \in X$ such that $x^{*T} Ay^* \leq \overline{x}^T Ay^*$ holds. By definition, we have

$$(x^{*T}(A - H)y^*, x^{*T}(A + H)y^*)^T \leq (\overline{x}^T(A - H)y^*, \overline{x}^T(A + H)y^*)^T \tag{11}$$

By rearranging (11), we have

$$(x^{*T} Ay^* - \overline{x}^T Ay^*, x^{*T} Ay^* - \overline{x} Ay^*)^T \leq (x^{*T} Hy^* - \overline{x} Hy^*, \overline{x}^T Hy^* - x^{*T} Hy^*)^T$$

this implies that $x^{*T} Ay^* < \overline{x}^T Ay^*$ holds. Therefore, for all $\alpha \in [0, 1]$, we have

$$(x^{*T}(A - (1 - \alpha)H)y^*, x^{*T}(A + (1 - \alpha)H)y^*)^T \leq (\overline{x}^T(A - (1 - \alpha)H)y^*, \overline{x}^T(A + (1 - \alpha)Hy^*)y^*)^T$$

this implies that $x^{*T}\tilde{A}y^* \preceq \overline{x}^T\tilde{A}y^*$ holds. This is a contradiction.

Next, we suppose that there exists a strategy $\overline{y} \in Y$ such that $x^{*T}By^* \leq x^{*T}B\overline{y}$ holds. By definition, we have

$$(x^{*T}(B-H')y^*,\ x^{*T}(B+H')y^*)^T \leq (x^{*T}(B-H')\overline{y},\ x^{*T}(B+H')\overline{y})^T \tag{12}$$

By rearranging (12), we have

$$(x^{*T}By^* - x^{*T}B\overline{y},\ x^{*T}By^* - x^{*T}B\overline{y})^T \leq (x^{*T}H'y^* - x^{*T}H'\overline{y},\ x^{*T}H'\overline{y} - x^{*T}H'y^*)^T$$

this implies that $x^{*T}By^* \leq x^{*T}B\overline{y}$ holds. Therefore, for all $\alpha \in [0,\ 1]$, we have

$$(x^{*T}(B-(1-\alpha)H')y^*,\ x^{*T}(B+(1-\alpha)H')y^*)^T \leq (x^{*T}(B-(1-\alpha)H')\overline{y},\ x^{*T}(B+(1-\alpha)H')\overline{y})^T$$

this implies that $x^{*T}\mathscr{B}y^* \preceq x^{*T}\mathscr{B}\overline{y}$. This is a contradiction.

Conversely, let $(x^*,\ y^*)$ be any strategy to Game $\tilde{\Gamma}$ such that conditions (1) and (2) hold.

First, we suppose that there exists strategy $\overline{x} \in X$ such that $x^{*T}\tilde{A}y^* \preceq \overline{x}^T\tilde{A}y^*$ holds. Then, by definition 2.2, we have

$$(x^{*T}(A-H)y^*,\ x^{*T}(A+H)y^*)^T \leq (\overline{x}^T(A-H)y^*,\ \overline{x}^T(A+H)y^*)^T$$

this is a contradiction.

Next, we suppose that there exists a strategy $\overline{y} \in Y$ such that $x^{*T}\tilde{B}y^* \preceq x^{*T}\tilde{B}\overline{y}$ holds. Then, by Definition 2.2, we have

$$(x^{*T}(B-H')y^*,\ x^{*T}(B+H')y^*)^T \leq (x^{*T}(B-H')\overline{y},\ x^{*T}(B+H')\overline{y})^T$$

this is a contradiction.

By a similar way, we have the following theorem.

Theorem 3.3 In order that a strategy $(x^*,\ y^*) \in X \times Y$ be a weak non-dominated minim ax equilibrium strategy to Game $\tilde{\Gamma}$, it is necessary and sufficient that the following conditions hold:

(1) there is no $x \in X$ such that $x^{*T}Ay^* < x^TAy^*$ holds.

(2) there is no $y \in Y$ such that $x^{*T}By^* < x^{*T}By$holds

Theorem 3.4 we have the following conclusion:

(1) There is a non-dominated minim ax equilibrium strategy to Game $\tilde{\Gamma}$ at least.

(2) There is a weak non-dominated minim ax equilibrium strategy to Game $\tilde{\Gamma}$ at least.

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Some Properties of Relative Regularity and Compactness

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Abstract

In this paper, some relative topological properties were studied, especially including relative regular and relative countable1-paracompact and the property of countable1-paracompact under the perfect mapping was also discussed.

Keywords: X is regular, X is lindeloff, Y is regular in X, Y is countable1-paracompact in X

1. Introduction

Relative topological properties are extension of classic topological invariants.In 1989,the relative topological properties were discussed by A.V.Archangel'skii and H.M.M.Genecli in (A.V.Arhangel'skii. 1996), and A.V.Arhangel'skii gave the first systematic text on relative topological properties in 1996. In recent years, some further new results of the relative topology were obtained respective by A.V.Arhangel'skii, J.Tartir and W.Just, O.Pavlov and M.Matveer, I.Yaschenko, V.V.Tkachuk, M.G.Tkachenko and R.G.Wilson, etc.

In my paper, some relative topological properties were studied and some results were given.

2. The properties of relative topology

X is a space, $Y \subset X$, the concept of X is regular, lindeloff were introduced in (R.Engelking, 1997) and the concept of Y is regular in X and the definition of countable1-paracompact was respectively introduced in (A.V.Arhangel'skii. 1996) and (Gartside P and Aneirin G, 2000) . In this part, some properties of them were discussed, and I gave two results.

Definition 2.1 X is regular: If for each y of X and each closed subset p of X, such that $y \notin p$, there are disjoint open subsets u and v of X, such that: $y \in u$ and $p \subset v$.

Definition 2.2 X is lindeloff: If for each open covering A of X, there exists an countable open subcovering of A.

Definition 2.3 Y is regular in X: If for each y of Y and each closed subset p of X, such that $y \notin p$, there are disjoint open subsets u and v of X, such that: $y \in u$ and $p \cap Y \subset v$.

Definition 2.4 Y is strongly regular in X: If for each x of X and each closed subset p of X, such that $y \notin p$, there are disjoint open subsets u and v of X, such that: $x \in u$ and $p \cap Y \subset v$.

Definition 2.5 Y is countable1-paracompact in X: If for each countable open covering A of X, there exists an open family covering $\Re$ of X, such that: $\Re$ refines A, $\cup\Re = X$ and $\Re$ is locally finite at each y of Y.

Theory 2.6 If X is regular. Then, Y is regular in X.

Proof. Let y is a arbitrary points of Y and an arbitrary closed subset p of X, such that $y \notin p$. Since X is regular, so there exist two disjoint open subsets u_1 and v_1 in X, such that: $y_1 \in u_1$ and $y_2 \in v_1$. Obviously, $y_1 \in u_1$ and $p \cap Y \subset v_1$. This is, Y is regular in X.

Theory 2.7 If X is regular and Y is Lindloff. Then, Y is in countable1-paracompact X.

Proof. Let $A = \{u_s : s \in S\}$ is an countable open covering of X, and y is an arbitrary point of Y. Then there is an $u \in A$, such that: $v \in u$. Since X is regular, so there exists an open set v_y of X, such that: $y \in v_y \subset \overline{v}_y \subset u$. So $X \setminus v_y$ is closed in X and which does not contain y. By the Theory 2.6, Y is regular in X, so, there are two disjiont open sets t_y and w_y, such that: $y \in t_y$ and $X \setminus v_y \cap Y \subset w_y$. It is obvious that $\bar{t}_y \cap (X \setminus v_y) \cap Y = \emptyset$, so we can get: $t_y \cap Y \subset v_y$. Then for each arbitrary points y of Y, there is an open set t_y of X, such that: $y \in t_y \subset v_y$, $t_y \cap Y \subset v_y \cap Y$. This is, $\Re = \{t_y : y \in Y\}$ is an countable open covering of Y. And since Y is Lindloff, so there exist an countable subcovering $\Re_1 = \{t_{y_j} : j = 1, 2\cdots\}$. We may also assume that $v'_{y_j} = v_{y_i} \setminus \bigcup_{i=1}^{j-1} \bar{t}_{y_i}$: for each j. It is obvious that: $\Re_2 = \{v_{y_j} : j = 1, 2\cdots\}$ is an open family of X and which refines A and for each $y \in Y$, since $Y \subset \bigcup_{j=1}^{\infty} t_{y_j}$.So there exist the most smallest integer j, such that: $y \in \bar{t}_{y_j}$, so, $y \in v'_{y_j}$, this is $Y \subset \bigcup_{j=1}^{\infty} v'_{y_j}$ Also since $Y \subset \bigcup_{j=1}^{\infty} t_{y_j}$, then there t_{y_i} which contains y. And since $v'_{y_j} = v_{y_j} \setminus \bigcup_{i=1}^{j-1} \bar{t}_{y_i}$, so when $j > i$ $t_{y_j} \cap v'_{y_j} = \emptyset$, then, $\Re_2$ is locally finite at each y of Y. This is, Y is in countable1-paracompact X.

3. The Property of Relative Compactness under the Perfect Mapping.

Some properties of topological spaces under the perfect mapping were given in (R.Engelking, 1997). In this part, I studied the property of countable1-paracompact under the perfect mapping, and gave a result about it.

Definition 3.1 f: $X \to Y$ is a perfect mapping: If f is a continuous mapping which is closed and for each $y \in Y$, the fiber $f^{-1}(y)$ is compact subset of X.

Theorem 3.2 Let f: $X \to Y$ is a perfect countable 1 mapping. If Y_1 is countable 11-paracompact in Y. Then, $f^{-1}(Y_1)$ is countable 1-paracompact in X.

Proof. Let $A = \{u_s : s \in S\}$ is an countable open covering of X. Since f is a perfect mapping, so for each $y \in Y$, the fiber $f^{-1}(y)$ is a compact subset of X. Thus, there exists a finite subset of $s(y)$ S, such that: $f^{-1}(y) \subset \bigcup_{s\in S(y)} u_s = u_{y(s)}$. Since f is a perfect mapping, by the TH1.4.13 in (R.Engelking, 1997), there exists an open neighborhood $w_{y(s)}$ of y, such that: $f^{-1}(y) \subset f^{-1}(w_{y(s)}) \subset u_{y(s)}$. We may also assume that: $f^{-1}(w_{y(s)})$ is $v_{y(s)}$. That is $v_{y(s)} = f^{-1}(w_{y(s)})$. Then, it is obvious that: $v_{y(s)}$ is open in X and such that: $f^{-1}(y) \subset v_{y(s)} = f^{-1}(f(v_{y(s)})) \subset u_{y(s)}$ and is an open subset of Y. Obviously, $\Re_1 = \{f(v_{y(s)}) : y \in Y\}$ is an open covering of Y. Since Y_1 is nearly1-paracompact in Y, so there exists an open family covering $\Re_2 = \{v_a : a \in A\}$ of Y by open subsets of Y, such that: $\Re_2$ refines $\Re_1$, $\cup\Re_2 = Y$ and $\Re_2$ is locally finite at each $y \in Y$. We may also assume that $f(v_{y(s)})$ which contains v_a is $f(y_a(s))$. Since f is perfect mapping, thus, $\Re_3 = \{f^{-1}(v_a), a \in A\}$ is an open family of X and locally Finite each $x \in f^{-1}(Y_1)$. Obviously. Let $\Re = \{f^{-1}(v_a) \cap u_s \cap Y : a \in A, s \in S_a(y)\}$. Then, $\Re$ is an open family covering of Y and such that $\Re$ refines A, $\cup\Re = X$ and $\Re$ is locally finite at each $x \in f^{-1}(Y_1)$. That is $f^{-1}(Y_s)$ is countable1-paracompact in X.

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Some Properties and Determination of Complex Metapositive Subdefinite Matrix

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Abstract

In this article, we introduced the concept of complex metapositve subdefinite matrix, studied its properties and subeigenvalue, put forward some necessary and sufficient conditions that the complex matrix was the complex metapositive subdefinite matrix, and obtained the positive definiteness of the metapositve matrix which is subcongruent to the complex metapositive subdefinite matrix.

Keywords: Complex metapositive subdefinite matrix, Subtransposed matrix, Subeigenvalue, Subcongruence, Necessary and sufficient condition

1. Introduction and symbols

The positive definite matrix is always the important topic for the research of mathematic theory. However, traditional researches about the real symmetric positive definite matrix have not fulfilled the demands of theoretic development and applied practice, so numerous literatures extensively studied the metapositve definite matrix, the subpositive definite matrix, the complex positive definite matrix, the metapositive subdefinite matrix, the complex metapositive definite matrix and complex subpositive definite matrix (Guo, 2005, P.135-136, Jia, 1995, P.40-41, Johnson, 1970, P.259-264, Shen, 2002, P.186-192, Tong, 1984, P.801-810, Tu, 1990, P.462-471, Tu, 1991, P.91-102, Yang, 2000, 134-138, Zhan, 2003, P.191-196), but there are few researches for the complex metapositve subdefinite matrix. In this article, we mainly study the properties of the complex metapositive subdefinite matrix, put forward some criterions of the complex metapositive subdefinite, which is the extension and deepening for the results of metapositve definite matrix, subpositive definite matrix, complex positive definite matrix, metapositive subdefinite matrix, complex metapositive definite matrix and complex subpositive definite matrix.

In the article, $C^{n\times n}$ denotes the n order matrix set in the complex number field, $C^{n\times 1}$ denotes n-dimensional complex vector, A^T denotes the transpose matrix of A, $\overline{A^T}$ denotes the conjugate transpose matrix of A, A^- denotes the subtransposed matrix, and $\overline{A^-}$ denotes the sub-conjugate transpose matrix of A.

2. Basic definitions

In the article, we extend some concepts in references to the complex matrix and obtain some new concepts about complex metapositive subdefinite matrix.

Definition 1 (Yang, 2000, P.134-138): Suppose $A = (a_{ij})$ is $m \times n$ complex matrix, $B = (b_{ij})(b_{ij} = a_{m-j+1,n-i+1})$ is $n \times m$ complex matrix, and B is the complex subtransposed matrix of A, i.e. $B = A^-$. And if $A^- = A$, so A is called as the complex symmetric matrix. And if $A^- = -A$, so A is called as the reverse complex symmetric matrix.

Yang's article had given some properties of sub-transposed matrix in the real number field, and we will cite

them in the next part.

$(A^-)^- = A$, $(A+B)^- = A^- + B^-$, $\quad (AB)^- = B^- A^-$

It is easily to show the above properties exist in the complex number field.

Definition 2. The matrix which all elements on the minor diagonal are 1 and other elements are 0 is called the sub-identity matrix, because it has many properties of identity matrix in many aspects, and it is denoted by J.

In the same way, we will cite some properties of the sub-identity matrix in Yang's article in the following part.

(1) $J^- = J$ (2) $J^T = J$ (3) $J^2 = E$ (4) $J^{-1} = J$ (5) $J_n A^- J_m = A^T$ (A is arbitrary $m \times n$ matrix)

Definition 3 (Guo, 2005, P.135-136 & Tu, 1991, P.91-102): Suppose $A \in C^{n\times n}$, $H_+(A) = \frac{A+\overline{A^T}}{2}$ and $G_+(A) = \frac{A-\overline{A^T}}{2}$, so $A = H_+(A) + G_+(A)$, we call $H_+(A)$ and $G_+(A)$ are respectively Hermite branch and reverse Hermite branch.

For $H_+(A)$ and $G_+(A)$, $\overline{H_+(A)^T} = H_+(A)$ and $\overline{G_+(A)^T} = -G_+(A)$ exist obviously.

Definition 4 (Guo, 2005, P.135-136 & Tu, 1991, P.91-102): Suppose $A \in C^{n\times n}$, if $X^T H_+(A)\overline{X} > 0$ to $\forall 0 \neq X \in C^{n\times 1}$, so A is called as the complex metapositive subdefinite matrix.

Definition 5: $A \in C^{n\times n}$, $H_-(A) = \frac{A+\overline{A^-}}{2}$, $G_-(A) = \frac{A-\overline{A^-}}{2}$, so $A = H_-(A) + G_-(A)$, we call $H_-(A)$ and $G_-(A)$ respectively are Hermite sub-branch and reverse Hermite sub-branch of A. For $H_-(A)$ and $G_-(A)$, $\overline{H_-(A)^-} = H_-(A)$ and $\overline{G_-(A)^-} = -G_-(A)$ exist obviously.

Definition 6: Suppose $A \in C^{n\times n}$, if $X^- H_-(A)\overline{X} > 0$ to $\forall 0 \neq X \in C^{n\times 1}$, so A is called as the complex metapositive subdefinite matrix.

Definition 7: Suppose $A \in C^{n\times n}$, the root of n order multinomial $det(\lambda J - A)$ is the subeigenvalue of A.

Because $det(\lambda E - JA) = det(\lambda JJ - JA) = detJ \times det(\lambda J - A) = (-1)^{\frac{n(n-1)}{2}} det(\lambda J - A)$, so λ is the subeigenvalue of A if and only if λ is the eigenvalue of JA.

3. Main results

Theorem 1. The necessary and sufficient condition that n orders complex matrix A is the complex metapositive subdefinite matrix is that JA is complex metapositve subdefinite matrix.

Proof. To $\forall 0 \neq X \in C^{n\times 1}$ and $A \in C^{n\times n}$, $X^- = J^- X^T J = X^T J$, $JA^- = A^T J$ and $A^- = J^{-1} A^T J = JA^T J^T$ exist, so we can obtain the necessary and sufficient condition that A is the complex metapositive subdefinite matrix is $X^- H_-(A)\overline{X} > 0$ to $\forall 0 \neq X \in C^{n\times 1}$, which is equivalent to $X^T J H_-(A)\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$, and is equivalent to $X^T J \frac{A+\overline{A^-}}{2}\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$, and is equivalent to $X^T \frac{JA+\overline{JA^-}}{2}\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$, and is equivalent to $X^T \frac{JA+JJA^T J^T}{2}\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$, and is equivalent to $X^T H_+(JA)\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$. From Definition 4, JA is complex metapositive subdefinite matrix.

Theorem 2. If n orders complex matrix A is the complex metapositive subdefinite matrix, all real parts of eigenvalue of JA are positive.

Proof. From Theorem 1, JA is the complex metapositive subdefinite matrix, and from the properties of complex metapositive subdefinite matrix, the real parts of eigenvalue JA of are positive.

Theorem 3. If A is complex metapositive subdefinite matrix and B is real reverse sub-symmetric matrix ($B^- = -B$, $\overline{B} = B$), so $A + B$ is complex metapositve subdefinite matrix.

Proof. Because A is complex metapositive subdefinite matrix, so JA is complex metapositive subdefinite matrix, i.e. for $\forall 0 \neq X \in C^{n\times 1}$, $X^T \frac{JA+\overline{(JA)^T}}{2}\overline{X} > 0$. And because B is real reverse sub-symmetric matrix, i.e. $B^- = -B$, so $JB^- = -JB$, $JB^- = B^T J$, $B^T J = -JB$, $(JB)^T = -JB$. So for $\forall 0 \neq X \in C^{n\times 1}$, $X^T \frac{J(A+B)+\overline{[J(A+B)]^T}}{2}\overline{X} = X^T \frac{JA+JB+\overline{(JA)^T}+\overline{(JB)^T}}{2}\overline{X} = X^T \frac{JA+\overline{(JA)^T}}{2}\overline{X} + X^T \frac{JB+\overline{(JB)^T}}{2}\overline{X} = X^T \frac{JA+\overline{(JA)^T}}{2}\overline{X} + X^T \frac{JB-\overline{JB}}{2}\overline{X} = X^T \frac{JA+\overline{(JA)^T}}{2}\overline{X} + X^T \frac{JB-JB}{2}\overline{X} = X^T \frac{JA+\overline{(JA)^T}}{2}\overline{X} > 0$.

So, $J(A + B)$ is complex metapositive subdefinite matrix, and according to Theorem 1, $A + B$ is complex metapositive subdefinite matrix.

Theorem 4. If A and B are n orders complex metapositive subdefinite matrixes, so $A+B$ is complex metapositive subdefinite matrix.

Proof: If A and B are n orders complex metapositive subdefinite matrixes, so JA and JB are complex metapositive definite matrixes, and $J(A+B) = JA + JB$ is complex metapositive subdefinite matrix, so $A+B$ is complex metapositive subdefinite matrix.

Theorem 5. Suppose A is n orders complex metapositive subdefinite matrix, so the real parts of subeigenvalue of A are positive.

Proof. From Theorem 2 and Definition 7, the conclusion comes into existence obviously.

