Bounds on Normal Approximation on a Half Plane in Multidimension

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Abstract

We give bounds on multidimensional Berry-Esseen theorem on a set $A_k(x) = \{(w_1, w_2, \ldots, w_k) \in \mathbb{R}^k \mid \sum_{i=1}^{k} w_i \leq x\}$ for $x \in \mathbb{R}$ by using the Berry-Esseen theorem in $\mathbb{R}$. The rates of convergence are $O(n^{-\frac{1}{2}})$. In addition, we give known constants in the bounds of the approximation.

Keywords: Berry-Esseen inequality, Central limit theorem, Uniform and non-uniform bounds

1. Introduction

For $n \in \mathbb{N}$, let X_i, $1 \leq i \leq n$ be independent and identically distributed random variables with zero means and $\sum_{i=1}^{n} EX_i^2 = 1$. Define

$$S_n = \sum_{i=1}^{n} X_i$$

and Φ_1 the standard normal distribution in $\mathbb{R}$. Suppose that $E|X_i|^3 < \infty$ for $1 \leq i \leq n$. The uniform and non-uniform versions of the Berry-Esseen inequality are

$$\sup_{x\in\mathbb{R}} |P(S_n \leq x) - \Phi_1(x)| \leq C_0 \sum_{i=1}^{n} E|X_i|^3$$

and

$$|P(S_n \leq x) - \Phi_1(x)| \leq \frac{C_1}{1 + |x|^3} \sum_{i=1}^{n} E|X_i|^3$$

respectively, where C_0 and C_1 are positive constants. The uniform version was independently discovered by (Berry, 1941, p. 122-136) and (Esseen, 1945, p. 1-125) and the non-uniform version was discovered by (Nagaev, 1965, p. 214-235). Without assuming the identically of X_i, the best constant C_0 and C_1 were given by (Shevtsova, 2010, p. 862-864) and (Paditz, 1989, p. 453-464), respectively. The results are as follows:

Theorem 1.1 (Shevtsova, 2010, p. 862-864) *Let X_i, $1 \leq i \leq n$, be independent random variables such that $EX_i = 0$ and $E|X_i|^3 < \infty$. Assume that $\sum_{i=1}^{n} EX_i^2 = 1$. Then*

$$\sup_{x\in\mathbb{R}} |P(S_n \leq x) - \Phi_1(x)| \leq 0.5600 \sum_{i=1}^{n} E|X_i|^3.$$

Theorem 1.2 (Paditz, 1989, p. 453-464) *Under the assumptions of theorem 1.1, we have*

$$|P(S_n \le x) - \Phi_1(x)| \le \frac{31.935}{1+|x|^3} \sum_{i=1}^{n} E|X_i|^3$$

for all real numbers x.

(Chen, 2001, p. 236-254) relaxed the condition to the finiteness of the second moments and gave uniform and non-uniform versions of the inequality. The constant of the non-uniform version was given by (Neammanee, 2007, p. 1-10). Here are the results.

Theorem 1.3 (Chen, 2001, p. 236-254) *Let $X_i, 1 \le i \le n$, be independent random variables such that $EX_i = 0$ and $\sum_{i=1}^{n} EX_i^2 = 1$. Then*

$$\sup_{x\in\mathbb{R}} |P(S_n \le x) - \Phi_1(x)| \le 4.1 \sum_{i=1}^{n} \{E|X_i|^2 I(|X_i| > 1) + E|X_i|^3 I(|X_i| \le 1)\}$$

and for all real numbers x,

$$|P(S_n \le x) - \Phi_1(x)| \le C \sum_{i=1}^{n} \left\{ \frac{E|X_i|^2 I(|X_i| > 1+|x|)}{1+|x|^2} + \frac{E|X_i|^3 I(|X_i| \le 1+|x|)}{1+|x|^3} \right\}.$$

Theorem 1.4 (Neammanee, 2007, p. 1-10) *Under the assumptions of theorem 1.3, we have*

$$|P(S_n \le x) - \Phi_1(x)| \le C \sum_{i=1}^{n} \left\{ \frac{E|X_i|^2 I(|X_i| > 1+|x|)}{1+|x|^2} + \frac{E|X_i|^3 I(|X_i| \le 1+|x|)}{1+|x|^3} \right\}$$

where

$$C = \begin{cases} 13.11 & \text{if } 0 \le |x| < 1.3, \\ 28.54 & \text{if } 1.3 \le |x| < 2, \\ 46.32 & \text{if } 2 \le |x| < 3, \\ 61.40 & \text{if } 3 \le |x| < 7.98, \\ 40.12 & \text{if } 7.98 \le |x| < 14, \\ 39.39 & \text{if } |x| \ge 14. \end{cases}$$

The reduction they make is truncation. This method make the random variables become bounded random varibles. In the case that each X_i is bounded, the uniform and non-uniform versions were given in (Chen, 2005, p. 1-59) and (Chaidee, 2005), respectively.

