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Type II Topp-Leone Exponentiated Exponential Distribution: Properties and Applications to Failure Data

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Abstract

This work presents a flexible three-parameter lifetime distribution with increasing, increasing and decreasing and non-monotonic hazard rate called Type II Topp-Leone Exponentiated Exponential ($TIITLEE$) distribution. The density function of $TIITLEE$ model has right-skewed and symmetrical shapes. Descriptive properties such as quantile function, moments, incomplete moments, probability weighted moments, moment generating functions, characteristics function, and Renyi and $\boldsymbol{\rho}$ -entropies are theoretically established. Parameters of $TIITLEE$ distribution are estimated using maximum likelihood method. The potentiality/tractability of $TIITLEE$ distribution is demonstrated by its applications to two lifetime data

Keywords: quantile, Renyi and Shannon entropies, probability weighted moments, incomplete moments

1. Introduction

A well-known cumulative distribution function which was used during the first half of the nineteenth century by Gompertz (1825) and Verhulst (1838, 1845, 1847) to describe human mortality tables and to represent population growth is given by

$$G(x) = \left(1 - \alpha e^{-(\lambda x)}\right)^{\nu} \quad ; \; x > \frac{1}{\lambda} ln\alpha \tag{1}$$

where v, α and λ are all positive real numbers according to Ahuja and Nash, (1967). The exponentiated exponential distribution, also known as the generalized exponential distribution, is defined as a particular case of Gompertz-Verhulst distribution function (1) when $\alpha = 1$. Therefore, X is a two parameter exponentiated exponential random variable if it has the following distribution function;

$$G(x) = \left(1 - e^{-(\lambda x)}\right)^{\nu} \qquad ; x > 0 \tag{2}$$

for $v, \lambda > 0$. Here v and λ are the shape and scale parameters respectively. Note that the exponentiated exponential distribution developed by Nadarajah and Kotz (2006), is a particular case of the exponentiated Weibull distribution, Mudholkar and Srivastava (1993). Gupta and Kundu (1999) observed that the two-parameter exponentiated exponential distribution can be used in analyzing several lifetime data effectively, particularly in place of two-parameter gamma or two-parameter Weibull distribution. If the shape parameter is one, then all the three distributions coincide with the one parameter exponential distribution. Therefore, all the three distributions can be considered to be a generalization of the exponential distribution in different ways. Several generalizations of exponential distribution have been studied over time, and this includes: Harris Extended Exponential studied by Pinho et al., (2015), the Marshall-Olkin exponential developed by Marshall and Olkin (1997), Kumaraswamy transmuted exponential proposed by Afify et. al., (2016), exponentiated exponential discussed by Gupta and Kundu (2001), Extended Exponential investigated by Lemonte et. al., (2016), α-power transformed generalized exponential developed by Dey et. al., (2017), Odd Exponentiated Half-Logistic Exponential proposed and studied by Aldahlan and Afify (2020), Exponentiated Additive Weibull studied by Ahmad and Ghazal (2020), Exponentiated Weibull-Exponential developed by Elgarhy et. al., (2017), Marshall-Olkin logistic-exponential by Mansoor et al. (2017), Extended Odd Weibull Exponential (EOWEx) investigated by Afify and Mohamed. (2020), modified exponential by Rasekhi et al. (2017), Marshall-Olkin alpha power exponential by Nassar et al., (2019), odd log-logistic Lindley exponential by Alizadeh et al., (2019), and Bimodal Exponential by Jimmy et. al., (2021) are distributions are extensions of the exponential distribution frequently used in survival analysis. Exponentiated distributions are obtained by exponentiating existing distributions. Because they have more parameters, their model fits are better compared to baseline distributions. The idea of exponentiated distributions was first introduced by Lehmann

E.L. (2012). Exponentiated gamma, exponentiated Weibull, exponentiated Gumbel, and exponentiated Frechet distributions are members of the class of distributions developed using exponentiation. Nadarajah and Kotz (2006). One of the widely used exponentiated distributions is the exponentiated exponential (EE) distribution studied by Nadarajah and Kotz (2006).

