Journal of Mathematics Research; Vol. 16, No. 6; December 2024
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Integer Cordial Labelling on Graphs $nF_{m_i}(n-1)P_t$ With m_i Even or Odd

Nur Wigi Saputra[1*], Siti Khabibah[2], Lucia Ratnasari[3], Robertus Heri Soelistyo Utomo[4]

[1] Mathematics Undergraduate Study Program, Faculty of Science and Mathematics, Universitas Diponegoro, Semarang, Indonesia

[2,3,4] Department of Mathematics, Faculty of Science and Mathematics, Universitas Diponegoro, Semarang, Indonesia

Correspondence: Siti Khabibah, Department of Mathematics, Faculty of Science and Mathematics, Universitas Diponegoro, Semarang, Indonesia. E-mail:khabibah.undip@gmail.com

Received: November 19, 2024 Accepted: January 8, 2025 Online Published: January 25, 2024

doi:10.5539/jmr.v16n6p29 URL: https://doi.org/10.5539/jmr.v16n6p29

Abstract

This paper discusses the integer cordial labeling on the graph $nF_{m_i}(n-1)P_t$ when m_i is even or odd. The graph $nF_m(n-1)P_t$ is obtained from n friendship graphs connected by $n-1$ paths P_t at vertices other than the central vertex of the graph for the i-th and $i+1$-th graphs where $1 \leq i \leq n$. Additionally, m_i denotes the number of copies of the cycle C_3 for each friendship graph. The main result is the integer cordial label for this graph. Furthermore, it was proved that $nF_m(n-1)P_t$ when m_i even or odd is a graph with integer cordial labeling.

Keywords: Integer Cordial Labelling, Friendship Graph, Path P_t, $nF_{m_i}(n-1)P_t$ Graph

1. Introduction

Graph theory was first introduced by Leonhard Euler in 1736 when Euler considered how to cross all the bridges in Konigsberg (now Kaliningrad, Russia) exactly once and return to the original place. Since then, graph theory has continued to evolve. Graph theory is a branch of discrete mathematics with a set of objects represented by vertexs (vertices or vertexs) and lines between these objects represented by edges (arcs or edges). Graph G, consisting of vertexs and edges, is sequentially symbolized by $V(G)$ and $E(G)$ (Chartrand & Zhang, 2012). One of the topics in graph theory, namely graph labeling, is mapping the elements in a graph consisting of a vertex set and/or an edge set into a set of numbers that are usually positive integers. If the function's domain is a set of vertexs, then the term used is vertex labeling, while if the function's domain is a set of edges, then the term used is edge labeling. In addition, if the function's domain includes $V(G) \cup E(G)$, the term used is total labeling. (Wallis, 2007). Graphs have various types of labeling; one type that has been widely researched and continues to develop is cordial labeling.. A graph with cordial labeling is called cordial graph. The concept of cordial graph originated from Cahit (Cahit, n.d.) in 1987 as a weaker version of graceful and harmonious graphs and was based on $\{0,1\}$ binary labeling of vertices. One of the extensions of cordial labeling is integer cordial labeling.

Sarah Surya has defined integer cordial labeling on a graph G. A graph $G(V,E)$ is said to have an integer cordial labeling if there exists an injective mapping f from the set of vertexs $V(G)$ to the set $\left\{-\frac{|V(G)|}{2}, -\frac{|V(G)|}{2}+1, \dots, -1,0,1, \dots, \frac{|V(G)|}{2}-1, \frac{|V(G)|}{2}\right\}$ for $|V(G)|$ even or $\left\{-\left\lfloor\frac{|V(G)|}{2}\right\rfloor, -\left\lfloor\frac{|V(G)|}{2}\right\rfloor+1, \dots, -1,0,1, \dots, \left\lfloor\frac{|V(G)|}{2}\right\rfloor-1, \left\lfloor\frac{|V(G)|}{2}\right\rfloor\right\}$ for $|V(G)|$ odd. The function f^* from the set of edges $E(G)$ to the set $\{0,1\}$ is defined by $f^*(v_iv_j) = 1$ if $f(v_i)+f(v_j) \geq 0$ and $f^*(v_iv_j) = 0$ if $f(v_i)+f(v_j) < 0$, such that the number of edges labeled 1 and the number of edges labeled 0 differ by at most 1. If graph G has an integer cordial labeling, it is called an integer cordial graph (Sarah Surya et al., 2020). For more information about the concept of integer cordial labeling the reader can refer to (Gondalia et al., 2021), (Shah & Parmar, 2022), (Godhasara & Sonchhatra, 2013), (Parameswari, Saradata Prita, & Rajeswari, 2021).

