The Finite Element Methods for a Nonlinear Problem

Adeeb A. A. Alrahamneh (Corresponding author)
Faculty of Planning & Management, Al – Balqa Applied University
E-mail: moheeb982005@yahoo.co.uk

Omar M. Hawamdeh
Faculty of Planning & Management, Al – Balqa Applied University
E-mail: omar_h1964@yahoo.com

Received: January 16, 2012 Accepted: January 30, 2012 Published: April 1, 2012
doi:10.5539/jmr.v4n2p80 URL: http://dx.doi.org/10.5539/jmr.v4n2p80

Abstract

In this paper we construct a nonlinear Sturm–Lioville problem of even degree for which we can apply the variation methods developed in (Peter, 2008) and the finite element methods approximation for generalized solution of problem.

Keywords: Non Linear Sturm-Liovile problem, Finite element approximation

1. Introduction

We consider the problem:

$$P_u = \sum_{k=0}^{m}(-1)^k \frac{d^k}{dx^k}[P_k(\frac{d^k u}{dx^k})\frac{d^k u}{dx^k}] = f,\ f \in L^2(a,b) \tag{1}$$

With conditions:

$$\begin{cases} u(a) = u^{'}(a) = ... = u^{(m-1)}(a) = 0, \\ u(b) = u^{'}(b) = ... = u^{(m-1)}(b) = 0 \end{cases} \tag{2}$$

and P_k for $K = \overline{0,m}$ are from $C^(k)$ class and P_k depends only on $\frac{d^k u}{dx^k}$ and a, b finite.

We assume that

1. functions P_k satisfy conditions

$$\frac{d}{dt}[tP_k(t)] \geq C_k > 0, \text{ for all k } = \overline{0,\mathrm{m}}$$

and there is at least one index k_0 that $\frac{d}{dt}$ $[t\mathrm{P}k0(t)] \geq C_{k0} > 0$. satisfy conditions em: oblem. pply the variational methods developed in (Peter, 2008) and the finite element methods approximation for

We consider the Hilbert space $H = L^2(a,b)$ and

$$Q = \{u \in C^n[a,b] \backslash u(a) = u^{'}(a) = ... = u^{(m-1)}(a)$$

$$= u(b) = u'(b) = ... = u^{(m-1)}(b) = 0\}, \text{ where n = 2m and P : Q} \to \mathrm{L}^2(a,b).$$

Theorem 1 *The operator P is potential operator with its Gateaux differential linear in h and positive definite.*

Proof: There exists a functional

$$\forall u \in Q,\ \ \mathrm{Fu} = \int_0^1 (P(tu), u)_{L^2(a,b)}dt - (f,u)_{L^2(a,b)}$$
$$= \int_0^1 dt \int_a^b \sum_{k=0}^m (-1)^k \frac{d^k}{dx^k}[Pk(tu^{(k)})\ (tu^{(k)}(x))]u(x)dx - \int_{\mathrm{a}}^{\mathrm{b}} f(x)u(x)dx \tag{3}$$

and using the formula integration by parts and the limit condition we obtain:

$$\forall u \in Q\ \ F(u) = \sum_{k=0}^m \int_a^b dx \int_0^{u^{(k)}} Pk(t)tdt - \int_a^b f(x)u(x)dx. \tag{4}$$

For every $u,\ h \in Q$, there exists $DP(u)h$ linear in h

$$DP(u)h = \sum_{k=0}^{m}(-1)^k \frac{d^k}{dx^k}\left[\left(Pk(\frac{d^k u}{dx^k}) + Pk(\frac{d^k u}{dx^k})\frac{d^k u}{dx^k}\right)\frac{d^k h}{dx^k}\right]$$

and the symmetry and the positive definiteness results respectively from:

$$(DP(u)h_1, h_2)_{L^2(a,b)} = \sum_{k=0}^{m}\int_a^b \left[Pk(\tfrac{d^k u}{dx^k}) + Pk(\tfrac{d^k u}{dx^k})\tfrac{d^k u}{dx^k}\right]\tfrac{d^k h_1}{dx^k}\tfrac{d^k h_2}{dx^k}dx = (DP(u)\ h_2, h_1)_{L^2(a,b)}, \quad \forall u, h_1, h_2 \in Q,$$

$$(DP(u)h, h)_{L^2(a,b)} = \sum_{k=0}^{m}\int_a^b \left[Pk(\tfrac{d^k u}{dx^k}) + Pk'(\tfrac{d^k u}{dx^k})\tfrac{d^k u}{dx^k}\right](\tfrac{d^k h}{dx^k})^2 dx \geq C_{k0}\left\|h^{(k0)}\right\|^2_{L^2(a,b)} \geq C_{k0}\left[\tfrac{2}{(b-a)}\right]^{k0}\|h\|^2_{L^2(a,b)}, \forall u, h \in Q.$$

