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Analysis of the Well-posedness of a SEIRDS Dynamic Model for the Spread of Infectious Diseases

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Abstract

The goal of this paper is to demonstrate the well-posedness of a nonlinear parabolic reaction-diffusion system modeling the spread of infectious diseases. The considered mathematical model is of the SEIRDS type. We prove the global existence of a weak solution by using an approximation system with a delay operator λ^{τ} (which we define in the subsection 3.1) , along with a priori estimates and compactness arguments. Additionally, we establish the uniqueness of the solution and its continuous dependence on the contagion rates using Gronwall's lemma. These results not only show the existence of a solution but also ensure that it is unique and responds stably to variations in the model parameters.

Keywords: reaction-diffusion, nonlinear parabolic system, infectious diseases, SEIRD model, weak solution, well-posedness

1. Introduction

Over the past few decades, numerous mathematical models have been developed to study the evolution of infectious diseases and control their spread. These models, whether statistical or mathematical, provide valuable insights and help policymakers implement effective policies (Manfredi, P., & D'Onofrio, A. 2013; Song, P., & Xiao, Y. 2022). Time series and compartmental models are commonly used to predict and simulate the dynamics of infectious diseases, offering essential tools for epidemic management (Ayoola, T. A. et al. 2021; Taboe, H. B. et al. 2020; Omame, A. et al. 2021). Indeed, mathematical models have provided new insights and helped policymakers make the best possible decisions regarding the effectiveness of implemented policies (Deressa, C. T., & Duressa, G. F. 2021; Stone, L., Shulgin, B., & Agur, Z. 2000).

The significant impact of infectious diseases on the development of human society underscores the need for robust prevention and control policies for public health. The recent COVID-19 pandemic highlighted the crucial role of global public surveillance and response systems, which can mitigate negative effects on socio-economic activities and human health. The pandemic has significantly slowed the global economy by disrupting many economic sectors (Garad, A et al.2021; Ibn-Mohammed, T. et al. 2021).

Since 2020, scientific literature on epidemic mathematical models has been enriched with numerous contributions, often based on compartmental models (Gatto, M. et al., 2020; Wang, Z., et al. 2020; Zohdi, T. 2020; Soma, S., Kambele, S., & Guiro, A. March 9, 2024 ; Soma, S. et al. March 18,2024). These models divide the population into different compartments based on qualitative characteristics, such as "susceptible", "infected", and "recovered".

These models have naturally led to the introduction of diffusion terms. For a recent overview of mathematical models of viral pandemics, we refer to (Bellomo, N., Brezzi, F., & Chaplain, M. A. J. 2021). It should be noted that epidemic models including spatial diffusion have been studied for a long time (De Mottoni, P., Orlandi, E., & Tesei, A. 1979; Fitzgibbon, W. E. et al. 2018; Webb, G. F. 1981). Very recently, a new epidemic diffusion model with nonlinear transmission rates and diffusion coefficients was introduced and tested (Viguerie, A., et al. 2020; Viguerie, A., et al. 2021), while in (Auricchio, F., Colli, P., Gilardi, G., Reali, A., & Rocca, E. 2023), the authors proved well-posedness results for an initial-boundary value problem associated with a variant of the compartmental model for COVID-19 studied in (Viguerie, A., et al. 2020;

Viguerie, A., et al. 2021).

However, most models are based on ordinary differential equations (ODEs), but here we explore a compartmental model using partial differential equations (PDEs) to better represent spatial variations. By exploring these models, we hope to contribute to a deeper understanding and more effective management of infectious diseases.

The following diagram shows the dynamics of contagion between the compartments in our model.

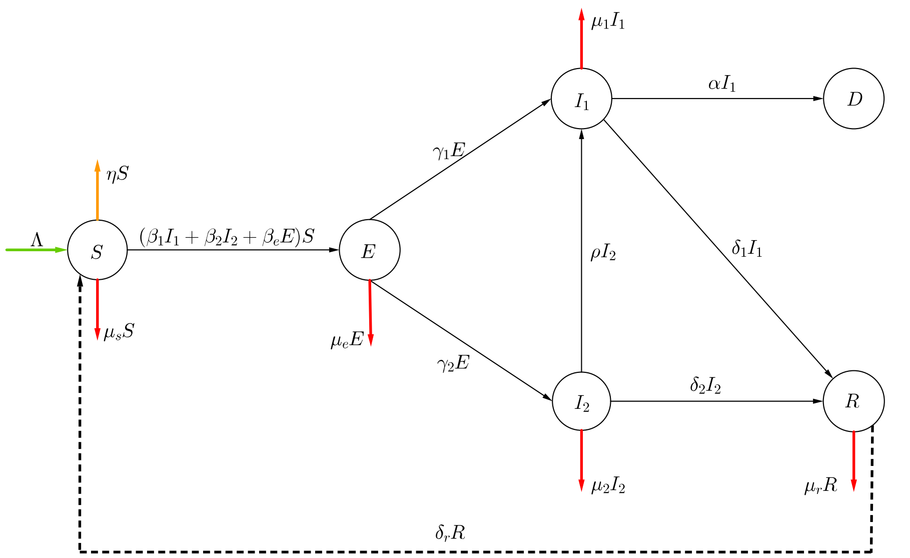

Figure 1. Flow chart describing the dynamics of contagion between the compartmental sub-groups considered in our model

