Higher Dimensional Irreducible Representations of the Pure Braid Group

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Abstract

The reduced Gassner representation is a multi-parameter representation of P_n, the pure braid group on n strings. Specializing the parameters $t_1, t_2, ..., t_n$ to nonzero complex numbers $x_1, x_2, ..., x_n$ gives a representation $G_n(x_1, \ldots, x_n) : P_n \to GL(\mathbb{C}^{n-1})$ which is irreducible if and only if $x_1 \ldots x_n \neq 1$. In a previous work, we found a sufficient condition for the irreducibility of the tensor product of two irreducible Gassner representations. In our current work, we find a sufficient condition that guarantees the irreducibility of the tensor product of three Gassner representations. Next, a generalization of our result is given by considering the irreducibility of the tensor product of k representations ($k \geq 3$).

Keywords: Pure braid group, Gassner representation, Irreducible

1. Introduction

The pure braid group, P_n, is a normal subgroup of the braid group, B_n , on n strings. It has a lot of linear representations. One of them is the Gassner representation which comes from the embedding $P_n \to Aut(F_n)$, by means of Magnus representation. According to Artin, the automorphism corresponding to the braid generator σ_i takes x_i to $x_i x_{i+1} x_i^{-1}$, $x_{i+1}$to x_i and fixes all other free generators. Applying this standard Artin representation to the generators of the pure braid group, we get a representation of the pure braid group by automorphisms. Such a representation has a composition factor, the reduced Gassner representation $G_n(t_1, \ldots, t_n) : P_n \to GL_{n-1}(\mathbb{C}[t_1^{\pm 1}, ..., t_n^{\pm 1}])$, where $t_1, ..., t_n$ are indeterminates. We specialize the indeterminates $t_1, ..., t_n$ to nonzero complex numbers $x_1, ..., x_n$ and we define a representation $G_n(x_1, \ldots, x_n) : P_n \to GL_{n-1}(\mathbb{C}) = GL(\mathbb{C}^{n-1})$ which is irreducible if and only if $x_1 \ldots x_n \neq 1$.

In section 2 of our work, we define the Gassner representation of a free normal subgroup of the pure braid group of rank $n-1$ denoted by U_r where $1 \leq r \leq n$. We consider $\mathbb{C}[U_r]$ to be the group algebra of U_r over $\mathbb{C}$, and let $\mathcal{A}$ be the augmentation ideal of $\mathbb{C}[U_r]$. On the other hand, if M is any P_n-module, then $\mathcal{A}M$ is a P_n-submodule of M. We first show that if $\mathbb{C}^{n-1}$ is made into a P_n-module via the specialization of the reduced Gassner representation $G_n(x_1, \ldots, x_n) : U_n \to GL(\mathbb{C}^{n-1})$, then $\mathcal{A}\mathbb{C}^{n-1}$ is its unique minimal nonzero P_n-submodule. Of course $\mathcal{A}\mathbb{C}^{n-1} = \mathbb{C}^{n-1}$ when $G_n(x_1, \ldots, x_n)$ is irreducible.

Our objective is to find sufficient conditions that guarantee the irreducibility of the tensor product of k irreducible representations:

$$G_n^{(1)}(x_{11}, \ldots, x_{n1}) \otimes ... \otimes G_n^{(k)}(x_{1k}, \ldots, x_{nk}) : P_n \to GL(\mathbb{C}^{n-1} \otimes ... \otimes \mathbb{C}^{n-1}).$$

Here $G_n^{(m)}(x_{1m}, \ldots, x_{nm})$ denotes the complex specialization of the reduced Gassner representation of P_n , where $x_{1m}, \ldots, x_{nm} \in \mathbb{C} - \{0, 1\}$ and $1 \leq m \leq k$.

The case $k = 2$ was handled in (Abdulrahim, 2009) and a sufficient condition for the irreducibility of the tensor product was determined. Shortly after, we improved the result (Abdulrahim, 2010).

In section 3, we deal with the case $k = 3$. Our main result is Theorem 1 that states that for $n \geq 3$ and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_n)$, the representation $G_n^{(1)}(x) \otimes G_n^{(2)}(y) \otimes G_n^{(3)}(z) : P_n \to GL(\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1})$ is irreducible if there exist some integers i, $j \in \{1, ..., n\}$ with $i \neq j$ such that $x_i x_j \neq y_i y_j, x_i x_j \neq z_i z_j, y_i y_j \neq z_i z_j, x_i x_j y_i y_j \neq 1, x_i x_j z_i z_j \neq 1, y_i y_j z_i z_j \neq 1, x_i x_j \neq y_i y_j z_i z_j, y_i y_j \neq x_i x_j z_i z_j, z_i z_j \neq x_i x_j y_i y_j$, $x_i x_j y_i y_j z_i z_j \neq 1, x_i \neq x_j, y_i \neq y_j$ and $z_i \neq z_j$.

In section 4, we generalize our result to include all values $k \geq 3$. The proof, in the general case, is almost the same as in the case $k = 3$. However, we expect the computations to be rather more difficult.

2. Notations and Preliminaries

Notation 1. The pure braid group, P_n, is defined as the kernel of the homomorphism $B_n \to S_n$, defined by $\sigma_i \to (i, i+1)$, $1 \leq i \leq n-1$. It has the following generators:

$$A_{i,r} = \sigma_{r-1}\sigma_{r-2} \ldots \sigma_{i+1}\sigma_i{}^2\sigma_{i+1}^{-1} \ldots \sigma_{r-2}^{-1}\sigma_{r-1}^{-1}, \ 1 \leq i < r \leq n$$

We will construct for each $r = 1, \ldots, n$ a free normal subgroup of rank $n-1$, namely, U_r. Let U_r be the subgroup generated by the elements

$$A_{1,r}, A_{2,r}, \ldots, A_{r-1,r}, A_{r,r+1}, \ldots, A_{r,n}$$

where $A_{i,r}$ are those generators of P_n that become trivial after the deletion of the r-th strand. For a fixed value of r, the image of $A_{i,r}$ under the reduced Gassner representation is denoted by $\tau_{i,r}$, where $\tau_{i,r} = I - P_{i,r}Q_{i,r}$. In other words, the generators of U_r are $A_{i,r}$ where $A_{i,r} = A_{r,i}$ whenever $i > r$. It is known that U_r generates a free subgroup of P_n which is isomorphic to the subgroup U_n freely generated by $\{A_{1,n}, A_{2,n}, \ldots, A_{n-1,n}\}$. This is intuitively clear because it is quite arbitrary how we assign indices to the braid "strings". For more details, see (Birman, 1975).

For simplicity, we denote $A_{i,r}$ by $\tau_{i,r}$. That is, we have

$$\tau_{1,r} = A_{1,r}, \ldots, \tau_{r-1,r} = A_{r-1,r}, \ \tau_{r+1,r} = A_{r,r+1}, \ \tau_{r+2,r} = A_{r,r+2}, \ldots, \tau_{n,r} = A_{r,n}$$

Definition 1. The reduced Gassner representation restricted to U_r is defined as follows: $\tau_{i,r} = I - P_{i,r}Q_{i,r}$ for $1 \leq i, r \leq n$. For $i < r$, $P_{i,r}$ is the column vector given by:

$$(1-t_1, \ldots, 1-t_{i-1}, \underbrace{1-t_it_r}_{i}, t_r(1-t_{i+1}), \ldots, t_r(1-t_{r-1}), \underbrace{t_{r+1}-1}_{r}, t_{r+2}-1, \ldots, t_n-1)^T,$$

and for $n \geq i > r$, $P_{i,r}$ is the column vector given by:

$$(t_r(t_1-1), \ldots, \ t_r(t_{r-1}-1), \underbrace{1-t_{r+1}, \ldots, 1-t_i}_{i-r}, 1-t_{i+1}t_r, t_r(1-t_{i+2}), \ldots, t_r(1-t_n)\ )^T.$$

Here T is the transpose and $Q_{i,r}$ is the row vector given by:

$$Q_{i,r} = (0, \ldots, 0, \underbrace{\ 1\ }_{i}, 0, ..., 0\ ), \quad 1 \leq i, r \leq n.$$

The definition of the reduced Gassner representation restricted to a free normal subgroup is the same, up to equivalence. Representations given by pseudoreflections $I - A_iB_i$ and $I - C_iD_i$ are equivalent if the inner products (B_iA_j) and (D_iC_j) are conjugate by a diagonal matrix. Here, A_i, C_i are column vectors and B_i, D_i are row vectors.

