

# Spatial Dynamics in Prey-Predator Model With Holling type II and Strong Allee Effect in Prey

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## Abstract

In this study, a spatiotemporal prey-predator mathematical model incorporating a Holling type II functional response and a strong Allee effect in the prey population was developed to explore the dynamics of species undergoing diffusion and migration. The stability of the model without diffusion and migration was analyzed, and the influence of the Allee effect on bifurcations was investigated. The spatial dynamics of the model were further examined to understand how diffusion and migration impact the onset of Turing instability under zero-flux boundary conditions. The findings reveal that the inclusion of the Allee effect can generate complex spatial patterns. While diffusion promotes the emergence of Turing instability, advection does not lead to such instabilities.

**Keywords:** Alle effect, diffusion, spatial, patterns, instability

## 1. Introduction

The study of prey-predator relation using non-linear differential equations in ecology emerges as long as 1927 when Lotka and Volterra simultaneously developed the first mathematical models to study ecological patterns (Naji, Al-Momen, & Raid, 2021)). The classic Lotka-Volterra model forms the foundation for numerous contemporary models used to study population dynamics (Zhang, Cai, Fu, & Wang, 2019). Several widely recognized classical models have been developed by researchers who have incorporated various environmental factors that influence the stability and persistence of these systems, such as prey refuges (Ghosh, Sahoo, & Poria, 2017), (Gakkhar, Singh, & Naji, 2007), disease (Sagamiko, Shaban, Nahonyo, & Makinde, 2015), delays, harvesting (Sinclair, Packer, Mduma, & Fryxell, 2008) and numerous other considerations.

In 1931, Allee (Allee, 1931) was the first to introduce the concept of the Allee effect. He examined evidence on how population density influences demographic and life-history traits, showing that growth rates are not always positive at low densities and may not decrease in line with the predictions of the logistic model. The Allee effect refers to a positive relationship between population density and individual average fitness when the population is small (Ma, Zhao, & Du, 2022). Ecologically, the Allee effect may happen due to the change of habitat (due migration), inbreeding depression, difficulties in finding a mate, avoidance of predators, environment conditions, social dysfunction in small population sizes, genetic drift, and food exploitation.

The Allee effect is generally categorized into two types: strong and weak (Wang & Kot, 2001). A strong Allee effect occurs when the per capita growth rate becomes negative if the population density falls below a specific threshold, known as the Allee threshold, and turns positive once the population density exceeds this level (Wang & Kot, 2001). This implies that the population needs to exceed this threshold to experience growth (below this threshold, the population cannot sustain itself and declines toward extinction). On the other hand, a weak Allee effect lacks this threshold requirement; the growth rate increases with population size but remains positive even at very low population sizes, allowing survival, albeit with slower growth.

(Ma, Zhao, & Du, 2022), (Courchamp, Clutton-Brock, & Grenfell, 1999) studied the effects of strong Allee effect in the prey-predator model. They compared the properties of the system with and without Allee effect, they showed how the strong Allee effect is of great importance to the dynamics of the systems of equations, that the Allee effect can increase the extinction risk of populations and has the ability to switch the system's stability to limit cycle oscillation from stable node.

Other studies of these biological systems to include spatial patterns emerged in 1952 when Allan Turing proposed that under certain conditions, the prey-predator non-linear system is asymptotically stable in the absence of diffusion and unstable in the presence of diffusion (Dutta, 2017), (Dutta, 2017) and (Courchamp, Clutton-Brock, & Grenfell, 1999). Since then, several reaction-diffusion equations have been proposed. For instance, Liu, 2010 used the reaction diffusion equations to show the effects of diffusion and migration and the emergence of Turing instability as a result of both

diffusion and migration

In this paper the Allee effect is studied in the prey predator model, the spatial dynamics of prey predator system was established to include diffusion and migration parameters.

**2. Methods**

*2.1 The Mathematical Model*

With the Holling type II functional response and strong Allee effect in prey, and logistic growth, the prey-predator model can be written as

$$\left. \begin{aligned} \frac{du}{dt} &= ru \left(1 - \frac{u}{K}\right) (u - A) - \frac{huv}{1+au} \\ \frac{dv}{dt} &= \frac{ehuv}{1+au} - dv \end{aligned} \right\} \tag{1}$$

Here,  $u(t)$  and  $v(t)$  stand for prey and predator population densities at any time  $t$ . The parameters  $r$  and  $K$  stand for the prey’s intrinsic growth rate and carrying capacity respectively,  $e$  is the conversion rate of the prey to predator and  $d$  is the mortality rate of the predator,  $h$  is the searching rate,  $a$  is the predator’s handling time on prey. The term  $\frac{u}{1+au}$  is the Holling type II functional response and the parameter  $A$  stands for the strong Allee effect in prey.

To reduce the number of parameters, the mathematical model (1) is non-dimensionalized as follows:

Let  $\bar{u} = \frac{u}{K}$ ,  $\bar{t} = Krt$ ,  $\bar{v} = \frac{a}{r}v$ ,  $\alpha = \frac{A}{K}$ ,  $\beta = \frac{h}{a}$ ,  $\omega = \frac{eh}{r}$ ,  $\gamma = \frac{d}{Kr}$ ,  $\mu = aK$ , and dropping the bars we get the following mathematical model:

$$\left. \begin{aligned} \frac{du}{dt} &= u(-u)(u - \alpha) - \frac{\beta uv}{1 + \mu u} \\ \frac{dv}{dt} &= \frac{\omega uv}{1 + \mu u} - \gamma v \end{aligned} \right\} \tag{2}$$

For a strong Allee effect occurs when  $0 < A < K$ . (3)

**Diffusion and migration parameters in the mathematical model**

The diffusion and migration components are introduced to the system of equations (2) to form the 2D reaction diffusion system (advection diffusion reaction system). This can be defined as:

$$\left. \begin{aligned} \frac{\partial u}{\partial t} &= f(u, v) - \frac{\partial}{\partial x}(C_1 u) - \frac{\partial}{\partial y}(C_1 u) + \frac{\partial}{\partial x}\left(D_1 \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_1 \frac{\partial u}{\partial y}\right) \\ \frac{\partial v}{\partial t} &= g(u, v) - \frac{\partial}{\partial x}(C_2 v) - \frac{\partial}{\partial y}(C_2 v) + \frac{\partial}{\partial x}\left(D_2 \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_2 \frac{\partial v}{\partial y}\right) \end{aligned} \right\} \tag{4}$$

The terms  $f(u, v)$  and  $g(u, v)$  are nonlinear dynamic systems (reaction kinetics) and are defined as:

$$\left. \begin{aligned} \frac{\partial u}{\partial t} &= f(u, v) - C_1 \nabla u + D_1 \nabla^2 u \\ \frac{\partial v}{\partial t} &= g(u, v) - C_2 \nabla v + D_2 \nabla^2 v \end{aligned} \right\} \tag{5}$$

The coefficients  $C_1$  and  $C_2$  are migration (advection) components for prey and predator respectively while  $D_1$  and  $D_2$  are diffusion coefficients for prey and predator respectively.