From the diagram, the equations representing the spatio-temporal variations of the compartments of our model are established as follows:

$$\begin{cases} \partial_t S - \operatorname{div}(k_s \nabla S) = \Lambda + \delta_r R - (\beta_1 I_1 + \beta_2 I_2 + \beta_e E) S - (\eta + \mu_s) S \quad \text{in} \quad Q \\ \partial_t E - \operatorname{div}(k_e \nabla E) = (\beta_1 I_1 + \beta_2 I_2 + \beta_e E) S - \gamma_1 E - \gamma_2 E - \mu_e E \quad \text{in} \quad Q \\ \partial_t I_2 - \operatorname{div}(k_2 \nabla I_2) = \gamma_2 E - \rho I_2 - \delta_2 I_2 - \mu_2 I_2 \quad \text{in} \quad Q \\ \partial_t I_1 - \operatorname{div}(k_1 \nabla I_1) = \gamma_1 E + \rho I_2 - \alpha I_1 - \delta_1 I_1 - \mu_1 I_1 \quad \text{in} \quad Q \\ \partial_t R - \operatorname{div}(k_r \nabla R) = \delta_1 I_1 + \delta_2 I_2 - \delta_r R - \mu_r R \quad \text{in} \quad Q \\ \partial_t D = \alpha I_1 \quad \text{in} \quad Q \end{cases} \tag{1.1}$$

in the space-time cylinder $Q := \Omega \times (0, T)$, satisfying

$$(S, E, I_1, I_2, R, D)(0) = (S_0, E_0, I_{1_0}, I_{2_0}, R_0, 0) \qquad \text{in} \quad \Omega, \tag{1.2}$$

and

$$\frac{\partial S}{\partial \nu} = \frac{\partial E}{\partial \nu} = \frac{\partial I_1}{\partial \nu} = \frac{\partial I_2}{\partial \nu} = \frac{\partial R}{\partial \nu} = 0, \quad \text{in} \quad \Sigma_T := \partial\Omega \times (0, T) \tag{1.3}$$

where
♦ $T > 0$, Ω is a bounded open spatial domain of $\mathbb{R}^d$, $d \geq 2$ and $\partial\Omega$ denotes its boundary, which is assumed to be regular, and ν the exterior normal to Ω. ♦ We denote by ν , the exterior normal vector to Ω. ♦ $S(x, t)$, $E(x, t)$, $I_1(x, t)$, $I_2(x, t)$, $R(x, t)$, and $D(x, t)$ are the respective densities at time $t \in [0, T)$ and location $x \in \Omega$ of susceptible individuals (those who can contract the disease), exposed individuals (those who carry the disease germs, do not show symptoms, but can still transmit the disease), detected infectious individuals (those who show symptoms, test positive, and can transmit the disease), undetected infectious individuals (those who are ill but unaware of their status and can transmit the

disease), recovered individuals (those who have recovered after an infectious period but are not necessarily immune), and individuals who have died from the disease.

Additionally, we define the following parameters, all of which are assumed to be positive:

Table1.DescriptionofParameters

Parameter	Description
β_1	Contribution of known infectious individuals to the force of infection
β_2	Contribution of unknown infectious individuals to the force of infection
β_e	Contribution of exposed individuals to the force of infection
η	Vaccination rate or immunity gain rate
δ_r	Rate of immunity loss depending on time
γ_1	Rate of progression of exposed to the detected infected compartment
γ_2	Rate of progression of exposed to the undetected infected compartment
ρ	Rate of progression of undetected infected to detected infected compartment
δ_1	Recovery rate of known infectious individuals
δ_2	Recovery rate of unknown infectious individuals
Λ	Natural birth rate
α	Disease-induced mortality rate for known infectious individuals
μ_k	Natural mortality rate in compartment $k = s, e, 1, 2, r$.