Theorem 1.5 (Chen, 2005, p. 1-59) *Let $X_i, 1 \le i \le n$, be independent random variables such that $EX_i = 0$, $\sum_{i=1}^{n} EX_i^2 = 1$ and $|X_i| \le \delta_0$, then*

$$\sup_{x\in\mathbb{R}} |P(S_n \le x) - \Phi_1(x)| \le 3.3\delta_0.$$

Theorem 1.6 (Chaidee, 2005) *Under the assumptions of theorem 1.5, there exists a constant C which does not depend on δ_0 such that for every real numbers x,*

$$|P(S_n \le x) - \Phi_1(x)| \le \frac{C\delta_0}{1+|x|^3}.$$

For multidimensional case, let $k \in \mathbb{N}$ and $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{ik})$ be independent and identically distributed random vectors in $\mathbb{R}^k$ with zero means and covariance identity matrices I_k. Define

$$W_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i.$$

Let F_n be the distribution function of W_n and Φ_k the standard Gaussian distribution in $\mathbb{R}^k$. (Bergström, 1945, p. 106-127) guaranteed that F_n converges weakly to Φ_k for large n. The uniform and non-uniform bounds of this convergence have been repeatedly refined over subsequent decades by many researchers such as (Esseen, 1945, p. 1-125), (Rao, 1961, p. 359-361) , (Bahr, 1967, p. 61-69), (Bahr, 1967, p. 71-88) and (Bhattacharya, 1970, p. 68-86), etc. For the assumption that

$$\sum_{j=1}^{k} E|Y_{1j}|^4 < \infty,$$

(Esseen, 1945, p. 1-125) gave a uniform bound on this convergence which is of the form

$$|F_n(B_r) - \Phi_k(B_r)| \leq \frac{C}{n^{\frac{k}{k+1}}}$$

where $B_r = \{(w_1, w_2, \ldots, w_k) \in \mathbb{R}^k \mid w_1^2 + w_2^2 + \cdots + w_k^2 \leq r^2\}$ for $r > 0$ and C is an absolute constant depending only on the moment. (Rao, 1961, p. 359-361) generalized Esseen's result to any measurable convex subset A of $\mathbb{R}^k$. His result is

$$|F_n(A) - \Phi_k(A)| \leq \frac{C}{\sqrt{n}}(\log n)^{\frac{k-1}{2(k+1)}}. \tag{1}$$

(Bahr, 1967, p. 71-88) assumed

$$E(\sum_{j=1}^{k} Y_{1j}^2)^{\frac{s}{2}} < \infty,$$

for an integer $s > k > 1$ and improved the rate of convergence in (1) by the inequality

$$|F_n(B) - \Phi_k(B)| \leq \frac{C}{\sqrt{n}}. \tag{2}$$

In the case that each Y_i may not be identically distributed random vectors, (Bhattacharya, 1970, p. 68-86) assumed

$$\sum_{j=1}^{k} E|Y_{ij}|^{3+\delta} < \infty \quad \text{for} \quad 1 \leq i \leq n \quad \text{where} \quad \delta > 0,$$

and gave a bound of the approximation as in (2) on any Borel subset of $\mathbb{R}^k$.
For a non-uniform version, (Bahr, 1967, p. 61-69) is the first one who investigated this version. He assumed the identically assumption on each Y_i and gave the rate of convergence on $B_k(r)$. Under the finiteness assumption of the s^{th} moments,

$$E(\sum_{j=1}^{k} Y_{1j}^2)^{\frac{s}{2}} < \infty,$$

for integer $s \geq 3$, the result is

$$|F_n(B_k(r)) - \Phi_k(B_k(r))| \leq \frac{C \cdot d(n)}{r^s n^{\frac{s-2}{2}}} \qquad \text{for} \qquad r \geq (\frac{5}{4}m(s-2)\log n)^{\frac{1}{2}}$$

where m is the largest eigenvalue of the covariance matrix of Y_i, $d(n)$ is bounded by one and $\lim_{n\to\infty} d(n) = 0$.
The aim of this paper is to find bounds on normal approximation to the distribution of W_n over the set

$$A_k(x) = \{(w_1, w_2, \ldots, w_k) \in \mathbb{R}^k \mid \sum_{i=1}^{k} w_i \leq x\}.$$