1.1 Motivation of Study

The main purpose of extending the Exponentiated Exponential distribution is to give room for better flexibility of the Exponentiated Exponential distribution to be able to model data showing different shapes of the hazard function which can be non-monotonic hazard rate, such as the bathtub, unimodal and modified unimodal hazard rate. Several modifications of the Exponential distribution achieved to described the purpose. On the other hand, unfortunately, the number of parameters has increased, the forms of the survival and hazard functions have been complicated and estimation problems have risen. This study present a simple closed form expression for an extended exponentiated exponential distribution.

1.2 EE Distribution; A Brief Review

The EE distribution is an extension of the Weibull family and was developed by Mudholkar and Srivastava (1993). The EE distribution exhibits a non-monotone failure rate which makes it a reliable model in modeling lifetime data. Mudholkar and Srivastava (1993), Mudholkar and Huston (1996), Gupta and Kundu (2001), Nassar and Eissa (2003) and Choudhury (2005) applied the EE model to modeling reliability and survival data. The random variable X follows an EE distribution if its cumulative density function (cdf) is given by

$$G(x;\lambda,\nu)=\left(1-e^{-(\lambda x)}\right)^{\nu} \quad ,x>0 \tag{3}$$

Where ν is a positive shape parameter and λ is a positive scale parameter. The associated probability density function (pdf) corresponding to (3) is given as

$$g(x;\lambda,\nu)=\nu\lambda e^{-(\lambda x)}\left(1-e^{-(\lambda x)}\right)^{\nu-1} \quad ,x>0 \tag{4}$$

The Survival function, $S(x;\lambda,\nu)$ and hazard rate function, $h(x;\lambda,\nu)$ functions of the EE distribution are respectively given as

$$s(x;\lambda,\nu)=1-\left(1-e^{-(\lambda x)}\right)^{\nu-1} . \tag{5}$$

And

$$h(x;\lambda,\nu)=\frac{\nu\lambda e^{-(\lambda x)}\left(1-e^{-(\lambda x)}\right)^{\nu-1}}{1-(1-e^{-(\lambda x)})^{\nu-1}}. \tag{6}$$

2. The Type II Topp-Leone Exponentiated Exponential Distribution

Using Elgarhy et al. (2018), the pdf of $TIITLEE$ distribution is represented by

$$\boldsymbol{f}(x;\rho,\lambda,\nu)=\lambda\rho\nu e^{-(\lambda x)}\left(1-e^{-(\lambda x)}\right)^{2\nu-1}\left[1-\left(1-e^{-(\lambda x)}\right)^{2\nu}\right]^{\rho-1} \tag{7}$$

The corresponding pdf to (7), is given by

$$F(x;\rho,\lambda,\nu)=1-\left[1-\left(1-e^{-(\lambda x)}\right)^{2\nu}\right]^{\rho}, \tag{8}$$

Where ρ and ν are positive shape parameters and λ is a positive scale parameter. The graph of the cdf and the pdf for various values of the parameters of the distribution are given in figures 1.