Theorem 2 *a) If problem (1), (2) has a unique solution, this is the unique solution and minimizes the functional (4), and conversely, if there is an $u_0 \in Q$ that minimizes the functional (4) on Q, then u_0 is the solution of problem (1), (2) and, according to the first part of the theorem, the solution is unique.*

b) The functional (4) is lower bounded on Q.

c) Any minimized sequence for the functional (4) has limit in L^2 (a,b), (the existence of generalized solution of problem (1), (2).

d) All minimized sequences have the same limit in L^2 (a, b).

Proposition 3 *If condition (A) is satisfied for all $k = \overline{0, m}$ then*

$$(DP(u)h, h)_{L^2(a,b)} \geq \frac{\min_k C_k}{\max_k Pk(0)}(DP(0)h, h)_{L^2(a,b)}, \forall u, h \in Q \tag{5}$$

and

$$(\min_k Pk(0)\,\|h\|^2_{w^{m,2}(a,b)}) \leq (DP(0)h, h)_{L^2(a,b)} \leq (\max_k Pk(0))\ \ \|h\|^2_{w^{m,2}(a,b)}, \forall h \in Q \tag{6}$$

Proof: The first inequality results from

$$(DP(u)h, h)_{L^2(a,b)} \geq \sum_{k=0}^{m} c_k \int_a^b (\tfrac{d^k h}{dx^k})^2 dx \geq (\min_k\ \mathrm{c_k})\sum_{k=0}^{m}\int_a^b(\tfrac{d^k h}{dx^k})^2 dx = (\min_k c_k)\,\|h\|^2_{w^{m,2}(a,b)}$$

and

$$(DP(0)h, h)_{L^2_{(a,b)}} = \sum_{k=0}^{m}\int_a^b Pk(0)(\tfrac{d^k h}{dx^k})^2 dx \leq (max_k Pk(0))\sum\int_a^b(\tfrac{d^k h}{dx^k})^2 dx$$
$$\leq (\max_k Pk(0))\sum_{k=0}^{m}\|h^{(k)}\|^2_{L^2_{(a,b)}} = (max_k Pk(0))\|h\|^2_{W^{m,2}_{(a,b)}} \tag{7}$$

The second results from (7) and

$$(DP(0)h, h)_{L^2(a,b)} = \sum_{k=0}^{m}\int_a^b Pk(0)(\tfrac{d^k h}{dx^k})^2 dx \geq (\min_k\ \mathrm{Pk}(0))\sum_{k=0}^{m}\int_a^b(\tfrac{d^k h}{dx^k})^2 dx$$

$$\geq (\min_k\ \mathrm{Pk}(0))\sum_{k=0}^{m}\left\|h^{(k)}\right\|^2_{L^2(a,b)} = (\min_k\ Pk(0))\ \ \|h\|^2_{w^{m,2}(a,b)}\ .$$

We consider H_0 the energetic space of problem (1), (2), the completion of Q by virtue of inner product

$$(u, v)_0 = (DP(0)u, v)_{L^2(a,b)}, \quad \forall u, v \in Q.$$

If condition (A) is true for all$k = \overline{0, m}$, then the corresponding norm $\|\ .\ \|_0$is equivalent to the norm $\|\ .\ \|_{w^{m,2}(a,b)}$ .This fact results from the inequalities (5) and (6) so

$$(\min_k\ \mathrm{Pk}(0))\,\|u\|^2_{w^{m,2}(a,b)} \leq \|u\|_0 \leq (\max_k\ Pk(0))\ \ \|u\|^2_{w^{m,2}(a,b)}, \forall u \in H0. \tag{8}$$

Proposition 4 *If condition (A) is true for all $k = \overline{0, m}$, then P is a strong monotone operator.*

Proof: The strong monotony results from relations

$$\begin{aligned}
&(Pu_1 - Pu_2, u_1 - u_2)_{L^2(a,b)} \\
&= \int_0^1 (DP(tu_1 + (1-t)u_2)\ (u_1 - u_2), u_1 - u_2)_{L^2(a,b)} dt \\
&\geq \frac{\min\limits_k c_k}{\max\limits_k Pk(0)} \int_0^1 (DP(0)(u_1 - u_2), u_1 - u_2) dt \\
&\geq \frac{\min\limits_k c_k}{\max\limits_k Pk(0)} \|u_1 - u_2\|_0^2 = \gamma^2 \parallel u_1 - u_2 \parallel_0^2 \\
&\geq C\,\|u_1 - u_2\|^2_{w^{m,2}(a,b)} \geq C\,\|u_1 - u_2\|_{L^2(a,b)}
\end{aligned}$$

where $\gamma^2 = \frac{\min\limits_k c_k}{\max\limits_k Pk(0)}$ and we have use the inequality (5).