In this work, assume only that $\frac{1}{nk}Var(\sum_{i=1}^{n}\sum_{j=1}^{k} Y_{ij}) = 1$ and give our results on various assumptions, the random variables Y_{ij} are bounded, $E|Y_{ij}|^3 < \infty$ and $E|Y_{ij}|^p < \infty$ for some $2 < p < 3$. Our results are as follows:

Theorem 1.7 *If $|Y_{ij}| \leq \delta_0$ for $1 \leq i \leq n$ and $1 \leq j \leq k$, then*

$$\sup_{x\in\mathbb{R}} |P(W_n \in A_k(x)) - \Phi_k(A_k(x))| \leq \frac{3.3\sqrt{k}\delta_0}{\sqrt{n}}$$

and there exists a constant C which does not depend on δ_0 such that for every real numbers x,

$$|P(W_n \in A_k(x)) - \Phi_k(A_k(x))| \leq \frac{Ck^2\delta_0}{\sqrt{n}[(\sqrt{k})^3 + |x|^3]}.$$

Theorem 1.8 *If $E|Y_{ij}|^p < \infty$ for $2 < p < 3, 1 \leq i \leq n$ and $1 \leq j \leq k$, then*

$$\sup_{x\in\mathbb{R}} |P(W_n \in A_k(x)) - \Phi_k(A_k(x))| \leq \frac{75(4)^{p-1}k^p}{(nk)^{\frac{p}{2}}} \sum_{i=1}^{n}\sum_{j=1}^{k} E|Y_{ij}|^p$$

and there exists an absulute constant C such that for $x \in \mathbb{R}$,

$$|P(W_n \in A_k(x)) - \Phi_k(A_k(x))| \leq \frac{5^pCk^p}{n^{\frac{p}{2}}(\sqrt{k} + |x|)^p} \sum_{i=1}^{n}\sum_{j=1}^{k} E|Y_{ij}|^p.$$

Theorem 1.9 *If $E|Y_{ij}|^3 < \infty$ for $1 \leq i \leq n$ and $1 \leq j \leq k$, then*

$$\sup_{x\in\mathbb{R}} |P(W_n \in A_k(x)) - \Phi_k(A_k(x))| \leq \frac{0.5600\sqrt{k}}{n^{\frac{3}{2}}} \sum_{i=1}^{n}\sum_{j=1}^{k} E|Y_{ij}|^3$$

and for all real numbers x

$$|P(W_n \in A_k(x)) - \Phi_k(A_k(x))| \leq \frac{31.935k^2}{n^{\frac{3}{2}}[(\sqrt{k})^3 + |x|^3]} \sum_{i=1}^{n}\sum_{j=1}^{k} E|Y_{ij}|^3.$$

The proofs of our main theorems are given in the next section.

2. Proof of Main Theorems

In the proofs of main theorems, we use the Berry-Esseen theorems in $\mathbb{R}$ in which the limit distribution is Φ_1. However, the limit distribution in our theorems is the standard Gaussian distribution Φ_k in $\mathbb{R}^k$. In the following proposition, we give a relation between Φ_1 and Φ_k.