2. Integer Cordial Labeling

Definition 1 Graph $G(V,E)$ with vertex set $V(G)$ and set of edges $E(G)$ is said to have cordial labeling if there exists a binary vertex mapping $f: V(G) \to \{0,1\}$ that induces labeling on the edge $v_i v_j$ expressed by $f^*: E(G) \to \{0,1\}$ and defined by $f^*(v_i v_j) = |f(v_i) - f(v_j)|$. The number of vertexs on G labeled 0 and 1 are denoted by $v_f(0)$ and $v_f(1)$. The number of edges in G labeled 0 and 1 are denoted by $e_f(0)$ and $e_f(1)$. A binary vertex labeling on graph G is called cordial labeling if it satisfies $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. (Cahit, n.d.).

Definition 2 *A graph $G(V,E)$ is said to have an integer cordial labeling if there exists an injective mapping f from the set of vertexs $V(G)$ to the set $\left\{-\frac{|V(G)|}{2}, -\frac{|V(G)|}{2}+1, \dots, -1, 0, 1, \dots, \frac{|V(G)|}{2}-1, \frac{|V(G)|}{2}\right\}$ for $|V(G)|$ even or $\left\{-\left\lfloor\frac{|V(G)|}{2}\right\rfloor, -\left\lfloor\frac{|V(G)|}{2}\right\rfloor+1, \dots, -1, 0, 1, \dots, \left\lfloor\frac{|V(G)|}{2}\right\rfloor-1, \frac{|V(G)|}{2}\right\}$ for $|V(G)|$ odd. The function f^* from the set of edges $E(G)$ to the set $\{0,1\}$ is defined by $f^*(v_i v_j) = 1$ if $f(v_i) + f(v_j) \geq 0$ and $f^*(v_i v_j) = 0$ if $f(v_i) + f(v_j) < 0$, such that the number of edges labeled 1 and the number of edges labeled 0 differ by at most 1. If graph G has integer cordial labeling (I.C.L.), then it is called integer cordial graph (I.C.G.).* (Sarah Surya et al., 2020).

Example 1 An image of an integer cordial labeling on graph C_5 is given.

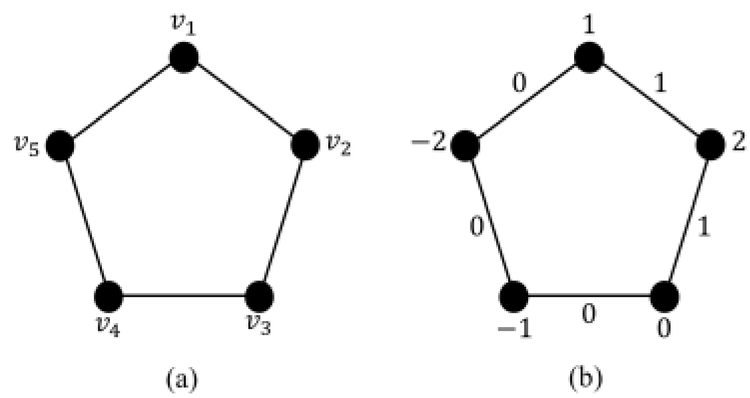

Figure 1. (a) Graph C_5, (b) Integer Cordial Labeling on Graph C_5

In Figure 1, the vertex labeling of the graph is defined as $f: V(C_5) \to \{-2,-1,0,1,2\}$ with vertex labels $f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 0$, $f(v_4) = -1$, and $f(v_5) = -2$. Furthermore, vertex labeling induces edge labeling. Define edge labeling $f^*: E(C_5) \to \{0,1\}$ with side labels as follows.

$f^*(v_1 v_2) = 1$, because $f(v_1) + f(v_2) = 3 \geq 0$.

In the same way, the other side label is:

$f^*(v_2 v_3) = 1$	$f^*(v_4 v_5) = 0$
$f^*(v_3 v_4) = 0$	$f^*(v_5 v_1) = 0$

It can be seen that $e_f(0) = 3$ and $e_f(1) = 2$. This shows that $|e_f(0) - e_f(1)| \leq 1$. Thus, it is proved that the graph C_5 is an integer cordial graph. The following discusses the definitions and theorems about integer cordial labeling on graphs $nF_{m_i}(n-1)P_t$ with m_i even or odd.

Definition 3 *Graph $nF_{m_i}(n-1)P_t$ is a graph obtained from n friendship graph connected by $n-1$ path P_t at a vertex other than the center vertex of the graph for the i-th graph and the $i+1$-th graph with $1 \leq i \leq n$. Additionally, m_i denotes the number of copies of the cycle C_3 for each friendship graph. The graph $nF_{m_i}(n-1)P_t$ has a vertex set*

$$V(nF_{m_i}(n-1)P_t) = \{u_1, u_2, \dots, u_n\} \cup \left(\bigcup_{i=1}^{n} \{v_1^i, v_2^i, \dots, v_{2m_i}^i\}\right) \cup \left(\bigcup_{j=1}^{n-1} \{d_2^j, d_3^j, \dots, d_{t-1}^j\}\right)$$

with $d_1^j = v_1^j$ and $d_t^j = v_2^{j+1}$ arranged anti-clockwise and the set of edges

$$E\left(nF_{m_i}(n-1)P_t\right) = \left(\cup_{i=1}^{n}\{u_iv_1^i, u_iv_2^i, \dots, u_iv_{2m_i}^i\} \cup \{v_1^iv_2^i, v_3^iv_4^i, \dots, v_{2m_i-1}^iv_{2m_i}^i\}\right) \cup \left(\cup_{j=1}^{n-1}\{v_1^jd_2^j, d_2^jd_3^j, \dots, d_{t-1}^jv_2^{j+1}\}\right)$$

with:

u_n = *n-th center vertexs on the friendship graph*

$v_{2m_i}^i$ = $2m_i$*-th vertexs other than the center vertex in the* i*-th friendship graph*

d_t^j = t*-th vertex on* j*-th path*

Example 2 Here is an example of a graph F_{m_1} with m_1 even and odd.