Theorem 6. Suppose A is n orders complex metapositive subdefinite matrix and B is n orders real symmetric positive definite matrix, so the real parts of subeigenvalue of AB are positive.

Proof. Because B is real symmetric positive definite matrix, so the real symmetric positive definite matrix P exists to make $B = P^2$, and $PJABP^{-1} = PJAP = P^T JAP$, so $P^T JAP$ is similar with JAB, and they have same eigenvalues. And because JA is complex metapositive subdefinite matrix, so $P^T JAP$ is complex subpositive definite matrix, and its real part of the eigenvalue is positive, so the real parts of eigenvalue of JAB is positive, i.e. the real part of subeigenvalue of AB is positive.

From above demonstrations, we can easily deduce following theorems.

Theorem 7. Suppose A is complex metapositive subdefinite matrix, so A is reversible.

Theorem 8. Suppose $A \in C^{n\times n}$, so following propositions are equivalent.

(1) A is complex metapositive subdefinite matrix.

(2) $X^- H_-(A)\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$.

(3) A^- is complex metapositive subdefinite matrix.

(4) $\overline{A^-}$ is complex metapositive subdefinite matrix.

(5) A^{-1} is complex metapositive subdefinite matrix.

(6) To arbitrary positive real number k, kA is complex metapositive subdefinite matrix.

Proof. (1)$\Leftrightarrow$ (2) can be directly proved by the definition. For (2)$\Leftrightarrow$ (3), because $X^- H_-(A)\overline{X} > 0$ and $X^- \frac{A+\overline{A^-}}{2}\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$, implement conjugation for both sides of the equipment, so $\overline{X^-}\frac{A^-+\overline{(A^-)^-}}{2}X > 0$, if $\overline{X} = Y$, $Y^- \frac{A^-+\overline{(A^-)^-}}{2}\overline{Y} > 0$, i.e. $Y^- H_-(A^-)\overline{Y} > 0$, so A^- is complex metapositive subdefinite matrix, vice versa.

For (3)$\Leftrightarrow$ (4), implement conjugation to A^-, it can be proved. For (2)$\Leftrightarrow$ (5), because $X^- H_-(A)\overline{X} > 0$, and $X^- \frac{A+\overline{A^-}}{2}\overline{X} > 0$ for $\forall 0 \neq X \in C^{n\times 1}$ implement matrix inverse to both sides of the equation, we can obtain $\overline{X}^{-1}\frac{A^{-1}+\overline{(A^{-1})^{-1}}}{2}(X^{-1})^{-1} > 0$, and if $Y = (\overline{X}^{-1})^-$, so $Y^- \frac{A^{-1}+\overline{(A^{-1})^-}}{2}\overline{Y} > 0$. So A^- is complex metapositive subdefinite matrix. (2) $\Leftrightarrow$ (6) comes into existence obviously.

Theorem 9: The necessary and sufficient condition that n orders complex matrix A is complex metapositive subdefinite matrix is that the all sequential principal minor of matrix A are complex metapositive subdefinite matrixes.

Proof. For the "necessity", suppose A_1 is the i'th sequential principal minor, and $A = \begin{pmatrix} A_3 & A_4 \\ A_1 & A_2 \end{pmatrix}$, $i = 1, 2, \cdots, n$, and take nonzero vector X_1, construct n-dimensional column vector $X = \begin{pmatrix} X_1 \\ 0 \end{pmatrix}$, so following conclusion comes into existence.

$$X^- H_-(A)\overline{X} = (0 \quad X_1^-)\frac{\begin{pmatrix} A_3 & A_4 \\ A_1 & A_2 \end{pmatrix} + \overline{\begin{pmatrix} A_2^- & A_4^- \\ A_1^- & A_3^- \end{pmatrix}}}{2}\overline{\begin{pmatrix} X_1 \\ 0 \end{pmatrix}} = X_1^- \frac{A_1+\overline{A_1^-}}{2}\overline{X_1} = X_1^- H_-(A_1)\overline{X_1} > 0$$

So, A_1 is complex metapositive subdefinite matrix, i.e. the all sequential principal minor of matrix A are complex metapositive subdefinite matrixes.

The "sufficiency" is obvious, so the proving process is omitted.

Theorem 10. If A is complex metapositive subdefinite matrix, B is complex subsymmetric matrix, so $BA\overline{B}$ is complex metapositive subdefinite matrix.

Proof. Because A is complex metapositive subdefinite matrix and B is complex subsymmetric matrix, so for $\forall 0 \neq X \in C^{n\times 1}$, $X^{-}\frac{A+\overline{A^{-}}}{2}\overline{X} > 0$ and $B^{-} = B$, and let $X = BY$, so $(BY)^{-}\frac{A+\overline{A^{-}}}{2}\overline{(BX)} = Y^{-}B^{-}\frac{A+\overline{A^{-}}}{2}\overline{BY} = Y^{-}B\frac{A+\overline{A^{-}}}{2}\overline{BY} = Y^{-}\frac{BA\overline{B}+\overline{(BA\overline{B})^{-}}}{2}\overline{Y} > 0$. So, $BA\overline{B}$ is complex metapositive subdefinite matrix.

Definition 8: To arbitrary complex matrix A, if the real inverse matrix C exists and makes $B = C^{-}AC$, so A is sub-congruent to B. For the subcongruent matrix, we can obtain following theorems.

Theorem 11. The matrix which is sub-congruent to complex metapositive subdefinite matrix still is complex metapositive subdefinite matrix.

Proof. Suppose A is complex metapositive subdefinite matrix, the matrix B is sub-congruent to the matrix A, so the real inverse matrix C exists and makes $A = C^{-}BC$, $\forall 0 \neq X \in C^{n\times 1}$, so $X^{-}\frac{A+\overline{A^{-}}}{2}\overline{X} = X^{-}\frac{C^{-}BC+\overline{(C^{-}BC)^{-}}}{2}\overline{X} = X^{-}\frac{C^{-}B\overline{C}+C^{-}\overline{B^{-}C}}{2}\overline{X} = (CX)^{-}\frac{B+\overline{B^{-}}}{2}\overline{(CX)} > 0$, and let $Y = CX$, so for $0 \neq Y \in C^{n\times 1}$, we have $(Y)^{-}\frac{B+\overline{B^{-}}}{2}\overline{Y} > 0$. So B is complex metapositive subdefinite matrix.

Theorem 12: If $A \in C^{n\times n}$ is sub-congruent to J, so A is complex metapositive subdefinite matrix.

Prove: Because J is complex metapositive subdefinite matrix, and from Theorem 11, it is easily to know A is complex metapositive subdefinite matrix.

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On the Proof of Bushell-Trustrum Inequality

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Abstract

Bushell and Trustrum (Bushell, 1990, p. 173-178) give the famous Bushell-Trustrum inequality, but their proof exists two main mistakes which make their proof process can not establish. This paper corrects these mistakes and gives the correct proof.

Keywords: Bushell-Trustrum inequality, Positive semi-definite Hermite matrix, Unitary matrix

1. Introduction

Let A and B be two positive semi-definite Hermite matrix with rank n, there eigenvalues are $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ and $\mu_1 \geq \cdots \geq \mu_n \geq 0$, respectively. Then for any positive integer k, (Marcus, 1956, p. 173-174. Marshall, 1979).

$$\sum_{i=1}^{n} \lambda_i^k \mu_{n-i+1}^k \leq tr(A^k B^k) \leq \sum_{i=1}^{n} \lambda_i^k \mu_i^k$$

And (Lieb and Thirring, 1976, see the third reference of (Bushell P J, 1990)).

$$tr(AB)^k \leq tr(A^k B^k)$$

In 1990, Bushell and Trustrum proved

$$\sum_{i=1}^{n} \lambda_i^k \mu_{n-i+1}^k \leq tr(AB)^k \leq tr(A^k B^k) \leq \sum_{i=1}^{n} \lambda_i^k \mu_i^k$$

Whereas the result proved by Lieb and Thirring, Bushell and Trustum only need to prove

$$\sum_{i=1} \lambda_i^k \mu_{n-i+1}^k \leq tr(AB)^k \leq \sum_{i=1} \lambda_i^k \mu_i^k$$

They construct $B_i = U_i B U_i^* (i = 1, 2)$ in their proof firstly, then $tr(AB_1)^k$, $tr(AB_2)^k$ are the smallest and largest values of $tr(AB)^k$, here U_i is unitary matrix. The mistakes in their proof are mainly in the following two points:

(1) Exist unitary matrix X with rank n, such that X^*AX, X^*B_1X, X^*B_2X become diagonal at the same time; (2)

$$tr(AB)^k = \sum_{i=1}^{n} \lambda_{\pi(i)}^k \mu_i^k \tag{1}$$

We will point out that unitary matrix X with rank n, which makes X^*AX, X^*B_1X, X^*B_2X become diagonal at the same time does not definitely exist, and for general positive semi-definite Hermite matrix, (1) does also not definitely exist.

We give the following conclusions:

Exist unitary matrixX with rank n, such that X^*AX and X^*B_1X become diagonal at the same time; Exist unitary matrixY with rank n, such that Y^*AY and Y^*B_2Y become diagonal at the same time. And for B_1, B_2,

$$tr(AB_1)^k = \sum_{i=1}^{n} \lambda_{\pi(i)}^k \mu_i^k \tag{2}$$

$$tr(AB_2)^k = \sum_{i=1}^{n} \lambda_{\pi(i)}^k \mu_i^k \tag{3}$$

Thus complete the certification of Bushell-Trustrum inequality. We need to use the following Lemma:

Lemma (Wang Song-Gui, 2006, p. 143) Assuming that $\alpha_1 \geq \cdots \geq \alpha_n$, $\mu_1 \geq \cdots \geq \mu_n$. If $\pi(1), \cdots, \pi(n)$ is any permutation of $1, \cdots, n$, then

$$\sum_{i=1}^{n} \alpha_i \mu_{n-i+1} \leq \sum_{i=1}^{n} \alpha_{\pi(i)} \mu_i \leq \sum_{i=1}^{n} \alpha_i \mu_i$$

2. Our proof

Suppose $A > 0$, $B > 0$, otherwise for any $c > 0$, There must be $A + cI > 0$, $B + cI > 0$, finally we take limit to the result obtained when $c \to 0$, then we conclude the proof.

Since entire unitary matrix with rank n constitutes a closed set and mapping $U \to tr(AUBU^*)^k$ is a continuous function defined on this closed set, so there must be the smallest and largest values in U_1 and U_2, Then

$$tr(AU_2BU_2^*)^k \leq tr(AUBU^*)^k \leq tr(AU_1BU_1^*)^k \tag{4}$$

Especially, in (4), take $U = I$, then we have

$$tr(AU_2BU_2^*)^k \leq tr(AB)^k \leq tr(AU_1BU_1^*)^k \tag{5}$$

If let $B_i = U_iBU_i^*(i = 1, 2)$, we will prove first: Exist unitary matrix X with rank n, such that X^*AX and X^*B_1X are diagonal.

Let

$$R = \begin{pmatrix} R_{12} & 0 \\ 0 & I \end{pmatrix} \tag{6}$$

$$R = \begin{pmatrix} F_{12} & 0 \\ 0 & 0 \end{pmatrix} \tag{7}$$

where

$$R_{12} = (1 + |\varepsilon|^2)^{-\frac{1}{2}} \begin{bmatrix} 1 & -\varepsilon \\ \overline{\varepsilon} & 1 \end{bmatrix} \tag{8}$$

$$F_{12} = \frac{1}{|\varepsilon|} \begin{bmatrix} 0 & -\varepsilon \\ \overline{\varepsilon} & 0 \end{bmatrix} \tag{9}$$

R, F are $n \times n$ rectangular matrix, 0, I are zero matrix and unit matrix on some degree.

Obviously, R is an unitary matrix, and to infinitely small $\varepsilon \neq 0$, R can denoted as

$$R = I + |\varepsilon|F + o(|\varepsilon|^2) \tag{10}$$

Here $o(|\varepsilon|^2)$ is $n \times n$ rectangular matrix, everyone of its element is infinitesimal of higher order of $|\varepsilon|$. For convenient we use $o(|\varepsilon|^2)$ to denote either matrix or number.

In fact,

$$R - I - |\varepsilon|F = \begin{bmatrix} R_{12} - I - |\varepsilon|F_{12} & 0 \\ 0 & 0 \end{bmatrix} \tag{11}$$

$$R_{12} - I - |\varepsilon|F_{12} = \begin{bmatrix} \frac{1}{\sqrt{1+|\varepsilon|^2}} - 1 & -\frac{\varepsilon}{\sqrt{1+|\varepsilon|^2}} - \varepsilon \\ \frac{\bar{\varepsilon}}{\sqrt{1+|\varepsilon|^2}} - \bar{\varepsilon} & \frac{1}{\sqrt{1+|\varepsilon|^2}} - 1 \end{bmatrix} \tag{12}$$

From mathematical analysis, when $x \to 0$,

$$1 - \frac{1}{\sqrt{1+x^2}} = \frac{\sqrt{1+x^2} - 1}{\sqrt{1+x^2}} = \frac{x^2}{\sqrt{1+x^2}(\sqrt{1+x^2}+1)} \sim x^2$$

so elements in (11) and (12) are infinitesimal of higher order of $|\varepsilon|$, thus (10) holds. For any unitary matrix T, define

$$\tilde{B} = (TRT^*)B(TR^*T^*) \tag{13}$$

Since R is unitary matrix, TRT^* is unitary matrix. Because B is positive semi-definite Hermite matrix, $TR^*T^* = (TRT^*)^*$, $\tilde{B}$ is positive semi-definite Hermite matrix too. From (10), we get

$$TRT^* = T(I + |\varepsilon|F + o(|\varepsilon|^2))T^* = I + |\varepsilon|TFT^* + o(|\varepsilon|^2) \tag{14}$$

Notice that $F^* = -F$,

$$TR^*T^* = T(I + |\varepsilon|F^* + o(|\varepsilon|^2))T^* = I - |\varepsilon|TFT^* + o(|\varepsilon|^2) \tag{15}$$

Then

$$\begin{aligned} \tilde{B} &= B + |\varepsilon|(TFT^*B - BTFT^*) + o(|\varepsilon|^2) \\ &= B + |\varepsilon|T(FT^*BT - T^*BTF)T^* + o(|\varepsilon|^2) \\ &= B + |\varepsilon|T(FC - CF)T^* + o(|\varepsilon|^2) \end{aligned} \tag{16}$$

Here

$$C = T^*BT \tag{17}$$

It is easy to prove that for any two unitary matrix with rank n P and Q, have

$$tr(P + |\varepsilon|Q)^k = trP^k + k|\varepsilon|trP^{k-1}Q + o(|\varepsilon|^2) \tag{18}$$

Then from (16), (18)

$$\begin{aligned} tr(A\tilde{B})^k &= tr(AB + |\varepsilon|AT(FC - CF)T^* + o(|\varepsilon|^2))^k \\ &= tr(AB)^k + k|\varepsilon|tr(AB)^{k-1}AT(FC - CF)T^* + o(|\varepsilon|^2) \\ &= tr(AB)^k + k|\varepsilon|tr[D(FC - CF)] + o(|\varepsilon|^2) \end{aligned} \tag{19}$$

Here

$$D = T^*(AB)^{k-1}AT \tag{20}$$

We can prove that $(AB)^{k-1}A \geq 0$

In fact, notice that A and B are both positive semi-definite Hermite matrixes.

When $k = 2$, $ABA = AB^{\frac{1}{2}}B^{\frac{1}{2}}A = (B^{\frac{1}{2}}A)^*B^{\frac{1}{2}}A \geq 0$.

When $k = 3$, $ABABA = ABA^{\frac{1}{2}}A^{\frac{1}{2}}BA = (A^{\frac{1}{2}}BA)^*A^{\frac{1}{2}}BA \geq 0$. It can be proved by induction.

In(20), because $(AB)^{k-1}A$ is positive semi-definite, T is any unitary matrix, so we can choose unitary matrix T, such that D becomes diagonal,

$$D = diag(d_1, \cdots, d_n), \quad d_1 \geq \cdots \geq d_n \geq 0 \tag{21}$$

Let $C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$, $D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$, where $C_1 = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$, $D_1 = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$; Notice that C_1 is Hermite matrix, $F_{12}^* = -F_{12}$, then

$$\begin{aligned} |\varepsilon| trD(FC - CF) &= |\varepsilon| tr \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \left[\begin{pmatrix} F_{12} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} - \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \begin{pmatrix} F_{12} & 0 \\ 0 & 0 \end{pmatrix} \right] \\ &= |\varepsilon| tr \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} F_{12}C_1 - C_1F_{12} & F_{12}C_2 \\ -C_2F_{12} & 0 \end{bmatrix} \\ &= |\varepsilon| tr \begin{bmatrix} D_1(F_{12}C_1 - C_1F_{12}) & D_1F_{12}C_2 \\ -D_2C_2F_{12} & 0 \end{bmatrix} \\ &= |\varepsilon| trD_1(F_{12}C_1 - C_1F_{12}) \\ &= |\varepsilon| trD_1(F_{12}C_1 + (F_{12}C_1)^*) \\ &= (d_2 - d_1)(\overline{\varepsilon}c_{12} + \varepsilon\overline{c}_{12}) \end{aligned} \tag{22}$$

The last equation is right because C_1 is Hermite matrix, and

$$|\varepsilon| F_{12}C_1 = \begin{pmatrix} 0 & -\varepsilon \\ \overline{\varepsilon} & 0 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} -\varepsilon c_{21} & -\varepsilon c_{22} \\ \overline{\varepsilon} c_{11} & \overline{\varepsilon} c_{12} \end{pmatrix}$$

By (22) and (19), then

$$tr(A\tilde{B})^k - tr(AB)^k = k(d_2 - d_1)(\overline{\varepsilon}c_{12} + \varepsilon\overline{c}_{12}) + o(|\varepsilon|^2) \tag{23}$$

This formula is correct on arbitrary semi-positive Hermite matrix B and infinitely small $\varepsilon \neq 0$.

Especially, set $B = B_1$, $\varepsilon = \eta c_{12}$, $\eta > 0$, If $d_2 \neq d_1$, then by definition of B_1 and (2), we obtain $\overline{\varepsilon}c_{12} + \varepsilon\overline{c}_{12} = \eta|c_{12}|^2 = 0$, then $c_{12} = \overline{c}_{21} = 0$.

Similarly, we take R, F such that their $i,\ j(i < j)$ row and column have form of (8), (9), and similar to the proof above, then it can be obtained.

$$tr(A\tilde{B})^k - tr(AB)^k = k(d_j - d_i)(\overline{\varepsilon}c_{ij} + \varepsilon\overline{c}_{ij}) + o(|\varepsilon|^2) \tag{24}$$

Set $\varepsilon = \eta c_{ij}$, $\eta > 0$, Use the same method $c_{ij} = \overline{c}_{ji} = 0$ can be obtained.

Suppose $c_1 > c_2 > \cdots > c_l$ are l different value of $d_1, \cdots, d_n$, here $D = diag(c_1 I_{n1}, \cdots c_l I_{nl})$. make $C = T^* B_1 T$ become block matrix

$$C = diag(C_1, \cdots, C_l) \tag{25}$$

Here C_i is positive semi-definite Hermite matrix with rank n_i. Let $V_i (i = 1, \cdots, l)$ is unitary matrix, such that $E_i = V_i^* C_i V_i$, $i = 1, \cdots, l$, becomes diagonal matrix.

Let

$$V = diag(V_1, \cdots, V_l) \tag{26}$$

$$E = diag(E_1, \cdots, E_l) \tag{27}$$

Set $X = TV$, then X is an unitary matrix, and

$$X^* B_1 X = V^* T^* B_1 T V = V^* C V = E \tag{28}$$

This is a diagonal matrix, its diagonal elements are eigenvalues of B_1, furthermore

$$X^*(AB_1)^{k-1}AX = V^*T^*(AB_1)^{k-1}ATV = V^*DV = D \tag{29}$$

The last equation is correct because V and D are block matrix with same degree. By (28) and (29) we know

$$(E^{\frac{1}{2}}(X^*AX)E^{\frac{1}{2}})^k = E^{\frac{1}{2}}X^*(AB_1)^{k-1}AXE^{\frac{1}{2}} = E^{\frac{1}{2}}DE^{\frac{1}{2}} \tag{30}$$

then

$$X^*AX = E^{-\frac{1}{2}}(E^{\frac{1}{2}}DE^{\frac{1}{2}})^{\frac{1}{k}}E^{-\frac{1}{2}} \tag{31}$$

It is a diagonal matrix. It be proved that exist $n \times n$ unitary matrix such that X^*AX, X^*B_1X are all diagonal matrix.

