

Some More Results on Harmonic Mean Graphs

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Abstract

A Graph $G = (V, E)$ with p vertices and q edges is called a harmonic mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, \dots, q + 1$ in such a way that when each edge $e = uv$ is labeled with $f(uv) = \left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil$ or $\left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$ then the edge labels are distinct. In this case f is called Harmonic mean labeling of G .

The concept of Harmonic mean labeling was introduced in (Somasundaram, Ponraj & Sandhya). In (Somasundaram, Ponraj & Sandhya) and (Sandhya, Somasundaram & Ponraj, 2012) we investigate the harmonic mean labeling of several standard graphs such as path, cycle comb, ladder, Triangular snakes, Quadrilateral snakes etc. In the present paper, we investigate the harmonic mean labeling for a polygonal chain, square of the path and dragon. Also we enumerate all harmonic mean graph of order ≤ 5 .

Keywords: Graph, Harmonic mean graph, Polygonal chain, Square of a path, Dragon

AMS subject classification: 05C 78

1. Introduction

The graph considered here will be finite, undirected and simple. Terms not defined here are used in the sense of Harary (Harary, 1988). The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G . The Cardinality of the graph G is called the order of G . The Cardinality of its edge set is called the size of G . The graph $G - e$ is obtained from G by deleting an edge e . The sum $G_1 + G_2$ of two graphs G_1 and G_2 has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. The Union of two graphs G_1 and G_2 is a graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The Square G^2 of the graph G has $V(G^2)$ with u, v adjacent in G^2 whenever $d(u, v) \leq 2$ in the graph G . A dragon is a graph formed by joining an end vertex of path P_m to the vertex of the cycle C_n . It is denoted $C_n @ P_m$. The graph $C_n \hat{\circ} K_{1,m}$ is obtained C_n and $K_{1,m}$ by identifying any vertex of C_n and the central vertex of $K_{1,m}$.

S. Somasundaram and R. Ponraj introduced mean labeling of graphs in (Somasundaram & Ponraj, 2003a) and investigate mean labeling for some standard graphs in (Somasundaram & Ponraj, 2003b) and in (Somasundaram & Ponraj, 2004). We introduce Harmonic mean labeling of graphs in (Somasundaram, Ponraj & Sandhya) and proved Harmonic mean labeling of some standard graphs in (Sandhya, Somasundaram & Ponraj, 2012). In this paper we prove that polygonal chains, square of a path and dragons are harmonic mean graphs. Finally we investigate all harmonic mean graph of order ≤ 5 .

We shall make frequent reference to the following results.

Theorem 1.1 (Somasundaram, Ponraj & Sandhya) *Any path is a harmonic mean graph.*

Theorem 1.2 (Somasundaram, Ponraj & Sandhya) *Any cycle C_n , $n \geq 3$ is a harmonic mean graph.*

Theorem 1.3 (Somasundaram, Ponraj & Sandhya) *The complete graph the K_n is a harmonic graph if and only if $n \leq 3$.*

Theorem 1.4 (Somasundaram, Ponraj & Sandhya) *The complete bipartite graph $K_{1,n}$ is a harmonic mean graph if and only if $n \leq 7$.*

Theorem 1.5 (Sandhya, Somasundaram & Ponraj, 2012) *Any wheel W_n is not a harmonic mean graph.*

Theorem 1.6 (Sandhya, Somasundaram & Ponraj, 2012) *$C_m \cup P_n$, $n > 1$, $C_m \cup C_n$, $m, n \geq 3$ is a harmonic mean graph.*

Theorem 1.7 (Sandhya, Somasundaram & Ponraj, 2012) *$K_n - e$, $n > 4$ is not a harmonic mean graph.*

Remark 1.8 (Sandhya, Somasundaram & Ponraj, 2012) *If $p > q + 1$, then the graph $G = (p, q)$ is not a harmonic mean graph.*

Remark 1.9 (Sandhya, Somasundaram & Ponraj, 2012) *If $n \leq 4$, $K_n - e$ is a harmonic mean graph.*

2. Harmonic Mean Labeling

Definition 2.1 A Graph G with p vertices and q edges is called a harmonic mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, \dots, q + 1$ in such a way that when each edge $e = uv$ is labeled with $f(uv) = \left\lceil \frac{2f(u)f(v)}{f(u)+f(v)} \right\rceil$ or $\left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$ then the edge labels are distinct. In this case f is called Harmonic mean labeling of G .