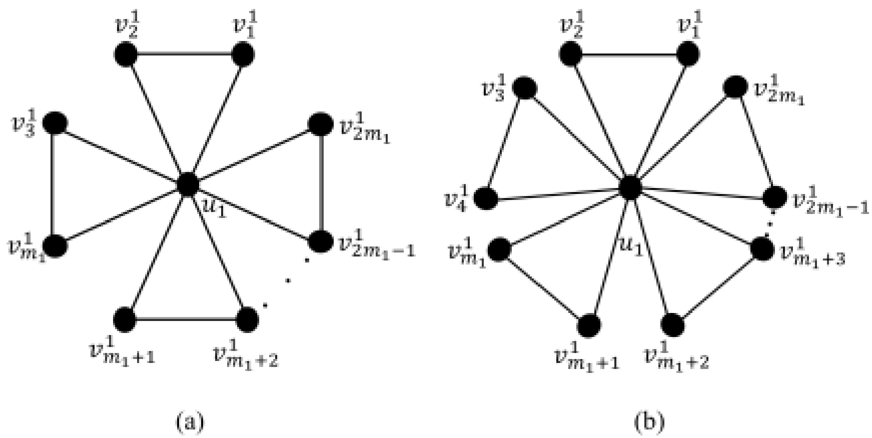

Figure 2. (a) Graph F_{m_1} with m_1 Even, (b) Graph F_{m_1} with m_1 Odd

Lemma 1 n *friendship graph on a graph* $nF_{m_i}(n-1)P_t$ *with* m_i *even or odd is an integer cordial graph.*

*Proof***.** Define the vertex labeling on the n friendship graph as follows.

1. To $n = 1$

$$\begin{aligned} &f(u_1) = 0 \\ &f(v_x^1) = x \\ &f\left(v_{m_1+x}^1\right) = -x \\ &\text{with } 1 \le x \le m_1 \end{aligned} \tag{1}$$

2. To $n > 1$

If n is even, then:

$$f(u_i) = \begin{cases} \frac{i+1}{2}, i = 1,3,5,\dots,n-1 \\ -\frac{i}{2}, i = 2,4,6,\dots,n \end{cases}$$

$$f\left(v_{x_i}^i\right) = \begin{cases} \frac{n}{2} + x_i + \sum_{r=0}^{i-1} m_r, i = 1,3,5,\dots,n-1 \\ -\left(\frac{n}{2} + x_i + \sum_{r=0}^{i-1} m_r\right), i = 2,4,6,\dots,n \end{cases} \tag{2}$$

$$f\left(v_{m_i+x_i}^i\right) = \begin{cases} -\left(\frac{n}{2} + x_i + \sum_{r=0}^{i-1} m_r\right), i = 1,3,5,\dots,n-1 \\ \frac{n}{2} + x_i + \sum_{r=0}^{i-1} m_r, i = 2,4,6,\dots,n \end{cases}$$

with $m_0 = 0$ and $1 \le x_i \le m_i$.

If n is odd, then:

$$f(u_i) = \begin{cases} \frac{i}{2}, i = 2,4,6, \dots, n-1 \\ -\left(\frac{i-1}{2}\right), i = 3,5,7, \dots, n \\ 0, i = 1 \end{cases}$$

$$f\left(v_{x_i}^i\right) = \begin{cases} \frac{n}{2} + x_i + \sum_{r=0}^{i-1} m_r, i = 1,3,5, \dots, n \\ -\left(\frac{n}{2} + x_i + \sum_{r=0}^{i-1} m_r\right), i = 2,4,6, \dots, n-1 \end{cases} \quad (3)$$

$$f\left(v_{m_i+x_i}^i\right) = \begin{cases} -\left(\frac{n}{2} + x_i + \sum_{r=0}^{i-1} m_r\right), i = 1,3,5, \dots, n-1 \\ \frac{n}{2} + x_i + \sum_{r=0}^{i-1} m_r, i = 2,4,6, \dots, n \end{cases}$$

with $m_0 = 0$ and $1 \le x_i \le m_i$.

Based on Lemma 1, the center vertex label has the smallest number of labels compared to other vertex labels. Vertexs other than the center vertex have labels with positive integers as many as $\sum_{i=1}^{n} m_i$ and negative integers as many as $\sum_{i=1}^{n} m_i$.