We identify $\mathbb{C}^{n-1}$ with $(n-1) \times 1$ column vectors. We let $e_1, \ldots, e_{n-1}$ denote the standard basis for $\mathbb{C}^{n-1}$, and we consider matrices to act by left multiplication on column vectors.

Definition 2. If $r = a_1e_1 + \cdots + a_{n-1}e_{n-1} \in \mathbb{C}^{n-1}$, the *support* of r, denoted $supp(r)$, is the set $\{e_i \mid a_i \neq 0\}$. If $s = \Sigma a_{ijk}(e_i \otimes e_j \otimes e_k) \in \mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$, the *support* of s, also denoted $supp(s)$, is the set $\{e_i \otimes e_j \otimes e_k \mid a_{ijk} \neq 0\}$, and a_{ijk} is called the *coefficient* of $e_i \otimes e_j \otimes e_k$ in s.

Definition 3. Given an integer r, $1 \leq r \leq n$ and a vector $t = (t_1, \ldots, t_n)$. We define $v_{i,r}(t) = e_i - \tau_{i,r}(t)(e_i) = (I - \tau_{i,r}(t))(e_i)$. In other words, we have the following:

For $1 \le i \le r-1$, $v_{i,r}(t) =$

$$(1-t_1,\ldots,1-t_{i-1},\underbrace{1-t_it_r}_{i},t_r(1-t_{i+1}),\ldots,t_r(1-t_{r-1}),\underbrace{t_{r+1}-1}_{r},t_{r+2}-1,\ldots,t_n-1\ )^T$$

and for $n \ge i > r$, $v_{i,r}(t) =$

$$(t_r(t_1-1),\ldots,\ t_r(t_{r-1}-1),\underbrace{1-t_{r+1},\ldots,1-t_i}_{i-r},1-t_{i+1}t_r,t_r(1-t_{i+2}),\ldots,t_r(1-t_n)\ )^T$$

Lemma 1. For $t=(t_1,\ldots,t_n)$, we have:

(1) $\tau_{i,r}(t)(v_{i,s}(t)) = v_{i,s}(t)+(t_it_s-1)v_{i,r}(t)$ $\quad$ *for* $1 \le i \le s-1$,

$\tau_{i,r}(t)(v_{i,s}(t)) = v_{i,s}(t)+(t_{i+1}t_s-1)v_{i,r}(t)$ $\quad$ *for* $1 \le s < i$,

(2) $\tau_{i,r}(t)(v_{j,s}(t)) = v_{j,s}(t)+(t_i-1)v_{i,r}(t)$ $\quad$ *for* $i<j<s$,

$\tau_{i,r}(t)(v_{j,s}(t)) = v_{j,s}(t)+t_s(t_i-1)v_{i,r}(t)$ $\quad$ *for* $j<i<s$,

$\tau_{i,r}(t)(v_{j,s}(t)) = v_{j,s}(t)+(1-t_{i+1})v_{i,r}(t)$ $\quad$ *for* $j<s<i$,

(3) $\tau_{i,r}(t)(v_{j,s}(t)) = v_{j,s}(t)+t_s(1-t_i)v_{i,r}(t)$ $\quad$ *for* $i<s<j$,

$\tau_{i,r}(t)(v_{j,s}(t)) = v_{j,s}(t)+(t_{i+1}-1)v_{i,r}(t)$ $\quad$ *for* $s<i<j$,

$\tau_{i,r}(t)(v_{j,s}(t)) = v_{j,s}(t)+t_s(t_{i+1}-1)v_{i,r}(t)$ $\quad$ *for* $s<j<i$.

For a fixed value of r, we use this Lemma to determine elements in the group algbera $\mathbb{C}(P_n)$ over $\mathbb{C}$ that send the vector $v_{i,r}$ to the vector $v_{i+1,r}$ and other elements that send the vector $v_{i,r}$ to $v_{i-1,r}$.

Definition 4. Given an integer r such that $1 \le r \le n$. Consider the following elements of the pure braid group algebra:

$$f_{i,r} = \begin{cases} \tau_{i,r}-(t_it_r)\tau_{i+1,r}, & 1 \le i < r-1 \\ \tau_{i,r}-(t_it_r)\tau_{i+2,r}, & i=r-1 \\ \tau_{i,r}-(t_{i+1}t_r)\tau_{i+1,r}, & 1 \le r < i \le n-1 \end{cases}$$

and

$$g_{i,r} = \begin{cases} \tau_{i,r}-(t_it_r)\tau_{i-1,r}, & 1 \le i \le r-1 \\ \tau_{i,r}-(t_{i+1}t_r)\tau_{i-2,r}, & i=r+1 \le n-1 \\ \tau_{i,r}-(t_{i+1}t_r)\tau_{i-1,r}, & r+1 < i \le n-1. \end{cases}$$

Lemma 2. *Fix an integer r, $1 \le r \le n$. For all integers i, $1 \le i \le n-1$, the action of the elements of the pure braid group algebra, namely, $f_{i,r}$ and $g_{i,r}$, on the vectors $v_{i,r}$ is given by:*

$$\text{(i) } f_{i,r}(v_{i,r}) = \begin{cases} -t_i{t_r}^2(t_{i+1}-1)v_{i+1,r}, & 1 \le i < r-1 \\ -t_{r-1}t_r(1-t_{r+2})v_{r+1,r}, & i=r-1 \le n-3 \\ -t_{i+1}{t_r}^2(t_{i+2}-1)v_{i+1,r}, & 1 \le r < i \le n-2 \end{cases}$$

and

$$\text{(ii) } g_{i,r}(v_{i,r}) = \begin{cases} -t_it_r(t_{i-1}-1)v_{i-1,r}, & 1 \le i \le r-1 \\ -t_{r+2}{t_r}^2(1-t_{r-1})v_{r-1,r}, & i=r+1 \le n-1 \\ -t_{i+1}t_r(t_i-1)v_{i-1,r}, & r+1 < i \le n-1. \end{cases}$$

Notation 2. Let $G_n(x_1,\ldots,x_n)$ denote the reduced Gassner representation of P_n under the specialization $t_i \to x_i$, where x_i is a non-zero complex number.

Lemma 3. Having U_r a free normal subgroup of the pure braid group, we *let $G_n(x_1,\ldots,x_n) : U_r \to GL(\mathbb{C}^{n-1})$ be a specialization of the reduced Gassner representation restricted to U_r making $\mathbb{C}^{n-1}$ into a U_r-module, where $n \ge 3$. Then*

(a) *Let $\mathcal{A}$ be the kernel of the homomorphism $\mathbb{C}[U_r] \to \mathbb{C}$ induced by $\tau_{i,r} \to 1$ (the augmentation ideal). Let x be the vector $(x_1,\ldots,x_n)$.Then $\mathcal{A}\mathbb{C}^{n-1}$ is equal to the $\mathbb{C}$-vector space spanned by $v_{1,r}(x),\ldots,v_{r-1,r}(x),v_{r+1,r}(x),\ldots,v_{n,r}(x)$.*

(b) *If M is a nonzero U_r-submodule of $\mathbb{C}^{n-1}$, then $\mathcal{A}\mathbb{C}^{n-1} \subseteq M$. Hence $\mathcal{A}\mathbb{C}^{n-1}$ is the unique minimal nonzero U_r-submodule of $\mathbb{C}^{n-1}$.*

(c) *If* $p(x_1, \dots, x_n) = (x_r - 1)^{n-2}(x_1 x_2 \dots x_n - 1) \neq 0$, *then* $\mathcal{A}\mathbb{C}^{n-1} = \mathbb{C}^{n-1}$, *and* $G_n(x_1, x_2, \dots, x_n)$ *restricted to* U_r *is irreducible.*

Proof. Here, we will take the free normal subgroup, U_r, of rank $n-1$. Notice that, in the proof of (b), we need the fact that if $v_{j,r} \in M$ for some j and r then all $v_{i,r} \in M$. This is due to Lemma 1. As for (c), the determinant of the matrix, whose columns are the vectors $v_{1,r}(x), \dots, v_{n,r}(x)$, is $p(x) = (x_r - 1)^{n-2}(x_1 x_2 \dots x_n - 1)$, so if $p(x) \neq 0$ then $v_{1,r}(x), \dots v_{n,r}(x)$ is a basis for $\mathbb{C}^{n-1}$ and $\mathcal{A}\mathbb{C}^{n-1} = \mathbb{C}^{n-1}$. □

Hence, $\mathcal{A}\mathbb{C}^{n-1}$ is its unique minimal nonzero U_r-submodule. Of course $\mathcal{A}\mathbb{C}^{n-1} = \mathbb{C}^{n-1}$ when $G_n(x_1, \dots, x_n)$ is irreducible. For more details, see (Formanek, 1996) and (Abdulrahim, 2005).