Definition 2.2 A Polygonal chain $G_{m,n}$ is a connected graph all of whose m blocks are polygons (C_n).

Theorem 2.3 *Polygonal chain $G_{m,n}$ are Harmonic mean graphs for all m and n .*

Proof: In $G_{m,n}$ chain, let $u_1 u_2 u_4 u_6 \dots u_{n-4} u_{n-2} u_{n+1} u_{n-1} u_{n-3} u_{n-5} \dots u_7 u_5 u_3 u_1$ be the first cycle. The second cycle is connected to the first cycle at the vertex u_{n+1} . Let $u_{n+1} u_{n+2} u_{n+4} \dots u_{2n+1} u_{2n-1} u_{2n-3} \dots u_{n+7} u_{n+5} u_{n+3} u_{n+1}$ be the second cycle. The third chain is connected to the second cycle at the vertex u_{2n+1} . Let the third cycle be $u_{2n+1} u_{2n+2} u_{2n+4} \dots u_{3n+1} u_{3n-1} u_{3n-3} \dots u_{2n+5} u_{2n+3} u_{2n+1}$. In general the r^{th} cycle is connected to the $(r - 1)^{th}$ cycle at the vertex u_{rn+1} . Let the r^{th} cycle be $u_{rn+1} u_{rn+2} u_{rn+4} u_{rn+6} \dots u_{(r+1)n-4} u_{(r+1)n-2} u_{(r+1)n+1} u_{(r+1)n-1} u_{(r+1)n-3} u_{(r+1)n-5} u_{(r+1)n-7} \dots u_{rn+5} u_{rn+3} u_{rn+1}$.

Magnified figure of the r^{th} cycle is given in Figure 1.

Assume the graph has m cycles. Define a function $f : V(G_{m,n}) \rightarrow \{1, 2, \dots, q + 1\}$ by $f(v_i) = i$, $1 \leq i \leq mn - 1$, $f(v_n) = mn + 1$. Then the label of the edge is given below

$$\begin{aligned} f(u_{mn+1} u_{mn+2}) &= mn + 1 \\ f(u_{mn+i} u_{mn+i+2}) &= mn + i + 1 \\ f(u_{(m+1)n-2} u_{(m+1)n+1}) &= (m + 1)n - 1 \\ f(u_{(m+1)n+1} u_{(m+1)n-1}) &= (m + 1)n \end{aligned}$$

Since the graph $G_{m,n}$ has distinct edge labels, $G_{m,n}$ is a harmonic mean graph.

Example 2.4 Harmonic mean labeling of $G_{4,9}$ chain is in Figure 2.

Now we investigate the square of a path.

Theorem 2.5 *The graph P_n^2 is a harmonic mean graph.*

Proof: Let P_n be the path $u_1, u_2, \dots, u_n$. Clearly P_n^2 has n vertices and $2n - 3$ edges. Define $f : V(P_n^2) \rightarrow \{1, 2, \dots, q + 1\}$ by $f(u_i) = 2i - 1$, $1 \leq i \leq n - 1$ and $f(u_n) = 2n - 2$. The label of the edge $u_i u_{i+1}$ is $2i - 1$, $1 \leq i \leq n - 1$. The label of the edge $u_i u_{i+2}$ is $2i$, $1 \leq i \leq n - 2$. The label of the edge $u_{n-1} u_n$ is $2n - 3$. Hence P_n^2 is a harmonic mean graph.

Example 2.6 Harmonic mean labeling of P_8^2 is given in Figure 3.