The edge labelling on n friendship graph is as follows.

1. To $n = 1$

   If m_1 is even, the $\frac{3m_1}{2}$ edge has label 1, and the $\frac{3m_1}{2}$ edge has label 0. Also, if m_1 is odd, the $\left\lfloor\frac{3m_1}{2}\right\rfloor + 1$ edge has label 1, and the $\left\lfloor\frac{3m_1}{2}\right\rfloor$ edge has label 0.

2. To $n > 1$

   a. If m_i is even, then the $\sum_{i=1}^{n} \frac{3m_i}{2}$ edge in the graph has label 1, and the $\sum_{i=1}^{n} \frac{3m_i}{2}$ edge has label 0.

   b. If m_i is odd:

      1) If the graph is even in number, then $\sum_{i=1}^{n/2}\left(\left\lceil\frac{3m_{2i-1}}{2}\right\rceil + \left\lfloor\frac{3m_{2i}}{2}\right\rfloor\right)$ edge has label 1 and $\sum_{i=1}^{n/2}\left(\left\lfloor\frac{3m_{2i-1}}{2}\right\rfloor + \left\lceil\frac{3m_{2i}}{2}\right\rceil\right)$ the edge has label 0.

      2) If the graph is odd in number, then $\sum_{i=1}^{(n-1)/2}\left(\left\lceil\frac{3m_{2i-1}}{2}\right\rceil + \left\lfloor\frac{3m_{2i}}{2}\right\rfloor\right) + \left\lfloor\frac{3m_n}{2}\right\rfloor + 1$ edge has label 1, and $\sum_{i=1}^{(n-1)/2}\left(\left\lfloor\frac{3m_{2i-1}}{2}\right\rfloor + \left\lceil\frac{3m_{2i}}{2}\right\rceil\right) + \left\lfloor\frac{3m_n}{2}\right\rfloor$ edge has label 0.

Since the number of edges labeled 1 and the number of edges labeled 0 differ by at most 1 in all possibilities, it is proven that the n friendship graph on the graph $nF_{m_i}(n-1)P_t$ with m_i even or odd is an integer cordial graph.

Lemma 2 *$n-1$ path P_t on the graph $nF_{m_i}(n-1)P_t$ with m_i even or odd is an integer cordial path.*

*Proof***:** Define vertex labeling on $n-1$ *path* P_t, as follows.

1. Path with Length One

   Because $d_1^j = v_1^j$ and $d_2^j = v_2^{j+1}$, then $f\left(d_1^j\right) = f\left(v_1^j\right)$ and $f\left(d_2^j\right) = f\left(v_2^{j+1}\right)$.

2. Paths with Even Length

   a. To n Even

      If $n = 2$, then:

$$f(d_s^1) = \begin{cases} f\left(v_{2m_2}^2\right) + (s-1), 2 \le s < \frac{t+1}{2} \\ -f\left(d^1_{s-\left(\frac{t+1}{2}-1\right)}\right), \frac{t+1}{2} < s \le t-1 \\ 0, s = \frac{t+1}{2} \end{cases} \tag{4}$$

If $n = 4$, then:

$$f(d_s^1) = f\left(v_{2m_4}^4\right) + (s-1), 2 \le s \le t-1$$
$$f(d_s^2) = \begin{cases} f(d_s^1) + (t-2), 2 \le s < \frac{t+1}{2} \\ -f\left(d^2_{s-\left(\frac{t+1}{2}-1\right)}\right), \frac{t+1}{2} < s \le t-1 \\ 0, s = \frac{t+1}{2} \end{cases} \tag{5}$$
$$f(d_s^3) = -f(d_s^1), 2 \le s \le t-1$$

If $n > 4$, then:

$$f(d_s^1) = f\left(v_{2m_n}^n\right) + (s-1), 2 \le s \le t-1$$
$$f\left(d_s^j\right) = \begin{cases} f\left(d_s^{j-1}\right) + (t-2), 2 \le j \le \left(\frac{n}{2}-1\right), 2 \le s \le t-1 \\ -f\left(d_s^{j-\frac{n}{2}}\right), \left(\frac{n}{2}+1\right) \le j \le (n-1), 2 \le s \le t-1 \end{cases}$$
$$f\left(d_s^{n/2}\right) = \begin{cases} f\left(d_s^{\frac{n}{2}-1}\right) + (t-2), 2 \le s < \frac{t+1}{2} \\ -f\left(d^{n/2}_{s-\left(\frac{t+1}{2}-1\right)}\right), \frac{t+1}{2} < s \le t-1 \\ 0, s = \frac{t+1}{2} \end{cases} \tag{6}$$

b. To n Odd

If $n = 3$, then:

$$f(d_s^1) = f\left(v_{m_3}^3\right) + (s-1)$$
$$f(d_s^2) = -f(d_s^1) \tag{7}$$
with $2 \le s \le t-1$.