The symbol  $\nabla$  is the gradient operator defined in two dimensions as:  $\nabla(\cdot) = \frac{\partial(\cdot)}{\partial x}i + \frac{\partial(\cdot)}{\partial y}j$ , and the Laplacian operator  $\nabla^2$

in two dimensions is defined as:  $\nabla^2 = \frac{\partial^2(\cdot)}{\partial x^2} + \frac{\partial^2(\cdot)}{\partial y^2}$ .

Equation (5) represents the reaction-diffusion model, which treats space as a continuous variable and describes the population densities (or concentrations) of interacting species over time. The dynamics of predator and prey interactions are governed by this equation.

### 2.2 Existence of Equilibrium Points

In the absence of diffusion ( $D_u = D_v = 0$ ) and migration ( $C_u = C_v = 0$ ) the system of equations (5) may reach the stable equilibrium as equation (2):

$$\left. \begin{aligned} \frac{du}{dt} &= u(1-u)(u-\alpha) - \frac{\beta uv}{1+\mu u} \\ \frac{dv}{dt} &= \frac{\omega uv}{1+\mu u} - \gamma v \end{aligned} \right\}$$

The system of equations (2) has three boundary equilibria  $E_0(0,0)$ ,  $E_1(1,0)$ ,  $E_2(\alpha,0)$  and one positive equilibrium point

$$E_3 \left( \frac{-\gamma}{\mu\gamma - \omega}, \frac{\omega(\gamma\mu + \gamma - \omega)(\alpha\gamma\mu - \alpha\omega + \gamma)}{\beta(\gamma\mu - \omega)^3} \right)$$

Where  $\gamma\mu \neq \omega$ . For the survival of the species  $\gamma\mu - \omega < 0$ . This is because the rate of growth of the prey must be greater than that of the predator and the predator's death rate must be less than the efficient rate of the predator to convert prey into new predators.

### 2.3 Local Stability of the Equilibrium Points

The local asymptotic stability of each equilibrium point is studied by computing the Jacobian matrix and finding the eigenvalues evaluated at each equilibrium point. For stability of the equilibrium points, the real parts of the eigenvalues of the Jacobian matrix must be negative. From Equations (3), the Jacobian matrix of the system is given by:

$$J(E_i) = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} \tag{6}$$

For  $i = 0, 1, 2, 3$

The Jacobian matrix of the system of equations (2) is

$$J(E_i) = \begin{bmatrix} (1-u)(u-\alpha) - u(u-\alpha) + u(1-u) - \frac{\beta v}{1+\mu u} + \frac{\beta \mu uv}{(1+\mu u)^2} & -\frac{\beta u}{1+\mu u} \\ \frac{\omega v}{1+\mu u} - \frac{\omega \mu uv}{(1+\mu u)^2} & \frac{\omega u}{1+\mu u} - \gamma \end{bmatrix}$$

The Jacobian matrix at  $E_0(0,0)$  is

$$J(E_0) = \begin{bmatrix} -\alpha & 0 \\ 0 & -\gamma \end{bmatrix}$$

Which has two eigenvalues negative eigen values  $\lambda_1(E_0) = -\alpha < 0$  and  $\lambda_2(E_0) = -\gamma < 0$ . This implies that the extinction of both prey and predator is stable. Therefore, for Bifurcation analysis, no bifurcation is expected here, as the

eigenvalues are negative and stable.

The Jacobian matrix at  $E_1(1,0)$  is

$$J(E_1) = \begin{bmatrix} -1 + \alpha & \frac{-\beta}{\mu + 1} \\ 0 & \frac{\omega}{\mu + 1} - \gamma \end{bmatrix}$$

Whose eigenvalues are  $\lambda_1(E_1) = -1 + \alpha < 0$  and  $\lambda_2(E_1) = \frac{\omega}{\mu + 1} - \gamma$ .

There is an instance of Transcritical Bifurcation: This occurs when two equilibrium points exchange stability. This can happen if eigenvalues change their signs. For this equilibrium point:  $\lambda_1 = -1 + \alpha$  changes sign at  $\alpha = 1$ . When  $\alpha$  passes through 1,  $\lambda_1$  changes from negative to positive indicating shift in stability (Fig 2).

For  $\frac{\omega}{\mu + 1} - \gamma$  changes sign at  $\frac{\omega}{\mu + 1} - \gamma$ .