Similarly, in(24), let $B = B_2$, $\varepsilon = \eta c_{ij}$, $\eta < 0$, then $c_{ij} = \overline{c}_{ji} = 0$. Notice that because $C = T^*BT$ and B_i are different, so we write as $G = T^*B_2T$.

Suppose that $g_1 > g_2 > \cdots > g_m$ are m different values of $d_1, \cdots, d_n$, here $D = diag(g_1I_{n1}, \cdots g_mI_{n_m})$, make $G = T^*B_2T$ become block matrix

$$G = diag(G_1, G_2, \cdots G_m) \tag{32}$$

Let $W_i(i = 1, 2, \cdots m)$ be an unitary matrix, such that $W_i^*G_iW_i(i = 1, 2, \cdots m)$ is diagonal matrix. Write

$$W = diag(W_1, W_2, \cdots W_m) \tag{33}$$

Set $Y = TW$, then Y is an unitary matrix, similar to the proof on (28)-(31), it be obtained that exist unitary matrixY such that Y^*AY, Y^*B_2Y are all diagonal matrix.

According to (28) and (31), X^*AX, X^*B_1X, Y^*AY, Y^*B_2Y are all diagonal matrixes. Notice that $X^*B_1X = X^*U_1B_1{U_1}^*X$, $Y^*B_2Y = Y^*U_2B_2{U_2}^*Y$, U_1, U_2, X, Y are all unitary matrix, so diagonal elements of X^*B_1X, Y^*B_2Y are eigenvalues of B. Thus

$$tr(AB_1)^k = tr(X^*(AB_1)^kX) = tr(X^*AXX^*B_1X)^k = \sum_{i=1}^{n} \lambda_{\pi(i)}^k \mu_i^k \tag{34}$$

$$tr(AB_2)^k = tr(Y^*(AB_2)^kY) = tr(Y^*AYY^*B_2Y)^k = \sum_{i=1}^{n} \lambda_{\pi'(i)}^k \mu_i^k \tag{35}$$

Here $\pi(i)$, $\pi'(i)$ is any permutation of $1, 2, \cdots, n$, respectively. From Lemma,

$$tr(AB_1)^k \leq \sum_{i=1}^{n} \lambda_i^k \mu_i^k \tag{36}$$

$$tr(AB_2)^k \geq \sum_{i=1}^{n} \lambda_i^k \mu_{n-i+1}^k \tag{37}$$

And using(5), then the proof is completed.

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The Research of Bidding Model Based on the Generalized Order Statistics

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Abstract

In the Seller Bidding of order statistics bidding function, the expected deviation of the last non-winner's price and the winner's price is a decrease function of the number of bidder. This paper analysis and know that the expected deviation of the last non-winner's price and the winner's price decrease with the increasing of the number of bidder on the Seller Bidding of the generalized order statistics.

Keywords: The Generalized Order Statistics, Bidding Price Function, Seller Bidding

1. Reviews

Anand Paul and Genaro Gutierrez prove that the expected returns of bidding proprietor in a seller's bidding is a monotonically decreasing function of the number of bidders if the population from which the valuations are sampled is characterized by a concave distribution function. BUIOW and Klemperer prove that expected revenue with N +1 bidders exceeds expected revenue with N bidders if the bidders' valuations are either independent. Now we study bidding model in the Generalized Order Statistics and analysis the relations between the number of bidder with the expected the expected deviation of the last non-winner's price and the winner's price.

2. Definition

The concept of generalized order statistics (gos) was given by Kamps (1995) as below (see (HASEEB, 2004)):

Let $F(\tilde{X})$ be an absolutely continuous distribution function (df) with probability density function (pdf)f($\tilde{X}$),Let $n \in N$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, \cdots, m_{n-1}) \in R^{n-1}$, $M_i = \sum_{j=i}^{n-1} m_j$, such that $\gamma_i = k + n - i + M_i > 0$ for all $i \in \{1, 2, \cdots, n-1\}$. Then $X(i, n, \tilde{m}, k)$, $i = 1, 2, \cdots, n$ are called generalized order statistics (gos) if their joint probability density function is given by

$$k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\prod_{i=1}^{n-1}[1-F(x_i)]^{k-1}f(x_i)[1-F(x_n)]^{k-1}f(x_n) \tag{1}$$

on the cone $F^{-1}(0+) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1)$ of R^n

For $m_1 = m_2 = \cdots = m_{n-1} = m$, the gos will be denoted as $X(i, n, m, k)$ and its pdf is given by Kamps (1995) as:

$$f_{X(i,n,m,k)}(x) = \frac{c_{i-1}}{(i-1)!}[1-F(x)]^{\gamma_i-1}f(x)g_m^{i-1}(F(x)) \tag{2}$$

Where $C_{i-1} = \prod_{j=1}^{i}\gamma_j$, $\gamma_j = k + (n-j)(m+1)$; $g_m(x) = \begin{cases} \frac{1-(1-x)^{m+1}}{m+1} & m \neq -1 \\ \log(\frac{1}{1-x}) & m = -1 \end{cases}$ $x \in [0, 1)$

3. Bidding Model Setup

We first analyze a special bidding. Denote the *i*th generalized order statistics in a sample of size N by $\tilde{X}_{i:N}$ and suppose there are N bidders competing to sell a good. Each bidder has a function valuation $\tilde{X}_{(i,N,m,k)}$. The buyer assumes that the function valuations of the bidders are *iid* random variables. Prices are bid in a descending sequence by individual bidders until only one bidder remains, the winner. The valuation of the winner is the first generalized order statistics.

$$E(\tilde{X}_{K+1:N} - \tilde{X}_{K:N}) = C_K^N \int_0^\infty F(\tilde{X})^K (1 - F(\tilde{X}))^{N-K} d\tilde{X} \tag{3}$$

which is a standard result when the parent distribution is positive valued (see Udo Kamps, 1995).

The expectation of the deviation between the second and first generalized order statistic of a random sample of size N in a seller's bidding by

$$E(\tilde{X}_{2:N} - \tilde{X}_{1:N}) = N \int_0^\infty F(\tilde{X})\{1 - F(\tilde{X})\}^{N-1} d\tilde{X} \tag{4}$$

Now we denote the deviation between expected of N bidders and $N + 1$ bidders by $D(N)$.

$$\begin{aligned} D(N) &= E(N) - E(N+1) \\ &= N \int_0^\infty F(\tilde{X})\{1 - F(\tilde{X})\}^{N-1} d\tilde{X} - (N+1) \int_0^\infty F(\tilde{X})\{1 - F(\tilde{X})\}^N d\tilde{X} \\ &= \int_0^\infty F(\tilde{X})\{1 - F(\tilde{X})\}^{N-1}\{(N+1)F(\tilde{X}) - 1\} d\tilde{X} \end{aligned}$$

Note that as N tends to infinity£this integral tends to zero by L*é*vesque's Dominated Convergence Theorem£We are, however, interested in the situation when N is finite. In order to guarantee $D(N) > 0$, we assume that the valuations are distributed on a compact interval $[a, b](0 \le a < b \le)$ on the positive real line and that the density function of the underlying random variable is continuous.

4. Important Result

Lemma 1: Let $m_1 = m_2 = \cdots = m_{i-1} = m$, the df of the i th uniform gos is denoted by $\varphi_{i,N}(x) = f_{U(i,N,m,k)}(x) = \frac{c_{i-1}}{(i-1)!}(1-x)^{\gamma_i - 1} g_m^{i-1}(x)$, $x \in (0, 1)$, $1 \le i \le n$ (see(Anand Paul, 2003) and (Udo Kamps, 1995))

Lemma 2 Let $m_1 = m_2 = \cdots = m_{i-1} = m$ and $i \in \{1, 2, \cdots, n\}$£then we have, $\phi_{i,N}(x) = 1 - C_{i-1}(1-x)^{k+N-i+M_i} + \sum_{j=0}^{i-1} \frac{1}{j! C_{i-j-1}} g_m^j(x)$, $x \in (0, 1)$ and $F_{X(i,N,m,k)}(x) = \phi_{i,N}(F(x))$ (see(CRAMLR,E, 2000)).

Lemma 3 $\int_a^b F(\tilde{X})[1 - F(\tilde{X})]^{N-1}[(N+1)F(\tilde{X}) - 1] dF(\tilde{X}) > 0$ for all function $F(\tilde{X})$ where $J(\tilde{X}) = F(\tilde{X})\{1 - F(\tilde{X})\}^{N-1}\{(N+1)F(\tilde{X}) - 1\}$ is obvious.

The Stieltjes integral $\int_a^b J(\tilde{X}) dF(\tilde{X})$ in Lemma 3 can be rewritten as $\int_a^b J(\tilde{X}) f_{(i,N,m,k)}(x) dx$ where $f_{(i,N,m,k)}(x)$ is pdf of the distribution of valuations

Theorem 1 The expected deviation of the last non-winner's price and the winner's price in a seller's bidding is a monotonically decreasing function of the number of bidders if the bidder from which the valuations are sampled is characterized by a concave distribution function in the Generalized Order Statistics.

In theorem 1, we only considerate two cases: case I $m_1 = m_2 = \cdots = m_{i-1} = m \ne -1$ and Case II $m_1 = m_2 = \cdots = m_{i-1} = m = -1$.

For case I: From Lemma 1, we have $\varphi_{2,N}(x) = \frac{\gamma_1 \gamma_2}{m+1}\{(1-x)^{\gamma_2 - 1} - (1-x)^{\gamma_1 - 1}\}$. So we know the probability density function and the distribution function of 2nd generalized order statistics (gos) as follows:

$$f(c) = \varphi_{2,N}(c) = \frac{\gamma_1 \gamma_2 [(1-c)^{\gamma_2 - 1} - (1-c)^{\gamma_1 - 1}]}{(m+1)} \tag{5}$$

$$F(c) = \int_a^c \varphi_{2,N}(x) dx = \frac{\gamma_1(1-c)^{\gamma_2} - \gamma_2(1-c)^{\gamma_1} + \gamma_2(1-a)^{\gamma_1} - \gamma_1(1-a)^{\gamma_2}}{m+1} \tag{6}$$

And from Definition and Lemma 2, we can rewrite the generalized order statistics distribution function $F_{X(i,N,m,k)(x)}$ and the probability density function $f_{X(i,N,m,k)}(x)$:

$$F_{X(2,N,m,k)}(x) = \frac{\gamma_2[1-F(x)]^{\gamma_1} - \gamma_1[1-F(x)]^{\gamma_2} + 1}{m+1}$$

$$f_{X(2,N,m,k)}(x) = \frac{\gamma_1\gamma_2}{m+1}\{[1-F(x)]^{\gamma_2-1} - [1-F(x)]^{\gamma_1-1}\}f(x)$$

From Lemma 3 we know if $f_{X(2,N,m,k)}(x)$ were a decreasing function such that $f_{X(2,n,m,k)}(c) = 1$ and $F_{X(2,n,m,k)}(c) = 1/(N+1)$, then $J(\tilde{X})$ is 0 at $x = a$, c and b, negative on the interval (a, c) and positive on the interval (c, b). we would have $\int_a^b J(\tilde{X})(1 - f_{(i,N,m,k)}(x))dx > 0$ in $[a, c]$ and $[c, b]$, then we have integral unequal $\int_a^b J(\tilde{X})dx > \int_a^b J(\tilde{X})f_(i, N, m, k)(x)dx > 0$ and Theorem 1 would be proved.

But if $f_{X(2,n,m,k)}(c) = 1$ is unknown here, So we reduce two equation through substituting (5) and (6) to

$$\left\{\frac{N}{N+1} + k\left[\frac{v(b)-v(c)}{m+1}\right]^k\right\}\frac{(1-c)^{\gamma_1} - (1-c)^{\gamma_2}}{1-c} + \frac{(v(b)-v(c))}{\gamma_1\gamma_2} = 0$$

where $v(t) = \gamma_1(1-t)^{\gamma_2} - \gamma_2(1-t)^{\gamma_1}$. Let

$$T_1(x) = \left\{\frac{N}{N+1} + k\left[\frac{v(b)-v(x)}{m+1}\right]^k\right\}\frac{(1-x)^{\gamma_1} - (1-x)^{\gamma_2}}{1-x} + \frac{(v(b)-v(x))}{\gamma_1\gamma_2}$$

. Where we would have

$$T_1(a) = \frac{-\gamma_1\lfloor(1-a)^{\gamma_2} - (1-a)^{\gamma_1}\rfloor}{(N+1)(1-a)} - 1 < 0$$

and $T_1(b) = \frac{N_{\gamma_1}\lfloor(1-b)^{\gamma_2}-(1-b)^{\gamma_1}\rfloor}{(N+1)(1-b)} > 0$. So we have at least a point c which make $T_{(c)}$ equal to 0 for $T_1(x)$ is a continuous function.

For Case II: From Lemma 1, we have $\varphi_{2,N}(x) = -\gamma_1\gamma_2(1-x)^{\gamma_1-1}\ln(1-x)$. So we obtain the density function and the distribution function of 2nd generalized order statistics (gos) as follows:

$$f_{2,N}(c) = \varphi_{2,N}(c) = -\gamma_1\gamma_2(1-c)^{\gamma_1-1}\ln(1-c) \tag{7}$$

$$F_{2,N}(c) = \int_a^c \phi_{2,N}(x)dx = \frac{(1-c)^{\gamma_1}}{\gamma_1^2}\ln\frac{(1-c)^{\gamma_1}}{e} - \frac{(1-a)^{\gamma_1}}{\gamma_1^2}\ln\frac{(1-a)^{\gamma_1}}{e} \tag{8}$$

From Definition and Lemma 2, the distribution function $F_{X(i,N,m,k)}$ and the probability density function $f_{X(i,N,m,k)}$ in generalized order statistics can be rewrote:

$$F_{X(z,N,m,k)}(x) = \int_0^{F(x)} \varphi_{i,N}(t)dt = [1-F(x)]^k\{k\ln[1-F(x)] - 1\} + 1$$

$$f_{X(2,N,m,k)}(x) = -k^2[1-F(x)]^{k-1}\ln[1-F(x)]f(x)$$

As case I, we analysis Case II and substitute (7) and (8) to above two equation and reduce them to

$$k\{\frac{N}{N+1} - [u(b)-u(c)]^k\}(1-c)^{\gamma_1-1}\ln(1-c) + \frac{u(b)-u(c)}{\gamma_1\gamma_2} = 0$$

where

$$u(t) = \frac{(1-t)^{\gamma_1}}{\gamma_1^2}\ln\frac{(1-t)^{\gamma^1}}{e}$$

Let

$$T_2(x) = k\{\frac{N}{N+1} - [u(b)-u(x)]^k\}(1-x)^{\gamma_1-1}\ln(1-x) + \frac{u(b)-u(x)}{\gamma_1\gamma_2}.$$

Then we would have

$$T_2(a) = \frac{k(1-a)^{\gamma_1}}{(N+1)(1-a)} \ln(1-a) - 1 < 0$$

and

$$T_2(b) = \frac{kN(1-b)^{\gamma_1}[1-(1-b)^{m+1}]}{(N+1)(1-b)} > 0$$

So we have at least a point c which make $T_2(c)$ equal to 0 for $T_2(x)$ is a continuous function also.

From case I and Case II, we would have $\int_a^b J(\tilde{X})dx > \int_a^b J(\tilde{X}) f_{(r,n,m,k)}(x)dx > 0$ if $m_1 = m_2 = \cdots = m_{i-1} = m$. So the result has been proved.

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Fuzzy Topological Transformation Groups

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Abstract

In this paper we have introduced the concept of Topological Transformation Groups in fuzzy setting as a natural transition from the corresponding crisp structure and study some properties thereof. Classical results on orbits, orbit closure, invariant subsets are investigated in this setting. Finally we have constructed some new fuzzy topological transformation groups from given ones.

Keywords: Fuzzy topological transformation group, Fuzzy orbit

1. Introduction

A classical Topological Transformation Group is a structure (π, G, X,) where G is a topological group, X is a topological space and π is a continuous function from $G \times X \to X$ satisfying $\pi(0, x) = x$ and $\pi(s, \pi(t, x)) = \pi(s + t, x)$, where 0 is the identity of G. In this paper we fuzzify the above concept as a natural transition from the corresponding crisp structure. For this fuzzification we will consider a fuzzy topological group (Chu-hai Yu, 1987), a fuzzy topological space and a fuzzy continuous map from $G \times X \to X$ satisfying the above stated conditions. Throughout our discussion the fuzzy topology on any set will contain all the constant fuzzy subsets. In other words we will use Lowen (R. Lowen, 1976) definition of fuzzy topology.

2. Preliminaries

In this section we recall some preliminary definitions and results to be used in the sequel.

Let X be a non-empty set. A fuzzy set in X is an element of the set I^X of all functions from X into the unit interval I. A fuzzy point of a set X is a fuzzy subset which takes non-zero value at a single point and zero at every other point. The fuzzy point which takes value $\alpha \neq 0$ at $x \in X$, and zero elsewhere is denoted by x_α . If $x \in X$, then the fuzzy point x_1 will be denoted simply by x. Let λ be a fuzzy subset of X. Suppose $\lambda(x) = \alpha$ for $x \in X$. Then λ can be expressed as union of all its fuzzy points, i.e, $\lambda = \vee_{x \in X} x_\alpha$. Here $\vee$ denote union. We will use the same notation $\vee$ to denote supremum of a set of numbers. Similarly $\wedge$ will be used to denote intersection of fuzzy sets as well as infimum of a set of real numbers.

Let λ and μ be fuzzy subsets of X, then we write $\lambda \subseteq \mu$ whenever $\lambda(x) \leq \mu(x)$. Let λ be a fuzzy subset of a group $(G, +)$. Then we define a fuzzy subset $-\lambda$ as $-\lambda(x) = \lambda(-x)$. If f is a function from X into Y and $\mu \in I^Y$, then $f^{-1}(\mu)$ is the fuzzy set in X defined by $f^{-1}(\mu)(x) = \mu(f(x))$. Equivalently, $f^{-1}(-\mu) = \mu \circ f$. Also, for $\rho \in I^X$, $f(\rho)$, is the member of I^Y which is defined by

$$f(\rho)(y) = \begin{cases} \sup\{\rho(x) : x \in f^{-1}[y]\} & \text{if} f^{-1}[y] \text{is not empty} \\ 0 & \text{otherwise} \end{cases}$$

For the definition of a fuzzy topology, we will use the one given by Lowen (1976) since his definition is more appropriate in our case. So, throughout this paper, by a fuzzy topology on a set X we will mean a sub-collection τ of I^X satisfying the following conditions:

(i) τ contains every constant fuzzy subset in X ;

(ii) If $\mu_1, \mu_2 \in \tau$, then $\mu_1 \wedge \mu_2 \in \tau$;

(iii) if $\mu_i \in \tau$ for each $i \in A$, then $\vee_{i\in A}\mu_i \in \tau$.

A fuzzy topological space is a set X on which there is given a fuzzy topology τ. The elements of τ are the open fuzzy sets in X. Complement of an open fuzzy set is called a closed fuzzy set. Interior of a fuzzy set λ is the union of all the open fuzzy set contained in λ and the closure of λ is the intersection of all fuzzy set containing λ. The interior and closure of λ will be denoted by λ^o and clλ respectively. A map f from a fuzzy topological space X to a fuzzy topological space Y, is called continuous if $f^{-1}(\mu)$ is open in X for each open fuzzy set μ in Y. Let X be a fuzzy topological space and $x \in X$. A fuzzy set μ in X is called a neighborhood of the fuzzy point x_α if there exists an open fuzzy set ρ with $\rho \subseteq \mu$ and $x_\alpha \in \rho \subseteq \mu$. Given a crisp topological space (X, T), the collection $\varpi(T)$, of all fuzzy sets in X which are lower semicontinuous, as functions from X to the unit interval $I = [0, 1]$ equipped with the usual topology, is a fuzzy topology on X (Lowen, 1976). We will refer to the fuzzy topology $\varpi(T)$ as the fuzzy topology generated by the usual topology T. If $(X, T_j)_{j\in J}$ is a family of crisp topological spaces and T the product topology on $X = \prod_{j\in J} X_j$, then $\varpi(T)$ is the product of the fuzzy topologies $\varpi(T_j)$, $j \in J$ (Lowen, 1977).

Result 2.1. (A. K. Katsaras, 1981) Let (X_i, T_i), $i = 1, 2, 3$, be crisp topological spaces, $X = X_1 \times X_2$, T the product of the topologies T_1 , T_2 and $f : (X, T) \to (X_3, T_3)$ a continuous map. If δ is the product of the fuzzy topologies $\varpi(T_1)$ and $\varpi(T_2)$, then

$$f : (X, \delta) \to (X_3, \varpi(T_3))$$

is fuzzy continuous.