3. The Tensor Product of Three Irreducible Representations

For $1 \leq j \leq n$, we consider the normal subgroup of rank $n-1$, namely, U_j, defined as before. We find a sufficient condition for the irreducibility of the tensor product of three irreducible representations of U_j:

$$G_n^{(1)}(x_{11}, \dots, x_{n1}) \otimes G_n^{(2)}(x_{12}, \dots, x_{n2}) \otimes G_n^{(3)}(x_{13}, \dots, x_{n3}) :$$

$$U_j \rightarrow GL(\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1})$$

We now introduce Proposition 1 that provides us with a sufficient condition for irreducibility. For simplicity, we write

$$x_{uv} = \begin{cases} x_u & \text{if } v = 1 \\ y_u & \text{if } v = 2 \\ z_u & \text{if } v = 3 \end{cases}$$

for $1 \leq u \leq n$.

Proposition 1. Suppose that $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ and $z = (z_1, \dots, z_n)$ $\in \mathbb{C}^n$, where $x_s, y_s, z_s \in \mathbb{C} - \{0, 1\}$ for $1 \leq s \leq n$. Suppose also that for some $i < j$, we have that

$$x_i x_j \neq y_i y_j, x_i x_j \neq z_i z_j, y_i y_j \neq z_i z_j, x_i x_j y_i y_j \neq 1, x_i x_j z_i z_j \neq 1, y_i y_j z_i z_j \neq 1,$$

$$x_i x_j \neq y_i y_j z_i z_j, y_i y_j \neq x_i x_j z_i z_j, z_i z_j \neq x_i x_j y_i y_j, x_i x_j y_i y_j z_i z_j \neq 1,$$

$$x_i \neq x_j, y_i \neq y_j, z_i \neq z_j.$$

Let M be a nonzero U_j-submodule of $\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$ under the action of $G_n^{(1)}(x) \otimes G_n^{(2)}(y) \otimes G_n^{(3)}(z) : U_j \rightarrow GL(\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1})$, where $n \geq 3$. For simplicity, we write $v_{p,j} = v_p$ for $p \in \{1, \dots, j-1, j+1, \dots, n\}$. Then M contains all $v_p(x) \otimes v_q(y) \otimes v_r(z)$ for $p, q, r \in \{1, \dots, j-1, j+1, \dots, n\}$. Thus M contains $\mathcal{A}\mathbb{C}^{n-1} \otimes \mathcal{A}\mathbb{C}^{n-1} \otimes \mathcal{A}\mathbb{C}^{n-1}$. Here, the action of U_j on the first factor is induced by $G_n^{(1)}(x_1, \dots, x_n)$, the action of U_j on the second factor is induced by $G_n^{(2)}(y_1, \dots, y_n)$ and the action of U_j on the third factor is induced by $G_n^{(3)}(z_1, \dots, z_n)$.

Proof.

Claim 1. There exists an $s \in \{1, \dots, n-1\}$ such that $e_s \otimes e_s \otimes e_s \in supp(m)$ for some $m \in M$.

Proof of Claim 1.

Case 1. Suppose that there exists an $s \in \{1, ..., n-1\}$ such that $e_s \otimes e_s \otimes e_s \in supp(m)$, then we are done.

Case 2. Suppose that there exists an $s \in \{1, ..., n-1\}$, with $s \neq i$, such that at least one of $e_i \otimes e_s \otimes e_s, e_s \otimes e_i \otimes e_s, e_s \otimes e_s \otimes e_i, e_i \otimes e_i \otimes e_s, e_i \otimes e_s \otimes e_i, e_s \otimes e_i \otimes e_i \in supp(m)$.

We write m as follows:

$m = ae_i \otimes e_s \otimes e_s + be_s \otimes e_i \otimes e_s + ce_s \otimes e_s \otimes e_i + de_i \otimes e_i \otimes e_s + ee_i \otimes e_s \otimes e_i + fe_s \otimes e_i \otimes e_i + W.$

Here, at least one of $a, b, c, d, e, f \in \mathbb{C}^*$ and $supp(W)$ does not contain any of $e_i \otimes e_s \otimes e_s, e_s \otimes e_i \otimes e_s, e_s \otimes e_s \otimes e_i, e_i \otimes e_i \otimes e_s, e_i \otimes e_s \otimes e_i, e_s \otimes e_i \otimes e_i$. We also assume that $supp(W)$ does not contain any of $e_\alpha \otimes e_\alpha \otimes e_\alpha$ for any α.

Applying τ_i on m, we obtain

$\tau_i(m) = a\tau_i(e_i) \otimes e_s \otimes e_s + be_s \otimes \tau_i(e_i) \otimes e_s + ce_s \otimes e_s \otimes \tau_i(e_i) + d\, \tau_i(e_i) \otimes \tau_i(e_i) \otimes e_s +$

$$e\tau_i(e_i)\otimes e_s\otimes\tau_i(e_i)+fe_s\otimes\tau_i(e_i)\otimes\tau_i(e_i)+\tau_i(W)$$
$$=a(M_se_s+x_ix_je_i+...)\otimes e_s\otimes e_s+be_s\otimes(N_se_s+y_iy_je_i+...)\otimes e_s+ce_s\otimes e_s\otimes(P_se_s+z_iz_je_i+...)+$$
$$d(M_se_s+x_ix_je_i+...)\otimes(N_se_s+y_iy_je_i+...)\otimes e_s+e(M_se_s+x_ix_je_i+...)\otimes e_s\otimes(P_se_s+z_iz_je_i+...)+$$
$$fe_s\otimes(N_se_s+y_iy_je_i+...)\otimes(P_se_s+z_iz_je_i+...)+\tau_i(W).$$

Here, M_s, N_s and P_s are all nonzero complex numbers given as follows:

If $s<i$ then $M_s=x_s-1,\ N_s=y_s-1$ and $P_s=z_s-1$.

If $i<s<j$ then $M_s=x_j(x_s-1),\ N_s=y_j(y_s-1)$ and $P_s=z_j(z_s-1)$.

If $s\geq j$ then $M_s=1-x_s,\ N_s=1-y_s$ and $P_s=1-z_s$.

$$\tau_i(m)=(aM_s+bN_s+cP_s+dM_sN_s+eM_sP_s+fN_sP_s)e_s\otimes e_s\otimes e_s+(ax_ix_j+$$
$$dN_sx_ix_j+eP_sx_ix_j)e_i\otimes e_s\otimes e_s+(by_iy_j+dM_sy_iy_j+fP_sy_iy_j)e_s\otimes e_i\otimes$$
$$e_s+(cz_iz_j+eM_sz_iz_j+fN_sz_iz_j)e_s\otimes e_s\otimes e_i+d\,x_ix_jy_iy_je_i\otimes e_i\otimes e_s+$$
$$ex_ix_jz_iz_je_i\otimes e_s\otimes e_i+fy_iy_jz_iz_je_s\otimes e_i\otimes e_i+...+\tau_i(W).$$

The coefficients of $e_s\otimes e_s\otimes e_s$ in $\tau_i(m),\dots,\tau_i^6(m)$ are given as follows:

In $\tau_i(m)$: the coefficient of $e_s\otimes e_s\otimes e_s$ is $aM_s+bN_s+cP_s+dM_sN_s+eM_sP_s+fN_sP_s$.