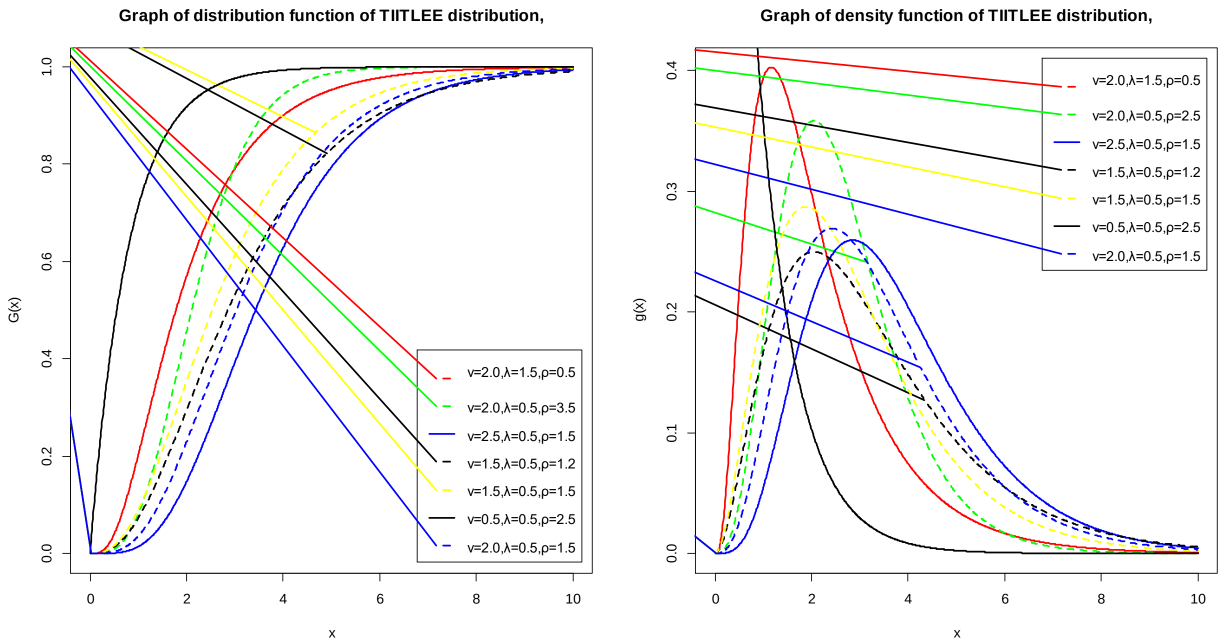

Figure 1. Graph of the distribution and density functions of $\boldsymbol{TIITLEE}$ distribution

An expression for the Reliability, hazard, reversed, and cumulative hazard functions is respectively given by

$$S(x;\rho,\lambda,v)=\left[1-\left(1-e^{-(\lambda x)}\right)^{2v}\right]^{\rho}, \tag{7}$$

$$h(x;\rho,\lambda,v)=\frac{\lambda\rho v e^{-(\lambda x)}\left(1-e^{-(\lambda x)}\right)^{2v-1}\left[1-\left(1-e^{-(\lambda x)}\right)^{2v}\right]^{\rho-1}}{[1-(1-e^{-(\lambda x)})^{2v}]^{\rho}}, \tag{8}$$

$$\Phi(x;\rho,\lambda,v)=\frac{\lambda\rho v e^{-(\lambda x)}\left(1-e^{-(\lambda x)}\right)^{2v-1}\left[1-\left(1-e^{-(\lambda x)}\right)^{2v}\right]^{\rho-1}}{1-[1-(1-e^{-(\lambda x)})^{2v}]^{\rho}}, \tag{9}$$

And

$$H(x;\rho,\lambda,v)=log\left(1-\left[1-\left(1-e^{-(\lambda x)}\right)^{2v}\right]^{\rho}\right). \tag{10}$$

2.1 Important Representation

In this subsection, an important tool for the expansion of the pdf and cdf for $TIITLEE$ is provided. From the generalized binomial series given by

$$(1-k)^{c}=\sum_{i=0}^{\infty}(-1)^{i}\binom{b}{i}k^{i} \tag{11}$$

For $|k|<1$ and c is a positive real non-integer. Then, by applying the binomial theorem (11) in (7) the density function of $TIITLEE$ distribution can be written as

$$g(x;\rho,\lambda,v)=2\rho v\lambda\sum_{i,j,k=0}^{\infty}(-1)^{i+j}\binom{\rho-1}{i}\binom{2v(i+1)-1}{j}e^{-(j+1)\lambda x} \tag{12}$$

This indicates that the $TIITLEE$ model can be written as an infinite mixture of the Exponential model.

2.2 The Quantiles, Median and the Upper Quartile

A mathematical expression for the quantile and the median of $TIITLEE$ model is obtained in this subsection.

The quantile x_u of the $TIITLEE$ is given as

$$x_u = \frac{1}{\lambda}\left(-log\left[1-\left(1-(1-u)^{1/\rho}\right)^{1/2v}\right]\right) \tag{13}$$

The median and the upper quartile of $TIITLEE$ are found by putting $q = 0.5$ and 0.75 in (13), respectively, as follows:

$$x_{0.5} = \frac{1}{\lambda}\left(-log\left[1-\left(1-(0.5)^{1/\rho}\right)^{1/2v}\right]\right) \tag{14}$$

and

$$x_{0.75} = \frac{1}{\lambda}\left(-log\left[1-\left(1-(0.25)^{1/\rho}\right)^{1/2v}\right]\right) \tag{15}$$