If $n > 3$, then:

$$f(d_s^1) = f\left(v_{m_n}^n\right) + (s-1)$$
$$f\left(d_s^j\right) = \begin{cases} f\left(d_s^{j-1}\right) + (t-2), 2 \le j < \left(\frac{n+1}{2}\right) \\ -f\left(d_s^{j-\frac{n-1}{2}}\right), \left(\frac{n+1}{2}\right) \le j \le (n-1) \end{cases} \tag{8}$$
with $2 \le s \le t-1$.

In addition, because $d_1^j = v_1^j$ and $d_t^j = v_2^{j+1}$, then $f\left(d_1^j\right) = f\left(v_1^j\right)$ and $f\left(d_t^j\right) = f\left(v_2^{j+1}\right)$.

3. Path P_t with Odd Length

a. To n Even

If $n = 2$, then:

$$f(d_s^1) = \begin{cases} f\left(v_{2m_2}^2\right) + (s-1), 2 \le s \le \frac{t}{2} \\ -f\left(d^1_{s-\left(\frac{t}{2}-1\right)}\right), \frac{t}{2} < s \le t-1 \end{cases} \tag{9}$$

If $n = 4$, then:

$$f(d_s^1) = f\left(v_{2m_4}^4\right) + (s-1), 2 \le s \le t-1$$
$$f(d_s^2) = \begin{cases} f(d_s^1) + (t-2), 2 \le s \le \frac{t}{2} \\ -f\left(d_{s-\left(\frac{t}{2}-1\right)}^2\right), \frac{t}{2} < s \le t-1 \end{cases} \tag{10}$$
$$f(d_s^3) = -f(d_s^1), 2 \le s \le t-1$$

If $n > 4$, then:

$$f(d_s^1) = f\left(v_{2m_n}^n\right) + (s-1), 2 \le s \le t-1$$
$$f\left(d_s^j\right) = \begin{cases} f\left(d_s^{j-1}\right) + (t-2), 2 \le j \le \left(\frac{n}{2}-1\right), 2 \le s \le t-1 \\ -f\left(d_s^{j-\frac{n}{2}}\right), \left(\frac{n}{2}+1\right) \le j \le (n-1), 2 \le s \le t-1 \end{cases} \tag{11}$$
$$f\left(d_s^{n/2}\right) = \begin{cases} f\left(d_s^{\frac{n}{2}-1}\right) + (t-2), 2 \le s \le \frac{t}{2} \\ -f\left(d_{s-\left(\frac{t}{2}-1\right)}^{\frac{n}{2}}\right), \frac{t}{2} < s \le t-1 \end{cases}$$

b. For n odd.

If $n = 3$, then:

$$f(d_s^1) = f\left(v_{m_3}^3\right) + (s-1)$$
$$f(d_s^2) = -f(d_s^1) \tag{12}$$
with $2 \le s \le t-1$.

If $n > 3$, then:

$$f(d_s^1) = f\left(v_{m_n}^n\right) + (s-1)$$
$$f\left(d_s^j\right) = \begin{cases} f\left(d_s^{j-1}\right) + (t-2), 2 \le j < \left(\frac{n+1}{2}\right) \\ -f\left(d_s^{j-\frac{n-1}{2}}\right), \left(\frac{n+1}{2}\right) \le j \le (n-1) \end{cases} \tag{13}$$
with $2 \le s \le t-1$.

In addition, because $d_1^j = v_1^j$ and $d_t^j = v_2^{j+1}$, then $f\left(d_1^j\right) = f\left(v_1^j\right)$ and $f\left(d_t^j\right) = f\left(v_2^{j+1}\right)$.

In Lemma 2, we can see the result of the edge labeling on the path P_t as follows.

1. For a path of length one.
   a. If the path *is* one, 1 vertex is labeled as a positive integer and 1 vertex as a negative integer. Therefore, the edge on the path is labeled 0.
   b. If there is more than one path, then $(n-1)$ vertexs are labeled with positive, and $(n-1)$ vertexs are labeled with negative integers. This results in:
      1) If the path is even in number, the $\frac{n-1}{2}$ edges have label 1, and the $\frac{n-1}{2}$ edge has label 0.
      2) If the path is odd in number, the $\frac{n}{2}-1$ edge has label 1, and the $\frac{n}{2}$ edge has label 0.
2. For paths of even length.
   a. If the path is one, then $\frac{(t-1)}{2}$ vertexs are labeled as positive integers, $\frac{(t-1)}{2}$ vertexs as negative integers, and 1 vertex as 0.
   b. If the path is more than one:
      1) If the path is an even number, then $\frac{(n-1)t}{2}$ vertexs are labeled positive integers, and $\frac{(n-1)t}{2}$ vertexs are labeled negative integers.

2) If the path is odd in number, $\frac{t(n-2)+(t-1)}{2}$ vertexs have a label of 1, $\frac{t(n-2)+(t-1)}{2}$ vertexs have a label of 0, and 1 vertex has a label of 0.

Thus, there are as many edges labeled 1 as $\frac{(n-1)(t-1)}{2}$ and as many edges labeled 0 as $\frac{(n-1)(t-1)}{2}$.