Proof. Let $\mu \in \varpi(\tau_3)$. Then μ is a lower semicontinuous function from (X_3, T_3) to the unit interval I. Since f is continuous with respect to the topologies T and T_3, it follows that the function $f^{-1}(\mu) = \mu \circ f$ is a lower semicontinuous function from (X, T) to the unit interval. Thus $f^{-1}(\mu) \in \varpi(T) = \delta$. This completes the proof.

Result 2.2. Let (π, G, X) be a classical topological transformation group. If we equip G and X with the induced fuzzy topologies and $G \times X$, with the corresponding product fuzzy topology, then the mapping π: $G \times X \to X$ is fuzzy continuous.

Proof. It follows from the previous result.

Definition 2.3(Liu Ying-Ming, 1997) : Let (X, δ) be a fuzzy topological space and $Y \subseteq X$. Then the collection $\delta/_Y = \{\sigma/_Y : \sigma \in \delta\}$ is a fuzzy topology on Y. Then $(Y, \delta/_Y)$ is called fuzzy subspace of (X, δ).

Result 2.4 : Let (X, δ) and (Y, μ) be two fuzzy topological spaces. If $f : X \to Y$ is fuzzy continuous, then for any subset $A \subseteq X$, $f/_A$ is fuzzy continuous. In particular an inclusion map is fuzzy continuous.

Result 2.5 (Liu Ying-Ming, 1997) Let (X, δ), (Y, τ) and (Z, κ) be fuzzy topological spaces and f : X$\to$Y and g : Y$\to$ Z be any mappings. Then f, g are fuzzy continuous $\Rightarrow$gof is fuzzy continuous.

Definition 2.6 If σ is a fuzzy subset of X and η is a fuzzy subset of Y, then the fuzzy subset $\sigma \times \eta$ on $X \times Y$ is defined as $(\sigma \times \eta)(x, y) = \min\{\sigma(x), \eta(y)\}$.

Definition 2.7 (Liu Ying-Ming, 1997) Let (X, δ) and (Y, τ) be two fuzzy topological spaces. Then f : X $\to$ Y is fuzzy open (closed) if the image of every fuzzy open(closed) subset of X is fuzzy open(closed) in Y.

Definition 2.8 (R. Lowen, 1976) Let X be a fuzzy topological space and λ a fuzzy subset of X. An open fuzzy cover of λ is a collection $\{\lambda_\alpha\}$ of open fuzzy subsets of X such that $\lambda \subseteq \vee\lambda_\alpha$. If every open cover of λ and $\varepsilon > 0$ there exists a finite sub-collection $\{\lambda_i : i = 1, 2 \cdots n\}$ such that $\vee\{\lambda_i : i = 1, 2 \cdots n\} \geq \lambda - \varepsilon$ then λ said to be fuzzy compact.

Result 2.9 (R. Lowen, 1976) (X, δ) and (Y, τ) be two fuzzy topological spaces and $f : X \to Y$ be a fuzzy continuous function. If λ is fuzzy compact subset of X, then $f(\lambda)$ is fuzzy compact in Y.

Result 2.10 (R. Lowen, 1977). Let (X, δ) and (Y, τ) be two fuzzy topological spaces and λ, μ are fuzzy compact subsets of X and Y respectively, then $\lambda \times \mu$ is fuzzy compact in $X \times Y$.

Definition 2.11 (Rajesh Kumar, 1993). Let (G, +) be a group. Then a fuzzy subset λ is said to be a fuzzy subgroup of G if $\lambda(x + y) \geq \min\{\lambda(x), \lambda(y))$ and $\lambda(-x) = \lambda(x)$

Remark : If λ is a fuzzy subgroup of G then supp is a crisp subgroup of G.

Definition 2.12 (N. Palaniappan, 2005) A fuzzy topological space (X, τ) is said to be product related to another fuzzy topological space (Y, δ) if for any fuzzy set υ of X and ζ of Y whenever $\lambda^c \not\geq \upsilon$ and $\mu^c \not\geq \zeta$ implies $(\lambda^c \times 1) \vee (1 \times \mu^c) \geq \upsilon \times \zeta$, where $\lambda \in \tau$ and $\mu \in \delta$, then there exist $\lambda_1 \in \tau$ and $\mu_1 \in \delta$ such that $\lambda_1^c \geq \upsilon$ or $\mu_1^c \geq \zeta$ and $(\lambda^c \times 1) \vee (1 \times \mu^c) = (\lambda_1^c \times 1) \vee (1 \times \mu_1^c)$.

Result 2.13. (N. Palaniappan, 2005) Let (X, τ) be product related to (Y, δ). Then for any fuzzy subset λ of X and a fuzzy subset μ of Y, $cl(\lambda \times \mu) = cl\lambda \times cl\mu$.

3. Fuzzy topological transformation groups

In this section we will introduce the concept of fuzzy topological transformation group and prove some properties.

Definition 3.1 Let X be fuzzy topological space, G be a fuzzy topological group. If $\pi : G \times X \to X$ satisfies

(FTG1) $\pi(0, x) = x$

(FTG2) $\pi(s, \pi(t, x)) = \pi(s + t, x)$

(FTG3) π is fuzzy continuous

then (π, G, X,) is called a fuzzy topological transformation group.

Definition 3.2 Let $t \in G$, then the t-transition of (π, G, X,) denoted by π^t is the mapping : $\pi^t : X \to X$ such that $\pi^t(x) = \pi(t, x)$.

Result 3.3 (i) π^0 is the identity mapping of X.

(ii) $\pi^s\pi^t = \pi^{s+t}$ for $s, t. \in G$.

(iii) π^t is one-to-one mapping of X onto X and $-(\pi^t) = \pi^{-t}$.

(iv) For $t \in G$, π^t is a fuzzy homomorphism of X onto X.

Proof. Straightforward.

Definition 3.4 The transition group of (π , G, X,) is the set G = $\{\pi^t : t \in G\}$. The transition projection of (G, X, π) is the mapping $\theta : G \to G$ defined as $\theta(t) = \pi^t$.

Definition 3.5 (G, X, π) is said to be effective if $t \in G$ with $t \neq 0 \Rightarrow \pi^t(x) \neq x$ for some x.

Result 3.6 (i) G is a group of fuzzy homeomorphisms of X onto X

(ii) θ is a group homomorphism of G onto G.

(iii) θ is one-one iff (π , G, X) is effective.

Proof. Straightforward.

Definition 3.7 Let $x \in X$, then the x-motion of (π , G, X) is the mapping $\pi_x : G \to X$ such that $\pi_x(t) = \pi(t, x)$.

Result 3.8 π_x is a fuzzy continuous mapping of G into X.

Proof. Straightforward.

Result 3.9 Let X, Y, Z be fuzzy topological spaces and f : $X \times Y \to Z$ be a fuzzy continuous map. If a_α, b_β be fuzzy points of X and Y respectively and γ be a fuzzy neighbourhood of $f(a_\alpha, b_\beta)$ then there exists fuzzy neighbourhoods η and ρ of a_α and b_β respectively such that $f(\eta \times \rho) \subseteq \gamma$.

Proof. Without loss of generality we can assume that γ is fuzzy open. As f is continuous $f^{-1}(\gamma)$ is a fuzzy open set containing $a_\alpha \times b_\beta$. So there exists basic fuzzy open sets say, η of a_α and ρ of b_β such that $a_\alpha \times b_\beta \in \eta \times \rho \subseteq f^{-1}(\gamma)$. Which gives $f(a_\alpha \times b_\beta) \in f(\eta \times \rho) \subseteq \gamma$.

Result 3.10 Let X, Y, Z be fuzzy topological spaces and f : $X \times Y \to Z$ be a fuzzy continuous map. If λ and μ are fuzzy compact subsets of X and Y respectively and γ is a fuzzy neighbourhood of $f(\lambda \times \mu)$, then for any $\varepsilon > 0$, there exists fuzzy open sets λ' and μ' such that $\lambda' \geq \lambda - \varepsilon$ and $\mu' \geq \mu - \varepsilon$ and $f(\lambda' \times \mu') \subseteq \gamma$.

Proof. Let $\varepsilon > 0$ be arbitrary. Let $x \in X$ be arbitrarily fixed and suppose $\lambda(x) = \alpha$. Then for any $y \in Y$ with $\mu(y) = \beta$, by previous result there exist fuzzy open sets $\lambda_y \ni x_\alpha$ and $\mu_y \ni y_\beta$ such that $f(x_\alpha \times y_\beta) \in f(\lambda_y \times \mu_y) \subseteq \gamma$. This is true for each $y \in Y$. Thus the collection $C_\mu = \{\mu_y : y \in Y\}$ is an open cover of μ. As μ is compact there is a finite sub-collection say S_μ of C_μ satisfying $\vee\{\mu_y : \mu_y \in S\} \geq \mu - \varepsilon$. Let μ_x denote the union of all members of S_μ and λ_x denote the intersection of the corresponding $\lambda_y's$. Then λ_x is a fuzzy open set containing x_α and μ_x is a fuzzy open set satisfying $\mu_x \geq \mu - \varepsilon$.

But this is true for each $x \in X$. Thus we get a collection $\{\mu_x : x \in X\}$ of fuzzy open sets each satisfying $\mu_x \geq \mu - \varepsilon$ and another collection $\{\lambda_x : x \in X\}$ of fuzzy open sets such that $x_\alpha \in \lambda_x (\alpha = \lambda(x))$. Then $C_\lambda = \{\lambda_x : x \in X\}$ is a cover of λ. As λ is compact there exists a finite collection S_λ of C_λ satisfying $\vee\{\lambda_x : \lambda_x \in S_\lambda\} \geq \lambda - \varepsilon$. Let λ' denote the union of the members of S_λ and μ' denote the intersection of the corresponding μ_x. Then λ' is a fuzzy open set satisfying $\lambda' \geq \lambda - \varepsilon$ and μ' is a fuzzy open set satisfying $\mu' \geq \mu - \varepsilon$. Further then $f(\lambda' \times \mu') \subseteq \gamma$.

Result 3.11

(i) For $t \in G$ and a fuzzy subset μ of X, $cl\pi(t \times \mu) = \pi(t \times cl\mu)$

(ii) Let G and X be product related, then for a fuzzy subset λ of G and a fuzzy subset μ of X, $\pi(cl\lambda \times cl\mu) \subseteq cl\pi(\lambda \times \mu)$ and $cl\pi(cl\lambda \times \mu) = cl\pi(\lambda \times cl\mu) = cl\pi(\lambda \times \mu)$.

(iii) If λ is a compact fuzzy subset of G and μ is a compact fuzzy subset of X, then $\pi(\lambda \times \mu)$ is a compact fuzzy subset of X.

(iv) If λ is a compact fuzzy subset of G and μ is a compact fuzzy subset of X, and γ is a fuzzy neighbourhood of $\pi(\lambda \times \mu)$, then for any $\varepsilon > 0$, there exists fuzzy open sets λ' and μ' such that $\lambda' \geq \lambda - \varepsilon$ and $\mu' \geq \mu - \varepsilon$ such that $f(\lambda' \times \mu') \subseteq \gamma$.

(v) $\pi^t\mu = \mu\pi^{-t}$ for any $t \in G$.

(vi) $\pi^t\mu^c = 1 - \pi^t\mu$

Proof. (i) Since π^t is a homeomorphism $cl\pi^t(\mu) = \pi^t(cl\mu)$, i.e., $cl\pi(t \times \mu) = \pi(t \times cl\mu)$

(ii) Since G and X are product related, $(cl\lambda \times cl\mu) = cl(\lambda \times \mu)$ which implies

$$\pi(cl\lambda \times cl\mu) = \pi\{cl(\lambda \times \mu)\}$$

$\Rightarrow \pi(cl\lambda \times cl\mu) = \pi\{cl(\lambda \times \mu)\} \subseteq cl\pi(\lambda \times \mu)$, since π is continuous.

Again $\pi(\lambda \times \mu) \subseteq \pi(cl\lambda \times \mu) \subseteq \pi(cl\lambda \times cl\mu) \subseteq cl\pi(\lambda \times \mu)$

and $\pi(\lambda \times \mu) \subseteq \pi(\lambda \times cl\mu) \subseteq \pi(cl\lambda \times cl\mu) \subseteq cl\pi(\lambda \times \mu)$ and consequently $cl\pi(cl\lambda \times \mu) = cl\pi(\lambda \times cl\mu) = cl\pi(\lambda \times \mu)$.

(iii) λ and μ are fuzzy compact, so $\lambda \times \mu$ is fuzzy compact. As continuous image of a fuzzy compact set is fuzzy compact $\pi(\lambda \times \mu)$ is fuzzy compact.

(iv) Follows from **Result 3.10**.

(v) We have for any $u \in X$, $\pi^t(\mu)(u) = \pi(t \times \mu)(u) = \sup\{(t \times \mu)(s, x) : \pi(s, x) = u\}$

$= \sup\{(t(s) \wedge \mu(x) : \pi(s, x) = u\}$

$= \sup\{(t(t) \wedge \mu(x) : \pi(t, x) = u\}$, since $t(s) \neq 0$ only when $t = s$.

$= \mu(x)$ where $\pi(s, x) = u$

$= \mu(\pi^{-t})(u)$.

(vi) From (v) we have $\pi^t(\mu) = \mu(\pi^{-t})$ for any $\mu \in I^X$ and $t \in G$.

Now for any $x \in X$, we have $(\pi^t\mu^c)(x) = (\mu^c\pi^{-t})(x) = \mu^c((\pi^{-t})(x)) = 1 - (\mu\pi^{-t})(x)$

$= (1 - \mu\pi^{-t})(x)$. Therefore $\pi^t\mu^c = 1 - \mu\pi^{-t}$.

Result 3.12. Let α be a constant fuzzy subset of G and $\mu \in I^X$ be fuzzy open. Then $\pi(\alpha \times \mu)$ is fuzzy open.

Proof. We have for any $u \in X$, $\pi(\alpha \times \mu)(u) = \sup\{(\alpha \times \mu)(t,\ x) : \pi(t,\ x) = u\}$

$= \sup\{(\alpha(t) \wedge \mu(x) : \pi(t,\ x) = u\} = \sup\{(\alpha \wedge \mu(x) : \pi(t,\ x) = u\}$

$= \alpha \wedge \sup\{\mu(x) : \pi^t(x) = u\} = \alpha \wedge \sup\{\mu(\pi^{-t}(u)) : \pi^{-t}(u) = x\}$

$= \alpha \wedge \sup\{\pi^t\mu(u) : \pi^{-t}(u) = x\}$, since $\mu\pi^{-t} = \pi^t\mu$.

$= \alpha \wedge \{\vee\{\pi^t\mu(u)\}$ where $\pi^{-t}(u) = x$

$= \{\alpha \wedge \{\vee(\pi^t\mu)\}\}(u)$, where $\pi^{-t}(u) = x$

Thus $\pi(\alpha \times \mu) = \alpha \wedge \{\vee(\pi^t\mu)\}$. Now each π^t is open and μ is open so $\pi^t\mu$ is open. Also by definition of fuzzy topology is open. Consequently $\alpha \wedge \{\vee(\pi^t\mu)\}$ is open. Hence $\pi(\alpha \times \mu)$ is open.

Corollary 3.13. Let μ be a fuzzy open subset of X. then for any fuzzy point t_α of G, $\pi(t_\alpha \times \mu)$ is fuzzy open.

Proof. We have for any $u \in X$, $\pi(t_\alpha \times \mu)(u) = \sup\{(t_\alpha \times \mu)(s,\ x) : \pi(s,\ x) = u\}$

$= \sup\{t_\alpha(s) \wedge \mu(x) : \pi(s,\ x) = u\} = \alpha \wedge \mu(x) : \pi(t,\ x) = u$

$= \alpha \wedge \mu(x) : \pi^t(x) = u$

$= \alpha \wedge \mu(\pi^{-t}(u))$

$= \alpha \wedge \pi^t\mu(u)$, since $\mu\pi^{-t} = \pi^t\mu$.

$= (\alpha \wedge \pi^t\mu)(u)$, considering α as a constant fuzzy subset on X.

Thus $\pi(\alpha \times \mu) = \alpha \wedge \pi^t\mu$. Now π^t is open and μ is open so $\pi^t\mu$ is open. Also by definition of fuzzy topology α is open. Consequently $\alpha \wedge \pi^t$ is fuzzy open. Hence $\pi(t_\alpha \times \mu)$ is open.

Corollary 3.14. Let λ be any fuzzy subset of G and $\mu \in I^X$ be fuzzy open, then $\pi(\lambda \times \mu)$ is fuzzy open.

Proof. We have $\lambda = \vee t_\alpha$, where $\alpha = \lambda(x)$. So $\pi(\lambda \times \mu) = \pi(\vee t_\alpha \times \mu) = \vee\pi(t_\alpha \times \mu)$. As already proved each $\pi(t_\alpha, \times\mu)$ is open and hence $\pi(\lambda \times \mu)$ is open.

Result 3.15 Let μ be a fuzzy closed subset of X. then for any fuzzy point t_α of G, $\pi(t_\alpha \times \mu)$ is fuzzy closed.

Proof. We have for any $u \in X$, $\pi(t_\alpha \times \mu)(u) = \sup\{(t_\alpha\mu)(s,\ x) : \pi(s,\ x) = u\}$

$= \sup\{t_\alpha(s) \wedge \mu(x) : \pi(s,\ x) = u\}$

$= \alpha \wedge \mu(x) : \pi(t,\ x) = u$, since $t_\alpha(s) \neq 0$ only when $s = t$.

$= \alpha \wedge \mu(x) : \pi^t(x) = u$

$= \alpha \wedge \mu(\pi^{-t}(u))$

$= \alpha \wedge \pi^t\mu(u)$, since $\mu\pi^{-t} = \pi^t\mu$.

$= (\alpha \wedge \pi^t\mu)(u)$, considering α as a constant fuzzy subset on X.

Thus $\pi(\alpha \times \mu) = \alpha \wedge \pi^t\mu$. Now π^t is closed and μ is closed so $\pi^t\mu$ is closed. Also by definition of fuzzy topology is closed. Consequently $\alpha \wedge \pi^t\mu$ is fuzzy closed. Hence $\pi(t_\alpha \times \mu)$ is closed.

Corollary 3.16. Let λ be any fuzzy subset of G and $\mu \in I^X$ be fuzzy closed. If suppλ is finite, then $\pi(\lambda \times \mu)$ is fuzzy closed.

Proof. We have $\lambda = \vee t_\alpha$, where $\alpha = \lambda(x)$. So $\pi(\lambda \times \mu) = \pi(\vee t_\alpha \times \mu) = \vee\pi(t_\alpha \times \mu)$. As already proved each $\pi(t_\alpha, \times\mu)$ is closed. Also since suppλ is finite, the union is over finite number of closed fuzzy subsets. Hence $\pi(\lambda \times \mu)$ is closed.

4. Invariant fuzzy subsets

In this section we will introduce the notion of invariance of a fuzzy subset of X under the action of a fuzzy subset of G.

Definition 4.1 Let λ a fuzzy subset of G and μ a fuzzy subset of X. Then μ is said to be invariant under λ or λ-invariant provided that $\pi(\lambda \times \mu) \subseteq \mu$. If $\lambda = \chi_G$ then μ is simply said to be invariant. If $\lambda = \chi_G$ and μ is a crisp

subset, then fuzzy invariance reduces to crisp invariance.

Result 4.2 (i) If μ is a fuzzy subset of X, and λ is a fuzzy subgroup of X satisfying $\lambda(0) = 1$, then the following statements are pairwise equivalent : $\pi(\lambda \times \mu) \subseteq \mu$. ; $\pi(\lambda \times \mu) = \mu$; $\pi(\lambda(t) \times \mu) \subseteq \mu \ \forall t \in G$.

(ii) 0 and 1 are invariant.

(iii) If $\pi(t \times \mu) \subseteq \mu$, then $\pi(-t \times \mu^c) \subseteq \mu^c$ and conversely

(iv) If μ is λ-invariant then intμ is λ-invariant and clμ is λ-invariant provided G and X are product related.

(v) If $\{\mu_i\}$ is a collection of λ-invariant fuzzy subset, then $\vee\mu_i$ and $\wedge\mu_i$ are λ-invariant.

(vi) Let A be a crisp subset of G and μ a fuzzy subset of X., then μ is χ_A-invariant iff μ^c is χ_{-A}- invariant.

Proof (i) First we show $\pi(\lambda \times \mu) \subseteq \mu \Leftrightarrow \pi(\lambda \times \mu) = \mu$.