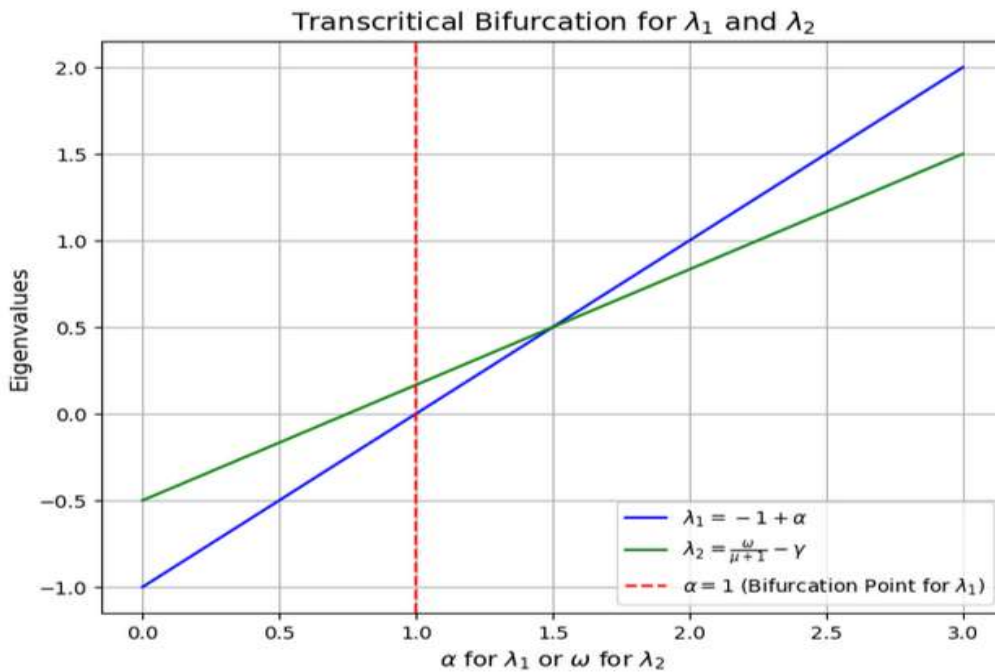


Figure 1. Emergence of Transcritical bifurcation for equilibrium point  $E_1$ .

The system is stable if  $\omega < \gamma(\mu + 1)$  and saddle if  $\omega > \gamma(\mu + 1)$ . Since  $0 < \alpha < 1$  and  $0 < \omega < 1$ , then the change of stability (Figure 1) by transcritical bifurcation is from stable to saddle.

The Jacobian matrix at  $E_2(\alpha,0)$  is

$$J(E_2) = \begin{bmatrix} -\alpha(-1 + \alpha) & \frac{-\beta\alpha}{\alpha\mu + 1} \\ 0 & \frac{\omega\alpha}{\alpha\mu + 1} - \gamma \end{bmatrix}$$

The eigenvalues of  $E_2$  are  $\lambda_1(E_2) = -\alpha(-1 + \alpha) > 0$  and  $\lambda_2(E_2) = \frac{\omega\alpha}{\alpha\mu + 1} - \gamma$ .

A transcritical bifurcation typically occurs at this point (Figure 2). Since  $0 < \alpha < 1$ , then the change of stability (Figure 2) by transcritical bifurcation is from saddle to unstable.

- For  $\lambda_1 = -\alpha^2 + \alpha$ , the transcritical bifurcation occurs at  $\alpha = 0$  and  $\alpha = 1$ .
- For  $\lambda_2 = \frac{\omega\alpha}{\alpha\mu+1} - \gamma$ , the transcritical bifurcation occurs at  $\alpha = \frac{\gamma}{\omega - \gamma\mu}$ .

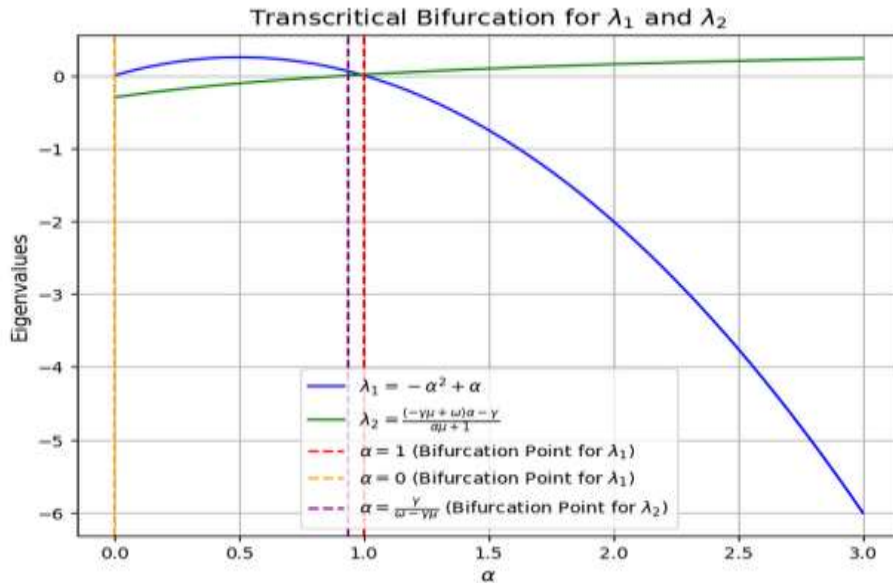


Figure 2 Emergence of Transcritical Bifurcation for equilibrium point  $E_2$ .

Thus,  $E_2$  is a saddle node if  $\alpha < \frac{\gamma}{\omega - \mu\gamma}$  and unstable node if  $\alpha > \frac{\gamma}{\omega - \mu\gamma}$ . Therefore, the change of stability by transcritical bifurcation is from saddle to unstable.

For the positive equilibrium point  $E_3$ , from the biological perspective, we are interested in studying the stability behavior of the interior equilibrium point  $E_3(u^*, v^*)$ . The Jacobian matrix corresponding to this equilibrium point is as follows:

$$J(E_3) = J(u^*, v^*) = \begin{bmatrix} \frac{-\gamma(\mu(\mu+1)(\alpha\mu+1)\gamma^2 + (-2\alpha\mu^2 + 2)\omega\gamma + \omega^2(\alpha\mu - \alpha - 1))}{\omega(\gamma\mu - \omega)^2} & -\frac{\gamma\beta}{\omega} \\ \frac{(\gamma\mu + \gamma - \omega)(\alpha\gamma\mu - \alpha\omega + \gamma)}{\beta(\gamma\mu - \omega)} & 0 \end{bmatrix}$$

The analysis of the Hopf is presented in Figure 3.

The determinant is  $\det(J(E_3)) = \frac{\gamma(\gamma\mu + \gamma - \omega)(\alpha\gamma\mu - \alpha\omega + \gamma)}{\omega(\gamma\mu - \omega)}$  and the trace is

$$\text{trace}(J(E_3)) = \frac{-\gamma(\mu(\mu+1)(\alpha\mu+1)\gamma^2 + (-2\alpha\mu^2 + 2)\omega\gamma + \omega^2(\alpha\mu - \alpha - 1))}{\omega(\gamma\mu - \omega)^2}$$

Considering the Routh–Hurwitz stability criterion:

- The positive equilibrium point  $E_3$  is locally asymptotically stable when trace  $< 0$  and the determinant is  $> 0$ .
- $E_3$  is unstable when the trace of the system is positive. That is when trace  $> 0$  and the determinant is  $> 0$ .
- $E_3$  is a center or weak focus when the determinant is  $> 0$  and  $(\text{trace}^2 - 4\det) < 0$ .