3. For paths of odd length.

a. If the path is one, then $\frac{t}{2}$ vertexs are positive, and $\frac{t}{2}$ vertexs have negative integer labels. So, the $\frac{t}{2}$ edge has label 1, and the $\frac{t}{2}-1$ edge has label 0.

b. If the path is more than one:

1) If the path is an even number, then $\frac{(n-1)t}{2}$ vertexs are labeled as positive integers and $\frac{(n-1)t}{2}$ vertexs are labeled with negative integers. So, the $\frac{(n-1)(t-1)}{2}$ edge has label 1, and the $\frac{(n-1)(t-1)}{2}$ edge has label 0.

2) If there is an odd number of paths, then $\frac{t(n-2)+t}{2}$ vertexs are labeled positive integers, and $\frac{t(n-2)+t}{2}$ vertexs are labeled negative integers. So the $\frac{(n-2)(t-1)+t}{2}$ edge has label 1 and $\frac{(n-2)(t-1)+t}{2}-1$ the edge has label 0.

Since the number of edges labeled 1 and the number of edges labeled 0 differ by at most 1 in all possibilities, it is proven that the $n-1$ path P_t on the graph $nF_{m_i}(n-1)P_t$ with m_i even or odd are integer cordial paths.

Theorem 1 *Graphs F_{m_1} with m_1 even or odd is an integer cordial graph.*

Proof: A function

$$f: V(F_{m_1}) \rightarrow \left\{-\left\lfloor\frac{|V(G)|}{2}\right\rfloor, -\left\lfloor\frac{|V(G)|}{2}\right\rfloor+1, \dots, -1,0,1, \dots, \left\lfloor\frac{|V(G)|}{2}\right\rfloor-1, \left\lfloor\frac{|V(G)|}{2}\right\rfloor\right\}$$

is defined by Equation (1). Furthermore, Lemma 1 shows that the number of edges labeled 1 and the number of edges labeled 0 differs by at most 1. Therefore, it has been sufficiently proved that the graph of $nF_{m_i}(n-1)P_t$ with $n=1$ and m_i is an integer cordial graph.

Example 3 Given a graph, F_4 is an integer cordial graph. The integer cordial labeling on the graph F_4 is illustrated in Figure 3

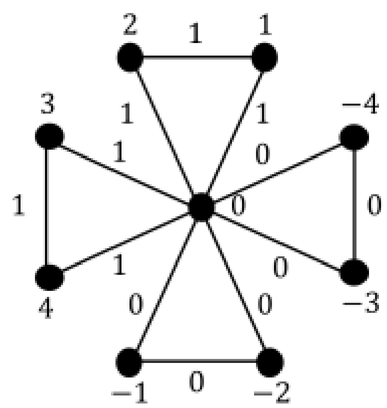

Figure 3. Integer Cordial Labeling of Graph F_4

Theorem 2 *Graphs $nF_{m_i}(n-1)P_t$ with m_i connected by paths of length one is an integer cordial graph.*

Proof: For n even, a function

$$f: V\left(nF_{m_i}(n-1)P_2\right) \rightarrow \left\{-\frac{|V(G)|}{2}, -\frac{|V(G)|}{2}+1, \dots, -1,0,1, \dots, \frac{|V(G)|}{2}-1, \frac{|V(G)|}{2}\right\}$$

defined by Equation (ii). Furthermore, for n odd, a function

$$f: V\left(nF_{m_i}(n-1)P_2\right) \rightarrow \left\{-\left\lfloor\frac{|V(G)|}{2}\right\rfloor, -\left\lfloor\frac{|V(G)|}{2}\right\rfloor+1, \dots, -1,0,1, \dots, \left\lfloor\frac{|V(G)|}{2}\right\rfloor-1, \left\lfloor\frac{|V(G)|}{2}\right\rfloor\right\}$$

is defined by Equation (3). Based on Lemma 1 and Lemma 2, for all possibilities, it can be seen that the number of edges labeled 1 and the number of edges labeled 0 differ by at most 1. Hence, it has been sufficiently proved that graphs $nF_{m_i}(n-1)P_t$ with m_i even or odd connected by paths of length one is an integer cordial graph.

Example 4 Given a graph $3F_{m_i}2P_2$ with $m_1 = m_2 = 2$ and $m_3 = 4$ is an integer cordial graph. The integer cordial labeling on graph $3F_{m_i}2P_2$ with $m_1 = m_2 = 2$ and $m_3 = 4$ is illustrated in Figure 4