It should be noted that the exposed population and the infectious population refer to asymptomatic and symptomatic people respectively. Consequently, as observed in the COVID-19 epidemic, exposed individuals can also spread the disease. Here, we neglect the participation of newborns, assuming that this population is small and does not essentially contribute to the transmission of the epidemic during the time period (0, T). Thus, in our model, the birth rate is considered to be zero. The positive coefficients β_1, β_2 and β_e may depend on space and time. The total living population is given by

$$N(x,t) = S(x,t) + E(x,t) + I_1(x,t) + I_2(x,t) + R(x,t).$$

As can be seen, the equation for D depends only on I_1 and does not influence the other equations. Consequently, D can be considered as an independent compartment because knowledge of I_1 allows knowledge of D. Thus, the study of the following SEIR-type auxiliary problem

$$\begin{cases} \partial_t S - \mathrm{div}(k_s \nabla S) = \Lambda + \delta_r R - (\beta_1 I_1 + \beta_2 I_2 + \beta_e E) S - (\eta + \mu_s) S \quad \text{in} \quad Q \\ \partial_t E - \mathrm{div}(k_e \nabla E) = (\beta_1 I_1 + \beta_2 I_2 + \beta_e E) S - \gamma_1 E - \gamma_2 E - \mu_e E \quad \text{in} \quad Q \\ \partial_t I_2 - \mathrm{div}(k_2 \nabla I_2) = \gamma_2 E - \rho I_2 - \delta_2 I_2 - \mu_2 I_2 \quad \text{in} \quad Q \\ \partial_t I_1 - \mathrm{div}(k_1 \nabla I_1) = \gamma_1 E + \rho I_2 - \alpha I_1 - \delta_1 I_1 - \mu_1 I_1 \quad \text{in} \quad Q \\ \partial_t R - \mathrm{div}(k_r \nabla R) = \delta_1 I_1 + \delta_2 I_2 - \delta_r R - \mu_r R \quad \text{in} \quad Q \end{cases} \tag{1.4}$$

in the space-time cylinder $Q := \Omega \times (0,T)$, satisfying

$$(S, E, I_1, I_2, R)(0) = (S_0, E_0, I_{1_0}, I_{2_0}, R_0) \qquad \text{in} \quad \Omega, \tag{1.5}$$

and

$$\frac{\partial S}{\partial \nu} = \frac{\partial E}{\partial \nu} = \frac{\partial I_1}{\partial \nu} = \frac{\partial I_2}{\partial \nu} = \frac{\partial R}{\partial \nu} = 0, \quad \text{in} \quad \Sigma_T := \partial\Omega \times (0,T) \tag{1.6}$$

the study is sufficient to understand the dynamics of the epidemic in the compartments S, E, I_1, I_2 and R. Consequently D can be determined as a post-processing by integrating the equation:

$$D(t) = D(0) + \alpha \int_Q I_1$$

where $D(0)$ is the initial condition for the compartment of deaths (generally $D(0) = 0$ if it is assumed that there are no deaths at the start of the epidemic). The article is structured into three distinct sections, each making a specific contribution to our research. The first section, as mentioned above, serves as an introduction to our study. In section 2, we list our assumptions and notations and state our results. The proofs of theorems 1 and 2 concerning the well-posedness of the state system (1.4)-(1.6) are given in section 3.

2. Hypotheses, Notation and Results of the Problem

In this section, we make some specific assumptions and present our results. First, we consider that the set $\Omega \subset \mathbb{R}^d, d = 1, 2, 3$, is bounded, connected and regular. Then, if X is a Banach space, $\|\cdot\|_X$ represents its norm, with the exception of the space $\mathbf{H}$ defined below and the spaces L^∞ constructed on $\Omega, (0,T)$ and Q, whose norms are denoted by $\|\cdot\|$ (i.e. without index) and $\|\cdot\|_\infty$ respectively. In addition, for simplicity, we use the same symbol for the norm in X and the norm in all powers of X. We also introduce

$$\mathbf{H} := L^2(\Omega) \quad \text{and} \quad V := H^1(\Omega)$$

and adopt the framework of the Hilbert triplet $(V, \mathbf{H}, V^*)$ obtained by identifying $\mathbf{H}$ with a subspace of the dual space V^* of V in the usual way, that is, so that

$$\langle z, v \rangle = \int_\Omega zv$$

for all $z \in \mathbf{H}$ and $v \in V$, where $\langle \cdot, \cdot \rangle$ is the dual coupling between V^* and V.

We assume that
♦ $k_s, k_e, k_1, k_2, k_r : Q \to \mathbb{R}$ are positive functions in $L^\infty(Q)$ satisfying

$$k_* \le k_s(x,t), k_e(x,t), k_1(x,t), k_2(x,t), k_r(x,t) \le k^* \quad \text{a.e. } (x,t) \in Q \tag{2.1}$$

where k_* and k^* are strictly positive constants.