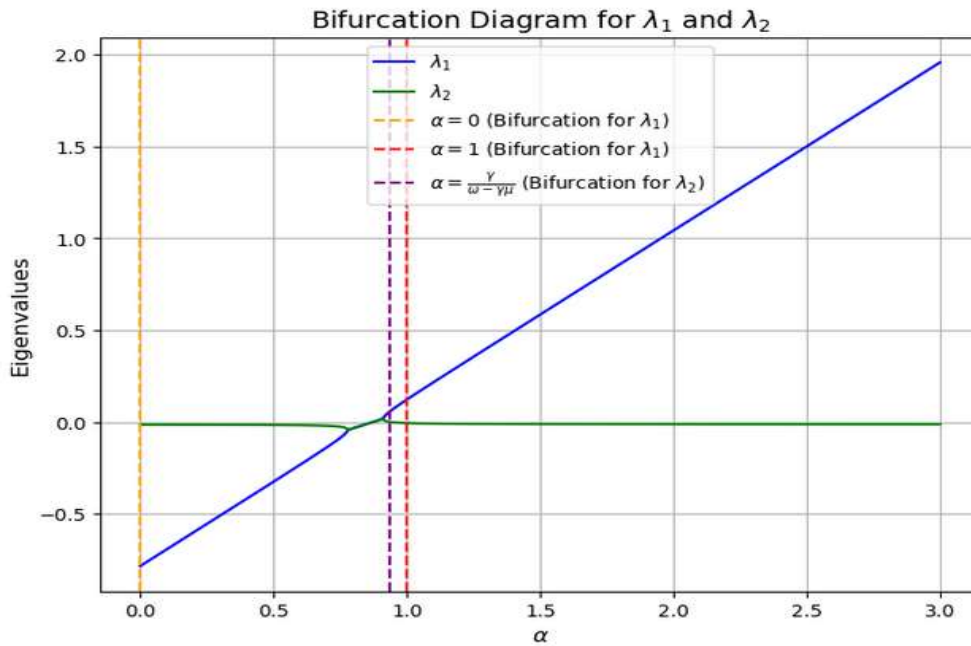


Figure 3. Bifurcation analysis for equilibrium point  $E_3$

All four equilibrium points are in the following phase portrait.

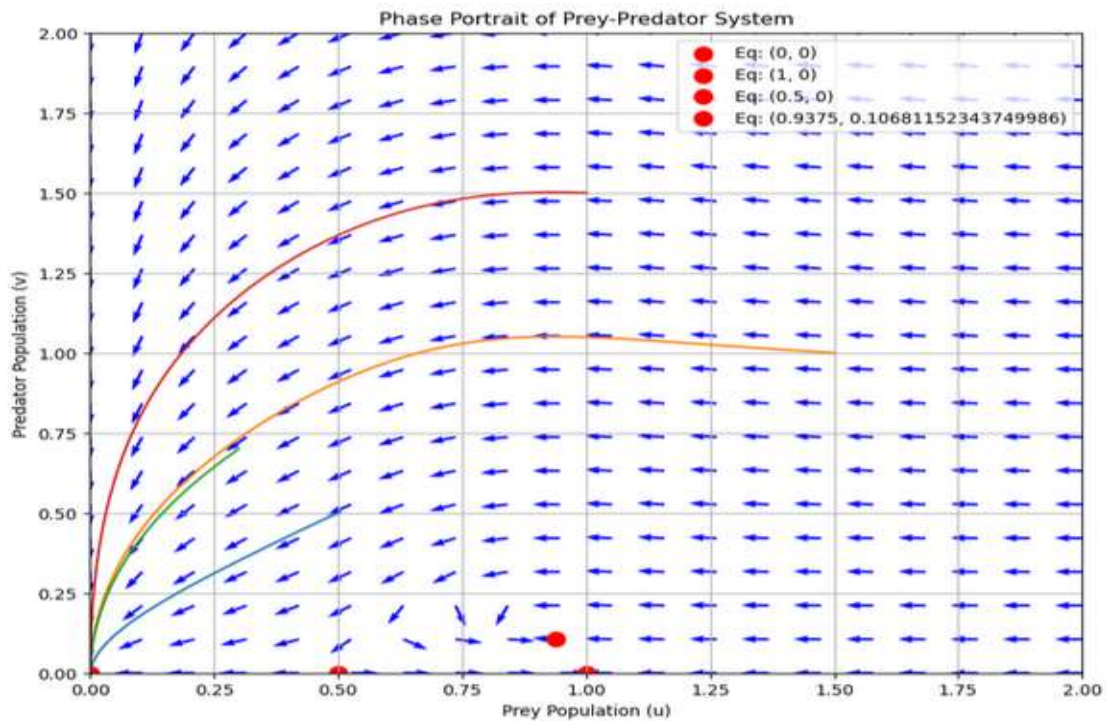


Figure 4. Phase portrait of the equilibrium points

#### 2.4 Impact of the Allee Effects on Population

The prey population cannot sustain itself below a critical threshold  $A$ . If the prey population drops below this level (figure 5), extinction becomes inevitable (Kisoma & Colin, 2023) which can lead to the predator population also declining

to extinction due to lack of food.