3. The r^{th} Ordinary Moment

If $X \sim TIITLEE(\gamma)$, then the r^{th} moment of X can be derived using

$$\mu_r' = E(X^r) = \int_0^\infty x^r g(x)dx. \tag{16}$$

By substituting from (12) in (17), we obtain the r^{th} moment as follows

$$\mu_r' = 2\rho v \sum_{i,j,k=0}^{\infty} (-1)^{i+j}\binom{\rho-1}{i}\binom{2v(i+1)-1}{j}(j+1)^{-(1+r)}\lambda^{-r}\Gamma(1+r) \tag{17}$$

Where $\Gamma(1+r)$ is a gamma function. By setting $r = 1$ in (17), we obtain the mean of X as

$$\mu_1' = 2\rho v \sum_{i,j,k=0}^{\infty} (-1)^{i+j}\binom{\rho-1}{i}\binom{2v(i+1)-1}{j}(j+1)^{-(1+1)}\lambda^{-1}\Gamma(1+1) \tag{18}$$

3.1 The r^{th} Incomplete Moments

If $X \sim TIITLEE(\gamma)$, then the r^{th} incomplete moments of X can be derived using

$$\varsigma_r(t) = \int_0^t x^r g(x)dx. \tag{19}$$

By substituting from (12) in (19), we obtain the r^{th} moment as follows

$$\varsigma_r(t) = 2\rho v \sum_{i,j,k=0}^{\infty} (-1)^{i+j}\binom{\rho-1}{i}\binom{2v(i+1)-1}{j}(j+1)^{-(1+r)}\lambda^{-r}\Gamma((1+r),(j+1)(\lambda t)) \tag{20}$$

Where $\Gamma((1+r),(j+1)(\lambda t))$ is an incomplete gamma function. Bet setting $r = 1$ in (20), we obtain the first incomplete moment of $TIITLEE$ model as

$$\varsigma_1(t) = 2\rho v \sum_{i,j,k=0}^{\infty} (-1)^{i+j}\binom{\rho-1}{i}\binom{2v(i+1)-1}{j}(j+1)^{-(1+1)}\beta^{-1}\Gamma(1+1,(j+1)(\lambda t)) \tag{21}$$

3.2 Moment Generating Function (MGF)

The MGF of $TIITLEE(\zeta)$, say $M_X(t)$ is found using

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tX} f(x)dx = \sum_{v=0}^{\infty}\frac{t^r}{r!}E(X^r) \tag{22}$$

Substituting (18) into (22), we obtain

$$M_X(t) = 2\rho v \sum_{i,j,r=0}^{\infty} \frac{t^r}{r!} (-1)^{i+j} \binom{\rho - 1}{i} \binom{2v(i+1) - 1}{j} (j+1)^{-(1+1)} \lambda^{-r} \Gamma(1+1) \qquad (23)$$

3.3 Characteristics Function

The characteristic function can be derived by replacing t with it in (23). Thus, the characteristic moments for $TIITLEE$ distribution is given as

$$\varphi_X t = E(e^{itx}) = \sum_{r=1}^{\infty} \frac{(it)^r}{r!} E(X^r)$$

Then we obtain

$$\varphi_X t = 2\rho v \sum_{i,j,r,1}^{\infty} \frac{(it)^r}{r!} (-1)^{i+j} \binom{\rho - 1}{i} \binom{2v(i+1) - 1}{j} (j+1)^{-(1+r)} \ \lambda^{-r} \Gamma(1+1) \ (24)$$

3.4 The Probability Weighted Moment (PWM)

Taking the expectation of a function of X, which can be used to obtain the parameters of a certain distribution for which the inverse form can be obtained, this is defined as the probability-weighted moment (PWM). The PWM of X cdf, $G(x)$, say $\zeta_{r,k}$, is obtained by

$$\zeta_{r,s} = E(x^r F^k(x)) = \int_0^{\infty} x^r G^k(x) g(x) dx \qquad (25)$$

If X $\sim TIITLEE(\zeta)$, then $\zeta_{r,k}$ is given by

$$\zeta_{r,k} = 2v\rho\lambda^{-r} \sum_{i,j}^{\infty} (-1)^{i+j} \binom{\rho(i+j) - 1}{i} \binom{2v(i+1) - 1}{j} (1+j)^{1-r-1} \Gamma(r+1) \quad (26)$$