In $\tau_i^2(m)$: the coefficients of $e_s\otimes e_s\otimes e_s$ is

$$aM_s(1+x_ix_j)+bN_s(1+y_iy_j)+cP_s(1+z_iz_j)+dM_sN_s(1+x_ix_j)(1+y_iy_j)+eM_sP_s(1+x_ix_j)(1+z_iz_j)+fN_sP_s(1+y_iy_j)(1+z_iz_j).$$

In τ_i^3: the coefficient of $e_s\otimes e_s\otimes e_s$ is

$$aM_s(1+x_ix_j+x_i^2x_j^2)+bN_s(1+y_iy_j+y_i^2y_j^2)+cP_s(1+z_iz_j+z_i^2z_j^2)+dM_sN_s(1+x_ix_j+x_i^2x_j^2)(1+y_iy_j+y_i^2y_j^2)+eM_sP_s(1+x_ix_j+x_i^2x_j^2)(1+z_iz_j+z_i^2z_j^2)+fN_sP_s(1+y_iy_j+y_i^2y_j^2)(1+z_iz_j+z_i^2z_j^2).$$

Likewise for $\tau_i^4(m),\tau_i^5(m)$ and $\tau_i^6(m)$. Therefore, we consider the 6×6 matrix whose first column corresponds to the coefficients of a, the second column corresponds to the coefficient of b and so on. More precisely, the matrix obtained is given by

$$A=[a_{\alpha\beta}]=\begin{pmatrix}a_{11}&a_{12}&a_{13}&a_{11}.a_{12}&a_{11}.a_{13}&a_{12}.a_{13}\\a_{21}&a_{22}&a_{23}&a_{21}.a_{22}&a_{21}.a_{23}&a_{22}.a_{23}\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\a_{61}&a_{62}&a_{63}&a_{61}.a_{62}&a_{61}.a_{63}&a_{62}.a_{63}\end{pmatrix}.$$

Here, $a_{\alpha1}=M_s(\sum_{k=0}^{\alpha-1}x_i^kx_j^k),\quad a_{\alpha2}=N_s(\sum_{k=0}^{\alpha-1}y_i^ky_j^k),\quad a_{\alpha3}=P_s(\sum_{k=0}^{\alpha-1}z_i^kz_j^k), 1\leq\alpha\leq 6.$

The determinant of the matrix above is

$$-M_s^3N_s^3P_s^3x_i^3x_j^3y_i^3y_j^3z_i^3z_j^3(x_ix_j-y_iy_j)^2(x_ix_j-z_iz_j)^2(y_iy_j-z_iz_j)^2(x_ix_jy_iy_j-1)(x_ix_jz_iz_j-1)(y_iy_jz_iz_j-1)(x_ix_j-y_iy_jz_iz_j)(y_iy_j-x_ix_jz_iz_j)(z_iz_j-x_ix_jy_iy_j).$$

Using the hypothesis, we get that the determinant of the matrix A is nonzero. Then, at least one of $\tau_i(m)$, $\tau_i^2(m)$, $\tau_i^3(m)$, $\tau_i^4(m)$, $\tau_i^5(m)$, $\tau_i^6(m)$ has $e_s\otimes e_s\otimes e_s$ in its support.

Case 3. Suppose that for every $\alpha\in\{1,...,n\}$, we have that $e_i\otimes e_i\otimes e_\alpha, e_i\otimes e_\alpha\otimes e_i, e_\alpha\otimes e_i\otimes e_i, e_i\otimes e_\alpha\otimes e_\alpha, e_\alpha\otimes e_i\otimes e_\alpha, e_\alpha\otimes e_\alpha\otimes e_i\notin supp(m)$, but there exist exist distinct β & γ with $\beta\neq i,\gamma\neq i$ such that at least one of $e_i\otimes e_\beta\otimes e_\gamma, e_\beta\otimes e_i\otimes e_\gamma, e_\beta\otimes e_\gamma\otimes e_i, e_\beta\otimes e_\beta\otimes e_\gamma, e_\beta\otimes e_\gamma\otimes e_\beta, e_\gamma\otimes e_\beta\otimes e_\beta\in supp(m)$. Here, i is the integer given in the hypothesis of Proposition 1.

Here, m can be written as:

$$m=ae_i\otimes e_\beta\otimes e_\gamma+be_\beta\otimes e_i\otimes e_\gamma+ce_\beta\otimes e_\gamma\otimes e_i+de_i\otimes e_\gamma\otimes e_\beta+ee_\gamma\otimes e_i\otimes e_\beta+$$
$$fe_\gamma\otimes e_\beta\otimes e_i+ge_\beta\otimes e_\beta\otimes e_\gamma+he_\beta\otimes e_\gamma\otimes e_\beta+ie_\gamma\otimes e_\beta\otimes e_\beta+je_\gamma\otimes e_\gamma\otimes e_\beta+$$
$$ke_\gamma\otimes e_\beta\otimes e_\gamma+le_\beta\otimes e_\gamma\otimes e_\gamma+W.$$

Here, at least one of $a,b,c,d,e,f,g,h,i,j,k,l\in\mathbb{C}^*$ and $supp(W)$ does not contain any of the previous tensors. We also assume that $supp(W)$ does not contain any of $e_\alpha \otimes e_\alpha \otimes e_\alpha$ for any α.

Applying τ_β on m, we obtain

$$\begin{aligned}
\tau_\beta(m) = {} & ae_i \otimes (N_ie_i + N_\gamma e_\gamma + y_\beta y_je_\beta + ...) \otimes e_\gamma + b(M_ie_i + M_\gamma e_\gamma + x_\beta x_je_\beta + ...)\otimes \\
& e_i \otimes e_\gamma + c(M_ie_i + M_\gamma e_\gamma + x_\beta x_je_\beta + ...) \otimes e_\gamma \otimes e_i + de_i \otimes e_\gamma \otimes (P_ie_i + P_\gamma e_\gamma \\
& + z_\beta z_je_\beta + ...) + ee_\gamma \otimes e_i \otimes (P_ie_i + P_\gamma e_\gamma + z_\beta z_je_\beta + ...) + fe_\gamma \otimes (N_ie_i + N_\gamma \\
& e_\gamma + y_\beta y_je_\beta + ...) \otimes e_i + g(M_ie_i + M_\gamma e_\gamma + x_\beta x_je_\beta + ...) \otimes (N_ie_i + N_\gamma e_\gamma + y_\beta \\
& y_je_\beta + ...) \otimes e_\gamma + h(M_ie_i + M_\gamma e_\gamma + x_\beta x_je_\beta + ...) \otimes e_\gamma \otimes (P_ie_i + P_\gamma e_\gamma + z_\beta z_j \\
& e_\beta + ...) + ie_\gamma \otimes (N_ie_i + N_\gamma e_\gamma + y_\beta y_je_\beta + ...) \otimes (P_ie_i + P_\gamma e_\gamma + z_\beta z_je_\beta + ...) \\
& + je_\gamma \otimes e_\gamma \otimes (P_ie_i + P_\gamma e_\gamma + z_\beta z_je_\beta + ...) + ke_\gamma \otimes (N_ie_i + N_\gamma e_\gamma + y_\beta y_je_\beta + \\
& ...) \otimes e_\gamma + l(M_ie_i + M_\gamma e_\gamma + x_\beta x_je_\beta + ...) \otimes e_\gamma \otimes e_\gamma + \tau_\beta(W) \\
= {} & (aN_i + bM_i + gM_iN_i)e_i \otimes e_i \otimes e_\gamma + (aN_\gamma + dP_\gamma + gM_iN_\gamma + hM_iP_\gamma + lM_i)e_i \\
& \otimes e_\gamma \otimes e_\gamma + (bM_\gamma + eP_\gamma + gM_\gamma N_i + iN_iP_\gamma + kN_i)e_\gamma \otimes e_i \otimes e_\gamma + (cM_i + dp_i \\
& + hM_iP_i)e_i \otimes e_\gamma \otimes e_i + (cM_\gamma + fN_\gamma + hM_\gamma P_i + iN_\gamma P_i + jP_i)e_\gamma \otimes e_\gamma \otimes e_i + \\
& (eP_i + fN_i + iN_iP_i)e_\gamma \otimes e_i \otimes e_i + (gM_\gamma N_\gamma + hM_\gamma P_\gamma + iN_\gamma P_\gamma + jP_\gamma + kN_\gamma \\
& + lM_\gamma)e_\gamma \otimes e_\gamma \otimes e_\gamma + ay_\beta y_je_i \otimes e_\beta \otimes e_\gamma + bx_\beta x_je_\beta \otimes e_i \otimes e_\gamma + cx_\beta x_je_\beta \otimes e_\gamma \\
& \otimes e_i + dz_\beta z_je_i \otimes e_\gamma \otimes e_\beta + ez_\beta z_je_\gamma \otimes e_i \otimes e_\beta + fy_\beta y_je_\gamma \otimes e_\beta \otimes e_i + gM_iy_\beta y_j \\
& e_i \otimes e_\beta \otimes e_\gamma + gM_\gamma y_\beta y_je_\gamma \otimes e_\beta \otimes e_\gamma + gx_\beta x_jN_ie_\beta \otimes e_i \otimes e_\gamma + gx_\beta x_jN_\gamma e_\beta \otimes \\
& e_\gamma \otimes e_\gamma + gx_\beta x_jy_\beta y_je_\beta \otimes e_\beta \otimes e_\gamma + hM_iz_\beta z_je_i \otimes e_\gamma \otimes e_\beta + hM_\gamma z_\beta z_je_\gamma \otimes e_\gamma \\
& \otimes e_\beta + hx_\beta x_jP_ie_\beta \otimes e_\gamma \otimes e_i + hx_\beta x_jP_\gamma e_\beta \otimes e_\gamma \otimes e_\gamma + hx_\beta x_jz_\beta z_je_\beta \otimes e_\gamma \otimes \\
& e_\beta + iN_iz_\beta z_je_\gamma \otimes e_i \otimes e_\beta + iN_\gamma z_\beta z_je_\gamma \otimes e_\gamma \otimes e_\beta + iy_\beta y_jP_ie_\gamma \otimes e_\beta \otimes e_i + iy_\beta \\
& y_jP_\gamma e_\gamma \otimes e_\beta \otimes e_\gamma + iy_\beta y_jz_\beta z_je_\gamma \otimes e_\beta \otimes e_\beta + jz_\beta z_je_\gamma \otimes e_\gamma \otimes e_\beta + ky_\beta y_je_\gamma \otimes \\
& e_\beta \otimes e_\gamma + lx_\beta x_je_\beta \otimes e_\gamma \otimes e_\gamma + W'.
\end{aligned}$$