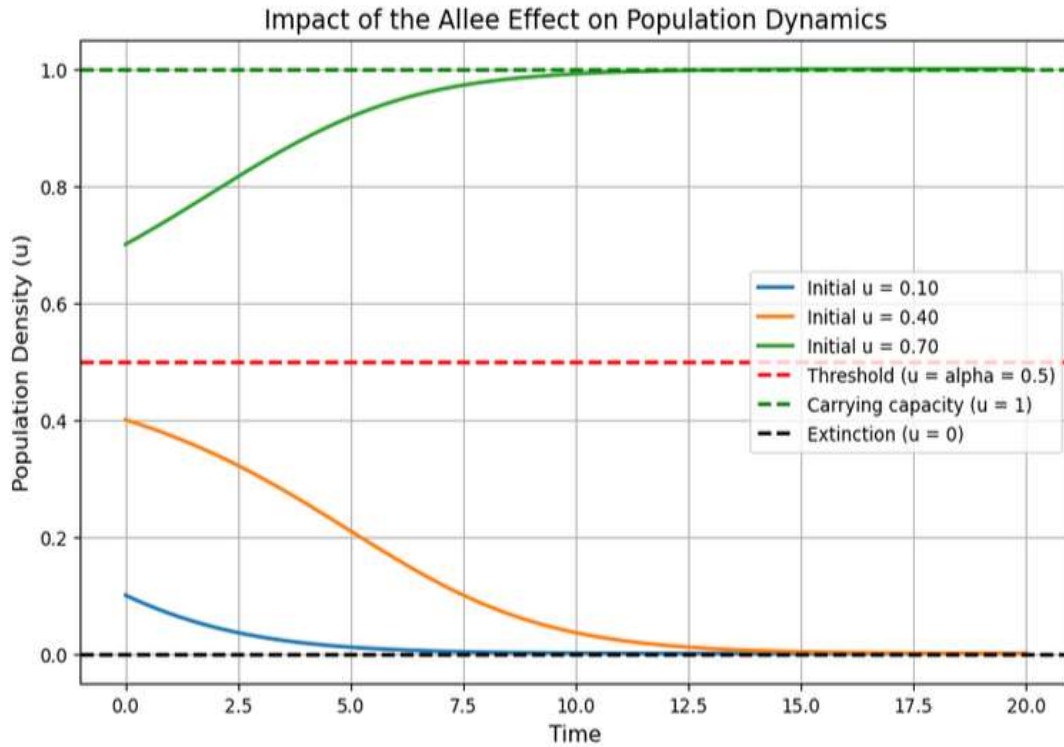


Figure 5. Impact of Allee effect on the prey population Dynamics

- For  $u < \alpha$  the population of prey declines towards  $u = 0$  (extinction)
- For  $u > \alpha$  the population of prey grows towards  $u = 1$  (carrying capacity)
- At  $u = \alpha$ , the equilibrium is unstable, and even small perturbations determine whether the population moves toward extinction or growth.

Predator-prey coexistence becomes precarious and depends on whether the prey population consistently remains above the Allee threshold.

### 2.5 Stability Analysis of the Model With Diffusion and Migration

From the mathematical model (3), assuming the diffusion and migration in each spatial dimension are the same, equation (5) can be written as

$$\left. \begin{aligned} \frac{\partial u}{\partial t} &= D_1 \nabla^2 u - C_1 \nabla u + u(1-u)(u-\alpha) - \frac{\beta uv}{1+\mu u} \\ \frac{\partial v}{\partial t} &= D_2 \nabla^2 v - C_2 \nabla v + \frac{\omega uv}{1+\mu u} - \gamma v \end{aligned} \right\} \quad (7)$$

$$\left. \begin{aligned} \frac{\partial u}{\partial t} &= f(u, v) - C_1 \nabla u + D_1 \nabla^2 u \\ \frac{\partial v}{\partial t} &= g(u, v) - C_2 \nabla v + D_2 \nabla^2 v \end{aligned} \right\} \quad (8)$$

Where  $f(u, v)$  and  $g(u, v)$  are nonlinear dynamic systems (reaction kinetics) (Liu, 2010; Wang *et al.*, 2016).

The homogeneous steady-state values of  $u$  and  $v$  are solutions to the reaction terms  $f(u, v) = 0$  and  $g(u, v) = 0$ .

### 2.6 Spatiotemporal Patterns Formed by Diffusion

Assume small perturbation (these are perturbation around the coexistence equilibrium point  $E_3$ ) due to diffusion  $\delta_u, \delta_v$

about the equilibrium stable state  $(u_0, v_0)$  as  $u = u_0 + \delta_u$  and  $v = v_0 + \delta_v$  and taking Taylor expansion of the non-linear functions  $f(u, v)$  and  $g(u, v)$  about the stationary state; thus, one can write:

$$\left. \begin{aligned} \frac{\partial \delta_u}{\partial t} &= \left( \frac{\partial f}{\partial u} \right)_{(u_0, v_0)} \delta_u + \left( \frac{\partial f}{\partial v} \right)_{(u_0, v_0)} \delta_v + D_u \nabla^2 \delta_u \\ \frac{\partial \delta_v}{\partial t} &= \left( \frac{\partial g}{\partial u} \right)_{(u_0, v_0)} \delta_u + \left( \frac{\partial g}{\partial v} \right)_{(u_0, v_0)} \delta_v + D_v \nabla^2 \delta_v \end{aligned} \right\} \quad (9)$$

The perturbation can be assumed to be harmonic in space, and the spatial variation can be expressed as  $e^{ik_n \chi}$  while the temporal variation which allows this perturbation to grow can be expressed as  $e^{\omega_n t}$ . The parameters  $k_n$  stands for wave number,  $\chi = (x, y)$  is the spatial vector in two dimensions,  $i$  is the imaginary unit and  $i^2 = -1$ ,  $\omega_n$  is the growth rate of perturbations. Thus, one can write:

$$\delta_u = \delta_{u_0} e^{ik_n \chi} e^{\omega_n t}, \quad \delta_v = \delta_{v_0} e^{ik_n \chi} e^{\omega_n t} \quad (10)$$

This leads to an eigenvalue Equation of the form:

$$A \delta \chi = \omega_n \delta \chi \quad (11)$$

With,

$$A = \begin{pmatrix} f_u - D_u k_n^2 f_v \\ g_u g_v - D_v k_n^2 \end{pmatrix}, \quad \delta \chi = \begin{pmatrix} \delta u_0 \\ \delta v_0 \end{pmatrix}$$

The perturbation amplitudes  $\delta u_0$  and  $\delta v_0$  can be non-zero if and only if  $\det(A - \omega_n I) = 0$ .

This gives the characteristic polynomial defined as:

$$\omega_n^2 + [(D_u + D_v)k_n^2 - f_u - g_v]\omega_n + D_u D_v k_n^4 - k_n^2(D_v f_u + D_u g_v) + f_u g_v - f_v g_u = 0 \quad (12)$$

A fluctuation associated with the eigenvalue  $\omega_n$  grows if:

$$Re(\omega_n) > 0.$$

Thus,

At the onset of Turing bifurcation, instability occurs when  $Re(\omega_n) = 0$ .