♦ $\beta_1, \beta_2, \beta_e : Q \to \mathbb{R}$ are positive functions in $L^\infty(Q)$ satisfying

$$0 \le \beta_1(x,t), \beta_2(x,t), \beta_e(x,t) \le \beta^*, \quad \text{a.e. } (x,t) \in Q \tag{2.2}$$

where β^* is a positive constant. ♦

$$\Lambda,\ \gamma_1,\ \gamma_2,\ \delta_1,\ \delta_2,\ \eta,\ \rho,\ \alpha,\ \mu_s,\ \mu_e,\ \mu_1,\ \mu_2, \text{ and } \mu_r \text{ are all positive constants.} \tag{2.3}$$

♦ $\delta_r \in L^\infty(0,T)$ and satisfies

$$0 \le \delta_r(t) \le \delta^* \quad \text{a.e. } t \in (0,T) \tag{2.4}$$

where δ^* is a positive constant. ♦ For the initial data, we assume that

$$S_0, E_0, I_{1_0}, I_{2_0}, R_0 \in L^\infty(\Omega) \quad \text{and} \quad D_0 = 0 \tag{2.5}$$

are positive functions.

We conclude this section by recalling some tools and stating a general rule regarding the constants that appear in the estimates we perform later on. Throughout this article, we will repeatedly use the Young's inequality

$$ab \le \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for all } a, b \in \mathbb{R} \text{ and } \delta > 0,$$

as well as the Hölder and Schwarz inequalities. Additionally, we will leverage the continuous embedding in three dimensions $V \hookrightarrow L^p(\Omega)$ for $p \in [1,6]$, which is compact if $p < 6$, and the corresponding Sobolev and compactness inequalities.

$$\|v\|_p \le C_\Omega \|v\|_V \quad \text{for all } v \in V \text{ et } p \in [1,6],$$
$$\|v\|_p \le \delta\|\nabla v\| + C_{\Omega,\delta,p}\|v\| \quad \text{for all } v \in V, p \in [1,6) \text{ and } \delta > 0,$$

where C_Ω is a constant depending only on Ω, while $C_{\Omega,\delta,p}$ also depends on p and δ.

Now, we provide the notion of solution to our control problem under assumptions (2.1)-(2.5).

Definition 1. *Assume* (2.1)-(2.5). *Given $S_0, E_0, I_{1_0}, I_{2_0}, R_0 \in L^\infty(\Omega)$, a weak solution of problem* (1.4)-(1.6) *is a quintuple of positive functions (S, E, I_1, I_2, R) satisfying the regularity properties*

$$S, E, I_1, I_2, R \in H^1(0,T;V^*) \cap L^2(0,T;V) \hookrightarrow C^0([0,T];H) \tag{2.6}$$
$$S, E, I_1, I_2, R \ge 0 \quad \textit{p.p. dans } Q \tag{2.7}$$
$$S, E, I_1, I_2, R \in L^\infty(Q) \tag{2.8}$$

and satisfying the variational equations

$$\langle \partial_t S, v\rangle + \int_\Omega k_s \nabla S \cdot \nabla v = \int_\Omega \left[\Lambda + \delta_r R - (\beta_1 I_1 + \beta_2 I_2 + \beta_e E) S - (\eta + \mu_s) S\right] v \tag{2.9}$$
$$\langle \partial_t E, v\rangle + \int_\Omega k_e \nabla E \cdot \nabla v = \int_\Omega \left[(\beta_1 I_1 + \beta_2 I_2 + \beta_e E) S - \gamma_1 E - \gamma_2 E - \mu_e E\right] v \tag{2.10}$$
$$\langle \partial_t I_2, v\rangle + \int_\Omega k_2 \nabla I_2 \cdot \nabla v = \int_\Omega (\gamma_2 E - \rho I_2 - \delta_2 I_2 - \mu_2 I_2)\, v \tag{2.11}$$
$$\langle \partial_t I_1, v\rangle + \int_\Omega k_1 \nabla I_1 \cdot \nabla v = \int_\Omega (\gamma_1 E + \rho I_2 - \delta_1 I_1 - \alpha I_1 - \mu_1 I_1)\, v \tag{2.12}$$
$$\langle \partial_t R, v\rangle + \int_\Omega k_r \nabla R \cdot \nabla v = \int_\Omega (\delta_1 I_1 + \delta_2 I_2 - \delta_r R - \mu_r R)\, v \tag{2.13}$$

a.e. in $(0,T)$ for all $v \in V$, and the initial condition

$$(S, E, I_1, I_2, R, D)(0) = (S_0, E_0, I_{1_0}, I_{2_0}, R_0, 0). \tag{2.14}$$

The first result of this paper concerns the existence and uniqueness of a solution to the problem (1.4)-(1.6) and also an estimate of stability on this solution.