Here, W' does contain any of the previous tensors and $M_i, N_i, P_i, M_\gamma, N_\gamma, P_\gamma$ are all nonzeros.

If the coefficient of $e_\gamma \otimes e_\gamma \otimes e_\gamma$ or at least one of the coefficients of $e_i \otimes e_i \otimes e_\gamma, e_i \otimes e_\gamma \otimes e_\gamma, e_\gamma \otimes e_i \otimes e_\gamma, e_i \otimes e_\gamma \otimes e_i, e_\gamma \otimes e_\gamma \otimes e_i, e_\gamma \otimes e_i \otimes e_i$ in $\tau_\beta(m)$ is not zero then we refer to cases 1 or 2 and so we are done; otherwise, we consider the following system:

$$\begin{cases}
aN_i + bM_i + gM_iN_i = 0 \\
aN_\gamma + dP_\gamma + gM_iN_\gamma + hM_iP_\gamma + lM_i = 0 \\
bM_\gamma + eP_\gamma + gM_\gamma N_i + iN_iP_\gamma + kN_i = 0 \\
cM_i + dp_i + hM_iP_i = 0 \\
cM_\gamma + fN_\gamma + hM_\gamma P_i + iN_\gamma P_i + jP_i = 0 \\
eP_i + fN_i + iN_iP_i = 0
\end{cases}$$

Computing $\tau_\beta^2(m)$, we find that the coefficient of $e_i \otimes e_i \otimes e_\gamma$ is

$$ay_\beta y_jN_i + bx_\beta x_jM_i + gM_iN_i(y_\beta y_j + x_\beta x_j + y_\beta y_jx_\beta x_j).$$

If this coefficient is nonzero then we refer to Case 2 and so we are done; otherwise, we work with $\tau_\gamma(m)$ to get that $a, b, c, d, e, f, g, h, i, j, k$ & l are all zeros using the system above, which is a contradiction.

Case 4. Suppose that there exist α, β & γ different from i such that $e_\alpha \otimes e_\beta \otimes e_\gamma \in supp(m)$.

We write m as follows:

$$\begin{aligned}
m = {} & ae_\alpha \otimes e_\beta \otimes e_\gamma + be_\alpha \otimes e_\gamma \otimes e_\beta + ce_\beta \otimes e_\alpha \otimes e_\gamma + de_\beta \otimes e_\gamma \otimes e_\alpha + ee_\gamma \otimes e_\alpha \otimes \\
& e_\beta + fe_\gamma \otimes e_\beta \otimes e_\alpha + W.
\end{aligned}$$

Here, at least one of $a, b, c, d, e, f \in \mathbb{C}^*$ and $supp(W)$ does not contain any of the previous tensors. We also assume that $supp(W)$ does not contain any of $e_\alpha \otimes e_\alpha \otimes e_\alpha$ for any α.

Applying τ_α on m, we obtain

$$\begin{aligned}\tau_\alpha(m) &= a(M_ie_i + x_\alpha x_je_\alpha + M_\beta e_\beta + M_\gamma e_\gamma + ...)\otimes e_\beta\otimes e_\gamma + b(M_ie_i + x_\alpha x_je_\alpha +\\ &\quad M_\beta e_\beta + M_\gamma e_\gamma + ...)\otimes e_\gamma\otimes e_\beta + ce_\beta\otimes(N_ie_i + y_\alpha y_je_\alpha + N_\beta e_\beta + N_\gamma e_\gamma\\ &\quad +...)\otimes e_\gamma + de_\beta\otimes e_\gamma\otimes(P_ie_i + z_\alpha z_je_\alpha + P_\beta e_\beta + P_\gamma e_\gamma + ...) + ee_\gamma\otimes(N_i\\ &\quad e_i + y_\alpha y_je_\alpha + N_\beta e_\beta + N_\gamma e_\gamma + ...)\otimes e_\beta + fe_\gamma\otimes e_\beta\otimes(P_ie_i + z_\alpha z_je_\alpha + P_\beta\\ &\quad e_\beta + P_\gamma e_\gamma + ...) + \tau_\alpha(W)\\ &= aM_ie_i\otimes e_\beta\otimes e_\gamma + bM_ie_i\otimes e_\gamma\otimes e_\beta + cN_ie_\beta\otimes e_i\otimes e_\gamma + dP_ie_\beta\otimes e_\gamma\otimes\\ &\quad e_i + eN_ie_\gamma\otimes e_i\otimes e_\beta + fP_ie_\gamma\otimes e_\beta\otimes e_i + ... + \tau_\alpha(W).\end{aligned}$$

At least one of $aM_i, bM_i, cN_i, dP_i, eN_i, fP_i$ is nonzero. So, by Case 3, we are done.

Claim 2. $e_i\otimes e_i\otimes e_i \in supp(m)$ for some $m\in M$.

Proof of Claim 2. We have, by Claim 1, that $e_s\otimes e_s\otimes e_s \in supp(m)$ for some $s\in\{1,\dots,n-1\}$ and $m\in M$. We write $m = \alpha_se_s\otimes e_s\otimes e_s + W$ and $supp(W)$ does not contain $e_s\otimes e_s\otimes e_s$. Here, $\alpha_s\in\mathbb{C}^*$. It follows that

$$\begin{aligned}\tau_s(m) &= \alpha_s\tau_s(e_s\otimes e_s\otimes e_s) + \tau_s(W)\\ &= \alpha_s((e_s - v_s)\otimes(e_s - v_s)\otimes(e_s - v_s)) + \tau_s(W)\\ &= \alpha_s(\sum_{l=1}^{n-1}A_le_l\otimes\sum_{l=1}^{n-1}B_le_l\otimes\sum_{l=1}^{n-1}C_le_l) + \tau_s(W).\end{aligned}$$

This implies that $e_l\otimes e_l\otimes e_l \in supp(\tau_s(m))$ for every $l\in\{1,\dots,n-1\}$. In particular, we let $l = i$. Then we get that $e_i\otimes e_i\otimes e_i \in supp(m), m\in M$.

Claim 3. $v_i(x)\otimes v_i(y)\otimes v_i(z)\in M$.