Theorem 2.7 *Dragon's $C_n @ P_m$ are harmonic mean graphs.*

Proof: Let $u_1 u_2 \dots u_n$ be the cycle C_n and $v_1 v_2 \dots v_m$ be the path P_m . Identify u_{n-1} with v_1 . Define a function $f : V(C_n @ P_m) \rightarrow \{1, 2, \dots, q + 1\}$ by $f(u_i) = i$, $1 \leq i \leq n - 3$, $f(u_{n-2}) = n - 1$, $f(u_{n-1}) = n$, $f(u_n) = n - 2$ and $f(v_{i+1}) = n + i$, $1 \leq i \leq m - 1$. Clearly f is a harmonic mean labeling

Example 2.8 A harmonic mean labeling of $C_6 @ P_7$ is given in Figure 4.

Theorem 2.9 The graph $C_n \hat{\circ} K_{1,1}$ is a harmonic mean graph.

Proof: We know that, $C_n \hat{\circ} K_{1,1} = C_n @ P_2$. Hence the proof follows from the Theorem 2.7.

Remark 2.10 The graph $C_n \hat{\circ} K_{1,2}$ is also a harmonic mean graph.

Theorem 2.11 Let C_n be the cycle $u_1 u_2 u_3 \dots u_n$. Let G be the graph with $V(G) = V(C_n) \cup \{v_1, v_2\}$ and $E(G) = E(C_n) \cup \{u_1 v_1, u_n v_2\}$. Then G is a harmonic mean graph.

Proof: Define a function $f : V(G) \rightarrow \{1, 2, \dots, q+1\}$ by $f(u_i) = i+1, 1 \leq i \leq n, f(v_1) = 1$ and $f(v_2) = n+2$. Then we get distinct edge labels from $1, 2, \dots, q$. Obviously f is a harmonic mean labeling for G .

Example 2.12 The harmonic mean labeling of G is given in Figure 5.

Theorem 2.13 Let G be the graph obtained from $K_4 - e$ by attaching P_n with the vertex of degree 3 in $K_4 - e$. Then G is a harmonic mean graph.

Proof: Let v_1, v_2, v_3 and v_4 be the vertices of $G = K_4 - e$ and $e = v_1 v_3$. Let P_n be the path $u_1 u_2 u_3 \dots u_n$. Identify u_1 with v_4 . Define $f : V(G) \rightarrow \{1, 2, \dots, q+1\}$ by $f(v_1) = 1, f(v_2) = 3, f(v_3) = 4, f(v_4) = 6, f(u_{i+1}) = 6+i, 1 \leq i \leq n-1$. Obviously G is a harmonic mean graph.

Example 2.14 Harmonic mean labeling of G is given in Figure 6.

Next we have

Theorem 2.15 Let G be the graph obtained from $K_4 - e$ by attaching P_n to the vertex of the degree 2 in the graph $K_4 - e$. Then G is a harmonic mean graph.

Proof: Let v_1, v_2, v_3 and v_4 be the vertices of $G = K_4 - e$ and $u_1 u_2 u_3 \dots u_n$ be the path P_n . Define $f : V(G) \rightarrow \{1, 2, \dots, q+1\}$ by $f(v_1) = 1, f(v_2) = 3, f(v_3) = 5, f(v_4) = 7, f(u_{i+1}) = 5+i, 1 \leq i \leq n-1$. Clearly f is a harmonic mean labeling.

Example 2.16 Harmonic mean labeling of G is given in Figure 7.

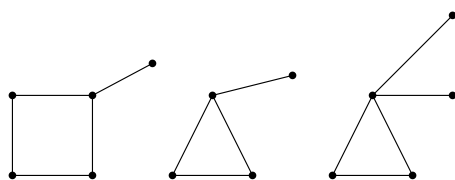
3. Harmonic Mean Graph of Order ≤ 5

Theorem 3.1 The following graphs of Order ≤ 5 are harmonic mean graphs

i) P_1, P_2, P_3, P_4, P_5

ii) C_3, C_4, C_5

iii)