3.5 Rényi Entropy Function and $\boldsymbol{\rho}$ *−Entropy*

The entropy function can be used to evaluate the level randomness or uncertainty related to X whose pdf $g(x)$. It plays a fundamental role in reliability, engineering, and others. The Rényi entropy of X, say $I_\rho(X)$, is determined by

$$I_d = \frac{1}{1-d} log \int_{-\infty}^{\infty} g^d(x) dx, \qquad (27)$$

If X $\sim TIITLEE$ (ζ), then $I_\delta(X)$ is obtained by

$$I_d = \frac{1}{1-d} log\left(2^d \rho^d v^d \theta^{d-1} \lambda^{d-1} H^* \Gamma(((d-1)+1)\right). \qquad (28)$$

where

$$H^* = \sum_{l=p=0}^{\infty} (-1)^{l+d} \binom{d(\rho - 1)}{l} \binom{2v(l+d) - d}{d} (d+p)^{-1}$$

Consequently, the p-entropy of X, say $H_\rho(X)$ is given by

$$H_d(X) = \frac{1}{1-d} log[1 - (1-d) I_d(X)]. \qquad (29)$$

Where an expression for $I_d(X)$ can be found in (29).

3.6 Stress Strength Reliability

Here, we derived an expression for the stress-strength parameter of $TIITLEE$ distribution. Suppose X_1 stand for the strength of a structure with stress X_2, and if X_1 follows $TIITLEE$ (v_1, ρ_1, θ) and X_2 follows $TIITLEE$ $(v_2,, \rho_2)$, given that $X_1$and X_2 are independent random variables, then the Stress-strength Reliability ($\mathfrak{K}$) of $TIITLEE$ is obtained as follows:

$$\mathfrak{K} = P(X_2 < X_1) = \int_0^\infty g_1(x; v_1, \lambda, \rho_1) G_2(x; v_2, \lambda, \rho_2) dx \tag{30}$$

If $X \sim TIITLEE(\zeta)$, then $\mathfrak{K}$ is given by

$$\mathfrak{K} = P(X_2 < X_1) = G_1(x; v_1, \lambda, \rho_1) - G^{i,j,k}$$

Where

$$G^{i,j,k} = 2v_1\rho_1 \sum_{i=0}^{\rho_1-1} \sum_{j=0}^{\rho_2} \sum_{k=0}^{\infty} \binom{\rho_1 - 1}{i} \binom{\rho_2}{j} \binom{2[v_1(i+1) + v_2 j] - 1}{k} (-1)^{i+j+k}$$

4. Maximum Likelihood Estimator of $TIITLEE$ Distribution

Let $X_1, X_2, \dots, X_n$ be a random sample drawn from $TIITLEE$ (v, λ, ρ) then the log-likelihood function is given by

$$l = \log(2) + \log(v) + \log(\rho) + \log(\lambda) - \sum_{i=1}^{n}(-\lambda x_i) + (2v - 1)\sum_{i=1}^{n} \log(1 - e^{-\lambda x_i})$$

$$+(\rho - 1)\sum_{i=1}^{n} \log\left(1 - \left(1 - e^{-\lambda x_i}\right)^{2v}\right) \tag{31}$$

We differentiate (31) with respect (v, β, ρ) to obtain the element of the score vector $\left(\rho_v = \frac{\partial l}{\partial v}, \rho_\lambda = \frac{\partial l}{\partial \lambda}, \rho_\rho = \frac{\partial l}{\partial \rho}\right)^T$.

The elements of the score vector is given by

$$\frac{dl}{dv} = \frac{n}{v} + 2\sum_{i=1}^{n} \log\left(1 - e^{-\lambda x_i}\right) + 2(\rho - 1)\sum_{i=1}^{n} \frac{\left(1 - e^{-\lambda x_i}\right)^{2v} \log(1 - e^{-\lambda x_i})}{(1 - (1 - e^{-\lambda x_i})^{2v})} \tag{32}$$

$$\frac{dl}{d\lambda} = \frac{n}{\lambda} + \lambda log\lambda - \lambda\sum_{i=1}^{n} \log(\lambda)\, x + 2(v - 1)\lambda \sum_{i=1}^{n} \frac{xe^{-\beta\lambda} \log(\lambda)}{(1 - e^{-\lambda x_i})}$$

$$+(\rho - 1)\sum_{i=1}^{n} \frac{\lambda \log(\lambda)\, x^\theta e^{-\lambda x_i}\left(1 - e^{-\lambda x_i}\right)^{2v}}{(1 - (1 - e^{-\beta\lambda})^{2v})(1 - e^{-\lambda x_i})} \tag{34}$$

$$\frac{dl}{d\rho} = \frac{n}{\rho} - \sum_{i=1}^{n} log\left[\left(1 - \left(1 - e^{-\beta x_i}\right)^{2v}\right)\right] \tag{35}$$