Proof of Claim 3. A calculation shows that:

$$\begin{aligned}&(\tau_i - x_ix_jz_iz_j)(\tau_i - x_ix_jy_iy_j)(\tau_i - y_iy_jz_iz_j)(\tau_i - z_iz_j)(\tau_i - y_iy_j)(\tau_i - x_ix_j)(\tau_i - 1)(e_i\otimes e_i\otimes e_i)\\ &= -x_i^3y_i^3z_i^3x_j^3y_j^3z_j^3(x_ix_jy_iy_j - 1)(x_ix_jz_iz_j - 1)(y_iy_jz_iz_j - 1)(x_ix_jy_iy_jz_iz_j - 1)(v_i(x)\otimes v_i(y)\otimes v_i(z))\end{aligned}$$

and

$$(\tau_i - x_ix_jz_iz_j)(\tau_i - x_ix_jy_iy_j)(\tau_i - y_iy_jz_iz_j)(\tau_i - z_iz_j)(\tau_i - y_iy_j)(\tau_i - x_ix_j)(\tau_i - 1)(e_u\otimes e_v\otimes e_w) = 0 \text{ if } (u, v, w)\neq(i, i, i).$$

Claim 4. For $l\in\{1, ..., j-1, j+1, ..., n\}$, we have that $v_l\otimes v_l\otimes v_l\in M$.

Proof of Claim 4. We have, by claim 3, that $v_i(x)\otimes v_i(y)\otimes v_i(z)\in M$. Here, $v_i = v_{i,j}$. Applying Lemma 2, we have that $f_{i,j}(v_i\otimes v_i\otimes v_i)\in M$, which implies that $v_{i+1}\otimes v_{i+1}\otimes v_{i+1}\in M$. Similarly, we also have that $g_{i,j}(v_i\otimes v_i\otimes v_i)\in M$, which implies that $v_{i-1}\otimes v_{i-1}\otimes v_{i-1}\in M$. After a consecuitive use of $f_{i,j}, f_{i+1,j},\dots$ and $g_{i,j}, g_{i-1,j},\dots$, we obtain that

$$v_l\otimes v_l\otimes v_l\in M \text{ for every } l\in\{1,\dots,j-1,j+1,\dots,n\}.$$

Claim 5. For $p, q, r\in\{1,\dots,j-1,j+1,\dots,n\}$, $v_p\otimes v_q\otimes v_r\in M$.

Proof of Claim 5. We consider the following cases:

Case 5. $p = q = r$, we are done.

Case 6. $p = q = i,\ r\neq i$.

Applying τ_i on $v_r\otimes v_r\otimes v_r\in M$, we obtain $(v_r + av_i)\otimes(v_r + bv_i)\otimes(v_r + cv_i)\in M$, and so

$$\begin{gathered}cv_r\otimes v_r\otimes v_i + bv_r\otimes v_i\otimes v_r + bcv_r\otimes v_i\otimes v_i + av_i\otimes v_r\otimes v_r + acv_i\otimes v_r\otimes v_i +\\ abv_i\otimes v_i\otimes v_r\in M.\end{gathered}\tag{1}$$

Applying τ_i again, we obtain

$$\begin{gathered}cz_iz_jv_r\otimes v_r\otimes v_i + by_iy_jv_r\otimes v_i\otimes v_r + ax_ix_jv_i\otimes v_r\otimes v_r + ab(x_ix_j + y_iy_j + x_ix_jy_iy_j)v_i\otimes v_i\otimes v_r +\\ ac(x_ix_j + z_iz_j + x_ix_jz_iz_j)v_i\otimes v_r\otimes v_i + bc(y_iy_j + z_iz_j + y_iy_jz_iz_j)v_r\otimes v_i\otimes v_i\in M.\end{gathered}\tag{2}$$

Combining (1) and (2), we get

$$(y_iy_j - z_iz_j)bv_r \otimes v_i \otimes v_r + (x_ix_j - z_iz_j)av_i \otimes v_r \otimes v_r + (x_ix_j + y_iy_j - z_iz_j + x_ix_jy_iy_j)abv_i \otimes v_i \otimes v_r + (x_ix_j + x_ix_jz_iz_j)acv_i \otimes v_r \otimes v_i + (y_iy_j + y_iy_jz_iz_j)bcv_r \otimes v_i \otimes v_i \in M. \tag{3}$$

Applying τ_i again, we obtain

$$\begin{gathered} y_iy_j(y_iy_j - z_iz_j)bv_r \otimes v_i \otimes v_r + x_ix_j(x_ix_j - z_iz_j)av_i \otimes v_r \otimes v_r + \\ ab[(x_ix_j + y_iy_j - z_iz_j + x_ix_jy_iy_j)x_ix_jy_iy_j + y_iy_j(y_iy_j - z_iz_j) + x_ix_j(x_ix_j - z_iz_j)]v_i \otimes v_i \otimes v_r + \\ ac[x_ix_j(x_ix_j - z_iz_j) + (x_ix_j + x_ix_jz_iz_j)x_ix_jz_iz_j]v_i \otimes v_r \otimes v_i + \\ bc[y_iy_j(y_iy_j - z_iz_j) + (y_iy_j + y_iy_jz_iz_j)y_iy_jz_iz_j]v_r \otimes v_i \otimes v_i \in M. \end{gathered} \tag{4}$$

Combining (3) and (4), we let

$\alpha = [a(x_ix_j - z_iz_j)(x_ix_j - y_iy_j)], \beta = [ab(x_i^2x_j^2y_iy_j - x_ix_jy_iy_jz_iz_j + x_i^2x_j^2y_i^2y_j^2 + x_i^2x_j^2 - x_ix_jz_iz_j - x_ix_jy_iy_j)], \gamma = [ac(x_i^2x_j^2 - x_ix_jz_iz_j + x_i^2x_j^2z_iz_j + x_i^2x_j^2z_i^2z_j^2 - x_ix_jy_iy_j - x_ix_jy_iy_jz_iz_j)], \delta = [bc(-y_iy_jz_iz_j + y_i^2y_j^2z_i^2z_j^2)]$
then

$$\alpha v_i \otimes v_r \otimes v_r + \beta v_i \otimes v_i \otimes v_r + \gamma v_i \otimes v_r \otimes v_i + \delta v_r \otimes v_i \otimes v_i \in M. \tag{5}$$

Applying τ_i again and simplifying, we obtain

$$\begin{gathered} \alpha x_ix_jv_i \otimes v_r \otimes v_r + (\alpha x_ix_jc + \gamma x_ix_jz_iz_j)v_i \otimes v_r \otimes v_i + \\ (\alpha x_ix_jb + \beta x_ix_jy_iy_j)v_i \otimes v_i \otimes v_r + \delta y_iy_jz_iz_jv_r \otimes v_i \otimes v_i \in M. \end{gathered} \tag{6}$$

Combining (5) and (6), we get

$$\begin{gathered} (\alpha x_ix_jb + \beta x_ix_jy_iy_j - \beta x_ix_j)v_i \otimes v_i \otimes v_r + (\alpha x_ix_jc + \gamma x_ix_jz_iz_j - \gamma x_ix_j)v_i \otimes v_r \otimes v_i + \\ (\delta y_iy_jz_iz_j - \delta x_ix_j)v_r \otimes v_i \otimes v_i \in M. \end{gathered} \tag{7}$$

Applying τ_i again and simplifying, we obtain

$$\begin{gathered} (\alpha x_ix_jb + \beta x_ix_jy_iy_j - \beta x_ix_j)x_ix_jy_iy_jv_i \otimes v_i \otimes v_r + (\alpha x_ix_jc + \gamma x_ix_jz_iz_j - \gamma x_ix_j) \\ x_ix_jz_iz_jv_i \otimes v_r \otimes v_i + (\delta y_iy_jz_iz_j - \delta x_ix_j)y_iy_jz_iz_jv_r \otimes v_i \otimes v_i \in M. \end{gathered} \tag{8}$$

Combining (7) and (8), we get

$$\begin{gathered} (\alpha x_ix_jb + \beta x_ix_jy_iy_j - \beta x_ix_j)(x_ix_jy_iy_j - y_iy_jz_iz_j)v_i \otimes v_i \otimes v_r + \\ (\alpha x_ix_jc + \gamma x_ix_jz_iz_j - \gamma x_ix_j)(x_ix_jz_iz_j - y_iy_jz_iz_j)v_i \otimes v_r \otimes v_i \in M. \end{gathered} \tag{9}$$