iv) $K_{1,3}, K_{1,4}$

v) $C_3 \cup P_2$

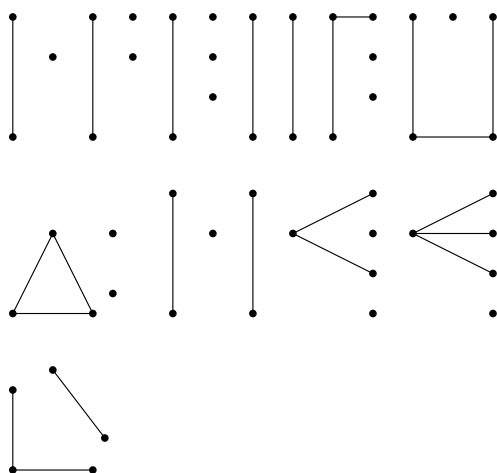
vi)



Proof: Since all paths are harmonic mean graphs, the graphs in case (i) are harmonic mean graphs by Theorem 1.1 The graphs in case (ii) are harmonic mean graphs by Theorem 1.2 The graph in case (iii) $C_3 \hat{\circ} K_{1,1}, C_3 \hat{\circ} K_{1,2}$ and $C_4 \hat{\circ} K_{1,1}$ are harmonic mean graphs by Theorem 2.9 and Remark 2.10. The graphs in case (iv) are harmonic mean graphs by Theorem 1.4. The graphs in case (v) is a harmonic mean graph by Theorem 1.6. The graph in case (vi) are harmonic mean graphs by Remark 1.9.

Theorem 3.2 The following graphs of order ≤ 5 are not harmonic mean graphs.

i) $K_2^c, K_3^c, K_4^c, K_5^c,$



ii) $K_4, K_5, K_4 \cup K_1$

iii) W_4, W_5

iv) $K_5 - e$

Proof: The graphs in case (i) are not harmonic mean graph by Remark 1.8 The graphs in case (ii) are not harmonic mean graphs by Theorem 1.3. The graphs in case (iii) are not harmonic mean graph by Theorem 1.5. The graph in case (iv) are not harmonic mean graph by Theorem 1.7.

Remark 3.3 The remaining 16 graphs of order ≤ 5 are given in Table 3.1 and Table 3.2. They are shown to be harmonic mean graphs by giving a specific harmonic mean labeling.

The Table 3.3 shows the number of graphs of order ≤ 5 which are harmonic mean and which are not harmonic mean.

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Table 3.1

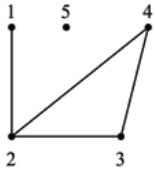
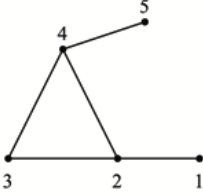
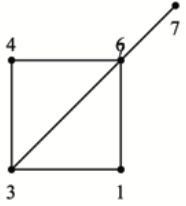
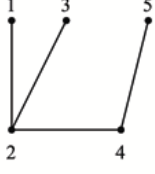
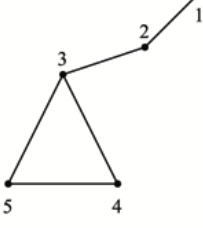
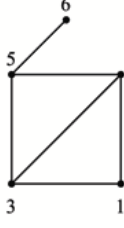
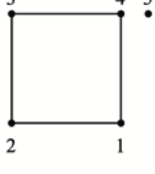
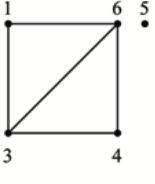
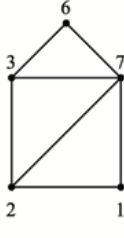
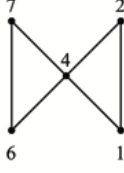
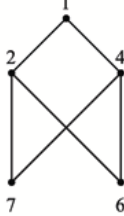
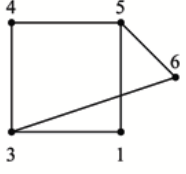
	q=4	q=5	q=6
p=5			
			
			
			
			
			