In that case, the terms independent of  $\omega_n$  in equations (12) are reduced to zero. This determines the critical wave number  $k_c$  for diffusion as:

$$k_c^2 = \frac{(D_v f_u + D_u g_v) \pm \sqrt{(D_v f_u + D_u g_v)^2 - 4D_u D_v (f_u g_v - f_v g_u)}}{2D_u D_v} \quad (13)$$

The critical wave numbers in turn determine the critical wave length  $L_c(n) = \frac{n\pi}{k_c}$ . Thus, the fluctuations associated with the frequency  $\omega_n$  are amplified in the system of length  $L$  if  $L > L_c(n)$ . This is a necessary condition for the development of an inhomogeneous state by the growth of fluctuations associated with wavenumber  $k_n$ . Further, the most unstable wavenumber  $k_{max}$ , can be obtained from the condition:

$$\frac{\partial}{\partial k} Re[\omega(k)] = 0 \text{ at } k = k_{max}$$



Giving the maximum wave number for diffusion as

$$k_{\max} = \left[ \frac{D_v f_u + D_u g_v}{2D_u D_v} \right]^{1/2} = \left[ \frac{f_u g_v - f_v g_u}{D_u D_v} \right]^{1/4} \tag{14}$$

Therefore, from equations (13) and (14) the necessary conditions for the onset of Turing instability for model (6) are as follows:

$$f_u + g_v < 0 \tag{15}$$

$$f_u g_v - f_v g_u > 0 \tag{16}$$

$$D_v f_u + D_u g_v > 0 \tag{19}$$

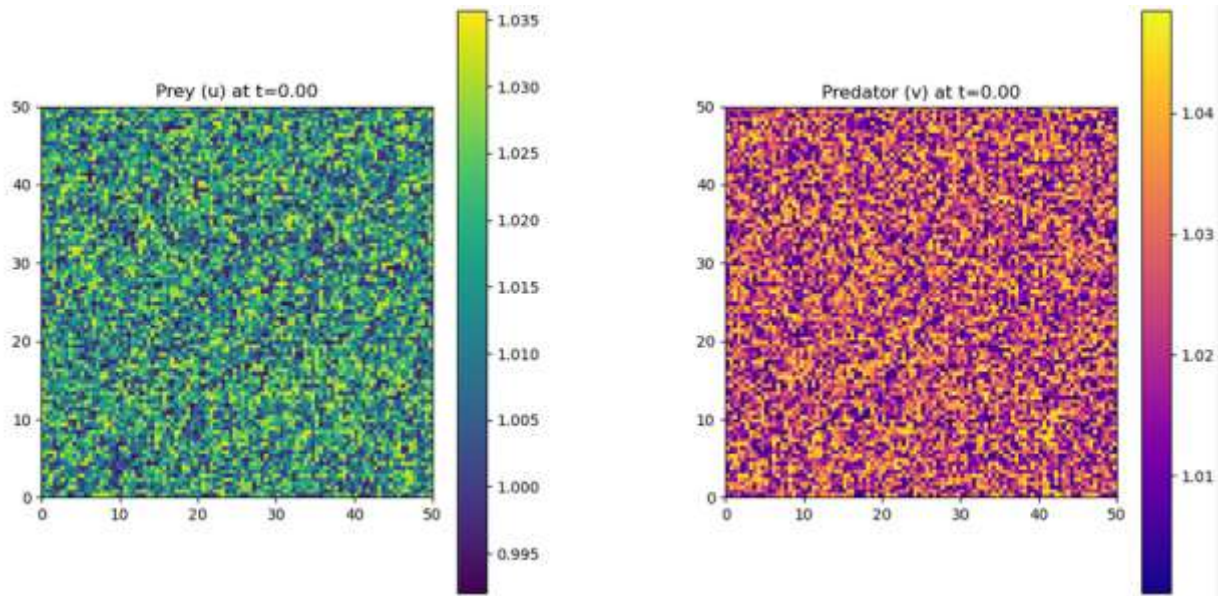
$$(D_v f_u + D_u g_v)^2 > 4D_u D_v (f_u g_v - f_v g_u) \tag{20}$$

It can be observed that the dispersion relation that drives the system of equations to instability depends on the variation of diffusion terms.

### 3. Numerical Simulations

#### 3.1 Turing Instability Due to Diffusion and Alle Effect

This section examines the influence of the Allee effect to explain the qualitative results of the mathematical model (7) in two-dimensional space. Numerical integration was performed using the explicit Euler method with a time step size of 0.1 over a time interval of 100 units. The domain size is  $100 \times 100$ . The initial conditions consist of a small random perturbation around the coexistence equilibrium point  $E_3(u^*, v^*)$ .



(Courchamp, Clutton-Brock, & Grenfell, 1999)

Figure 6(A). Homogeneous spatially periodic patterns due to diffusion and Allee effect at  $t = 0$

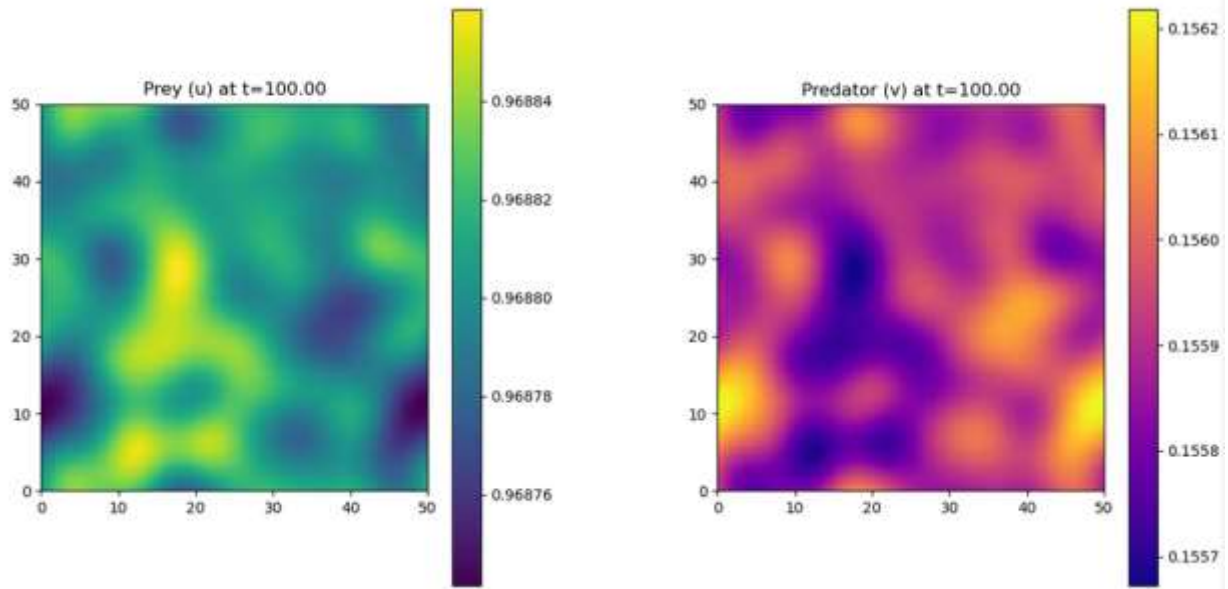


Figure 6 (B). Spatially periodic patterns due to diffusion and Allee effect after  $t = 100$