Applying τ_i again and simplifying, we obtain

$$\begin{gathered} (\alpha x_ix_jb + \beta x_ix_jy_iy_j - \beta x_ix_j)(x_ix_jy_iy_j - y_iy_jz_iz_j)x_ix_jy_iy_jv_i \otimes v_i \otimes v_r + \\ (\alpha x_ix_jc + \gamma x_ix_jz_iz_j - \gamma x_ix_j)(x_ix_jz_iz_j - y_iy_jz_iz_j)x_ix_jz_iz_jv_i \otimes v_r \otimes v_i \in M. \end{gathered} \tag{10}$$

Combining (9) and (10), we get

$(\alpha x_ix_jb + \beta x_ix_jy_iy_j - \beta x_ix_j)(x_ix_jy_iy_j - y_iy_jz_iz_j)(x_ix_jy_iy_j - x_ix_jz_iz_j)v_i \otimes v_i \otimes v_r \in M.$

Since $(\alpha x_ix_jb + \beta x_ix_jy_iy_j - \beta x_ix_j)(x_ix_jy_iy_j - y_iy_jz_iz_j)(x_ix_jy_iy_j - x_ix_jz_iz_j) = abx_i^2x_j^2y_i^2y_j^2(x_ix_j - z_iz_j)(y_iy_j - z_iz_j)(x_ix_jy_iy_j - 1)(x_ix_jy_iy_j - z_iz_j) \neq 0$, it follows that $v_i \otimes v_i \otimes v_r \in M$.

Similarly, we can prove that $v_i \otimes v_q \otimes v_i$ and $v_p \otimes v_i \otimes v_i \in M$ since $acx_i^2x_j^2z_i^2z_j^2(y_iy_j - z_iz_j)(x_ix_j - y_iy_j)(x_ix_jz_iz_j - 1)(x_ix_jz_iz_j - y_iy_j) \neq 0$ and $bcy_i^2y_j^2z_i^2z_j^2(x_ix_j - z_iz_j)(x_ix_j - y_iy_j)(y_iy_jz_iz_j - 1)(y_iy_jz_iz_j - x_ix_j) \neq 0$.

Case 7. $p = q \neq i,\ r = i$.

Applying τ_p on $v_i \otimes v_i \otimes v_i \in M$, we obtain

$$bcv_i \otimes v_p \otimes v_p + acv_p \otimes v_i \otimes v_p + abv_p \otimes v_p \otimes v_i \in M. \tag{11}$$

Applying τ_i and simplifying, we obtain

$$bcx_ix_jv_i \otimes v_p \otimes v_p + acy_iy_jv_p \otimes v_i \otimes v_p + abz_iz_jv_p \otimes v_p \otimes v_i \in M. \tag{12}$$

Combining (11) and (12), we get

$$ac(y_iy_j - x_ix_j)v_p \otimes v_i \otimes v_p + ab(z_iz_j - x_ix_j)v_p \otimes v_p \otimes v_i \in M. \tag{13}$$

Applying τ_i again and simplifying, we obtain

$$acy_iy_j(y_iy_j - x_ix_j)v_p \otimes v_i \otimes v_p + abz_iz_j(z_iz_j - x_ix_j)v_p \otimes v_p \otimes v_i \in M. \tag{14}$$

Combining (13) and (14), we get

$$ab(z_iz_j - x_ix_j)(z_iz_j - y_iy_j)v_p \otimes v_p \otimes v_i \in M.$$

Since $ab(z_iz_j - x_ix_j)(z_iz_j - y_iy_j) \neq 0$, it follows that $v_p \otimes v_p \otimes v_i \in M$. Similarly, we prove that $v_p \otimes v_i \otimes v_p$ and $v_i \otimes v_q \otimes v_q \in M$ since $ac(y_iy_j - x_ix_j)(y_iy_j - z_iz_j) \neq 0$ and $bc(x_ix_j - y_iy_j)(x_ix_j - z_iz_j) \neq 0$.

Case 8. $p \neq q \neq i,\ r = i.$

Applying τ_p on $v_i \otimes v_q \otimes v_i \in M$, we obtain

$$(v_i + av_p) \otimes (v_q + b'v_p) \otimes (v_i + cv_p) \in M.$$

Simplifying, we obtain

$$cv_i \otimes v_q \otimes v_p + av_p \otimes v_q \otimes v_i + acv_p \otimes v_q \otimes v_p \in M. \tag{15}$$

Applying τ_i again and simplifying, we obtain

$$c(x_ix_j + aa''')v_i \otimes v_q \otimes v_p + a(z_iz_j + cc''')v_p \otimes v_q \otimes v_i + acv_p \otimes v_q \otimes v_p \in M. \tag{16}$$

Combining (15) and (16), we get

$$c(x_ix_j + aa''' - 1)v_i \otimes v_q \otimes v_p + a(z_iz_j + cc''' - 1)v_p \otimes v_q \otimes v_i \in M. \tag{17}$$

Applying τ_i and simplifying, we obtain

$$cx_ix_j(x_ix_j + aa''' - 1)v_i \otimes v_q \otimes v_p + az_iz_j(z_iz_j + cc''' - 1)v_p \otimes v_q \otimes v_i \in M. \tag{18}$$

Combining (17) and (18), we get

$$a(z_iz_j - x_ix_j)(z_iz_j + cc''' - 1)v_p \otimes v_q \otimes v_i \in M.$$

So, $v_p \otimes v_q \otimes v_i \in M$ since $a(z_iz_j - x_ix_j)(z_iz_j + cc''' - 1) \neq 0$.

Note that if $z_iz_j + cc''' - 1 = 0$, then this contradicts $x_i \neq x_j, y_i \neq y_j, z_i \neq z_j$. Similarly, we can prove that $v_p \otimes v_i \otimes v_r$ and $v_i \otimes v_q \otimes v_r \in M$.

Case 9. $p = q,\ \ p \neq i,\ q \neq i$ and $r \neq i$

Applying τ_p on $v_i \otimes v_i \otimes v_r \in M$, we obtain

$$(v_i + av_p) \otimes (v_i + bv_p) \otimes (v_r + c'v_p) \in M.$$

Simplifying, we get that $abv_p \otimes v_p \otimes v_r \in M$ and so $v_p \otimes v_p \otimes v_r \in M$. Also, we prove that $v_p \otimes v_q \otimes v_p$ and $v_p \otimes v_q \otimes v_q \in M$.

Case 10. p, q and r are different from i, $p \neq q, p \neq r, q \neq r$.

Applying τ_r on $v_p \otimes v_q \otimes v_i \in M$, we obtain

$$(v_p + av_r) \otimes (v_q + b'v_r) \otimes (v_i + c''v_r) \in M.$$

Simplifying, we get that $c''v_p \otimes v_q \otimes v_r \in M$ and so $v_p \otimes v_q \otimes v_r \in M$. $\square$

Consider the representation $G_n(t_1, \ldots, t_n) : P_n \to GL_{n-1}(\mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}])$,
where $t_1, \ldots, t_n$ are indeterminates. Specializing $t_1, \ldots, t_n$ to nonzero complex numbers $x_1, \ldots, x_n$ defines a representation $G_n(x_1, \ldots, x_n) : U_j \to GL_{n-1}(\mathbb{C}) = GL(\mathbb{C}^{n-1})$ which is irreducible if and only if $q(x_1, \ldots, x_n) = x_1 \ldots x_n - 1 \neq 0$. Next, we get our main theorem.