Proposition 2.1 *For $k \in \mathbb{N}$ and $x \in \mathbb{R}$, we have*

$$\Phi_k(A_k(x)) = \Phi_1(\frac{x}{\sqrt{k}}).$$

Proof: To prove the proposition, we let $B = \{b_1, b_2, \dots, b_k\}$ be an orthonorrmal basis for $\mathbb{R}^k$ with $b_1 = \frac{1}{\sqrt{k}}(1, 1, \dots, 1)$ and $w = (w_1, w_2, \dots, w_k) \in A_k(x)$. Set

$$t_1 = \langle b_1, w\rangle \text{ and } t_i = \langle b_i, w\rangle \text{ for } t = 2, 3, \dots, k.$$

Then

$$t_1 = \frac{1}{\sqrt{k}}\sum_{i=1}^{k} w_i \leq \frac{x}{\sqrt{k}}, -\infty < t_i < \infty, \text{ for } t = 2, 3, \dots, k, \text{ and}$$

$$\sum_{i=1}^{k} \langle b_i, w\rangle\, b_i = w = \sum_{i=1}^{k} \langle e_i, w\rangle\, e_i$$

where $\{e_1, e_2, \dots, e_k\}$ is a usual orthonormal basis for $\mathbb{R}^k$. Thus, we have

$$\begin{aligned}\sum_{i=1}^{k} w_i^2 &= \|\sum_{i=1}^{k} w_ie_i\|^2 = \|\sum_{i=1}^{k} \langle e_i, w\rangle\, e_i\|^2 = \|\sum_{i=1}^{k} \langle b_i, w\rangle\, b_i\|^2 = \|\sum_{i=1}^{k} t_ib_i\|^2 \\ &= \sum_{i=1}^{k} t_i^2. \end{aligned} \tag{3}$$

Let J be the Jacobian matrix,

$$J = \begin{bmatrix} \frac{\partial w_1}{\partial t_1} & \frac{\partial w_2}{\partial t_1} & \cdots & \frac{\partial w_k}{\partial t_1} \\ \frac{\partial w_1}{\partial t_2} & \frac{\partial w_2}{\partial t_2} & \cdots & \frac{\partial w_k}{\partial t_2} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial w_1}{\partial t_k} & \frac{\partial w_2}{\partial t_k} & \cdots & \frac{\partial w_k}{\partial t_k} \end{bmatrix}.$$

Thun $|\det(J)| = 1$. Then by (3), we have

$$\begin{aligned} \Phi_k(A_k(x)) &= \frac{1}{(2\pi)^{\frac{k}{2}}} \int\int \cdots \int_{A_k(x)} e^{-\frac{1}{2}\sum_{i=1}^{k} w_i^2} dw_1 dw_2 \cdots dw_k \\ &= \frac{1}{(2\pi)^{\frac{k}{2}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}\int_{-\infty}^{\frac{x}{\sqrt{k}}} e^{-\frac{1}{2}\sum_{i=1}^{k} t_i^2} |\det J| dt_1 dt_2 \cdots dt_k \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sqrt{k}}} e^{-t^2} dt \\ &= \Phi_1(\frac{x}{\sqrt{k}}). \end{aligned}$$

Hence, the proposition is proved.

The proof of theorem 1.7 is completed by applying theorem 1.5-1.6. The proof of theorem 1.8, we use the propostion 2.1 and theorems in (Chen, 2004, p. 1985-2028). Theorem 1.9 is proved by applying theorem 1.1-1.2, respectively.

Proof of theorem 1.7

Proof: For each $1 \le i \le n$ and , $1 \le j \le k$, we define

$$W_{jn} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_{ij} \quad \text{and} \quad T_{in} = \frac{1}{\sqrt{n}} \sum_{j=1}^{k} Y_{ij}.$$

Thus $T_{1n}, T_{2n}, \ldots, T_{nn}$ are independent,

$$E(T_{in}) = 0, \quad |T_{in}| \le \frac{k\delta_0}{\sqrt{n}}, \tag{4}$$

$$W_n = (W_{1n}, W_{2n}, \ldots, W_{kn}) \quad \text{and} \quad \sum_{j=1}^{k} W_{jn} = \sum_{i=1}^{n} T_{in}. \tag{5}$$

Since Y_i has zero mean and covariance matrix I_k,

$$Var(Y_{ij}) = 1 \quad \text{and} \quad Cov(Y_{ij}, Y_{ik}) = 0 \quad \text{for} \quad j \ne k.$$

Therefore

$$Var(\frac{1}{\sqrt{k}} \sum_{i=1}^{n} T_{in}) = \frac{1}{k} Var \sum_{i=1}^{n} T_{in} = \frac{1}{nk} Var \sum_{i=1}^{n}\sum_{j=1}^{k} Y_{ij} = 1. \tag{6}$$