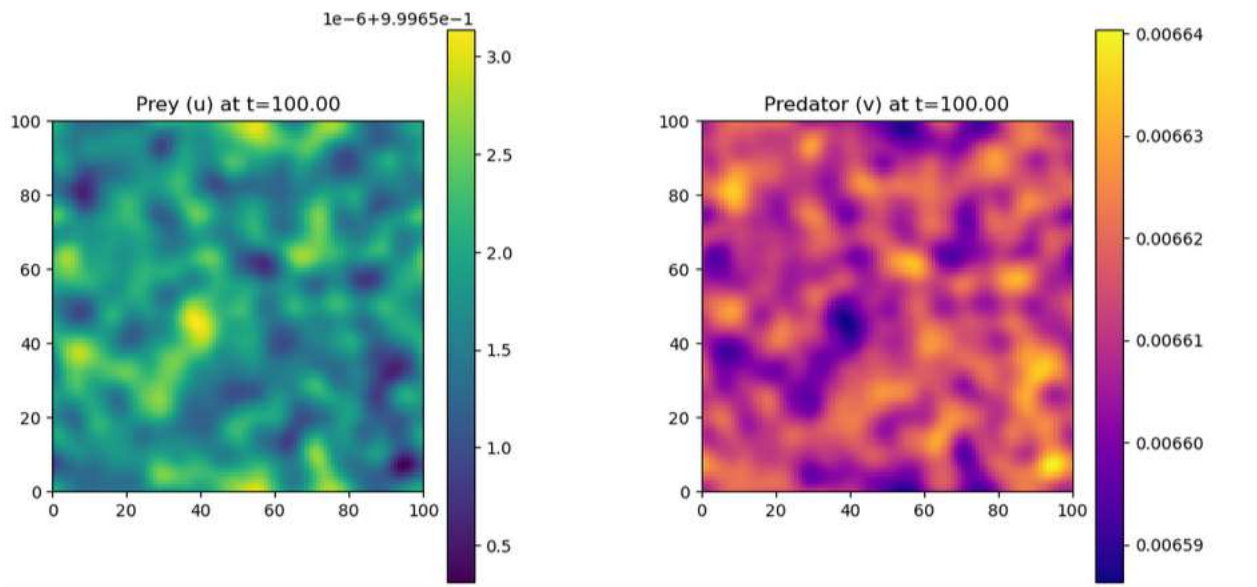


Figure 6 (C). Spatially periodic patterns due to diffusion with no Allee effect after  $t = 100$

The parameters used are  $\alpha = 0.5, \beta = 0.1, \gamma = 0.1, \mu = 1, \omega = 0.1, D_1 = 0.01, D_2 = 0.2$ .

A homogeneous equilibrium of the reaction-diffusion system is stable without diffusion (Kisoma & Colin, 2023). Introducing diffusion (with or without advection terms) destabilizes the system, leading to spatially periodic patterns.

Without diffusion, the system remains homogeneous (Figure 6(A) at  $t = 0$ ). With diffusion and specific parameter choices ( $D_1 \neq D_2$ ) spatial patterns emerge due to Turing instabilities.

From Figure 6B, with the diffusion difference and nonlinear reactions, the Turing patterns forming are observed over time. It can be observed that the spatiotemporal dynamics of the model lead to diffusion-controlled growth patterns, including holes, stripes, and spots. Furthermore, numerical simulations conducted in the absence of the Allee effect reveal a sequence of spatial patterns such as holes, hole- spot- mixtures.

### 3.2 Advection Parameters and Their Role in Model Behavior

Advection plays a crucial role in understanding migration patterns, particularly in systems where populations move under external influences (Kisoma, Kuznetsov, Colin, & Treydte, 2020). By excluding diffusion in the model, the study focuses

solely on the impact of migration dynamics, which leads to unique insights into the behavior of populations.

The mathematical model was simulated without incorporating diffusion, allowing advection (migration) to dominate the dynamics. The simulation involved setting up parameters that govern advection processes ( $C_1=0.1$  and  $C_2=0.05$ ) while excluding any diffusion-related terms. This approach helped isolate the effects of migration.