Theorem 5 If condition (A) is true for all $k = \overline{0, m}$, then:

a) Any minimized sequence for functional (4) has a limit in H_0,

b) All the minimized sequences for functional (4) have the same limit in H_0,

c) The limit in H_0 for any minimized sequence for functional (4) is the generalized solution of problem (1), (2),

d) The generalized solution of problem (1), (2) has generalized derivatives to and including m degree.

2. Finite Element Approximation

We introduce on the interval [a,b] a finite element mesh consisting of nodes

$$a = x_0 < x_1 < ... < x_N = b$$

and finite element $T_i = (x_i, x_{i+1}), i = \overline{0, N-1}$ with conditions:

$$\bar{T}_i \bigcap \bar{T}_j = \begin{cases} x_i & \text{if } j = i+1 \\ x_{i+1} & \text{if } j = i-1 \\ \phi & \text{if } j \neq i+1, i-1 \end{cases}$$

$$\bigcup_{i=0}^{N-1} \bar{T}_i = [a, b].$$

We introduce on the interval [a,b] a finite element mesh consisting of nodes

$$a = x_0 < x_1 < ... < x_N = b$$

and finite element $T_i = (x_i, x_{i+1}), i = \overline{0, N-1}$ with conditions:

$$\bar{T}_i \bigcap \bar{T}_j = \begin{cases} x_i & \text{if } j = i+1 \\ x_{i+1} & \text{if } j = i-1 \\ \phi & \text{if } j \neq i+1, i-1 \end{cases}$$

$$\bigcup_{i=0}^{N-1} \bar{T}_i = [a, b].$$

We consider the minimized problem: find a function $u \in H_0$ such that:

$$F(u) = \min_{v \in H_0} F(v) \tag{9}$$

and the discrete problem: find a function $u_h \in H_h$ so that:

$$F(u_h) = \min_{v_h \in H_h} F(v_h). \tag{10}$$

Theorem 6 The minimization problems (9) and (10) both have one and only one solution. Their respective solution $u \in H_0$ and $u_h \in H_h$ are also the unique solutions of the variational equations:

$$\sum_{k=0}^{m} \int_a^b P_k\left(\frac{d^k u}{dx^k}\right) \frac{d^k u}{dx^k} \frac{d^k v}{dx^k} dx = \int_a^b f v dx,\ \forall\ v \in H_0 \tag{11}$$

$$\sum_{k=0}^{m}\int_{a}^{b}P_k(\frac{d^k u_h}{dx^k})\frac{d^k u_h}{dx^k}\frac{d^k v_h}{dx^k}dx=\int_{a}^{b}fv_h dx \tag{12}$$

Proof: The functional F is Gateaux differentiable and

$$(DF(u))v=(Pu-f,v)=\sum_{k=0}^{m}\int_{a}^{b}Pk(\frac{d^k u}{dx^k})\frac{d^k u}{dx^k}\frac{d^k v}{dx^k}dx-\int_{a}^{b}f\ v\ dx,$$

so the solution u and $u_h$of the minimization problem (9) and (10) must satisfy relations (11) and (12) respectively. In view of the strict convexity of functional F, these relations are also sufficient for the existence of the unique solution.

Theorem 7 *The discrete solutions uh are bounded independently of subspace H_h.*

Proof: Using the strong monotone propriety of the operator P we obtain

$$(Pu_{\mathrm{h}}-Pu,u_{\mathrm{h}}-u)_{L^2(a,b)}\geq\gamma^2\,\|u_{\mathrm{h}}-u\|_0^2\geq C\,\|u_{\mathrm{h}}-u\|^2_{w^{m,2}(a,b)}$$

an we use the inequality.

$$\|u_h-u\|^2_{w^{m,2}(a,b)}\leq(Pu_h-Pu,u_h-u)_{L^2(a,b)}\leq\|Pu_L-Pu\|^*\|u_h-u\|_{L^2(a,b)}\leq\|Pu_h-Pu\|^*\|u_h-u\|_{w^{m,2}(a,b)}$$

Taking $v=\theta$ and u_h is the solution of discrete problem we obtain

$$\|u_h\|_{w^{m,2}(a,b)}\leq\frac{1}{\gamma^2}\,\|f\|^*$$

Theorem 8 *If* $\mathrm{u_h}$ *is the discrete solution of equation (12), the condition*

(A) is satisfied for all $k=\overline{o,m}$ and if

$$P^k\subset\hat{P}\subset H_0(\hat{T}) \tag{13}$$

$$W^{m,2}(\hat{T})\to C^{\sigma}(\hat{T}) \tag{14}$$

where σ is the greatest derivation degree that appears in definition of $\hat{\mathrm{T}}$, the reference finite element, then $(\mathrm{u_h})_{h\in H}$ is the minimization sequence of functional F and has an unique limit in H_0 .