Theorem 1. *Under the assumptions* (2.1)-(2.4) *on the structure of the system and* (2.5) *on the initial data, there exists a unique solution (S, E, I_1, I_2, R) satisfying the regularities* (2.6)-(2.8) *which solves the variational problem* (2.9)-(2.14) *and also satisfies the stability estimate*

$$\|(S, E, I_1, I_2, R)\|_{H^1(0,T;V^*)\cap C^0([0,T];H)\cap L^2(0,T;V)\cap L^\infty(Q)} \le K_1 \tag{2.15}$$

with a positive constant $K_1 > 0$ which depends only on Ω, T, the constants $k_*, k^*, \beta^*, \delta^*, \gamma_1, \gamma_2, \delta$ and ρ, as well as the initial data.

the second result of this paper concerns the estimation of the continuous dependence of the solution to the problem (1.4)-(1.6) on the different contact rates β_1, β_2 and β_e.

Theorem 2. *Suppose* (2.1)-(2.4) *on the structure of the system and* (2.5) *on the initial data. Let $\beta_1^{(j)}, \beta_2^{(j)}, \beta_e^{(j)}$, $j = 1,2$, be positive functions in $L^\infty(Q)$ whose norms are bounded by $\beta^\star$, and let $\left(S^{(j)}, E^{(j)}, I_1^{(j)}, I_2^{(j)}, R^{(j)}\right)$ be the corresponding solutions. Then the inequality*

$$\left\|\left(S^{(1)}, E^{(1)}, I_1^{(1)}, I_2^{(1)}, R^{(1)}\right) - \left(S^{(2)}, E^{(2)}, I_1^{(2)}, I_2^{(2)}, R^{(2)}\right)\right\|_{C^0([0,T];H)\cap L^2(0,T;V)}$$
$$\le K_2 \left\|\left(\beta_1^{(1)}, \beta_2^{(1)}, \beta_e^{(1)}\right) - \left(\beta_1^{(2)}, \beta_2^{(2)}, \beta_e^{(2)}\right)\right\|_{L^2(0,T;H)}$$

holds with a positive constant K_2 that depends only on the structure of the system, Ω, T, the initial data, and the constant $\beta^\star$.

3. State System

This section is dedicated to the study of the state system. We address the topic in two steps: first, we examine the existence of solutions and uniform bounds, then we focus on uniqueness and continuous dependence estimation.

3.1 Existence and Stability Estimation of at Least One Solution

Proof. **Proof of Theorem 1**

This subsection addresses the proof of the existence and uniqueness of the solution to problem (1.4)-(1.6) and its stability estimation (2.15) given by Theorem 1. To do this, we first solve an approximation problem. Let n be a positive integer, we set $\tau := T/n$ and introduce the delay operator.

$$\lambda^\tau : L^2(-\tau, T; H) \to L^2(0, T; H) \quad \text{defined by}$$
$$(\lambda^\tau u)(t) := u(t-\tau) \quad \text{for almost every } t \in (0, T).$$

The approximation problem then consists in finding a quintuple $(S_\tau, E_\tau, I_{1_\tau}, I_{2_\tau}, R_\tau)$ satisfying the regularities of (2.6)-(2.8) with an obvious variant concerning E_τ, which is the following:

$$E_\tau(t) = E_0 \text{ for all } t \in [-\tau, 0], \tag{3.1}$$

and satisfying the variational equations,

$$\langle \partial_t S_\tau, v\rangle + \int_\Omega (\beta_1 I_{1_\tau} + \beta_2 I_{2_\tau} + \beta_e \lambda^\tau E_\tau + \eta + \mu_s) S_\tau v + \int_\Omega k_s \nabla S_\tau \cdot \nabla v - \int_\Omega (\Lambda + \delta_r R_\tau) v = 0 \tag{3.2}$$

$$\langle \partial_t E_\tau, v\rangle + \int_\Omega (\gamma_1 + \gamma_2 + \mu_e) E_\tau v + \int_\Omega k_e \nabla E_\tau \cdot \nabla v - \int_\Omega (\beta_1 I_{1_\tau} + \beta_2 I_{2_\tau} + \beta_e \lambda^\tau E_\tau) S_\tau v = 0 \tag{3.3}$$

$$\langle \partial_t I_{2_\tau}, v\rangle + \int_\Omega (\rho + \delta_2 + \mu_2) I_{2_\tau} v + \int_\Omega k_2 \nabla I_{2_\tau} \cdot \nabla v - \int_\Omega \gamma_2 \lambda^\tau E_\tau \, v = 0 \tag{3.4}$$

$$\langle \partial_t I_{1_\tau}, v\rangle + \int_\Omega (\delta_1 + \alpha + \mu_1) I_{1_\tau} v + \int_\Omega k_1 \nabla I_{1_\tau} \cdot \nabla v - \int_\Omega (\gamma_1 \lambda^\tau E_\tau + \rho I_{2_\tau}) v = 0 \tag{3.5}$$

$$\langle \partial_t R_\tau, v\rangle + \int_\Omega (\delta_r + \mu_r) R_\tau v + \int_\Omega k_r \nabla R_\tau \cdot \nabla v - \int_\Omega (\delta_1 I_{1_\tau} + \delta_2 I_{2_\tau}) v = 0 \tag{3.6}$$

a.e. in $(0, T)$ for all $v \in V$, as well as the initial condition

$$(S_\tau, I_{1_\tau}, I_{2_\tau}, R_\tau)(0) = (S_0, I_{1_0}, I_{2_0}, R_0) \quad \text{and} \quad E_\tau(t) = E_0 \quad \text{for } t \in [-\tau, 0]. \tag{3.7}$$