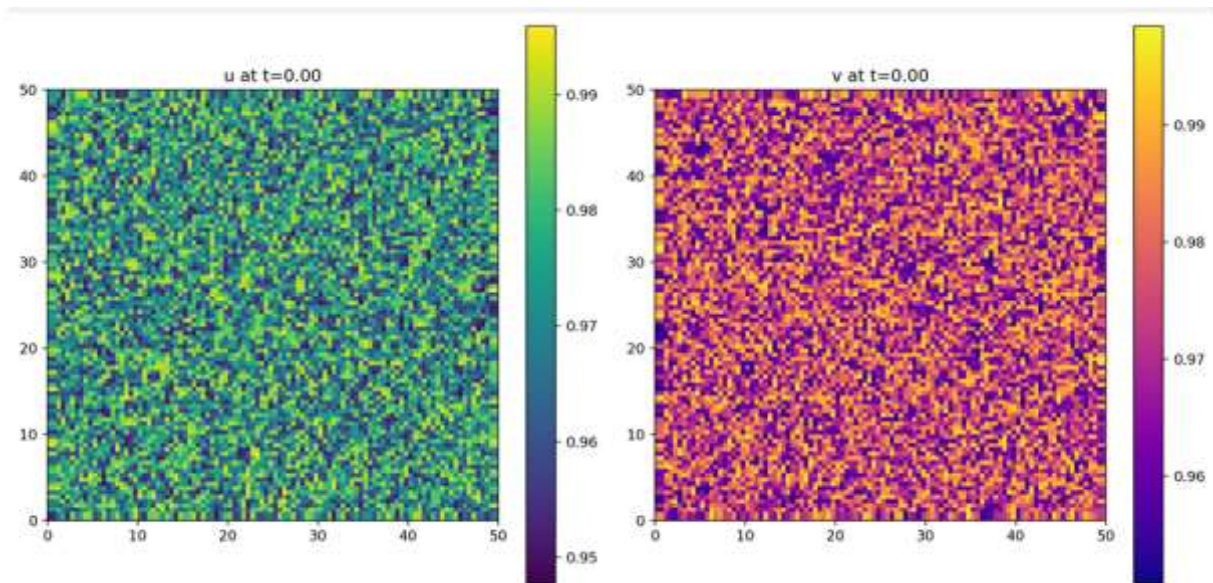


Figure 7(A). Homogeneous spatially periodic patterns formed due to advection at  $t = 0$

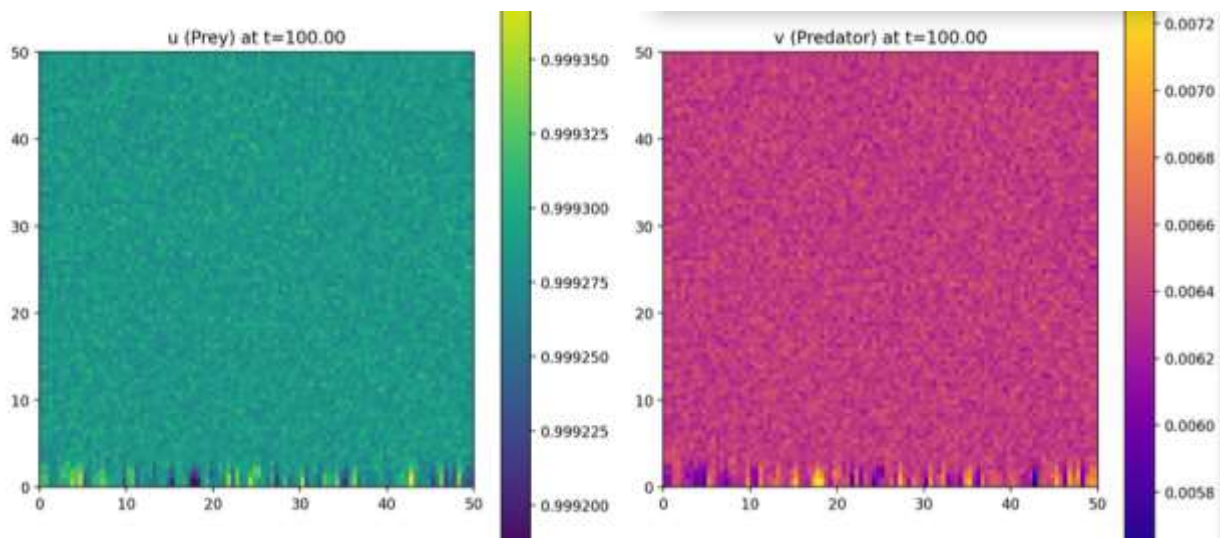


Figure 7(B). No spatially periodic patterns formed due to advection after  $t = 100$

The parameters are  $\alpha = 0.5$ ,  $\beta = 0.1$ ,  $\gamma = 0.1$ ,  $\mu = 1$ ,  $\omega = 0.1$ ,  $C_1 = 0.1$ ,  $C_2 = 0.05$

The simulation results revealed no significant formation of spatially periodic patterns (figure 7B). This outcome suggests that the absence of diffusion ( $D_1 = D_2 = 0$ ), combined with the presence of advection  $C_1$  and  $C_2$ , leads to uniform population distributions rather than localized concentrations (Figure s7A and 7B).

For the Pattern Formation, it is observed Turing patterns as a result of the competition between diffusion and the Allee effect. The predator population  $V$  interacts with the prey  $U$ , and the diffusion terms creates spatial heterogeneity, which leads to pattern formation.

Diffusion plays a crucial role to drive the stable system to instability (Dutta, 2017). For instance as the prey species migrate from place to place looking for better forage refuge they perform both advection and diffusion processes

(Kisoma & Colin, 2023), (Kisoma, Kuznetsov, Colin, & Treydte, 2020). Through this, they lead to diffusion-induced instability. In particular, the instability criteria of the spatially homogeneous state determines the pattern forming processes in a non-equilibrium system.

Advection plays a critical role in prey-predator models by capturing the directed movement of individuals in response to environmental stimuli. This directional movement is typically influenced by gradients such as resource availability (e.g., food or water) or threats (e.g., predator presence), which can significantly impact both migration patterns and population dynamics (Robert & Chris, 2003).

Prey often exhibit advection-driven movement toward regions with abundant resources, such as areas with higher vegetation density (Kisoma, Kuznetsov, Colin, & Treydte, 2020). This behavior can result in localized aggregations of prey, creating spatial heterogeneity.

Predators typically respond to prey density gradients, moving toward areas with higher prey abundance. This pursuit behavior leads to predator aggregation in prey-rich regions, intensifying local predation pressure (Liu, 2010), (Robert & Chris, 2003).

In some cases, predators are influenced by habitat features such as terrain or water availability, which can limit or direct their movement patterns.

#### 4. Discussion and Conclusion

Migration can help maintain prey populations above the Allee threshold, supporting predator-prey coexistence. Predator migration might stabilize or destabilize the system depending on prey availability and diffusion rates.

Strong Allee effects can lead to fragmented prey distributions, which may create isolated pockets of predator-prey coexistence or extinction zones. These patterns depend on diffusion rates and environmental heterogeneities.

This study has described the emergence of spatiotemporal natural patterns due to the coordination of events (diffusion and migration) occurring in interacting species as prey and predator. It has suggested and tested mechanisms by which complex mechanisms that may arise in biological patterns. The results presented in this study clearly highlight the crucial role of the Allee effect, in conjunction with diffusion, in the pattern formation of the predator-prey model, emphasizing their significance in the field of (mathematical) ecology. (Kisoma & Colin, 2023)

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