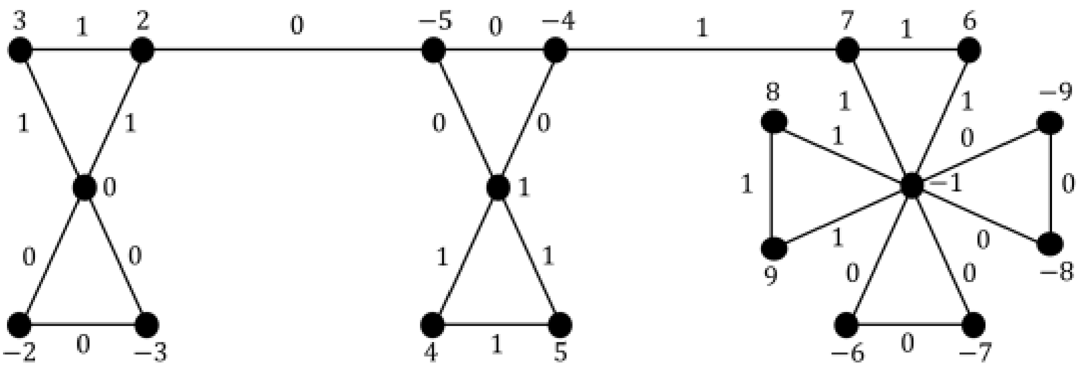

Figure 4. Integer Cordial Labeling of Graph $3F_{m_i}2P_2$ with $m_1 = m_2 = 2$ and $m_3 = 4$

Theorem 3 *Graphs $nF_{m_i}(n-1)P_t$ with m_i even or odd connected by paths of even length are integer cordial graphs.*

Proof: For n even, a function

$$f: V\left(nF_{m_i}(n-1)P_t\right) \rightarrow \left\{-\left\lfloor\frac{|V(G)|}{2}\right\rfloor, -\left\lfloor\frac{|V(G)|}{2}\right\rfloor+1, \dots, -1,0,1, \dots, \left\lfloor\frac{|V(G)|}{2}\right\rfloor-1, \left\lfloor\frac{|V(G)|}{2}\right\rfloor\right\}$$

defined by Equations (2), (4), (5), and (6). Furthermore, for n odd, a function

$$f: V\left(nF_{m_i}(n-1)P_t\right) \rightarrow \left\{-\left\lfloor\frac{|V(G)|}{2}\right\rfloor, -\left\lfloor\frac{|V(G)|}{2}\right\rfloor+1, \dots, -1,0,1, \dots, \left\lfloor\frac{|V(G)|}{2}\right\rfloor-1, \left\lfloor\frac{|V(G)|}{2}\right\rfloor\right\}$$

defined by Equations (3), (7), and (8). Based on Lemma 1 and Lemma 2, for all possibilities, it can be seen that the number of edges labeled 1 and the number of edges labeled 0 differ by at most 1. Therefore, it has been proved that the graphs $nF_{m_i}(n-1)P_t$ with m_i even or odd connected by paths of even length is an integer cordial graph.

Example 5 Given a graph, $3F_{m_i}2P_5$ with $m_1 = m_3 = 3$ and $m_2 = 5$ is an integer cordial graph. The integer cordial labeling on graph $3F_{m_i}2P_5$ with $m_1 = m_3 = 3$ and $m_2 = 5$ is illustrated in Figure 5

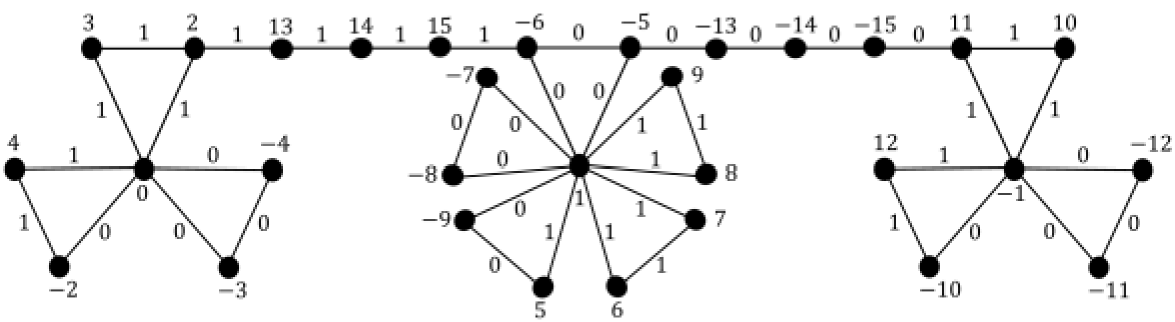

Figure 5. Integer Cordial Labeling of Graph $3F_{m_i}2P_5$ with $m_1 = m_3 = 3$ and $m_2 = 5$

Theorem 4 *Graphs $nF_{m_i}(n-1)P_t$ with m_i connected by paths of odd length are integer cordial graphs.*