4.1 Applications of $\boldsymbol{TIITLEE}$ *Model to Lifetime Data*

In this section, the $TIITLEE$ model is compared with Exponentiated Exponential (EE), Type II and Exponential distributions. Different goodness of fit measures like Cramer-von Mises (W), Anderson Darling (A), Akaike Information Criterion (AIC), consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), and Hannan-Quinn Information Criterion (HQIC). Two lifetime data were used to demonstrate the flexibility of TIITLEE model over all other models considered in the study. Data I consist 65 of consecutive eruption of the Kiama Blowhole and had been used by Pinho et al. (2012). The second data sets is the failure times of 84 Aircraft windshield taken from Tahir et al. (2015). Table 1 gives the exploratory data analysis of the data set I, which shows that the data is positively skewed, over-dispersed and leptokurtic. Figure 2 shows that the data exhibits an increasing failure rate. Table 3 shows the exploratory data analysis for data set II which shows that the data is moderately positively skewed, under-dispersed and mesokurtic.

Table 1. Exploratory data analysis for data set I

$Min.$	$Max.$	$Median$	q_1	$Mean$	q_3	$Skewness$	$Kurtosis$	$Var.$
7.00	169.00	28.00	14.75	39.83	60.00	1.55	5.77	1,139.10

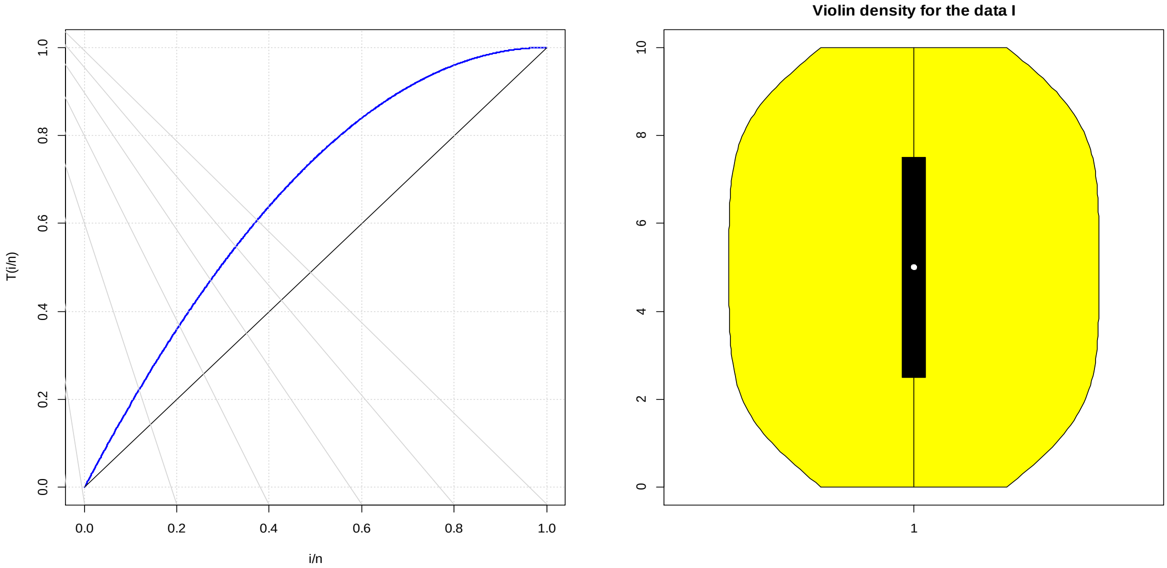

Figure 2. TTT and Violin Plots for data I

Table 2. Measures of Goodness of Fit for Data I

$Model$	A	B	C	$-l$	AIC	$CAIC$	$HQIC$	BIC	W
$TIITLEE$	2.483 (0.044)	0.216 (0.002)	0.131 (0.017)	292.71	591.41	591.81	593.96	597.89	0.088
EE	1.739 (0.322)	0.035 (0.005)	$-$ $(-)$	295.67	595.33	595.53	597.04	599.65	0.129
E	$-$ $(-)$	0.025 (0.003)	$-$ $(-)$	299.81	601.63	601.69	602.48	603.78	0.132