Suppose $\pi(\lambda \times \mu) \subseteq \mu$. We have for any u in X, $\pi(\lambda \times \mu)(u) = \sup\{(\lambda \times \mu)(t, x) : \pi(t, x) = u\}$

$= \sup\{(\lambda(t) \wedge \mu(x) : \pi(t, x) = u\}$

$\geq \lambda(0) \wedge \mu(u)$, since $\pi(0, u) = u$

$= \mu(u)$ since $\lambda(0) = 1$.

Thus $\pi(\lambda \times \mu) = \mu$. Consequently $\pi(\lambda \times \mu) \subseteq \mu. \Leftrightarrow \pi(\lambda \times \mu) = \mu$.

Next we show $\pi(\lambda(t) \times \mu) \subseteq \mu \Leftrightarrow \forall t \in G.\pi(\lambda \times \mu) \subseteq \mu$

Given $\pi(t_\alpha \times \mu) \subseteq \mu$ for all $t_\alpha : \alpha = \lambda(t)$.

Now $\pi(\lambda \times \mu) = \pi(\vee t_\alpha \times \mu) = \vee\pi(t_\alpha \times \mu) \subseteq \mu$

(ii) Trivial

(iii) We have $\pi(t \times \mu) \subseteq \mu \Rightarrow \pi^t(\mu) \subseteq \mu$

And $\pi(-t \times \mu^c) = \pi^{-t}(\mu^c)$

Now $1 - \mu \subseteq 1 - \pi^t(\mu) = \mu^c\pi^{-t} \Rightarrow \mu^c \subseteq \pi^t\mu^c \Rightarrow \pi^{-t}\mu^c \subseteq \mu^c \Rightarrow \pi(-t \times \mu^c) \subseteq \mu^c$

(iv) We have $\pi(\lambda \times \mu) \subseteq \mu$. Now $\pi(\lambda \times \mu^o) \subseteq \pi(\lambda \times \mu) \subseteq \mu$. Now μ^o is open so $\pi(\lambda \times \mu^o)$ is open and contained in μ. But μ^o is the largest open fuzzy set contained in μ.

Hence $\pi(\lambda \times \mu^o) \subseteq \mu^o$.

Since G and X are product related $\lambda \times cl\mu \subseteq cl\lambda \times cl\mu = cl(\lambda \times \mu)$

$\Rightarrow \pi(\lambda \times cl\mu) \subseteq \pi\{cl(\lambda \times \mu)\} \subseteq cl\pi(\lambda \times \mu)$, since π is continuous

$\subseteq cl\mu$.

(v) We have $\pi(\lambda \times \vee\mu_i) = \pi\{\vee(\lambda \times \mu_i)\} = \vee\pi(\lambda \times \mu_i) \subseteq \mu$.

Similarly $\pi(\lambda \times \wedge\mu_i) = \pi\{\wedge(\lambda \times \mu_i)\} \subseteq \wedge\pi(\lambda \times \mu_i) \subseteq \mu$.

(vi) It is sufficient to show that $\pi(t \times \mu) \subseteq \mu$, then $\pi(-t \times \mu^c) \subseteq \mu^c$ and conversely for $t \in A$. Hence it follows from (iii)

5. Fuzzy Orbits

In this section we will introduce the notion of orbits in fuzzy setting and extend some classical results.

Definition 5.1 Let $x \in X$ and λ a fuzzy subgroup of G. Then the fuzzy orbit of x under λ or the λ- orbit of x is defined to be fuzzy subset $\pi(\lambda \times x)$. The fuzzy orbit closure of x under λ or the λ- orbit closure of x is defined to be the fuzzy subset $cl\pi(\lambda \times x)$.

When $\lambda = \chi_G$, then the fuzzy orbit coincides with the crisp orbit. We will denote the orbit of x under λ by λ_x. We assume that G and X are product related and that $\lambda(0) = 1$.

Remark : $\lambda_x(u) = \sup\{\lambda(t) : \pi(t, x) = u\}$.

We have $\lambda_x(u) = \pi(\lambda, x)(u) = \sup\{(\lambda, x)(t, y) : (t, y) = u\}$

$= \sup\{(\lambda(t) \wedge x(y) : \pi(t, y) = u\}$

$= \sup\{(\lambda(t) : \pi(t, x) = u\}$, since $x(y) \neq 0$ iff $x = y$.

Result 5.2 Let $x, y \in X$, then

(i) $\lambda_x(y) = \lambda_y(x)$

Proof. We have $\lambda_x(y) = \sup\{\lambda(t) : \pi(t, x) = y\} = \sup\{\lambda(t) : \pi(-t, y) = x\}$

$= \sup\{\lambda(-t) : \pi(-t, y) = x\} = \lambda_y(x)$.

(ii) λ-orbit of x is λ-invariant.

Proof. We have for any $u \in X$,

$\pi(\lambda \times \lambda_x)(u) = \sup\{(\lambda \times \lambda_x)(t, y) : \pi(t, y) = u\} = \sup\{\lambda(t) \wedge \lambda_x(y) : \pi(t, y) = u\}$.

$= \sup[\lambda(t) \wedge \sup\{\lambda(s) : \pi(s, x) = y\} : \pi(t, y) = u\}]$

$= \sup_t \sup_s\{\lambda(t) \wedge \lambda(s) : \pi(t + s, x) = u\}\}$

$\leq \sup_t \sup_s\{\lambda(t + s) : \pi(t + s, x) = u\}$, since λ is a fuzzy subgroup of G

$= \sup\{\lambda(r) : \pi(r, x) = u\} = \lambda_x(u)$. Hence $\pi(\lambda \times \lambda_x) \subseteq \lambda_x$.

(iii) Let $\eta \in I^G$ be λ-invariant and $\eta(x) = 1$, then $\lambda_x \subseteq \eta$. In other words fuzzy orbit of x is the least λ-invariant fuzzy subset containing the fuzzy point x.

Proof. We have for any $u \in X$,

$\lambda_x(u) = \sup\{\lambda(t) : \pi(t, x) = u\} = \sup\{\lambda(t) \wedge \eta(x) : \pi(t, x) = u\}$, since $\eta(x) = 1$.

$= \pi(\lambda \times \eta)(u) \subseteq \eta(u)$.

(iv) The closure of λ-orbit of x is λ-invariant.

Proof. Since G and X are product related, we have $\pi(\lambda \times cl\lambda_x) \subseteq \pi(cl(\lambda \times \lambda_x))$

$\subseteq cl\pi(\lambda \times \lambda_x)$, since π is continuous

$\subseteq cl\lambda_x$, since $\pi(\lambda \times \lambda_x) \subseteq \lambda_x$.

(v) Let $\eta \in I^G$ be closed, λ-invariant and $\eta(x) = 1$, then $cl\lambda_x \subseteq \eta$. In other words fuzzy orbit of x is the least λ-invariant fuzzy closed subset containing the fuzzy point x.

Proof. Since λ_x is λ-invariant, $\pi(\lambda \times \lambda_x) = \lambda_x$ by Result 2(i). So $\lambda_x = \pi(\lambda \times x) \subseteq \pi(\lambda \times \eta)$, since $\eta(x) = 1$,

$\subseteq \eta \Rightarrow \lambda_x \subseteq \eta \Rightarrow cl\lambda_x \subseteq cl\eta \Rightarrow cl\lambda_x \subseteq \eta$, since η is closed.

(vi) If $cl\lambda_x(y) = 1$, then $cl\lambda_y \subseteq cl\lambda_x$.

Proof. Since $cl\lambda_y$ is the least closed λ-invariant fuzzy subset containing y, the result follows.

(vii) The collection $\{\lambda_x : x \in X\}$ is a cover of 1.

Proof. This is because for each $x \in X$, $\lambda_x(x) = 1$.

(viii) Let x, y ∈ X such that $\lambda_x(y) > 0$. Then, $\lambda_x(u) > 0 \Leftrightarrow \lambda_y(u) > 0$ for any u in X.

Proof. Given $\lambda_x(y) > 0$. So $\exists t \in G$ with $\lambda(t) > 0 : \pi(t, x) = y$. This implies $\pi(-t, y) = x$. Let $\lambda_x(u) > 0$, then there exist $s \in G$ with $\lambda(s) > 0 : \pi(s, x) = u$.

So $\pi(s - t, y) = u$ and $\lambda(s - t) \geq \lambda(s) \wedge \lambda(-t) = \lambda(s) \wedge \lambda(t) > 0$, since λ is a fuzzy subgroup $\Rightarrow \lambda_y(u) > 0$.

Similarly $\lambda_y(u) > 0 \Rightarrow \lambda_x(u)$.

(ix) If $\lambda_x(y) = 0$, then $\lambda_x(u) = 0$, for all u in X such that $\lambda_y(u) > 0$

Proof. We have $\lambda_x(y) = 0$, so there exists no $t \in G$ with $\lambda(t) > 0 : \pi(t, x) = y - -(i)$ Let $u \in X$ such that $\lambda_y(u) > 0 \Rightarrow \exists r \in G$ with $\lambda(r) > 0 : \pi(r, y) = u. - -(ii)$

Suppose $\lambda_x(u) > 0$, then $\exists s \in G$ with $\lambda(s) > 0 : \pi(s, x) = u \Rightarrow \pi(-r, (s, x)) = \pi(-r, u) = y$ using (ii)

$\Rightarrow \pi(-r + s, x)) = y$, where $\lambda(-r + s) \geq \lambda(-r) \wedge \lambda(s) = \lambda(r) \wedge \lambda(s) > 0$, since λ is a fuzzy subgroup But this

contradicts (i). Hence the result.

We consider the collection of all λ-orbits and define a relation on it as $\lambda_x \sim \lambda_y$ if $\lambda_x(y) > 0$. Then it can be easily verified that this relation is an equivalence relation, where the equivalence class of λ_x is $[\lambda_x] = \{\lambda_y : \lambda_x(y) > 0\}$. So $[\lambda_x]$ is a fuzzy subset of X defined as $[\lambda_x](y) = \lambda_x(y)$.

The collection $\{supp[\lambda_x] : x \in X\}$ is a crisp partition of X.

Let us denote the set of all equivalence classes by $X/_\lambda$, i.e., $X/_\lambda = \{[\lambda_x] : x \in X\}$. Define a map $f : X \to X/_\lambda$ given by $f(x) = [\lambda_x]$. We equip $X/_\lambda$ with the corresponding quotient topology.

Result 5.3 The map $f : X \to X/_\lambda$ given by $f(x) = [\lambda_x]$ is an open map.

Proof. Let μ be an open fuzzy subset of X. To show $f(\mu)$ is fuzzy open in $X/_\lambda$. Since $X/_\lambda$ has quotient topology with respect to f, it is sufficient to show $f^{-1}\{f(\mu)\}$ is open in X.

We have $f^{-1}\{f(\mu)\}(x) = f(\mu)(f(x)) = f(\mu)[\lambda_x] = \sup\{\mu(y) : f(y) = [\lambda_x]\}$

$= \sup\{\mu(y) : [\lambda_y] = [\lambda_x]\} = \sup\{\mu(y) : \lambda_y \in [\lambda_x]\}$

$= \sup\{\mu(y) : \lambda_x(y) > 0\} = \sup\{\mu(y) : \pi(t, x) = y \text{ for some } t \in G \text{ with } \lambda(t) > 0\}$

$= \sup\{\mu(\pi^{-t}(x)) : \pi(t, x) = y \text{ for some } t \in G \text{ with } \lambda(t) > 0\}$

$= \sup\{\pi^t\mu(x) : \pi(t, x) = y \text{ for some } t \in G \text{ with } \lambda(t) > 0\}$

$= \vee\pi^t\mu(x)$, where $t \in G$ with $\lambda(t) > 0$

Hence $f^{-1}\{f(\mu)\} = \vee\pi^t\mu$.

Now each π^t is a homeomorphism and μ is open and so $\pi^t\mu$ is open. Consequently $\vee\pi^t\mu$. is open. Hence $f^{-1}\{f(\mu)\}$ is open.

Consequently f is a fuzzy open map.

Corollary 5.4 : If λ has finite support, then the map $f : X \to X/_\lambda$ given by $f(x) = \lambda_x$ is a closed map.

Proof. Let μ be a closed fuzzy subset of X. To show $f(\mu)$ is fuzzy closed in $X/_\lambda$. Since $X/_\lambda$ has quotient topology with respect to f, it is sufficient to show $f^{-1}\{f(\mu)\}$ is closed in X.

We have $f^{-1}\{f(\mu)\}(x) = f(\mu)(f(x)) = f(\mu)[\lambda_x] = \sup\{\mu(y) : f(y) = [\lambda_x]\}$

$= \sup\{\mu(y) : [\lambda_y] = [\lambda_x]\} = \sup\{\mu(y) : \lambda_y \in [\lambda_x]\}$

$= \sup\{\mu(y) : \lambda_x(y) > 0\} = \sup\{\mu(y) : \pi(t, x) = y \text{ for some } t \in G \text{ with } \lambda(t) > 0\}$

$= \sup\{\mu(\pi^{-t}(x)) : \pi(t, x) = y \text{ for some } t \in G \text{ with } \lambda(t) > 0\}$

$= \sup\{\pi^t\mu(x) : \pi(t, x) = y \text{ for some } t \in G \text{ with } \lambda(t) > 0\}$

$= \vee\pi^t\mu(x)$, where $t \in G$ and $\lambda(t) > 0$

Hence $f^{-1}\{f(\mu) = \vee\pi^t\mu$. Now each π^t is a homeomorphism and μ is closed and so each $\pi^t\mu$ is closed. As suppλ is finite supremum is over finite number of $t's$.

Consequently $\vee\pi^t\mu$ being union of finite number of closed fuzzy sets is closed.

Hence $f^{-1}\{f(\mu)\}$ is closed.

Consequently f is a fuzzy closed map.

6. Construction of new fuzzy topological transformation groups from given ones

In this section we will construct new fuzzy topological transformation groups from given ones.

Result 5.1 Let (π, G, X) and $(\varphi$, G, Y) be two fuzzy topological transformation groups. Define Ψ: $G\times(X\times Y) \to X \times Y$ as $\Psi(t, (x, y)) = (\pi(t, x), (t, y))$. Then $(\Psi, G, (X \times Y))$ is a fuzzy topological transformation group.

Proof. As π is fuzzy continuous Ψ is fuzzy continuous.

Also Ψ(0, (x, y)) = (π (0, x), π(0, y)) = (x, y) and

Ψ (s + t, (x, y)) = (π(s + t, x), π(s + t, y))

= (π (s, π(t, x), π(s, π(t, y)) = Ψ (s, (π(t, x), π(t, y))) = Ψ(s, Ψ(t, (x, y)).

Thus (FGT1), (FGT2) and FGT3) are satisfied and hence (Ψ, G, (X×Y)) is a fuzzy topological transformation group.

Result 5.2 Let λ be a fuzzy subgroup of G and $\mu \in I^X$ such that $\pi(\lambda \times \mu) \subseteq \mu$. Then (supp$\lambda$, supp$\mu$, $\pi/_{supp\lambda \times supp\mu}$) is fuzzy topological transformation group.

Proof. Since λ is fuzzy subgroup, suppλ is an ordinary group Since subgroup of a fuzzy topological group is a fuzzy topological group, suppλ is a fuzzy topological group. Since $\pi(\lambda \times \mu) \subseteq \mu$, range of $\pi/_{supp\lambda \times supp\mu}$) is contained in supp$\mu$. Also restriction of fuzzy continuous function if fuzzy continuous. Hence the result.

Corollary 5.3 Let λ be a fuzzy subgroup of G then (suppλ, X, $\pi/_{supp\lambda \times X}$ is fuzzy topological transformation group.

Result 5.4 Consider the map $f : X \to X/_\lambda$ given by $f(x) = [\lambda_x]$. Define a map $\phi : G \times X \to X/_\lambda$ as $\phi(t, [\lambda_x]) = f(\pi(t, x))$. Then ($\phi$, G, $X/_\lambda$) is a fuzzy topological transformation group.

Proof. f and π being continuous, we have ϕ is continuous. Now $\phi(0, [\lambda_x]) = f(\pi(0, x)) = f(x) = [\lambda_x]$. Also $\phi(s, \phi(t, [\lambda_x])) = \phi\{s, f(\pi(t, x))\} = \phi\{s, [\lambda_{\pi(t,x)}]\} = f\{\pi(s, \pi(t, x)\} = f\{\pi(s + t, x)\} = \phi(s + t, [\lambda_x])$. Thus (FGT1), (FGT2) and FGT3) are satisfied and hence (ϕ, G, $X/_\lambda$) is a fuzzy topological transformation group.

In X define a relation $x \sim y$ if $[\lambda_x] = [\lambda_y]$. Then this relation is an equivalence. Denote the equivalence class of x by $[x]$. Let g be the canonical mapping from X to $\{[x] : x \in X\}$, i.e, $g(x) = [x]$. Equip $\{[x] : x \in X\}$ with the corresponding quotient topology. Then clearly this space is fuzzy homemorphic with the already introduced space $X/_\lambda$, where the corresponding homeomorphism is $x \to [\lambda_x]$. We can denote both the spaces by the same notation $X/_\lambda$. Consequently (ϕ, G, $X/_\lambda$), where ϕ: $G \times X \to X/_\lambda$ defined as $\phi(x) = g(\pi(t, x))$ is a fuzzy topological transformation group. Here $X/_\lambda$ stands for the quotient space $\{[x] : x \in X\}$.

Conclusion :. In this paper we have developed the notion of topological transformation group in fuzzy setting. We have attempted to extend most of the results of classical topological transformation group to this fuzzy setting. A topological transformation group is the basic structure in the study of topological dynamics. As the concept of orbit and orbit closure are fuzzified, it is expected that the other concepts of topological dynamics can be worked on in this setting.

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On P-nipotence of Finite Groups

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Abstract

A subgroup H is said to be weakly c^*-normal in a group G if there exists a subnormal subgroup K of G such that $HK = G$ and $H \cap K$ is s-quasinormally embedded in G. We give some results which generalize some authors' results.

Keywords: Weakly c^*-normality, p-nilpotence, s-quasinormally embedded

1. Introduction

In this paper the word group is always finite. Ore (1937, p150) gives quasinormality of subgroups. A subgroup H is said to be quasinormal in G if for every subgroup K of G such that $HK = KH$. A subgroup H of a group G is said to be s-quasinormal in G if H permutes with every Sylow subgroup of G. This concept was introduced by Kegel (1962, p 205),and extensively studied (Deskins, 1963, p126-131). Ballester-Bolinches and Pedraza-Aguilera (1998, p114) introduce the conception of s-quasinormally embedded in G if for each prime divisor p of H, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-quasinormal subgroup of G. Wei and Wang (2007, p212)introduced the notion of c^*-normality, a subgroup H of G is said to be c^*-normal in G if there exists a subgroup $K \leq G$ such that $G = HK$ and $H \cap K$ is s-quasinormally embedded in G.

For some notions and notations,the reader is referred to Robinson (1995)and Huppert (1968).

2. Some definitions and preliminary results

A subgroup H is called weakly c-normal in a group G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the largest normal subgroup of G contained in H. The conception of weakly c-normality was introduced by Lu, Guo, and Shum (2002, p 5506).

Definition 2.1 A subgroup H is said to be weakly c^*-normal in G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_sG$, where H_sG is s-quasinormally embedded subgroup of G contained in H.

Lemma 2.1 (Ballester-Bolinches and Pedraza-Aguilera, 1998, Lemma 1) Suppose that U is s-quasinormally embedded in a group G, and that $H \leq G$ and $K \lhd G$.

(1) If $U \leq H$, then U is s-quasinormally embedded in H.

(2) If UK is s-quasinormally embedded in G, then UK/K is s-quasiormally embedded in G/K.

(3) If $K \lhd H$ and H/K is s-quasinormally embedded in G/K, then H is s-quasinormally embedded in G.

Lemma 2.2 Let G be a group. Then the following statements hold.

(1) If H is weakly C^*-normal in G and $H \leq M \leq G$, then H is weakly c^*-normal in M.

(2) Let $N \lhd G$ and $N \leq H$. Then H is weakly c^*-normal in G if and only if H/N is weakly c^*-normal in G/N.

(3)Let π be a set of primes. H is a π-subgroup of G and N a normal π'-subgroup of G, if H is weakly c^*-normal in G, then HN/N is weakly c^*-normal in G/N.

(4)Let $L \leq G$ and $H \leq \Phi(L)$ If H is weakly c^*-normal in G, then H is s-quasinormally embedded in G.

(5)Let H is c^*-normal in G, then H is weakly c^*-normal in G.