Now we recall the weak maximum principle which is as follows: An element $f \in L^2(0, T; V^*)$ is positive if $\int_0^T \langle f(t), v(t)\rangle dt \geq 0$ for each positive $v \in L^2(0, T; V)$, and for all $f, g \in L^2(0, T; V^*)$, the inequality $g \leq f$ means that $f - g$ is positive.

Lemma 3.1. *Suppose $k \in L^\infty(Q)$ is positive, $\phi \in \mathbb{R}, f \in L^2(0, T; V^*)$, and $u_0 \in H$ are all positive, and let $u \in H^1(0, T; V^*) \cap L^2(0, T; V)$ be a solution to the problem*

$$\begin{cases} \langle \partial_t u, v\rangle + \int_\Omega \phi u v + \int_\Omega k \nabla u \cdot \nabla v = \langle f, v\rangle \quad \text{a.e. in } (0, T), \quad \forall v \in V \\ u(0) = u_0. \end{cases}$$

Then $u \geq 0$ a.e. in Q.

From this, we deduce the following lemma.

Lemma 3.2. *Let $\kappa \in L^\infty(Q)$ such that $\kappa_* \leq \kappa \leq \kappa^*$ a.e. in Q, and let $f, g \in L^2(0, T; V^*)$ be given, as well as $u_0 \in L^\infty(\Omega)$. Suppose that $u, w \in H^1(0, T; V^*) \cap L^2(0, T; V)$ satisfy*

$$\langle \partial_t u, v\rangle + \int_\Omega \kappa \nabla u \cdot \nabla v = \langle f, v\rangle \quad \text{a.e. in } (0, T), \text{ for all } v \in V,$$
$$\langle \partial_t w, v\rangle + \int_\Omega \kappa \nabla w \cdot \nabla v = \langle g, v\rangle \quad \text{a.e. in } (0, T), \text{ for all } v \in V,$$
$$u(0) = w(0) = u_0.$$

Then, we have:

i) If $f \in L^\infty(0,T;H)$, then $u \in L^\infty(Q)$ and its L^∞ norm is bounded by a constant depending only on Ω, T, the constants κ_* and κ^*, and the norms of the data f and u_0.

ii) If $f \in L^\infty(0,T;H)$, $g \le f$ and $w \ge 0$ a.e. in Q, the same conclusions apply for w.

Proof. The first assertion can be proved with minor modifications by following the argument used in (Ladyzhenskaia, O. A., Solonnikov, V. A., & Ural'tseva, N. N. 1968) to establish (Ladyzhenskaia, O. A., Solonnikov, V. A., & Ural'tseva, N. N. 1968, Thm. 7.1, p. 181), which deals with the case of Dirichlet boundary conditions. For the second assertion, the inequality $g \le f$ and the weak maximum principle imply that $w \le u$. Since $w \ge 0$ by assumption and u is bounded by (i), we deduce that w is also bounded and $\|w\|_\infty \le \|u\|_\infty$, from which the estimation of $\|w\|_\infty$ follows as desired.

Theorem 3. *The problem* (3.2)-(3.7) *has a unique solution* $(S_\tau, E_\tau, I_{1_\tau}, I_{2_\tau}, R_\tau)$.

Proof. We solve the problem over the intervals $J_k := [k\tau, (k+1)\tau]$ for $0 \le k \le n-1$. For each resolution of the problem over J_k, we continue to use the notation $(S_\tau, E_\tau, I_{1_\tau}, I_{2_\tau}, R_\tau)$ for the solution defined over J_k. For $k = 0$, we have $J_0 := [0, \tau]$ on which we construct a solution $(S_\tau, E_\tau, I_{1_\tau}, I_{2_\tau}, R_\tau)$ of the system by solving the linear parabolic variational equations (3.4), (3.5), (3.6), (3.2), and (3.3) (and thus find $I_{2_\tau}, I_{1_\tau}, R_\tau, S_\tau$, and E_τ, in that order).

Note that for all $t \in J_0$, by condition (3.7), we have $\lambda^\tau E = E_0$. Therefore, the only unknown in the variational equation (3.4) is the function I_{2_τ}. Hence, I_{2_τ} satisfies the variational equation from Lemma 3.1. Hence its existence and positivity. From Lemma 3.2, it follows that I_{2_τ} is bounded.