Table 3. Exploratory data analysis for data set II

$Min.$	$Max.$	$Median$	q_1	$Mean$	q_3	$Skewness$	$Kurtosis$	$Var.$
0.04	4.663	2.385	1.866	2.563	3.376	0.087	2.365	1.239

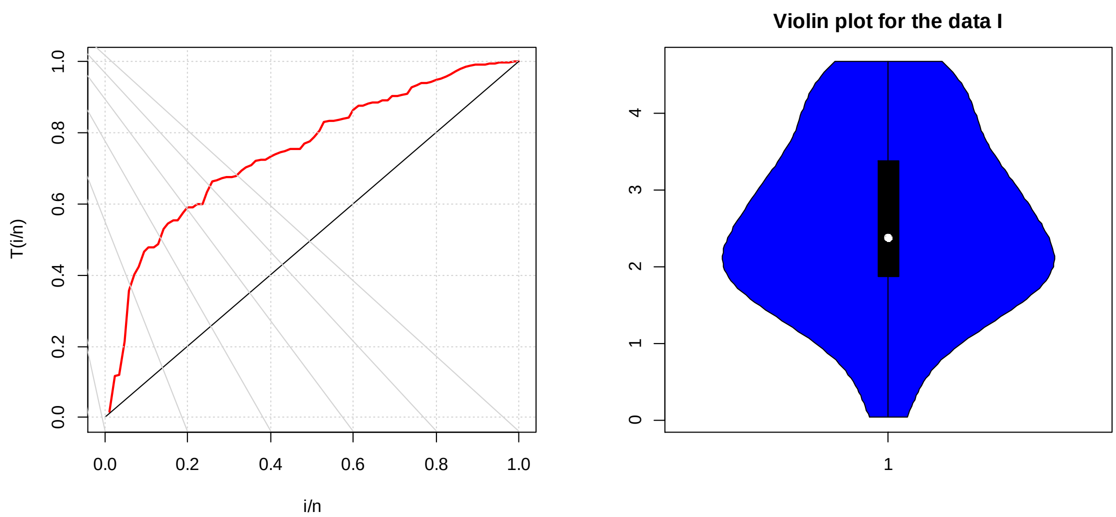

Figure 3. TTT and Violin Plots for data II

Table 4. Measures of goodness of fit for data II

$Model$	A	B	C	$-l$	AIC	$CAIC$	$HQIC$	BIC	W
$TIITLEE$	1.427 (0.214)	0.240 (0.134)	7.214 (6.966)	134.61	275.22	275.51	278.16	282.54	0.095
EE	3.590 (0.612)	0.760 (0.076)	− (−)	141.40	286.79	286.94	288.76	291.68	0.219
E	− (−)	0.390 (0.042)	− (−)	164.98	331.98	332.02	332.96	334.42	0.166

It could be observed from Table 2 and 4 that the TIITLEE possess the smallest AIC, BIC, CAIC, HQIC and W, as a result it can be regarded as providing a better than the exponentiated exponential and exponential distribution for the two lifetime data sets considered in this study.

5. Conclusion

A new three-parameter distribution called the Type II Topp-Leone Exponentiated Exponential distribution is developed. A characteristic of the distribution is that its failure rate function can be decreasing, increasing, bathtub-shaped and unimodal depending on its parameter values. Several statistical properties of the new distribution such as its probability density function, its cumulative density function, quantiles, moments, incomplete moments, moments generating functions, probability weighted moments, stress-strength reliability function, Renyi and ρ-entropies are obtained. Fitting the Type II Topp-Leone Exponentiated Exponential model to two lifetime data sets indicates the model is flexible and provides a better fit than other models considered in the study.

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Author’s Contribution

Victoria E. Laoye was responsible for study design and revising and also for data collection. She drafted the manuscript and also revised it. Author read and approved the final manuscript.

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Competing interests

The author declare that there is no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability statement

The data that support the findings of this study are available on request from the corresponding author.