Proof. (1) If H is weakly c^*-normal in G, that is, there exists a subnormal subgroup T of G such that $HT = G$ and $H \cap T$ is s-quasinormally embedded in G, then $M = M \cap G = (M \cap T)H$. Since T is subnormal in G, then $M \cap T$ is subnormal in M, and $H \cap (M \cap T)$ is s-quasinormally embedded in M. So we have H is weakly c^*-normal in M.

(2) If H is weakly c^*-normal in G, then there exists a subnormal T of G such that $G = HT$ and $H \cap T$ is s-quasinormally embedded in G. Then $G/N = (H/N)(TN/N)$, where TN/N is subnormal in G/N and $(H/N) \cap (TN/N)$ is s-quasinormally embedded in G/N. Then H/N is weakly c^*-normal in G/N. The converse part can be proved similarly.

(3) If H is weakly c^*-normal in G, then there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T$ is s-quasinormally embedded in G. Since $|G|_{\pi'} = |T|_{\pi'} = |TN|_{\pi'}$, then $N \leq T$. Clearly $(HN/N)(T/N) = G/N$ and $(HN/N) \cap (T/N) = (H \cap T)N/N$ is s-quasinormally embedded in G/N.

(4) Since H is weakly c^*-normal in G, then there exists a subnormal subgroup T such that $G = HT$ and $H \cap T$ is s-quasinormally embedded in $G.L = L \cap (HT) = H(T \cap L)$. Since $H \leq \Phi(L)$, then $L = T \cap L$ and so $L \leq T$, then $T = G$ and $H = H \cap T$ is s-quasinormally embedded in G.

(5) The result is obvious.

Lemma 2.3 Let M be a maximal subgroup of G and P a normal Sylow p-subgroup of G such that $G = PM$, where p is a prime, then $P \cap M$ is a normal subgroup of G.

Lemma 2.4 (Wei and Wang, 2007, Lemma 2.5) Let G be a group, K an s-quasinormal subgroup of G, P a Sylow p-subgroup of K where p is a prime divisor of $|G|$. If either $P \leq O_P(G)$ or $K_G = 1$, then P is s-quasinormal in G.

Lemma 2.5 (Li, Wang and Wei, 2003, Lemma 2.2) Let G be a group and P is s-quasinormal p-subgroup of G where p is a prime, then $O_P(G) \leq N_G(P)$.

Lemma 2.6 (Wei and Wang, 2007, Lemma 2.8) Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.

(1) If N is normal in G of order p, then N is in $Z(G)$.

(2) If G has cyclic Sylow p-subgroups, then G is p-nilpotent.

(3)If $M \leq G$ and $|G : M| = p$, then $M \lhd G$.

Lemma 2.7 (Huppert, 1968, IV, 5.4) Suppose that G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Then G is a group which is not nilpotent but whose proper subgroups are all nilpotent.

Lemma 2.8 (Robinson, 1995, III, 5.2) Suppose that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then

(1) G has a normal Sylow p-subgroup for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.

(2) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

(3) If P is non-abelian and $p \neq 2$, then $exp(P) = p$.

(4) If P is non-abelian and $p = 2$, then $exp(P) = 4$.

(5)If P is abelian, then $exp(P) = p$.

Lemma 2.9 Let H be a subgroup of G. Then H is weakly c^*-normal in G if and only if there exists a subgroup K such that $G = HK$ and $H \cap K = H_{sG}$.

Proof. $\Leftarrow$ It is clear.

$\Rightarrow$ By definition 2.1, there exists a subnormal subgroup L of G such that $G = HL$ and $H \cap L \leq H_{sG}$ If $H \cap L < H_{sG}$, note that $K = LH_{sG}$, then $HK = HLH_{sG} = LHH_{SG} = LH = G$ and hence $H \cap K = H \cap LH_{sG} = (H \cap L)H_{sG} = H_{sG}$.

3. Main results

Theorem 3.1 Let G be a group, P a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If all maximal subgroups of P are weakly c^*-normal in G, then G is p-nilpotent.

Proof. Suppose that the result is false, then we chose a minimal order G as a counterexample.We will prove by the following steps:

Steps 1. For every proper subgroup of G is p-nilpotent, thus G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent.

Let M be a maximal subgroup of G, Then $P \cap M$ is a maximal p-subgroup of P. By hypothesis, $P \cap M$ is weakly c^*-normal in G and so $P \cap M$ is weakly c^*-normal in M by lemma 2.2(1). Thus M, $P \cap M$ satisfies the hypotheses of the theorem, the minimal choice of G implies that M is p-nilpotent. Then we have that G is not p-nilpotent but all proper subgroups are p-nilpotent. Then, by lemma 2.7 and lemma 2.8(1), G has a normal Sylow p-subgroup for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.

Steps 2. Let L be a minimal normal subgroup of G contained in P, then G/L is p-nilpotent, L is the unique minimal normal of G and $L \not\leq \Phi(G)$.

Since P/L is a Sylow p-subgroup of G/L, we have M/L is a maximal subgroup in P/L, where M is a maximal subgroup of P. Since M is weakly c^*-normal in G, by lemma 2.2(2) M/L is weakly c^*-normal in G/L. Thus G/L, P/L satisfies the hypotheses of the theorem and so we have G/L is p-nilpotent by the minimal choice of G. If L_1 is an another minimal normal subgroup, then $G/1 \cong G/L \times G/L_1$ is p-nilpotent and so L is unique. If $L \leq \Phi(G)$, then $G/\Phi(G)$ is p-nilpotent, and so is G, a contradiction.

Steps 3. $\Phi(P) \neq 1$.

If $\Phi(P) = 1$, then P is abelian. By steps 1 and lemma 2.8(5), $exp(P) = p$. If $|P/\Phi(P)| = P^n$ and $P/\Phi(P) =< x_1\Phi(P), x_2\Phi(P), \cdots, x_n\Phi(P) >$, then $P =< x_1, x_2 \cdots, x_n >$. So we have $| < x_1 > | = p$, and $< x_i >$ char P, where i is nature number. And since P is normal in G, then $< x_i >$ are normal p-subgroup of G of order p. Thus by lemma 2.6(1), we have $< x_i > \leq Z(G)$ for all $i = 1, 2, \cdots, n$, then $P \leq Z(G)$, then G is p-nilpotent, a contradiction. Thus $\Phi(P) \neq 1$.

Steps 4. L is a Sylow p-subgroup of G.

By steps 3, $\Phi(P) \neq 1$, then $L \leq P$. If $L < P$, then for a maximal subgroup M of P, M is weakly c^*-normal in G and so there exists a subnormal subgroup K such that $G = MK$ and $M \cap K$ is s-quasinormally embedded in G. We consider the following cases.

1) $M \cap K = 1$.

Since $|K|_p = |G : M|_p = |PQ : MQ|_p = |P : M| = p$, then K has a normal p-complement Q_1 which is also a Sylow q-subgroup of G. By Sylow theorem, there exists an element $g \in G \setminus Q$ such that $Q_1^g = Q$. Since $M \lhd P$, then $G = MK = (MK)^g = MK^g$. Since $K^g \cong K$ and $Q = Q_1^g \leq K^g$, this implies $K^g \leq N_G(Q)$ in this case Q is not normal in G. So we have $G = MK = (MK)^g = MN_G(Q)$. So we have $M \cap N_G(Q) = 1$ and $N_G(Q) \leq K^g$. Thus $K^g = N_G(Q^g) = N_G(Q)$. If H be a sylow subgroup of $N_G(Q)$, then $K = HQ$ and $HP = PH = P$. This implies that H is s-quasinormal in G, then by lemma 2.5 we have $O_p(H) \leq N_G(H)$, and H is normal in G. Then $L = H \leq M$ since the minimality of L and L is unique. But $L \not\leq M$, a contradiction.

2) $M \cap K = M = M_{sG}$.

Then we have $M \leq K$, then $G = K$ is p-nilpotent, a contradiction.

3) $1 < M_{sG} < M$.

Let $S = M \cap K$. Then S is s?quasinormally embedded in G. Thus there exists an s-quasinormal subgroup R such that S is a Sylow p-subgroup of R. Then by lemma 2.5, we have $O^p(G) \leq N_G(S)$ and so S is normal in G, then we have, $S = P$ or $S = L$. If $S = P$. On the other hand, $|M| < |P|$, a contradiction. Then $S = L$ is a minimal normal Sylow p-subgroup of some s-quasinormal subgroup R of G, then for any sylow q-subgroup Q, we have $RQ = QR$ is a subgroup of G and, if $QS < G$, $Q \lhd RQ$ by (1), and so $LQ = L \times Q$. By steps 1 and Burnside's theorem, we have G is solvable. Thus $Q \leq C_G(L) \leq L$, a contradiction. Then $QS = G$, then

$G = PQ = QS$ and so $P = S^g$ for some $g \in Q$, a contradiction.

Steps 5. Conclusions.

By steps 4, $L = P$ is a Sylow p-subgroup of G, then, by hypothesis, maximal subgroup M of $L = P$ is weakly -normal in G. Then by lemma 2.10 there exists a subnormal subgroup K of G such that $G = MK$ and $M \cap K \leq M_{sG}$. Since $M_{sG} < L = P < L = P$, then $M_{sG} = 1$ and LQ/L is p-nilpotent since G/L is p-nilpotent by steps 2, where Q is a Hall p'-subgroup of G, then $LQ/L \lhd G/L$ and so $LQ \lhd G$. It follows from Q char $LQ \lhd G$ that Q is normal in G. Therefore G is p-nilpotent.

Corollary 3.1 (Wei and Wang, 2007, Theorem 3.1) Let G be a group, P a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p - 1) = 1$. If all maximal subgroups of P are c^*-normal in G, then G is p-nilpotent.

Theorem 3.2 Let G be a group, P a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If all cyclic subgroups of P of order p or 4 (if p = 2) are weakly c^*-normal in G, then G is p-nilpotent.

Proof. Suppose that the result is false, then we chose a minimal order G as a counterexample. We will prove by the following steps:

Steps 1. Let M be a proper subgroup of G, then M is p-nilpotent. So G is not p-nilpotent but all proper subgroups are p-nilpotent. Thus $G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non-normal cyclic Sylow q-subgroup of G. And so by Burnside's theorem G is solvable. Then $M \cap P$ is a Sylow p-subgroup of M. By hypothesis, for every cyclic subgroup of P of order p or 4 (if p = 2) is weakly -normal in G, then By lemma 2.2(1), for every cyclic subgroup of P M of order p or 4 (if p = 2) is weakly c^*-normal in M. Then M, $M \cap P$ satisfies the hypotheses of the theorem, M is p-nilpotent by the minimal choice of G, so we have: G is not p-nilpotent but all proper subgroups are p-nilpotent and so by lemma 2.7 and lemma 2.8(1), $G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non-normal cyclic Sylow q-subgroup of G.

Steps 2. Let L be a minimal normal subgroup of G contained in P, then L is unique minimal normal p-subgroup for some prime of $|G|$, G/L is p-nilpotent and $L \nleq \Phi(G)$. Furthermore, $L = F(G) = C_G(L)$.

Since all cyclic subgroups of P of order p or 4(if p = 2) is weakly c^*-normal in G, then by lemma 2.2(2) all cyclic subgroups of P/L with order p or 4 (if p = 2) is weakly c^*-normal in G/L, then the minimal choice of G implies that G/L is p-nilpotent. If $L \leq \Phi(G)$, then $G/\Phi(G)$ is p-nilpotent and G is p-nilpotent, a contradiction. By lemma 2.6 (Li, etc, 2003), $F(G) = L$. By steps 1, solubility of G implies that $L \leq C_G(F(G)) \leq F(G)$ and so $C_G(L) = F(G) = L$ as L is abelian.

Steps 3. Conclusions.

By steps 2 $C_G(L) = F(G) = L$. But on the other hand, for $x \in P, < x >$ is weakly c^*-normal in G, then there exists a subnormal subgroup T of G such that $G =< x > T$ and $< x > \cap T$ is s-quasinormally embedded in G. By lemma 2.7 and lemma 2.8, we have if p is odd or P is abelian, then $exp(P) = p$ or if $p = 2exp(P) = 4$. Since $F(G) =< x_1, x_2, \cdots, x_n >= L, | < x_i > | = p$ or 4 and $< x_i >$ char P since P is normal in G. Thus $F(G) = L =< x_i >$. So we have $LQ = QL = L \times Q$, Then $Q \leq N_G(L)$, then $G = P \times Q$ is nilpotent, a contradiction.

Corollary 3.2 (Li and Wang, 2004, Theorem 4.1) Suppose G is a group, p is a fixed prime number. If every element of $P_p(G)$ is contained in $Z_\infty(G)$. If $p = 2$, every cyclic subgroup of order 4 of G is s-quasinormal in G, then G is p-nilpotent.

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Generalized Kronecker Product and Its Application

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Abstract

In this paper, we promote the definition of Kronecker product, and give its corresponding properties. As the application of generalized Kronecker product, this paper shows the determination method that the algebraic operation in finite set suits the associative law.

Keywords: Generalized Kronecker product, Assoziativitaet, Decision condition

1. Introduction

Kronecker product expresses a special product of matrix. The product of a matrix A by $m \times n$ and a matrix B by $p \times q$ can be denoted by $A \otimes B$, which is a matrix by $mp \times nq$.

Definition 1.(Kronecker product)(*BellmanR.*,1970) The product of a matrix A by $m \times n$ and a matrix B by $p \times q$ can be denoted by $A \otimes B$, which is defined as follows:

$$A \otimes B = [a_{i,j}B] = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

Kronecker product also can be called direct product or tensor product.

Kronecker product has the following properties:

1). For $A_{m\times n}$ and $B_{p\times q}$, generally $A \otimes B \neq B \otimes A$.

2). The Kronecker product of arbitrary matrix and zero matrix equals zero matrix, i.e. $A \otimes 0 = 0 \otimes A = 0$.

3). If α and β are constant, $\alpha A \otimes \beta B = \alpha\beta(A \otimes B)$.

4). For $A_{m\times n}$, $B_{n\times k}$, $C_{l\times p}$ and $D_{p\times q}$, $AB \otimes CD = (A \otimes C)(B \otimes D)$.

5). For $A_{m\times n}$, $B_{p\times q}$, $C_{p\times q}$, $A \otimes (B \pm C) = (A \otimes B) \pm (A \otimes C)$, $(B \pm C) \otimes A = (B \otimes A) \pm (C \otimes A)$.

Note: properties 1)-5) is referred from (Rao c R,1971).

6).For $A_{m\times n}$ and $B_{p\times q}$, $(A \otimes B)^T = A^T \otimes B^T$.

7). For $A_{m\times n}$ and $B_{p\times q}$, $rank(A \otimes B) = rank(A)rank(B)$.

8). For $A_{m\times m}$ and $B_{n\times n}$, $det(A \otimes B) = (detA)^n(detB)^n$.

9). For $A_{m\times m}$ and $B_{n\times n}$, $tr(A \otimes B) = tr(A)tr(B)$.

10). For $A_{m\times n}$, $B_{m\times n}$, $C_{p\times q}$ and $D_{p\times q}$, $(A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D$.

11). For $A_{m\times n}$, $B_{k\times l}$, $C_{p\times q}$ and $D_{r\times s}$, $(A\otimes B)\otimes(C\otimes D) = A\otimes B\otimes C\otimes D$.

12). For $A_{m\times n}$, $B_{k\times l}$, $C_{p\times q}$, $(A\otimes B)\otimes C = A\otimes(B\otimes C)$.

13). For $A_{m\times n}$, $B_{p\times q}$, $C_{n\times r}$ and $D_{q\times s}$, $(A\otimes B)(C\otimes D) = (A\otimes C)(B\otimes D)$.

Note: properties 6)-13) is referred from (Brewer j w, 1978, 772-781).

But for the need of the real life and the mathematics development, the element of matrix may not limit to numbers, thus we introduce a new concept, also namely promoting the matrix concept.

2. Definition of generalized Kronecker product

Definition 2. Suppose S be a nonempty set, $(S,\circ,+)$ be an algebra system, then A is matrix in algebraic system $(S,\circ,+)$ if and only if $A = [a_{ij}]_{m\times n}, a_{ij}\in S(i = 1,2,\cdots,m; j = 1,2,\cdots,n)$.

Definition of Kronecker product can be promoted as the below definition.

Definition 3(generalized Kronecker product). The Kronecker product of a matrix A by $m\times n$ and a matrix B by $p\times q$ in algebra system $(S,\circ,+)$ can be denoted by $A\otimes B$, which is defined as follows:

$$A\otimes B = [a_{ij}\circ B] = \begin{bmatrix} a_{11}\circ B & a_{12}\circ B & \cdots & a_{1n}\circ B \\ a_{21}\circ B & a_{22}\circ B & \cdots & a_{2n}\circ B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\circ B & a_{m2}\circ B & \cdots & a_{mn}\circ B \end{bmatrix}.$$

Note: The generalized product, addition and number product of matrix are similar with the usual product, addition and number product of matrix.

3. Properties of generalized Kronecker product

Theorem 1. For $A_{m\times n}$ and $B_{p\times q}$, generally $A\otimes B\neq B\otimes A$.

Since the number operate is a special algebraic operator, for restricted untenable proposition, it is also untenable in generalized condition.

Theorem 2. The Kronecker product of arbitrary matrix and zero matrix equals zero matrix, i.e. $A\otimes 0 = 0\otimes A = 0$.

The reason is as that of theorem 1.

Theorem 3. If $(S,\circ)$ is a commutative semi-group, and for arbitrary α and β , $(\alpha\circ A)\otimes(\beta\circ B) = (\alpha\circ\beta)\circ(A\otimes B)$.

This theorem is equivalent the operate of elements: $(\alpha\circ a_{i,j})\circ(\beta\circ b_{kl}) = (\alpha\circ\beta)\circ(a_{ij}\circ b_{kl})$, where $a_{ij} = [A]_{ij}, b_{kl} = [B]_{kl}$.

Theorem 4. If $(S,\circ)$ is a commutative semi-group, and for $A_{m\times n}$, $B_{n\times k}$, $C_{l\times p}$ and $D_{p\times q}$, $(A\circ B)\otimes(C\circ D) = (A\otimes C)\circ(B\otimes D)$.

This theorem is equivalent the operate of elements: $(a_{ij}\circ b_{kl})\circ(c_{tu}\circ d_{wv}) = (a_{i,j}\circ c_{tu})\circ(b_{kl}\circ d_{wv})$, where $a_{ij} = [A]_{ij}, b_{kl} = [B]_{kl}, c_{tu} = [C]_{tu}, d_{wv} = [D]_{wv}$.

Theorem 5. If $(S,+,\circ)$ is a ring, and for $A_{m\times n}$, $B_{p\times q}$, $C_{p\times q}$, $A\otimes(B\pm C) = (A\otimes B)\pm(A\otimes C)$, $(B\pm C)\otimes A = (B\otimes A)\pm(C\otimes A)$.

Theorem 6. If $(S,\circ)$ is a commutative algebraic system, and for $A_{m\times n}$ and $B_{p\times q}$, $(A\otimes B)^T = A^T\otimes B^T$.

Theorem 7. If $(S,+,\circ)$ is a ring and for $A_{m\times n}$, $B_{m\times n}$, $C_{p\times q}$ and $D_{p\times q}$, $(A+B)\otimes(C+D) = A\otimes C+A\otimes D+B\otimes C+B\otimes D$.

This theorem is equivalent the operate of elements: $(a_{ij}+b_{ij})\circ(c_{kl}+d_{kl}) = a_{ij}\circ c_{kl}+a_{ij}\circ d_{kl}+b_{ij}\circ c_{kl}+b_{ij}\circ d_{kl}$, where $a_{ij} = [A]_{ij}, b_{ij} = [B]_{ij}, c_{kl} = [C]_{kl}, d_{kl} = [D]_{kl}$.

Theorem 8. If $(S,\circ)$ is a commutative semi-group, and for $A_{m\times n}$, $B_{k\times l}$, $C_{p\times q}$ and $D_{r\times s}$, $(A\otimes B)\otimes(C\otimes D) = A\otimes B\otimes C\otimes D$.

This theorem is equivalent the operate of elements: $(a_{ij}\circ b_{kl})\circ(c_{tu}\circ d_{wv}) = a_{ij}\circ b_{kl}\circ c_{tu}\circ d_{wv}$, where $a_{ij} = [A]_{ij}, b_{kl} = [B]_{kl}, c_{tu} = [C]_{tu}, d_{wv} = [D]_{wv}$.

Theorem 9. If $(S,\circ)$ is a semi-group , and for $A_{m\times n}$, $B_{k\times l}$, $C_{p\times q}$, $(A\otimes B)\otimes C = A\otimes(B\otimes C)$.