Table 3.2

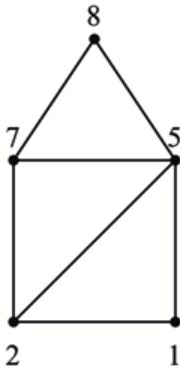
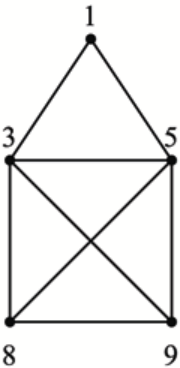
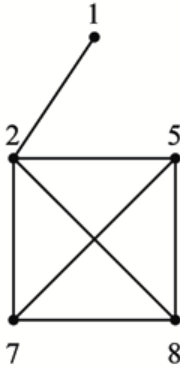
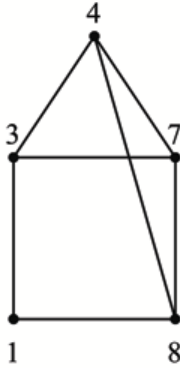
	q=7	q=8
p=5		
		
		

Table 3.3

Order of the graph	Harmonic mean	Not Harmonic mean
1	1	0
2	1	1
3	2	2
4	5	5
5	22	13

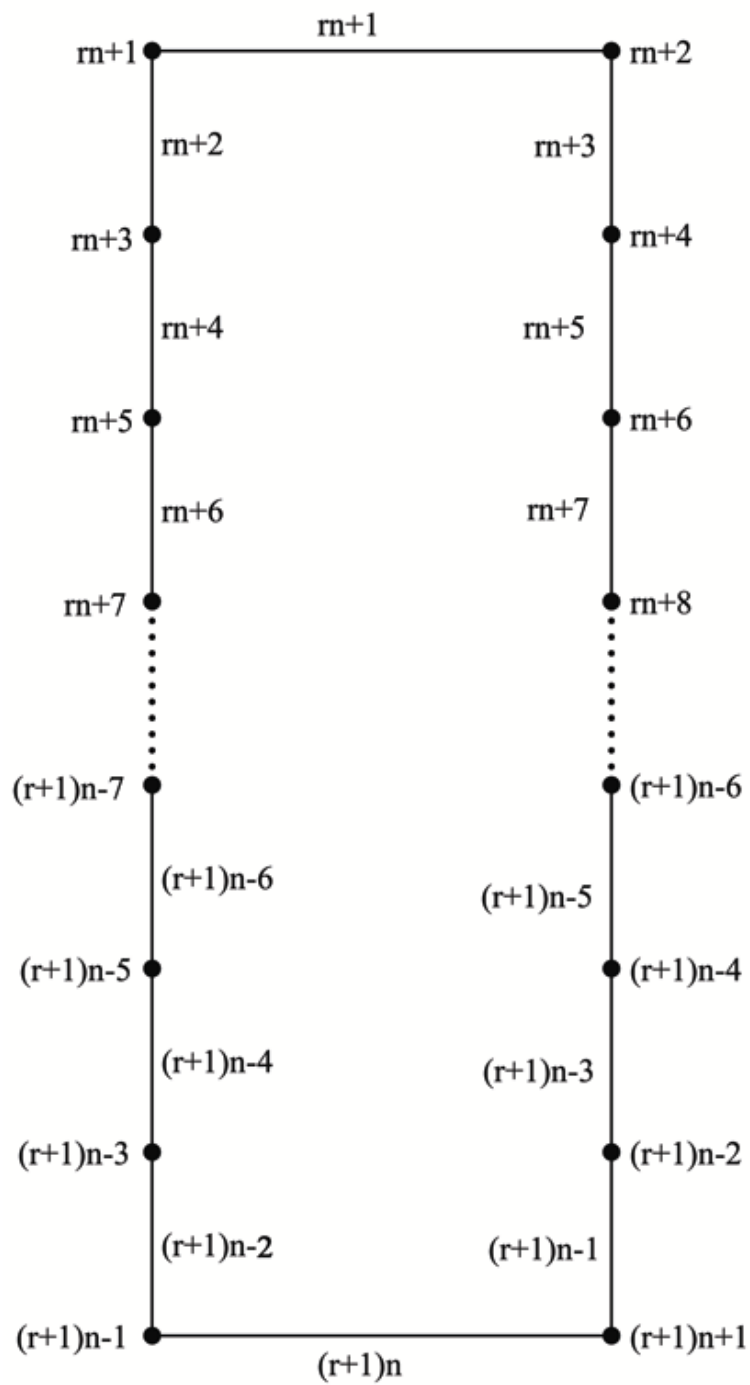


Figure 1

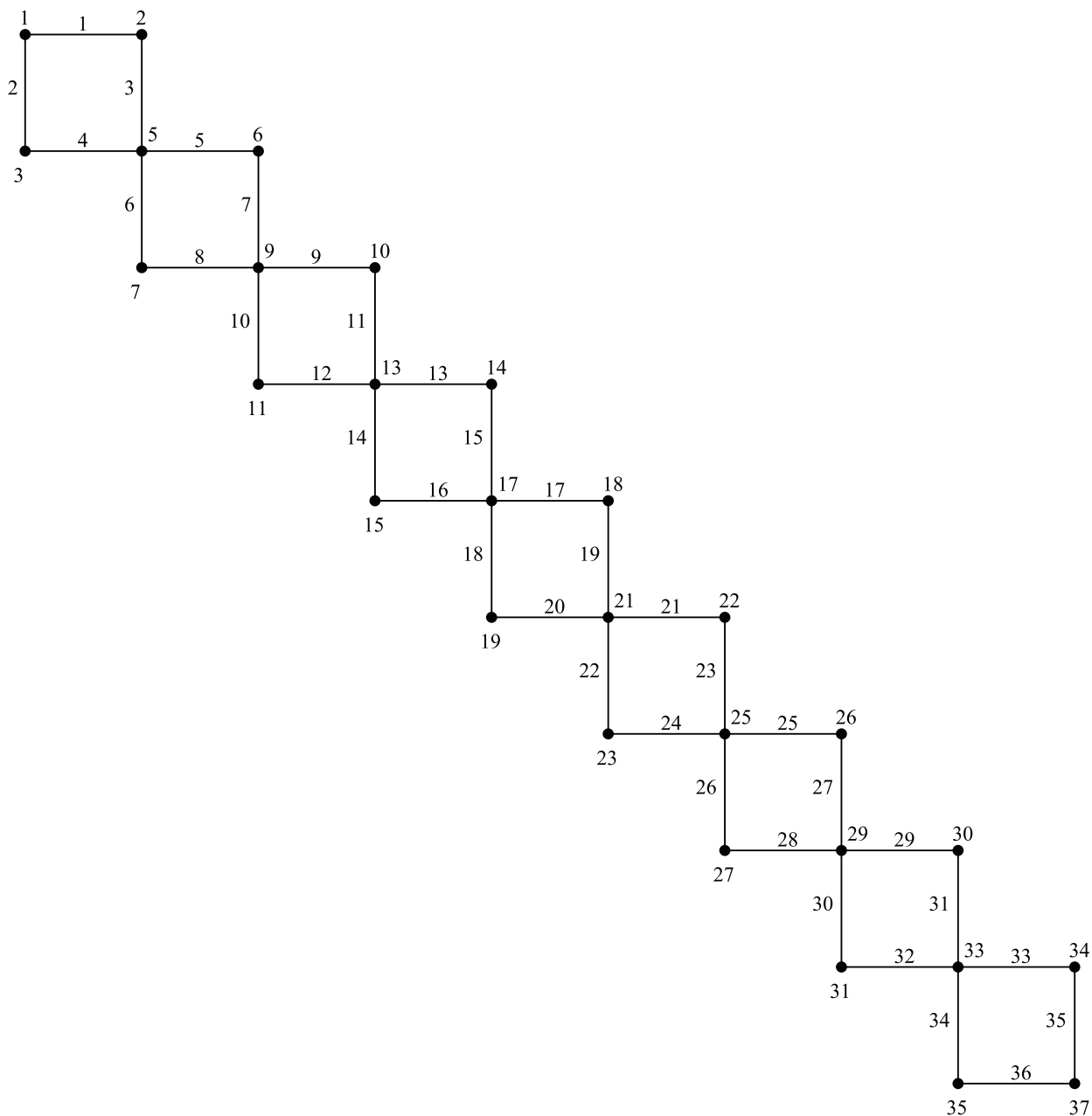


Figure 2

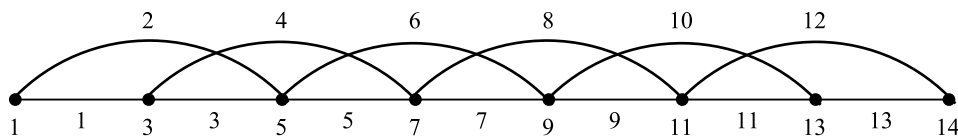