Subsequently, we use this function I_{2_τ} (now known) in (3.5), and still thanks to Lemmas 3.1 and 3.2, we also prove the existence, positivity, and boundedness of the component I_{1_τ}. Thus, by following the same process in order for (3.6), (3.2), and (3.3), we manage to prove the existence, positivity, and boundedness of each component of the solution $(S_\tau, E_\tau, I_{1_\tau}, I_{2_\tau}, R_\tau)$ to the problem (3.2)-(3.7) over J_0.

Once the problem is solved over $[0, \tau]$, it is possible to construct, using the same approach, the solution over the other intervals $[k\tau, (k+1)\tau]$ with $k \le n-1$ by induction on k, still considering (3.4), (3.5), (3.6), (3.2), and (3.3) in this order. Indeed, at each step, $\lambda^\tau E_\tau$ is a known function belonging to $L^\infty(k\tau, (k+1)\tau; H)$, and the initial value at $t = k\tau$ can be taken as the final value at $k\tau$ of the solution constructed in the previous step, whose components belong to $C^0([(k-1)\tau, k\tau]; H)$. Thus, the quintuplet $(S_\tau, E_\tau, I_{1_\tau}, I_{2_\tau}, R_\tau)$ constructed at the k^{th} step belongs to $\left(H^1(k\tau, (k+1)\tau; V^*) \cap L^2(k\tau, (k+1)\tau; V)\right)^5$, and therefore to $\left(C^0([k\tau, (k+1)\tau]; H)\right)^5$. Moreover, each component is positive and bounded, once again due to Lemmas 3.1 and 3.2 (where $(0, T)$ is replaced by a subinterval of the form $[k\tau, (k+1)\tau]$). This procedure constructs a quintuplet $(S_\tau, E_\tau, I_{1_\tau}, I_{2_\tau}, R_\tau)$ defined over the entire interval $[0, T]$ that is piecewise regular and globally continuous in H (due to the choice of initial conditions at each step). Furthermore, it solves the system in each subinterval J_k and satisfies the initial conditions. It follows that $(S_\tau, E_\tau, I_{1_\tau}, I_{2_\tau}, R_\tau)$ satisfies the required global regularity and is a global solution of (3.2)-(3.7). Finally, uniqueness follows since each step provides a unique solution. □

Now we need to let τ tend towards zero (or n towards infinity) in order to generate at least one solution to the variational problem (2.9)-(2.14). For this, we need some a priori estimates.

First a priori estimate

We introduce the auxiliary function

$$\varphi_\tau := S_\tau + E_\tau.$$

From (3.2) and (3.3), we observe that

$$\begin{aligned} \langle \partial_t \varphi_\tau, v \rangle &+ \int_\Omega \left[(\gamma_1 + \gamma_2 + \mu_e)(\varphi_\tau - s_\tau) + (\eta + \mu_s) S_\tau \right] v \\ &+ \int_\Omega \left(k_s \nabla S_\tau + k_e \nabla (\varphi_\tau - S_\tau) \right) \cdot \nabla v \\ &= \int_\Omega (\Lambda + \delta_r R_\tau) v \quad \text{a.e. in } (0, T), \text{ for every } v \in V. \end{aligned} \tag{3.8}$$

In (3.8), taking $v = \varphi_\tau$ as the test function, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\varphi_\tau|^2 + \int_\Omega k_e|\nabla\varphi_\tau|^2 + \int_\Omega (\gamma_1 + \gamma_2 + \mu_e)\,|\varphi_\tau|^2$$
$$= -\int_\Omega (k_s - k_e)\,\nabla S_\tau \cdot \nabla\varphi_\tau + \int_\Omega (\gamma_1 + \gamma_2 + \mu_e - \eta - \mu_s)\,S_\tau\varphi_\tau + \int_\Omega (\Lambda + \delta_r R_\tau)\,\varphi_\tau. \tag{3.9}$$

Now, in (3.2), (3.4), (3.5), and (3.6), taking $\zeta S_\tau,\ I_{1_\tau},\ I_{2_\tau}$, and R_τ respectively as test functions (where ζ is a positive parameter), we obtain