This theorem is equivalent the operate of elements: $(a_{ij} \circ b_{kl}) \circ c_{tu} = a_{ij} \circ (b_{kl} \circ c_{tu})$, where $a_{ij} = [A]_{ij}, b_{kl} = [B]_{kl}, c_{tu} = [C]_{tu}$.

Theorem 10. If $(S, \circ)$ is a commutative semi-group, and for $A_{m\times n}$, $B_{p\times q}$, $C_{n\times r}$ and $D_{q\times s}$, $(A \otimes B) \circ (C \otimes D) = (A \otimes C) \circ (B \otimes D)$, where $a_{ij} = [A]_{ij}, b_{kl} = [B]_{kl}, c_{tu} = [C]_{tu}, d_{wv} = [D]_{wv}$.

This theorem is equivalent the operate of elements: $(a_{ij} \circ b_{kl}) \circ (c_{tu} \circ d_{wv}) = (a_{ij} \circ c_{tu}) \circ (b_{kl} \circ d_{wv})$.

4. Application of generalized Kronecker product[7],[8],[9],[10],[11]

Theorem 12. Suppose $S = \{a_1, a_2, \cdots, a_n\}$ and $(S, \circ)$ is an algebraic system, then the table of algebraic operator in S is as follows:

<Figure1>

Construct $A = [a_1 \quad a_2 \quad \cdots a_n]^T$, then $(S, \circ)$ is a semi-group if and only if $A \otimes (A \otimes A^T) = (A \otimes A) \otimes A^T$.

Theorem 13. Suppose $S = \{a_1, a_2, \cdots, a_n\}$ and $(S, \circ)$ is an algebraic system, then the table of algebraic operator in S is as follows:

<Figure1>

Construct $A = [a_1 \quad a_2 \quad \cdots a_n]^T$, then the algebraic operator $\circ$ of S satisfies associative law if and if only $A \otimes (A \otimes A^T) = (A \otimes A) \otimes A^T$.

Sample: Suppose set $S = \{a, b, c, d\}$, the algebraic operator in S is as follows:

<Figure2>

then whether is the algebraic operator $\circ$ in S associative?

Proof: Easily , algebraic operator $\circ$ in finite set S is closed, then $(S, \circ)$ is a algebraic system. Construct matrix $A = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, and the definition of generalized Kronecker product is as definition 3, then

$$A\otimes(A\otimes A^T) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \otimes \left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \otimes [a \quad b \quad c \quad d] \right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \otimes \begin{bmatrix} a \circ [a & b & c & d] \\ b \circ [a & b & c & d] \\ c \circ [a & b & c & d] \\ d \circ [a & b & c & d] \end{bmatrix} \otimes [a \quad b \quad c \quad d] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \otimes \begin{bmatrix} a\circ a & a\circ b & a\circ c & a\circ d \\ b\circ a & b\circ b & b\circ c & b\circ d \\ c\circ a & c\circ b & c\circ c & c\circ d \\ d\circ a & d\circ b & d\circ c & d\circ d \end{bmatrix}$$

$$= \begin{bmatrix} a \circ \begin{bmatrix} a\circ a & a\circ b & a\circ c & a\circ d \\ b\circ a & b\circ b & b\circ c & b\circ d \\ c\circ a & c\circ b & c\circ c & c\circ d \\ d\circ a & d\circ b & d\circ c & d\circ d \end{bmatrix} \\ b \circ \begin{bmatrix} a\circ a & a\circ b & a\circ c & a\circ d \\ b\circ a & b\circ b & b\circ c & b\circ d \\ c\circ a & c\circ b & c\circ c & c\circ d \\ d\circ a & d\circ b & d\circ c & d\circ d \end{bmatrix} \\ c \circ \begin{bmatrix} a\circ a & a\circ b & a\circ c & a\circ d \\ b\circ a & b\circ b & b\circ c & b\circ d \\ c\circ a & c\circ b & c\circ c & c\circ d \\ d\circ a & d\circ b & d\circ c & d\circ d \end{bmatrix} \\ d \circ \begin{bmatrix} a\circ a & a\circ b & a\circ c & a\circ d \\ b\circ a & b\circ b & b\circ c & b\circ d \\ c\circ a & c\circ b & c\circ c & c\circ d \\ d\circ a & d\circ b & d\circ c & d\circ d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a\circ(a\circ a) & a\circ(a\circ b) & a\circ(a\circ c) & a\circ(a\circ d) \\ a\circ(b\circ a) & a\circ(b\circ b) & a\circ(b\circ c) & a\circ(b\circ d) \\ a\circ(c\circ a) & a\circ(c\circ b) & a\circ(c\circ c) & a\circ(c\circ d) \\ a\circ(d\circ a) & a\circ(d\circ b) & a\circ(d\circ c) & a\circ(d\circ d) \\ b\circ(a\circ a) & b\circ(a\circ b) & b\circ(a\circ c) & b\circ(a\circ d) \\ b\circ(b\circ a) & b\circ(b\circ b) & b\circ(b\circ c) & b\circ(b\circ d) \\ b\circ(c\circ a) & b\circ(c\circ b) & b\circ(c\circ c) & b\circ(c\circ d) \\ b\circ(d\circ a) & b\circ(d\circ b) & b\circ(d\circ c) & b\circ(d\circ d) \\ c\circ(a\circ a) & c\circ(a\circ b) & c\circ(a\circ c) & c\circ(a\circ d) \\ c\circ(b\circ a) & c\circ(b\circ b) & c\circ(b\circ c) & c\circ(b\circ d) \\ c\circ(c\circ a) & c\circ(c\circ b) & c\circ(c\circ c) & c\circ(c\circ d) \\ c\circ(d\circ a) & c\circ(d\circ b) & c\circ(d\circ c) & c\circ(d\circ d) \\ d\circ(a\circ a) & d\circ(a\circ b) & d\circ(a\circ c) & d\circ(a\circ d) \\ d\circ(b\circ a) & d\circ(b\circ b) & d\circ(b\circ c) & d\circ(b\circ d) \\ d\circ(c\circ a) & d\circ(c\circ b) & d\circ(c\circ c) & d\circ(c\circ d) \\ d\circ(d\circ a) & d\circ(d\circ b) & d\circ(d\circ c) & d\circ(d\circ d) \end{bmatrix}$$

$$= \begin{bmatrix} a\circ a & a\circ c & a\circ d & a\circ a \\ a\circ b & a\circ c & a\circ a & a\circ d \\ a\circ c & a\circ a & a\circ b & a\circ d \\ a\circ d & a\circ b & a\circ a & a\circ c \\ b\circ a & b\circ c & b\circ d & b\circ a \\ b\circ b & b\circ c & b\circ a & b\circ d \\ b\circ c & b\circ a & b\circ b & b\circ d \\ b\circ d & b\circ b & b\circ a & b\circ c \\ c\circ a & c\circ c & c\circ d & c\circ a \\ c\circ b & c\circ c & c\circ a & c\circ d \\ c\circ c & c\circ a & c\circ b & c\circ d \\ c\circ d & c\circ b & c\circ a & c\circ c \\ d\circ a & d\circ c & d\circ d & d\circ a \\ d\circ b & d\circ c & d\circ a & d\circ d \\ d\circ c & d\circ a & d\circ b & d\circ d \\ d\circ d & d\circ b & d\circ a & d\circ c \end{bmatrix} = \begin{bmatrix} a & d & a & a \\ c & d & a & a \\ d & a & c & a \\ a & c & a & d \\ b & a & d & b \\ c & a & b & d \\ a & b & c & d \\ d & c & b & a \\ c & b & d & c \\ a & b & c & d \\ b & c & a & d \\ d & a & c & b \\ d & a & c & d \\ b & a & d & c \\ a & d & b & c \\ c & b & d & a \end{bmatrix}.$$

But

$$(A\otimes A)\otimes A^T = (\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \otimes \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}) \otimes [a \quad b \quad c \quad d]$$

$$= \begin{bmatrix} a\circ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \\ b\circ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \\ c\circ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \\ d\circ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \end{bmatrix} \otimes [a \quad b \quad c \quad d] = \begin{bmatrix} a\circ a \\ a\circ b \\ a\circ c \\ a\circ d \\ b\circ a \\ b\circ b \\ b\circ c \\ b\circ d \\ c\circ a \\ c\circ b \\ c\circ c \\ c\circ d \\ d\circ a \\ d\circ b \\ d\circ c \\ d\circ d \end{bmatrix} \otimes [a \quad b \quad c \quad d] = \begin{bmatrix} a \\ c \\ d \\ a \\ b \\ c \\ a \\ d \\ c \\ a \\ b \\ d \\ d \\ b \\ a \\ c \end{bmatrix} \otimes [a \quad b \quad c \quad d]$$

$$= \begin{bmatrix} a\circ[a & b & c & d] \\ c\circ[a & b & c & d] \\ d\circ[a & b & c & d] \\ a\circ[a & b & c & d] \\ b\circ[a & b & c & d] \\ c\circ[a & b & c & d] \\ a\circ[a & b & c & d] \\ d\circ[a & b & c & d] \\ c\circ[a & b & c & d] \\ a\circ[a & b & c & d] \\ b\circ[a & b & c & d] \\ d\circ[a & b & c & d] \\ d\circ[a & b & c & d] \\ b\circ[a & b & c & d] \\ a\circ[a & b & c & d] \\ c\circ[a & b & c & d] \end{bmatrix} = \begin{bmatrix} a\circ a & a\circ b & a\circ c & a\circ d \\ c\circ a & c\circ b & c\circ c & c\circ d \\ d\circ a & d\circ b & d\circ c & d\circ d \\ a\circ a & a\circ b & a\circ c & a\circ d \\ b\circ a & b\circ b & b\circ c & b\circ d \\ c\circ a & c\circ b & c\circ c & c\circ d \\ a\circ a & a\circ b & a\circ c & a\circ d \\ d\circ a & d\circ b & d\circ c & d\circ d \\ c\circ a & c\circ b & c\circ c & c\circ d \\ a\circ a & a\circ b & a\circ c & a\circ d \\ b\circ a & b\circ b & b\circ c & b\circ d \\ d\circ a & d\circ b & d\circ c & d\circ d \\ d\circ a & d\circ b & d\circ c & d\circ d \\ b\circ a & b\circ b & b\circ c & b\circ d \\ a\circ a & a\circ b & a\circ c & a\circ d \\ c\circ a & c\circ b & c\circ c & c\circ d \end{bmatrix} = \begin{bmatrix} a & c & d & a \\ c & a & b & d \\ d & b & a & c \\ a & c & d & a \\ b & c & a & d \\ c & a & b & d \\ a & c & d & a \\ d & b & a & c \\ c & a & b & d \\ a & c & d & a \\ b & c & a & d \\ d & b & a & c \\ d & b & a & c \\ b & c & a & d \\ a & c & d & a \\ c & a & b & d \end{bmatrix}.$$

Because

$$\begin{bmatrix} a & d & a & a \\ c & d & a & a \\ d & a & c & a \\ a & c & a & d \\ b & a & d & b \\ c & a & b & d \\ a & b & c & d \\ d & c & b & a \\ c & b & d & c \\ a & b & c & d \\ b & c & a & d \\ d & a & c & b \\ d & a & c & d \\ b & a & d & c \\ a & d & b & c \\ c & b & d & a \end{bmatrix} \neq \begin{bmatrix} a & c & d & a \\ c & a & b & d \\ d & b & a & c \\ a & c & d & a \\ b & c & a & d \\ c & a & b & d \\ a & c & d & a \\ d & b & a & c \\ c & a & b & d \\ a & c & d & a \\ b & c & a & d \\ d & b & a & c \\ d & b & a & c \\ b & c & a & d \\ a & c & d & a \\ c & a & b & d \end{bmatrix}.$$

$A \otimes (A \otimes A^T) \neq (A \otimes A) \otimes A^T$. From theorem 13, the algebraic operator $\circ$ in S is suit for associative law.

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$\circ$	a_1	a_2	$\cdots\cdots$	a_n
a_1	$a_1 \circ a_1$	$a_1 \circ a_2$	$\cdots$	$a_1 \circ a_n$
a_2	$a_2 \circ a_1$	$a_2 \circ a_2$	$\cdots$	$a_2 \circ a_n$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
a_n	$a_n \circ a_1$	$a_n \circ a_2$	$\cdots$	$a_n \circ a_n$

Figure 1

$\circ$	a	b	c	d
a	a	c	d	a
b	b	c	a	d
c	c	a	b	d
d	d	b	a	c

Figure 2

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The Lax-Wendroff Theorem of Entropy Dissipation Method for Scalar Conservation Laws in One Space Dimension

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Abstract

In this paper, we present the Lax-Wendroff theorem of entropy dissipation method for scalar conservation laws in one space dimension. Suppose that $u_l(x,t)$ the numerical solution computed by the entropy dissipation method converges to a function $u(x,t)$ as $l \to \infty$,then u(x,t) is a weak solution that satisfying the entropy condition of the conservation law.

Keywords: Conservation law, Entropy condition, Lax-Wendroff theorem

1. Introduction

In this paper we continue to consider entropy dissipating method developed in(Li, Hong-xia, 2004), (Second-order entropy dissipation scheme for scalar conservation laws in one space dimension, Master's thesis, No.11903-99118086)for scalar conservation laws in one space dimension

$$u_t + f(u)_x = 0 \quad u(x,0) = u_0(x) \tag{1}$$

In this paper, we propose and prove a Lax-Wendroff theorem of entropy dissipation method for scalar conservation laws in one space dimension.

2. The Basic Definitions

In this section, we give the basic definitions of the theorem. We will consider the general form of the scheme.The numerical solution is computed by:

$$u_j^{n+1} = u_j^n - \lambda(\hat{f}^n_{j+\frac{1}{2}} - \hat{f}^n_{j-\frac{1}{2}}) \tag{2}$$

where the numerical flux is:

$$\hat{f}^n_{j+\frac{1}{2}} = \hat{f}(u^n_{j-k+1}, \cdots, u^n_{j+k}; U^n_{j-p+1}, \cdots, U^n_{j+p}) \tag{3}$$

The numerical entropy is computed by:

$$U_j^{n+1} = U_j^n - \lambda(\hat{F}^n_{j+\frac{1}{2}} - \hat{F}^n_{j-\frac{1}{2}}) - D_j^n \tag{4}$$

where the numerical entropy flux is:

$$\hat{F}^n_{j+\frac{1}{2}} = \hat{F}(u^n_{j-k+1}, \cdots, u^n_{j+k}; U^n_{j-p+1}, \cdots, U^n_{j+p}) \tag{5}$$

$$D_j^n = D(u_{j-l}^n, \cdots, u_{j+l}^n) \tag{6}$$

k, p, l are the positive integers.

Definition 2.1 (consistence): If all the $u_i^n (i = \max(j-k+1, j-l), \cdots, \max(j+k, j+l))$ in (3), (5), (6) are $\overline{u}$ and all the $U_{i'}^n (i' = j-p+1, \cdots, j+p)$ are $U(\overline{u})$, (where $\overline{u} \in R$), then:

$$\hat{f}(\overline{u}, \cdots, \overline{u}; U(\overline{u}), \cdots, U(\overline{u})) = f(\overline{u}) \tag{7}$$

$$\hat{F}(\overline{u}, \cdots, \overline{u}; U(\overline{u}), \cdots, U(\overline{u})) = F(\overline{u}) \tag{8}$$

$$D(\overline{u}, \cdots, \overline{u}) = 0 \tag{9}$$

If $u_i \to \overline{u}$, $U_{i'} \to U(\overline{u})$, then $\hat{f}$, $\hat{F}$ and D convergence to $f(\overline{u})$, $F(\overline{u})$ and 0 in the following: for $0 < q \leq 1$ there is a constant K (maybe dependent $\overline{u}$) such that at $\overline{u}$:

$$\begin{aligned} &|\hat{f}(u_{j-k+1}, \cdots, u_{j+k}; U_{j-p+1}, \cdots, U_{j+p}) - f(\overline{u})| \\ &\leq K \max_{\substack{j-k+1 \leq i \leq j+k \\ j-p+1 \leq i' \leq j+p}} (|u_i - \overline{u}|, |U_{i'} - U(\overline{u})|^q) \end{aligned} \tag{10}$$

$$\begin{aligned} &|\hat{F}(u_{j-k+1}, \cdots, u_{j+k}; U_{j-p+1}, \cdots, U_{j+p}) - F(\overline{u})| \\ &\leq K \max_{\substack{j-k+1 \leq i \leq j+k \\ j-p+1 \leq i' \leq j+p}} (|u_i - \overline{u}|, |U_{i'} - U(\overline{u})|^q) \end{aligned} \tag{11}$$

$$|D(u_{j-l}, \cdots, u_{j+l}) - D(\overline{u}, \cdots, \overline{u})| \leq K \max_{j-l \leq i'' \leq j+l} (|u_{i''} - \overline{u}|) \tag{12}$$

then the scheme is consistent.

We are going to discuss the theorem as the form in (LeVeque, R.J. , 2002), (LeVeque, R.J. , 1990). First we define two piecewise constant function $u_l(x,t)$, $U_l(x,t)$ for all x and t from the discrete values $\{u_j^n\}$ and $\{U_j^n\}$:

$$u_l(x,t) = u_j^n, U_l(x,t) = U_j^n, \quad x_{j-\frac{1}{2}} < x \leq x_{j+\frac{1}{2}} \quad , \quad t_n < t \leq t_{n+1} \tag{13}$$

3. The New Lax-Wendroff Theorem

Theorem 3.1 (Lax-Wendroff): Consider a sequence of g rids indexed by $l = 1, 2, \ldots \cdots$, with mesh parameters k_l, $h_l \to 0$ as $l \to \infty$. Let $u_l(x,t)$, $U_l(x,t)$ are the numerical approximation computed with the scheme (2)~(5). Suppose that $u_l(x,0)$, $u_l(x,t)$, $U_l(x,t)$ are uniformly bounded functions and converge to the functions $u(x,0)$, $u(x,t)$, $U(u(x,t))$ as $l \to \infty$, in the sense made precise below. Then $u(x,t)$ is a entropy satisfying weak solution of the conservation law.