Proof: The sequence $(u_h)_{h\in H}$ is bounded in a reflexive space $W^{m,2}(a,b)$.

So, there exists a subsequence (u_{hi}) which weakly converges to the same element$u\in W^{m,2}(a,b)$. We shall prove that u in minimum point of functional (4) in $\mathrm{W^{m,2}}$ (a, b), so $(u_h)_{h\in H}$ is a minimized sequence.

Let be $\varphi\in\mathrm{D}(a,b)$, the space of testing function. By definition of the discrete problem we have, in particular

$$\forall i\geq 1: F(u_{hi})\leq F(\Pi_{hi}\varphi)$$

Where Π_{hi} is the operator of finite element approximation. Since the functional F is continuous and convex we have

$$F(u)=\lim_{i\to\infty}F(u_{hi})\leq\lim_{i\to\infty}F(\Pi_{hi}\varphi). \tag{15}$$

Since σ is the maximum order of derivatives occurring in interpolation than, using Theorem 2.10.3 from (Titus Petrila & Calin John Cheorghiu, 1987) and from (13) and (14) we get

$$\|\,\Pi_{hi}\varphi-\varphi\,\|_{w^{m,2}(\hat{T})}\leq Ch^m_{\hat{T}}\,|\,\varphi\,|_{w^{m,2}(\hat{T})}$$

and summation after all T we have

$$\|\Pi_{hi}\varphi-\varphi\|\,w^{m,2}(a,b)\leq Ch^m_{\hat{T}}\,|\,\varphi\,|\,w^{m,2}(a,b),$$

so:

$$\lim_{i\to\infty}\|\Pi_{hi}\varphi-\varphi\|_{w^{m,2}(a,b)}=0.$$

This last relation and the continuity of the functional (4) imply that

$$\lim_{i\to\infty}F(\Pi_{hi}\varphi)=F(\varphi)$$

and so, using (15) it results that

$$\forall\varphi \in D(a,b) \quad F(u) \leq F(\varphi).$$

The space D (a, b) being dense in $W^{m,2}$ (a, b), it implies that

$$\forall v \in W^{m,2}(a,b) \quad F(u) \leq F(v)$$

and therefore the function u is the unique solution of minimized problem (10), and $(u_h)_{h\in H}$is a minimized sequence.

Theorem 9 *If $((u_h)_{h\in H}$is a minimization sequence (solution of discrete problem) and if $(u_h)_{h\in H}$ weakly converges to the same element u, then $u \in H_0$ and u has generalized derivatives up to the aforementioned order m inclusively.*

Proof: This result is obtained from Theorem 5.

In order to have an approach similar to that of the linear case we have obtained minimization sequence (solution of discrete problem) and if lem (10), and an analogous result as Cea's Lemma.

Theorem 10 *If Pk for all $k = \overline{0,m}$ are bounded functions on [a,b], then there exists a constant C, independent of the space H_h, such that:*

$$\| u - u_h\|_{W^{m,2}(a,b)} \leq C \inf_{v_h\in H_h} \| u - v_h\|_{W^{m,2}(a,b)} \tag{16}$$