Figure 3

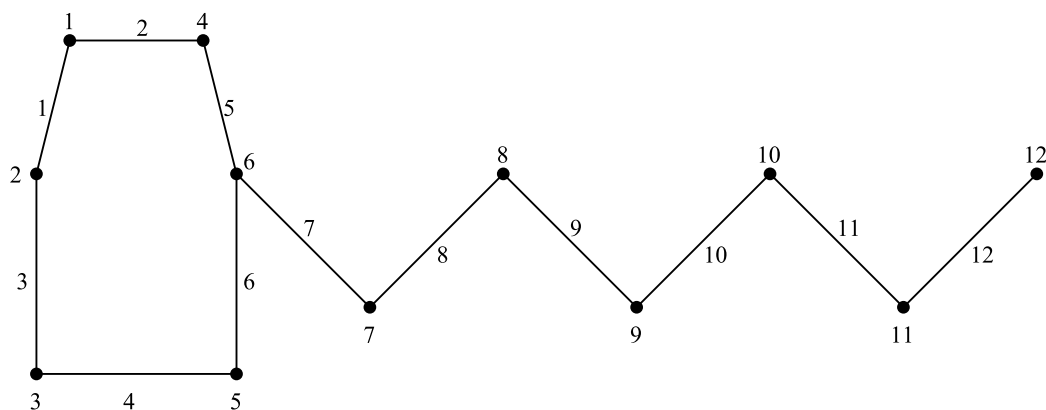


Figure 4

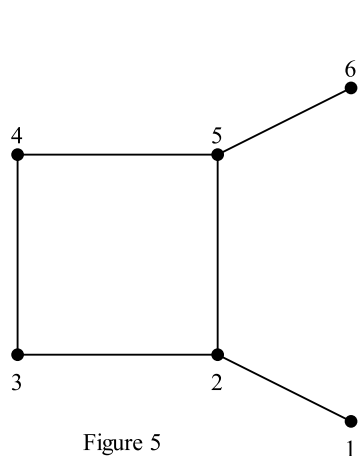


Figure 5

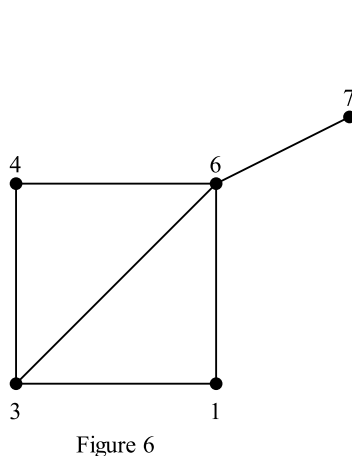


Figure 6

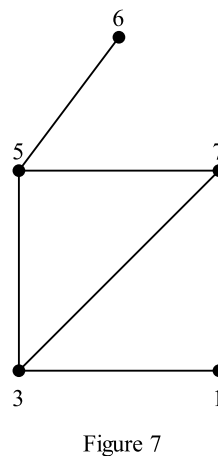


Figure 7