Theorem 1. *For $n \geq 3$, consider the tensor product of irreducible representations $G_n^{(1)}(x_1, \ldots, x_n) \otimes G_n^{(2)}(y_1, \ldots, y_n) \otimes G_n^{(3)}(z_1, \ldots, z_n) : U_j \to GL(\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1})$, where $q(x_1, \ldots, x_n) \neq 0, q(y_1, \ldots, y_n) \neq 0$ and $q(z_1, \ldots, z_n) \neq 0$. If for some $i \neq j$, $x_i x_j \neq y_i y_j, x_i x_j \neq z_i z_j, y_i y_j \neq z_i z_j, x_i x_j y_i y_j \neq 1, x_i x_j z_i z_j \neq 1, y_i y_j z_i z_j \neq 1, x_i x_j \neq y_i y_j z_i z_j, y_i y_j \neq x_i x_j z_i z_j, z_i z_j \neq x_i x_j y_i y_j$, $x_i x_j y_i y_j z_i z_j \neq 1, x_i \neq x_j, y_i \neq y_j, z_i \neq z_j$, then the above representation is irreducible.*

Proof. By Proposition 1, we have that $\mathcal{A}\mathbb{C}^{n-1} \otimes \mathcal{A}\mathbb{C}^{n-1} \otimes \mathcal{A}\mathbb{C}^{n-1}$ is the unique minimal non zero U_j-submodule of $\mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1}$. In particular, it is an irreducible U_j-module. The fact that $q(x_1, \ldots, x_n) \neq 0, q(y_1, \ldots, y_n) \neq 0$ and $q(z_1, \ldots, z_n) \neq 0$ implies that the first factor $\mathcal{A}\mathbb{C}^{n-1}$ corresponds to the representation $G_n^{(1)}(x_1, \ldots, x_n)$, the second factor $\mathcal{A}\mathbb{C}^{n-1}$ corresponds to the representation $G_n^{(2)}(y_1, \ldots, y_n)$ and the third factor $\mathcal{A}\mathbb{C}^{n-1}$ corresponds to the representation $G_n^{(3)}(z_1, \ldots, z_n)$ $\square$.

Since irreducibility on a subgroup implies irreducibility on the group itself, it follows that Theorem 1 is also true for the tensor product of specializations of the Gassner representation of the pure braid group.

4. The Tensor Product oF k Irreducible Gassner Representations ($k \geq 3$)

We now introduce Proposition 2 that provides us with a sufficient condition for the irreducibility of the tensor product of k irreducible Gassner representations $G_n^{(1)}(x_{11}, \ldots, x_{n1}) \otimes ... \otimes G_n^{(k)}(x_{1k}, \ldots, x_{nk}) : U_j \to GL(\mathbb{C}^{n-1} \otimes ... \otimes \mathbb{C}^{n-1})$, where $n \geq 3$ and $k \geq 3$.

Proposition 2. *Given $k \geq 3$ and $x_1 = (x_{11}, \ldots, x_{n1}), ..., x_k = (x_{1k}, \ldots, x_{nk}) \in \mathbb{C}^n$, where $x_{rs} \in \mathbb{C} - \{0, 1\}$. Here, $1 \leq r \leq n$ and $1 \leq s \leq k$. Suppose that for some $i < j$ and every integer s with $1 \leq s \leq k$, we have that*

$$x_{is} x_{js} \neq (\prod_{\substack{\alpha=1 \\ l_\alpha \neq s}}^{\substack{m \\ m \leq k-1}} x_{il_\alpha} x_{jl_\alpha})^{\pm 1} \text{ and } x_{is} \neq x_{js}.$$

(We might have the same condition repeated more than once. Here, $1 \leq \alpha \leq k-1$, $1 \leq l_\alpha \leq k$ and l_α's are taken to be distinct for different values of α.)

Let M be a nonzero U_j-submodule of $\mathbb{C}^{n-1} \otimes ... \otimes \mathbb{C}^{n-1}$ under the action of $G_n^{(1)}(x_{11}, \ldots, x_{n1}) \otimes ... \otimes G_n^{(k)}(x_{1k}, \ldots, x_{nk}) : U_j \to GL(\mathbb{C}^{n-1} \otimes ... \otimes \mathbb{C}^{n-1})$, where $n \geq 3$. For simplicity, we write $v_{p,j} = v_p$ for $p \in \{1, \ldots, j-1, j+1, \ldots, n\}$. Then M contains all $v_{p_1}(x_1) \otimes ... \otimes v_{p_k}(x_k)$, where $p_1, \ldots, p_k \in \{1, \ldots, k\}$. Thus M contains $\mathcal{A}\mathbb{C}^{n-1} \otimes ... \otimes \mathcal{A}\mathbb{C}^{n-1}$. Here, the action of U_j on the first factor is induced by $G_n^{(1)}(x_{11}, \ldots, x_{n1})$, ..., the action of U_j on the last factor is induced by $G_n^{(k)}(x_{1k}, \ldots, x_{nk})$.

Proof. For $1 \leq j \leq n$, we consider the normal free subgroup of rank $n-1$, namely, U_j. Almost the same proof, as in the case $k = 3$, is applied here. However, in the general case, we have a lot of tidious computations with large matrices and a lot of cases to handle. We will not repeat the argument, but we will rather generalize some of the crucial matrix forms and equations used in section three.

To generalize Case 2 of Claim 1 in section 3, we write m along the same lines as before and we apply $\tau_{ij}, \tau_{ij}^2, ..., \tau_{ij}^{2^k-2}$ on m. If one of the coefficients of $e_s \otimes ... \otimes e_s$ is nonzero, then we are done; otherwise, we consider the $(2^k-2) \times (2^k-2)$ matrix $A = [a_{\alpha\beta}]$ defined as follows:

$$\begin{pmatrix}
a_{1,1} & \cdots & a_{1,k} & a_{1,1}.a_{1,2} & \cdots & a_{1,k-1}.a_{1,k} & \cdots & a_{1,2}...a_{1,k} \\
a_{2,1} & \cdots & a_{2,k} & a_{2,1}.a_{2,2} & \cdots & a_{2,k-1}.a_{2,k} & \cdots & a_{2,2}...a_{2,k} \\
\vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
a_{2^k-2,1} & \cdots & a_{2^k-2,k} & a_{2^k-2,1}.a_{2^k-2,2} & \cdots & a_{2^k-2,k-1}.a_{2^k-2,k} & \cdots & a_{2^k-2,2}...a_{2^k-2,k}
\end{pmatrix}$$

The entries in the matrix A are given by

$$a_{\alpha,\beta} = M_{s\beta}(\sum_{k=0}^{\alpha-1} x_{i\beta}^k x_{j\beta}^k), \quad 1 \leq \alpha \leq 2^k-2, \quad 1 \leq \beta \leq k.$$

Also the non zero complex numbers $M_{s\beta}$'s are given as follows:

If $s < i$ then $M_{s\beta} = x_{s\beta} - 1$.

If $i < s < j$ then $M_{s\beta} = x_{j\beta}(x_{s\beta} - 1)$.

If $s \geq j$ then $M_{s\beta} = 1 - x_{s\beta}$.

Claim 3 of section three is generalized as follows:

$$A(e_i \otimes ... \otimes e_i) = \alpha_{ij}(v_i(x_1) \otimes \ldots \otimes v_i(x_k))$$

and

$$A(e_{l_1} \otimes ... \otimes e_{l_k}) = 0 \quad \text{if} \quad (l_1, ..., l_k) \neq (i, \ldots, i)$$

Here the pure braid element A of U_j, in the case $k \geq 3$, is given by

$$A = (\tau_i - 1) \prod_{m=1}^{k-1} \prod_{\alpha_1,...,\alpha_m \in \{1,...,k\}} [\tau_i - (x_{i\alpha_1} x_{j\alpha_1}) \ldots (x_{i\alpha_m} x_{j\alpha_m})].$$

For a fixed integer m, we have C_m^k possible products for $[\tau_i - (x_{i\alpha_1} x_{j\alpha_1}) \ldots (x_{i\alpha_m} x_{j\alpha_m})]$ because $\alpha_1, \ldots, \alpha_m$ are m different integers in $\{1, \ldots, k\}$. Here C_m^k denotes the number of m-combinations from a set of k objects. □

Since irreducibility on U_j implies irreducibility on P_n, we get a similar theorem to that of section 3.

Theorem 2. *For $n \geq 3$ and $k \geq 3$, consider the tensor product of irreducible representations $G_n^{(1)}(x_{11}, \ldots, x_{n1}) \otimes ... \otimes G_n^{(k)}(x_{1k}, \ldots, x_{nk}) : P_n \to GL(\mathbb{C}^{n-1} \otimes ... \otimes \mathbb{C}^{n-1})$, where $q(x_{11}, \ldots, x_{n1}) \neq 0, ..., q(x_{1k}, \ldots, x_{nk}) \neq 0$. If for some $i \neq j$ and every integer s with $1 \leq s \leq k$, we have that*

$$x_{is} x_{js} \neq \Big(\prod_{\substack{\alpha=1 \\ l_\alpha \neq s}}^{\substack{m \\ m \leq k-1}} x_{il_\alpha} x_{jl_\alpha}\Big)^{\pm 1} \text{ and } x_{is} \neq x_{js}$$

then the above representation is irreducible. (We might have the same condition repeated more than once)

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