As in (R.J. LeVeque, 2002), (R.J. LeVeque, 1990), we assume that we have convergence of $u_l(x,t)$, $U_l(x,t)$ to $u(x,t)$, $U(u(x,t))$ in the following sense:

on $\Omega = [a,b] \times [0,t]$ $(a \leq b, t \geq 0)$, as $l \to \infty$:

$$\int_0^t \int_a^b |u_l(x,t) - u(x,t)| dx dt \to 0 \tag{14}$$

$$\int_0^t \int_a^b |U_l(x,t) - U(u(x,t))| dx dt \to 0 \tag{15}$$

As $l \to \infty$:

$$\|u_l - u\|_{1,\Omega} \to 0 \tag{16}$$

$$\|U_l - U\|_{1,\Omega} \to 0 \tag{17}$$

Proof: We will show that the limit function $u(x,t)$ satisfies the weak form, for all $\phi \in C_0^1(R^2)$, $u(x,t)$:

$$\int_0^\infty \int_{-\infty}^{+\infty} (\phi_t u + \phi_x f(u)) dx dt = -\int_{-\infty}^{\infty} \phi(x,0) u(x,0) dx \tag{18}$$

Let ϕ be a $C_0^1(R^2)$ test function and multiply the numerical method (2) by $\phi(x_j, t_n)$ and sum it over all j and $n \geq 0$. We obtain

$$\sum_{n=0}^{\infty}\sum_{j=-\infty}^{\infty}\phi(x_j,t_n)(u_j^{n+1}-u_j^n) = -\frac{k}{h}\sum_{n=0}^{\infty}\sum_{j=-\infty}^{\infty}\phi(x_j,t_n)(\hat{f}^n_{j+\frac{1}{2}}-\hat{f}^n_{j-\frac{1}{2}}) \tag{19}$$

we now use "summation by parts", and multiply it by h:

$$hk\{\sum_{n=1}^{\infty}\sum_{j=-\infty}^{\infty}\frac{\phi(x_j,t_n)-\phi(x_j,t_{n-1})}{k}u_j^n + \sum_{n=1}^{\infty}\sum_{j=-\infty}^{\infty}\frac{\phi(x_{j+1},t_n)-\phi(x_j,t_n)}{h}\hat{f}^n_{j+\frac{1}{2}}\}$$

$$= -h\sum_{j=-\infty}^{\infty}\phi(x_j,0)u_j^0 \tag{20}$$

By our assumption that ϕ has compact support, and hence each of the sums is in fact a finite sum.
Since $u_l(x,0)$, $u_l(x,t)$ are converge to $u(x,0)$, $u(x,t)$ in L^1, and $\phi(x,t)$ is smooth, we get the first term of (20) is converges to $\int_0^{\infty}\int_{-\infty}^{+\infty}\phi_t(x,t)u(x,t)dxdt$ and the third term converges to $-\int_{-\infty}^{\infty}\phi(x,0)u(x,0)dx$. The second term can be written as:

$$hk\sum_{n=1}^{\infty}\sum_{j=-\infty}^{\infty}\frac{\phi(x_{j+1},t_n)-\phi(x_j,t_n)}{h}\hat{f}^n_{j+\frac{1}{2}} = hk\sum_{n=1}^{\infty}\sum_{j=-\infty}^{\infty}\frac{\phi(x_{j+1},t_n)-\phi(x_j,t_n)}{h}f(u_j^n)$$

$$+hk\sum_{n=1}^{\infty}\sum_{j=-\infty}^{\infty}\frac{\phi(x_{j+1},t_n)-\phi(x_j,t_n)}{h}(\hat{f}^n_{j+\frac{1}{2}}-f(u_j^n)) \tag{21}$$

Since f is continuous and the above conditions, the first term of (21) converges to $\int_0^{\infty}\int_{-\infty}^{\infty}\phi_x f(u(x,t))dxdt$ as $l \to \infty$. Next we will prove that the second term of the right (3.8) converges to 0. Because of $\hat{f}$'s consistence, and ϕ has compact support. ϕ is continuous different, e.t. there is a $N > 0$, such that $|\frac{\partial\phi(x,t)}{\partial x}| \leq N$, $(x,t) \in R^2$. So:

$$|hk\sum_{n=1}^{\infty}\sum_{j=-\infty}^{\infty}\frac{\phi(x_{j+1},t_n)-\phi(x_j,t_n)}{h}(\hat{f}^n_{j+\frac{1}{2}}-f(u_j^n))|$$

$$\leq KN\{\sum_{i=j-k+1}^{j+k}(hk\sum_{n=1}^{\infty}\sum_{j=-\infty}^{\infty}|u_i^n-u_j^n|) + \sum_{i'=j-p+1}^{j+p}(hk\sum_{n=1}^{\infty}\sum_{j=-\infty}^{\infty}|U_{i'}^n-U(u_j^n)|^q)\}$$

Due to (13), (14), the right of the above formulas is:

$$\begin{aligned} &= KN\{\sum_{i=j-k+1}^{j+k}(\int_0^t\int_a^b|u_l(x,t)-u_l(x+(i-j)h,t)|dxdt) \\ &+ \sum_{i'=j-p+1}^{j+p}(\int_0^t\int_a^b|U_l(x,t)-U(u_l(x+(i'-j)h,t))|^q dxdt)\}, \end{aligned}$$

it includes: $I_{h_1} = \int_0^t\int_a^b|u_l(x,t)-u_l(x+s_1h,t)|dxdt$, $I_{h_2} = \int_0^t\int_a^b|U_l(x,t)-U(u_l(x+s_2h,t))|^q dxdt$. where$-k+1 \leq s_1 \leq k$, $-p+1 \leq s_2 \leq p$is the positive integer, note:

$$\begin{aligned} I_{h_1} &\leq \int_0^t\int_a^b|u_l(x,t)-u(x,t)|dxdt + \int_0^t\int_a^b|u(x,t)-u(x+s_1h,t)|dxdt \\ &+ \int_0^t\int_a^b|u(x+s_1h,t)-u_l(x+s_1h,t)|dxdt, \end{aligned}$$

Since$u_l(x,t)$ converging to $u(x,t)$ in L^1, as$l \to \infty$, the right term of the above formulas $\to 0$.
Using $|a+b+c|^q \leq 3^q(|a|^q+|b|^q+|c|^q)$, $q \geq 0$ we get:

$$\begin{aligned} I_{h_2} &\leq 3^q\{\int_0^t\int_a^b|U_l(x,t)-U(u(x,t))|^q dxdt + \int_0^t\int_a^b|U(u(x,t))-U(u(x+s_2h,t))|^q dxdt \\ &+ \int_0^t\int_a^b|U(u(x+s_2h,t))-U(u_l(x+s_2h,t))|^q dxdt\}. \end{aligned}$$

Due to Hölder inequality , $\int_0^t \int_a^b |f(x,t)|^q dxdt$ can be controlled by $\int_0^t \int_a^b |f(x,t)|dxdt$. Since$U_l(x,t) \to U(u(x,t))$in L^1, right of the above inequality $\to 0$, as $l \to \infty$.

Above all the second term of the right of (21) $\to 0$, as $l \to \infty$. So $u(x,t)$satisfies:

$$\int_0^\infty \int_{-\infty}^\infty (\phi_t u(x,t) + \phi_x f(u(x,t)))dxdt = -\int_{-\infty}^\infty \phi(x,0)u(x,0)dxdt.$$

The limit function $u(x,t)$ is a weak solution of the conservation law.
We can prove the solution also satisfies the entropy condition in the same way. Note $D_j^n \geq 0$ and $\phi(x,t) > 0 \in C_0^1(R_+^2)$.

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Study of FAE Explosive Image Analysis Based on the Fractional Dimension

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Abstract

The method of box dimension is used in this paper for calculating the fractal dimension of the fuel air explosive(FAE) explosive images. The relationship between fractal dimension and the expansion law of fireball is discussed. The change of explosion image fractal dimension all the time is studied, and its change law is inferred, with the reason analyzed, which lays the foundation for the further study of explosive process.

Keywords: FAE, Fractal, Box Dimension, Cloud

1. Introduction

Fractal geometry is a non-linear science which studies irregular physique attribute by American nationality France mathematician B.B.Mandelbrot in 1973. As a result of its object of studying widespread, at present it goes thorough natural social sciences in each domain day by day , and develops unceasingly. Fractal has the following characteristics: Fractal geometry graph 's irregularity everywhere and different criterion graph's regularity, and the fractal dimension is bigger than the topology dimension.Fractal dimension plays important role in the fractal fundamental research, which is an important parameter in describing image fractal characteristic. Fractal dimension analysis of fractal image has the vital significance for the characteristic and rule of exploration fractal image change.

The detonation image can record the fuel physical disperser, characteristics of burning and detonation in detonation process, which can reflect each performance in the knocking fuel. At present the domestic and foreign research scholars have done massive work in method of researching the detonation image processing. Dong Yucai , in literature (Dong Yucai, 2007), introduces the fractal theory in the detonation image processing, calculates fractal dimension in the liquid fuel detonation process in various times using the box dimension method, summarizes the fractal dimension change rule in the liquid fuel detonation process, and analyzes the detonation spread change near field. Liu Gengran used the high speed colored camera in literature (Liu Gengran, 2007) to record process picture of 0.5kg solid state FAE detonation diffusion. In experiment, camera's photography frequency is 2000f/s, each gap is 0.5ms. The picture size is 640×480, which is preserved by the JPG document format. This paper has calculated mean radius and the duration for the solid state air fuel (SFAE) detonation fireball, summarize dissemination rule for explosive shock wave. On the foundation of the above experimental result, using fractal theory and through processing experimental image of exploding the spread, this paper obtains the detonation image fractal curve, calculates dimension value of border curve in various times and the dimension change rule. Moreover, this paper has analyzed the reason for dimension change, which lays foundation for analyzing the denotation process.

2. Computation for Dimension of Detonation Image Fractal

2.1 Extract detonation border curve

This paper applies the MATLAB software to process detonation image, and obtains the detonation curve gradually. The detonation image extraction divides into six steps.

First we will read the detonation image in procedure, then the image will be preserved by 480×640×3 matrix.In order to be advantageous for processing, transforms the image to the gradation image preserved, by 480×640, where each element is a picture element spot and the different element value represents different brightness or the gradation level. The value 0 represents the black and the value 255 represents the white. That the value is bigger indicated the brightness of picture element in this position is higher. In order to eliminate the irrelevant picture element disturbance in the graph, we need remove the graph background, and carry on screening the graph matrix through the selection threshold value. The value in this paper is taken by 240, the original graph transformation is two value charts, the graph matrix transformation is 480×640 binary matrices, where element only has 0 and 1. The digital 0 represents the position picture whose element in original graph matrix is smaller than or equal to 240, but digital 1 is bigger than 240, which is the same to the gradation image. When demonstrating graph, the value 0 represents the black and the value 1 represents the white. While cutting the background image, this paper constitutes a matrix with the last image, and carries on each image to spot ride with this matrix, then the curve obtained is the boundary curve of detonation image.

The above is a detonation image in the 15ms time. Figure (1) is the primitive image photographed, figure (2) is the gradation chart, figure (3) is two value charts, figure (4) is the initial boundary curves, figure (5) is the image removing the impurity background, figure (6) namely is the boundary diagram of curves finally obtained

2.2 Calculate curve fractal dimension using box dimension

Fractal dimension is the most core content in the fractal theory, which is discussed most much in the fractal theory. Also it is studied most widespread and can reflect the complex degree of fractal image. The computation methods of fractal dimension quite are many, where the box dimension is most commonly used.

Define box dimension: Supposes $A \subset R_n$ is a non-spatial set, under Euclid distance, A is contained close with the small box for length of side δ. Let $N(A)$ be the smallest box number which contains A . Then the fractal box dimension is:

$$D = \lim_{\delta \to 0} \frac{lnN(A)}{-ln\delta}$$

This paper uses MATLAB to carry on the programming. First read boundary curve in procedure, then the boundary curve will be preserved by a 480×640 binary matrix. Structure length of side with $r_1 = 1$(picture element), cover the image with this check, judge whether the covered small square does contain the boundary curve, and record the check number K_1 containing the image. Then take the length of side $r_2 = 2$ and record the check number $K_2, \cdots\cdots$. Take the length of side $r_n = 2 * r_{n-1}$ and record the check number K_n containing the image. If $r_n > 480$, withdraw the circulation, then we will obtain two groups of data $r_1, r_2, \cdots, r_n$ and $K_1, K_2, \cdots, K_n$, which are taken the logarithm and linearly fit. In the last the inverse of the fitted straight line slope is namely the fractal dimension of boundary curve.

3. Fractal dimension change analysis

Through contrasting the picture , flame brightness has changes from bright to dark, then to bright again in the detonation process. When exploding in 0.5ms, since the explosion products have extremely high pressure, the rate of the denotion fireball outward expansion is extremely quick, whose interior radius can achieve 4.9m in the extremely short time. Along with further proliferation of the explosion product, the explosion center pressure drops rapidly, the fireball edge flame starts to extinguish and the fireball radius changes rapidly small. But in this time massive nondetonation product in explosion center starts to burn, the flame starts to change bright, and diffuses outward, fireball radius slow fill-out. When fireball burning finished, flame brightness starts to change dark until extinguishment.

Figure 5 is the change tendency of various frequently fractal dimension in the SFAE fuel detonation process.

The dimension value change may divide into two stages: One is rise stage for the dimension value, two is the dimension tends to the steady stage.

First stage is the rise stage for dimension. Under explosive shock wave function fuel is staved and diffuses outward. These staved explosion product starts burning while diffusing outward, and produces many small burning pellets in the detonation fireball edge; Meanwhile as a result of the explosion center pressure, the fireball inflates unceasingly distorts, thus making the fireball edge to be more and more complex and the corresponding curve dimension assumes escalation trend.

Fractal dimension of the second stage tends to steady. After the fireball expands to the limit radius, the shock-wave has already been separated from the fireball contact surface, alone disseminates outward. In this time no new detonation and burning in the fireball edge occur . The fireball maintains the original shape and the fractal dimension also tends to steady.

Overall, the fractal dimension may divide into the rise stage and the steady stage, but from partial condition, the detonation dimension continuously is in during, which is showed in the following chart:

The reason of this phenomenon is that in detonation process fuel proliferation explodes, which can produce some burning explosion in the detonation fireball edge. These products interior can produce pressure and inflate while detonating, which causes the detonation fireball edge to become complex and the fractal dimension to increase; Meanwhile as a result of the detonation fireball internal pressure, those diffused explosion product can be separated, until burning extinguishes, thus making the entire fractal dimension to have a drop value. In the entire detonation process, the detonation fireball edge can have many such products, therefore the fractal dimension can be in during the shake.

4. Conclusion

Through comparison and analysis, the tendency of fractal dimension change and the rule of boundary curve is similar, which can reflect the change tendency of the boundary curve. Researching the detonation image and analyzing fireball change rule of detonation process has laid certain foundation for further researching detonation image and analyzing fireball change rule of detonation process foundation.But when withdrawing the boundary curve, the method of processing the edge impurity need to further study, which can obtain the more accurate dimension value.

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Table 1. Number of check contained by boundary curves in time 15ms

Check radius r	check quantity K	r takes logarithm	K takes logarithm	Check radius r	check quantity K	r takes logarithm	K takes logarithm
1	745	0	2.8722	32	25	1.5051	1.3979
2	443	0.3010	2.6464	64	11	1.8062	1.0414
4	242	0.6021	2.3838	128	5	2.1072	0.6990
8	120	0.9031	2.0792	256	2	2.4082	0.3010
16	55	1.2041	1.7404				

Table 2. Border curve dimension value from150ms to 163ms

time	dimension	time	dimension	time	dimension	time	dimension
150	1.1341	153.5	1.1597	157	1.1684	160.5	1.1847
150.5	1.1412	154	1.1625	157.5	1.1668	161	1.1874
151	1.1445	154.5	1.1664	158	1.1716	161.5	1.1819
151.5	1.1493	155	1.1649	158.5	1.1643	162	1.1888
152	1.1516	155.5	1.1625	159	1.1739	162.5	1.1895
152.5	1.1517	156	1.1678	159.5	1.1802	163	1.1922
153	1.1594	156.5	1.1662	160	1.1876	163.5	1.1909

Table 3. border curve dimension value from 250ms to 263ms

time	dimension	time	dimension	time	dimension	time	dimension
250	1.2856	253.5	1.2718	257	1.2748	260.5	1.2764
250.5	1.2789	254	1.2748	257.5	1.2717	261	1.276
251	1.2834	254.5	1.2772	258	1.2746	261.5	1.276
251.5	1.2672	255	1.2763	258.5	1.2759	262	1.2761
252	1.2727	255.5	1.2792	259	1.2723	262.5	1.2762
252.5	1.2735	256	1.2757	259.5	1.2757	263	1.2778
253	1.2769	256.5	1.2759	260	1.2786	263.5	1.2789

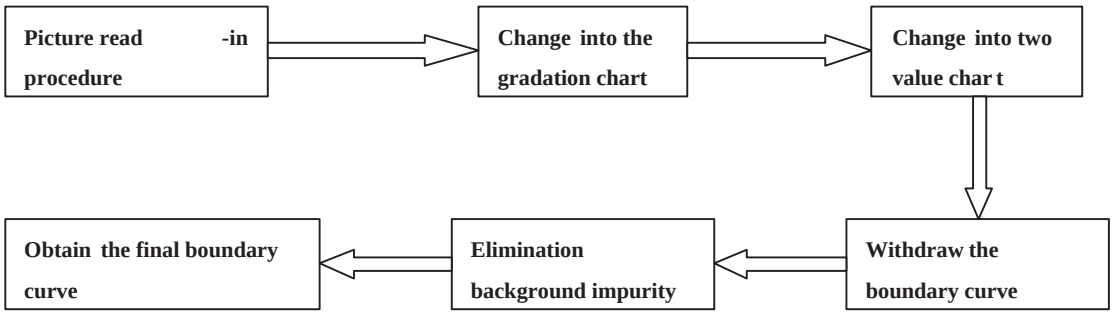


Figure 1. Flow chart for border curve extraction

Figure (1) Figure (2) Figure (3)

Figure (4) Figure (5) Figure (6)

Figure 2. Fitting chart for border curve fractal dimension in time 15ms

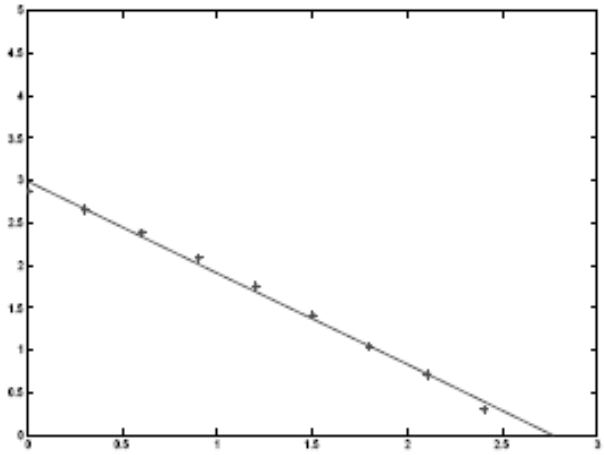


Figure 3. Fractal dimension fitting chart of the border curve in time 15ms

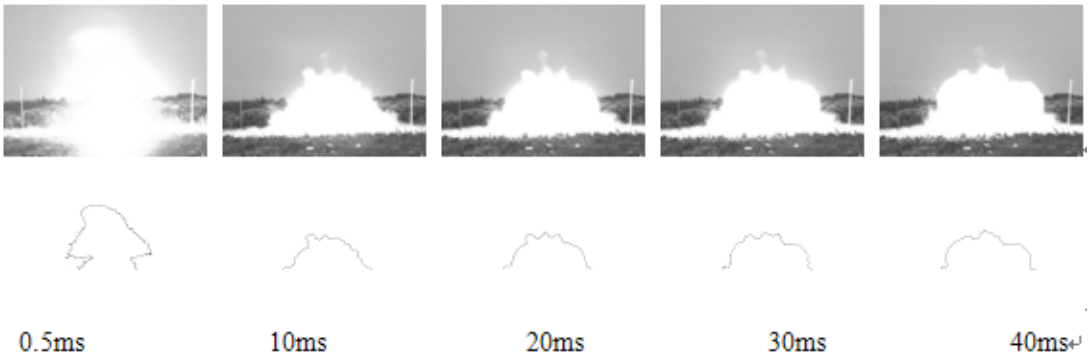


Figure 4. Detonation chart and corresponding boundary of solid state FAE

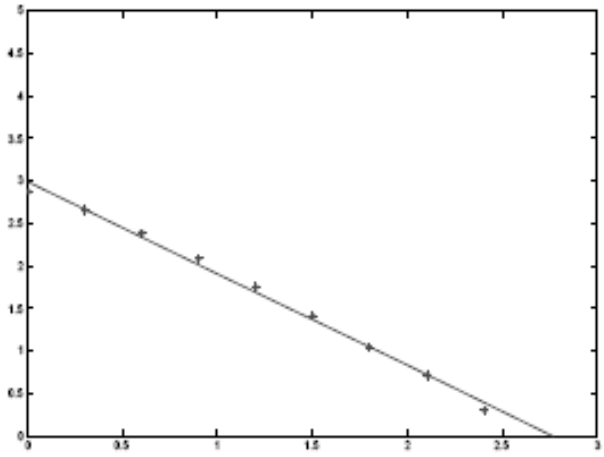


Figure 5. Fractal dimension change chart

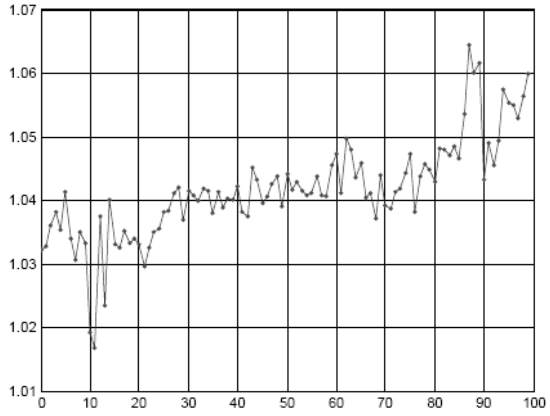
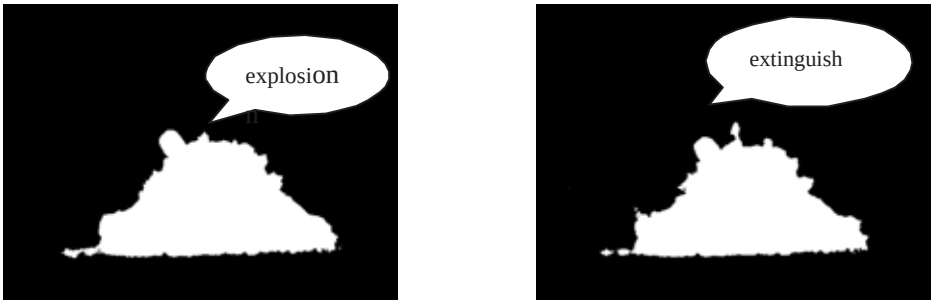


Figure 6. Fractal dimension change chart



Dimension Value 1.0428 Dimension Value 1.0134

Figure 7. two value chart of detonation image in 3.5-4.5ms

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