Proof: From $H_h \subset W^{m,2}(a,b)$ it results

$$(P(u), w_h)_{L^2(a,b)} = (f, w_h)_{L^2(a,b),\ \forall w_h \in H_h}.$$

$$(P(u_h), w_h)_{L^2(a,b)} = (f, w_h)_{L^2(a,b),\ \forall w_h \in H_h}.$$

We subtract these equalities and we get

$$P\ (u) - P\ (u_h), w_h)_{L^2(a,b)} = 0.$$

Let be $w_h = u_h - v_h$ and we get

$$(P\ (u) - P\ (u_h), u_h - v_h)_{L^2(a,b)} = 0$$

and so

$$\|u - u_h\|^2_{W^{m,2}(a,b)} \leq (P(u) - P(u_h), u - u_h)_{L^2(a,b)} \leq (P(u) - P(u_h)\ , u - v_h + v_h - u_h)_{L^2(a,b)}$$
$$= (P(u) - P(u_h), u - v_h)_{L^2(a,b)} \leq \left| \left(\sum_{k=0}^m \int_a^b \left[Pk(\tfrac{d^k u}{dx^k})\tfrac{d^k u}{dx^k} - Pk(\tfrac{d^k u_h}{dx^k})\tfrac{d^k u_h}{dx^k} \right]\ , \tfrac{d^k(u-v_h)}{dx^k} \right)_{L^2(a,b)} \right|$$
$$\leq C \sum_{k=0}^m \int_a^b \left| \tfrac{d^k u}{dx^k} - \tfrac{d^k u_h}{dx^k} \right| \ \left| \tfrac{d^k(u-u_h)}{dx^k} \right| dx \leq C \sqrt{\sum_{k=0}^m \left\| \tfrac{d^k(u-u_h)}{dx^k} \right\|^2_{L^2(a,b).}} \quad \cdot \quad \sqrt{\sum_{k=0}^m \left\| \tfrac{d^k(u-v_h)}{dx^k} \right\|^2_{L^2(a,b)}}$$
$$= C \| u - u_h\|_{W^{m,2}(a,b)} \| u - v_h\|_{W^{m,2}(a,b)} .$$

It results that

$$\| u - u_h\|_{W^{m,2}(a,b)} \leq C \| u - v_h\|_{W^{m,2}(a,b)}\ , \forall v_h \in H_h and \| u - u_h\|_{W^{m,2}(a,b)} \leq C \inf_{v_h\in H_h} \| u - v_h\|_{W^{m,2}(a,b).}$$

We can obtain an error evaluation:

Theorem 11 *In the above conditions if u is in $W^{m+k,2}$(a,b), than there is a constant C independent of the space H_h such that:*

$$\|u - u_h\|_{W^{m,2}(a,b)} \leq Ch^k |u|_{W^{m+k,2}(a,b)}. \tag{17}$$

Proof: Using the Theorem 10 we get

$$\| u - u_h\|_{W^{m,2}(a,b)} \leq \inf_{w_h\in H_h} \|u - v_h\|_{W^{m,2}(a,b)} \leq \| u - \Pi_h u \|_{W^{m,2}(a,b)} \leq Ch^k\ |u|_{W^{m+k,2}(a,b)}\ .$$

Let be $w_1,\ w_2,\ ...,\ w_s$ a basis in H_h. Than we have $u_h = \sum_{k=0}^s \alpha_k w_k$. We solve the nonlinear system:

$$\sum_{k=0}^s \int_a^b Pk \left(\sum_{i=0}^s \alpha_i \tfrac{d^k w_i}{dx^k} \right) \ \left(\sum_{i=0}^s \alpha_i \tfrac{d^k w_i}{dx^k} \right) \tfrac{d^k w_h}{dx^k} dx = \int_a^b f\ w_h dx$$

with the unknowns $\alpha_1, \alpha_2, ..., \alpha_s$. We shall approximate the integrals on each subinterval, so we have

$$\sum_{k=0}^m \sum_{j=0}^{N-1} \int_{x_j}^{x_j+1} Pk \left(\sum_{i=0}^s \alpha_i \frac{d^k w_i}{dx^k} \right) \ \left(\sum_{i=0}^s \alpha_i \frac{d^k w_i}{dx^k} \right) \frac{d^k w_h}{dx^k} dx = \int_a^b f\ w_h dx,$$

for all $w_h \in \{w_1, w_2..., w_s\}$ and for each subinterval the integrals are approximated through the process of numerical integration, using a quadrature scheme. This scheme must be compatible with the error estimation of finite element method. We obtain a nonlinear system for which we can apply Newton – Kantorovici method.

Remark 1 As usual, the same latter C stands for various constants independent of h and the various functions involved.

Remark 2 An analogous problem can be solved when operator P is

$$P(u) = \sum_{k=0}^{m}(-1)^k \frac{d^k}{dx^k}[gk(\frac{d^k u}{dx^k})]$$

and the corresponding conditions

$$g'(t) \geq C_{k+1} \geq 0, \forall k = \overline{0, m}.$$

References

Ciarlet, G. Philippe. (1980). *The Finite Element Method for Elliptic Problems*. North Holland.

Dinca, George. (1980). *Variational Method and Applications*. Bucharest.

Peter. W. (2008). *Nonlinear Finite Element Methods*. Berlin: Springer Science.

Titus Petrila, & Calin John Cheorghiu. (1987). *Finite Element Method and Applications*. RSRAcademy, Bucharest.