By applying theorem 1.5, proposition 2.1 and (4)-(6), we have

$$\begin{aligned} \sup_{x\in\mathbb{R}} |P(W_n \in A_k(x)) - \Phi_k(A_k(x))| &= \sup_{x\in\mathbb{R}} |P(\sum_{j=1}^{k} W_{jn} \le x) - \Phi_1(\frac{x}{\sqrt{k}})| \\ &= \sup_{x\in\mathbb{R}} |P(\sum_{i=1}^{n} T_{in} \le x) - \Phi_1(\frac{x}{\sqrt{k}})| \\ &= \sup_{x\in\mathbb{R}} |P(\frac{1}{\sqrt{k}} \sum_{i=1}^{n} T_{in} \le \frac{x}{\sqrt{k}}) - \Phi_1(\frac{x}{\sqrt{k}})| \qquad (7) \\ &\le \frac{3.3\sqrt{k}\delta_0}{\sqrt{n}}. \end{aligned}$$

For the second part, we apply theorem 1.6. The result is

$$\begin{aligned}|P(W_n \in A_k(x)) - \Phi_k(A_k(x))| &= |P(\frac{1}{\sqrt{k}}\sum_{i=1}^{n} T_{in} \leq \frac{x}{\sqrt{k}}) - \Phi_1(\frac{x}{\sqrt{k}})| \\ &\leq \frac{C\sqrt{k}\delta_0}{\sqrt{n}(1+|\frac{x}{\sqrt{k}}|^3)} \\ &= \frac{Ck^2\delta_0}{\sqrt{n}[(\sqrt{k})^3+|x|^3]}.\end{aligned}$$

for all real numbers x.

In theorem 1.8, we give the bounds by applying theorem 2.4 and theorem 2.5 in (Chen, 2004, p. 1985-2028). To prove this, we need the proposition 2.2. This proposition gives us that the random field $\{Y_{i,j} \mid i = 1,2,\ldots,n, j = 1,2,\ldots,k\}$ satisfied $(LD4^*)$ in (Chen, 2004, p. 1985-2028). This condition is proposed as follows:

Let $\mathcal{J}$ be a finite index set of cardianality n, and let $\{X_i, i \in \mathcal{J}\}$ be a random field with zero means and finite variances. For $A \subset \mathcal{J}$, let X_A denote $\{X_i, i \in A\}$,$A^c = \{j \in \mathcal{J} : j \notin A\}$ and $|A|$ the cardinality of A. The random field $\{X_i, i \in \mathcal{J}\}$ satisfied $(LD4^*)$ if for each $i \in \mathcal{J}$ there exists $A_i \subset B_i \subset B_i^* \subset C_i^* \subset D_i^* \subset \mathcal{J}$ such that X_i is independent of $X_{A_i^c}$, X_{A_i} is independent of $X_{B_i^c}$ and then X_{A_i} is independent of $\{X_{A_j}, j \in B_i^{*c}\}$, $\{X_{A_l}, l \in B_i^*\}$ is independent of $\{X_{A_j}, j \in C_i^{*c}\}$ and $\{X_{A_l}, l \in C_i^*\}$ is independent of $\{X_{A_j}, j \in D_i^{*c}\}$.

Proposition 2.2 *For $k, n \in \mathbb{N}$, let $Y_i = (Y_{i1}, Y_{i2}, ..., Y_{ik}), i = 1,2,...,n$ be independent random vectors in $\mathbb{R}^k$ with zero means. Then $\{Y_{ij} \mid i = 1,2,\ldots,n, j = 1,2,\ldots,k\}$ satisfies $(LD4^*)$.*

Proof: This proposition is completed by setting $A_{ij} \subset B_{ij} \subset B_{ij}^* \subset C_{ij}^* \subset D_{ij}^*$ for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,k$ as follows:

$$\begin{gathered}A_{ij} = \{il \mid l = 1,2,\ldots,k\} \text{ for } i = 1,2,\ldots,n,\\ B_{ij} = \{il, (i+1)l \mid l = 1,2,\ldots,k\} \text{ for } i = 1,2,\ldots,n-1 \text{ and } B_{nj} = B_{(n-1)j},\\ B_{ij}^* = C_{ij} = \{il,, (i+1)l, (i+2)l \mid l = 1,2,\ldots,k\} \text{ for } i = 1,2,\ldots,n-2 \text{ and}\\ B_{(n-m)j}^* = C_{(n-m)j} = B_{(n-2)j} \text{ for } m = 1,2,\\ C_{ij}^* = \{il, (i+1)l, \ldots, (i+3)l \mid l = 1,2,\ldots,k\} \text{ for } i = 1,2,\ldots,n-3 \text{ and}\\ C_{(n-m)j}^* = C_{(n-3)j}^* \; m = 1,2,3,\\ D_{ij}^* = \{il, (i+1)l, \ldots, (i+4)l \mid l = 1,2,\ldots,k\} \text{ for } i = 1,2,\ldots,n-4 \text{ and } D_{(n-m)j}^* = D_{(n-4)j}^* \; m = 1,2,3,4.\end{gathered}$$

So, we have the proposition.

From the sets defined in the above proposition, we can compute directly that for each $i = 1,2,\ldots,n$,

$$\max(|N(C_i)|, |\{j : i \in C_j\}| \leq 4 \tag{8}$$

and

$$\max_{1\leq i\leq n}\max(|D_i^*|, |\{j : i \in D_j^*\}|) \leq 5 \tag{9}$$

where $N(C_i)$ is defined in theorem 2.3 in (Chen, 2004, p. 1985-2028).
The condition $(LD4^*)$ implies the condition $(LD3)$ in (Chen, 2004, p. 1985-2028). Thus $\{Y_{ij} \mid i = 1,2,\ldots,n, j = 1,2,\ldots,k\}$ satisfies $(LD3)$.

Proof of theorem 1.8

Proof: For each $1 \leq i \leq n$, define T_{in} as in the proof of theorem 1.7. By the inequality

$$|\sum_{j=1}^{k} Y_{ij}|^p \leq k^p \sum_{j=1}^{k} |Y_{ij}|^p, \tag{10}$$

we have

$$E|T_{in}|^p = \frac{1}{n^{\frac{p}{2}}} E|\sum_{j=1}^{k} Y_{ij}|^p \leq \frac{k^p}{n^{\frac{p}{2}}} \sum_{j=1}^{k} E|Y_{ij}|^p < \infty.$$

So, by (4), (6), (8), (10), proposition 2.2 and theorem 2.4 in (Chen, 2004, p. 1985-2028), we have

$$
\begin{aligned}
\sup_{x\in\mathbb{R}} |P(W_n \in A_k(x)) - \Phi_k(A_k(x))| &= \sup_{x\in\mathbb{R}} |P(\frac{1}{\sqrt{k}}\sum_{i=1}^{n} T_{in} \leq \frac{x}{\sqrt{k}}) - \Phi_1(\frac{x}{\sqrt{k}})| \\
&\leq 75(4)^{p-1}\sum_{i=1}^{n} E|\frac{T_i}{\sqrt{k}}|^p \\
&= \frac{75(4)^{p-1}}{(nk)^{\frac{p}{2}}}\sum_{i=1}^{n} E|\sum_{j=1}^{k} Y_{ij}|^p \\
&\leq \frac{75(4)^{p-1}k^p}{(nk)^{\frac{p}{2}}}\sum_{i=1}^{n}\sum_{j=1}^{k} E|Y_{ij}|^p.
\end{aligned}
$$

Applying theorem 2.5 in (Chen, 2004, p. 1985-2028) and (9) to non-uniform case, we have for $x \in \mathbb{R}$,

$$
\begin{aligned}
|P(W_n \in A_k(x)) - \Phi_k(A_k(x))| &= |P(\sum_{i=1}^{n} T_{in} \leq x) - \Phi_1(\frac{x}{\sqrt{k}})| \\
&\leq \frac{5^p C}{(1+|\frac{x}{\sqrt{k}}|)^p}\sum_{i=1}^{n} E|\frac{T_i}{\sqrt{k}}|^p \\
&\leq \frac{(5k)^p C}{n^{\frac{p}{2}}(\sqrt{k}+|x|)^p}\sum_{i=1}^{n}\sum_{j=1}^{k} E|Y_{ij}|^p.
\end{aligned}
$$

Proof of theorem 1.9

Proof: By theorem 1.1, 1.2 and the same argument as in theorem 1.7, we have the theorem.

Remark The above theorems include the case that each Y_i has an indicator covariance matrix I_k.

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