*Proof***:** Given a graph $nF_{m_i}(n-1)P_t$. If n is even, a function

$$f: V\left(nF_{m_i}(n-1)P_t\right) \to \left\{-\frac{|V(G)|}{2}, -\frac{|V(G)|}{2}+1, \dots, -1,0,1, \dots, \frac{|V(G)|}{2}-1, \frac{|V(G)|}{2}\right\}$$

defined by Equations (2), (9), (10), and (11). Furthermore, for n odd, a function

$$f: V\left(nF_{m_i}(n-1)P_t\right) \to \left\{-\left\lfloor\frac{|V(G)|}{2}\right\rfloor, -\left\lfloor\frac{|V(G)|}{2}\right\rfloor+1, \dots, -1,0,1, \dots, \left\lfloor\frac{|V(G)|}{2}\right\rfloor-1, \left\lfloor\frac{|V(G)|}{2}\right\rfloor\right\}$$

defined by Equations (3), (12), and (13). Based on Lemma 1 and Lemma 2, for all possibilities, it can be seen that the number of edges labeled 1 and the number of edges labeled 0 differ by at most 1. Therefore, it has been proved that the graphs $nF_{m_i}(n-1)P_t$ with m_i even or odd connected by paths of odd length is a graph with integer cordial labeling.

Example 6 Given a graph $2F_{m_i}P_6$ with $m_1 = 5$ and $m_2 = 3$ is an *integer cordial graph*. The integer *cordial* labelling on the graph $2F_{m_i}P_6$ with $m_1 = 5$ and $m_2 = 3$ is illustrated by Figure 6

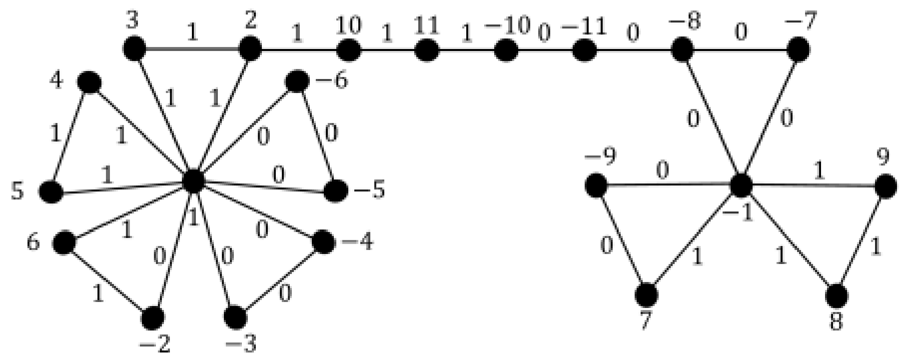

Figure 6. Integer Cordial Labeling of Graph $2F_{m_i}P_6$ with $m_1 = 5$ and $m_2 = 3$

3. Conclution

Based on the definitions that have been given in the discussion of the previous chapter, it can be concluded that determining the integer cordial labeling by giving a label that satisfies the integer cordial labeling where the set of vertex labels can be expressed by $f: V\left(nF_{m_i}(n-1)P_t\right) \to \left\{-\frac{|V(G)|}{2}, -\frac{|V(G)|}{2}+1, \dots, -1,0,1, \dots, \frac{|V(G)|}{2}-1, \frac{|V(G)|}{2}\right\}$ or $\left\{-\left\lfloor\frac{|V(G)|}{2}\right\rfloor, -\left\lfloor\frac{|V(G)|}{2}\right\rfloor+1, \dots, -1,0,1, \dots, \left\lfloor\frac{|V(G)|}{2}\right\rfloor-1, \left\lfloor\frac{|V(G)|}{2}\right\rfloor\right\}$. Some graphs that are graphs with integer cordial labeling include:

1. Graph $nF_{m_i}(n-1)P_t$ with $n = 1$ and m_i even or odd.
2. Graph $nF_{m_i}(n-1)P_t$ with m_i even or odd connected by paths of length one.
3. Graph $nF_{m_i}(n-1)P_t$ with m_i even or odd connected by paths of even length.
4. Graph $nF_{m_i}(n-1)P_t$ with m_i even or odd connected by paths of odd length.

References

Cahit, I. (1987). Cordial Graph: a Weaker Version of Graceful and Harmonious Graph. *Ars Combin, 23*, 201-207.

Chartrand, G., & Zhang, P. (2012). *A First Course in Graph Theory.* New York: Dover Publications, Inc.

Godhasara, G., & Sonchhatra, S. (2013). Cordial Labeling of Fan Related Graphs. *International Journal of Scientific & Engineering Research*, *4*(8), 470-476.

Parameswari, R., Saradata Prita, K., & Rajeswari, R. (2021). Integer Cordial and Face Integer Cordial Labeling of Some Flower Graphs. *Journal of Physics: Conference Series,* 1-10. https://doi.org/10.1088/1742-6596/1770/1/012079

Shah, P., & Parmar, D. (2022). Integer Cordial Labeling of Alternate Snake Graph and Irregular Snake Graph. *Applications and Applied Mathematics*, 59-71.

Surya, S. S., Arul, S. M., & Mathew, L. (2020). Integer cordial labelling for certain families of graphs. *Adv. Math. Sci. J.*

9(9), 7483-7489. https://doi.org/10.37418/amsj.9.9.95

Wallis, W. D. (2007). *A Beginner's Guide to Graph Theory.* Boston: Birkhauser. https://doi.org/10.1007/978-0-8176-4580-9