$$\frac{\zeta}{2}\frac{d}{dt}\int_\Omega |S_\tau|^2 + \zeta\int_\Omega (\beta_1 I_{1_\tau} + \beta_2 I_{2_\tau} + \beta_e\lambda^\tau E_\tau + \eta + \mu_s)\,|S_\tau|^2 + \zeta\int_\Omega k_s|\nabla S_\tau|^2 - \zeta\int_\Omega (\Lambda + \delta_r R_\tau)S_\tau = 0 \tag{3.10}$$
$$\frac{1}{2}\frac{d}{dt}\int_\Omega |I_{2_\tau}|^2 + \int_\Omega (\rho + \delta_2 + \mu_2)\,|I_{2_\tau}|^2 + \int_\Omega k_2|\nabla I_{2_\tau}|^2 - \int_\Omega \gamma_2\lambda^\tau E_\tau I_{2_\tau} = 0 \tag{3.11}$$
$$\frac{1}{2}\frac{d}{dt}\int_\Omega |I_{1_\tau}|^2 + \int_\Omega (\gamma_1 + \alpha + \mu_1)\,|I_{1_\tau}|^2 + \int_\Omega k_1|\nabla I_{1_\tau}|^2 - \int_\Omega (\gamma_1\lambda^\tau E_\tau + \rho I_{2_\tau})I_{1_\tau} = 0 \tag{3.12}$$
$$\frac{1}{2}\frac{d}{dt}\int_\Omega |R_\tau|^2 + \int_\Omega (\delta_r + \mu_r)\,|R_\tau|^2 + \int_\Omega k_r|\nabla R_\tau|^2 - \int_\Omega (\delta_1 I_{1_\tau} + \delta_2 I_{2_\tau})R_\tau = 0. \tag{3.13}$$

By summing (3.10) through (3.13), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\varphi_\tau|^2 + \zeta|S_\tau|^2 + |I_{1_\tau}|^2 + |I_{2_\tau}|^2 + |R_\tau|^2 \\
&\quad + \int_\Omega \left(k_e|\nabla\varphi_\tau|^2 + \zeta k_s|\nabla S_\tau|^2 + k_2|\nabla I_{2_\tau}|^2 + k_1|\nabla I_{1_\tau}|^2 + k_r|\nabla R_\tau|^2\right) \\
&\quad + \int_\Omega (\gamma_1 + \gamma_2 + \mu_e)|\varphi_\tau|^2 + \zeta\int_\Omega (\beta_1 I_{1_\tau} + \beta_2 I_{1_\tau} + \beta_e\lambda^\tau E_\tau + \eta + \mu_s)\,|S_\tau|^2 \\
&\quad + \int_\Omega (\rho + \delta_2 + \mu_2)\,|I_{2_\tau}|^2 + \int_\Omega (\delta_1 + \alpha + \mu_1)\,|I_{1_\tau}|^2 + \int_\Omega + \int_\Omega (\delta_r + \mu_r)\,|R_\tau|^2 \\
&\quad = -\int_\Omega (k_s - k_e)\,\nabla S_\tau . \nabla\varphi_\tau + \int_\Omega (\gamma_1 + \gamma_2 + \mu_e - \eta - \mu_s)\,S_\tau\varphi_\tau \\
&\qquad + \int_\Omega (\Lambda + \delta_r R_\tau)(\varphi_\tau + \zeta S_\tau) + \int_\Omega \gamma_2\lambda^\tau E_\tau I_{2_\tau} + \int_\Omega (\gamma_1\lambda^\tau E_\tau + \rho I_{2_\tau})I_{1_\tau} \\
&\qquad + \int_\Omega (\delta_1 I_{1_\tau} + \delta_2 I_{2_\tau})R_\tau.
\end{aligned} \tag{3.14}$$

Using Young's inequality, the first term on the right-hand side of equality (3.14) can be estimated as follows.

$$\begin{aligned}
-\int_\Omega (k_s - k_e)\,\nabla S_\tau \cdot \nabla\varphi_\tau &\le \frac{1}{2}\int_\Omega k_e\,|\nabla\varphi_\tau|^2 + \frac{1}{2}\int_\Omega \frac{|k_s - k_e|^2}{k_e}\,|\nabla S_\tau|^2 \\
&\le \frac{1}{2}\int_\Omega k_e\,|\nabla\varphi_\tau|^2 + \frac{1}{2}\int_\Omega \frac{4\,(k^*)^2}{k_*}\,|\nabla S_\tau|^2 \\
&\le \frac{1}{2}\int_\Omega k_e\,|\nabla\varphi_\tau|^2 + \frac{1}{2}\int_\Omega \zeta k_s\,|\nabla S_\tau|^2
\end{aligned} \tag{3.15}$$

where $\zeta = \dfrac{4\,(k^*)^2}{k_*}$. Now, from (3.14) and (3.15), we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\varphi_\tau|^2 &+ \zeta|S_\tau|^2 + |I_{1_\tau}|^2 + |I_{2_\tau}|^2 + |R_\tau|^2 \\
&+ \int_\Omega \left(\frac{k_e}{2}|\nabla\varphi_\tau|^2 + \frac{\zeta k_s}{2}|\nabla S_\tau|^2 + k_2|\nabla I_{2_\tau}|^2 + k_1|\nabla I_{1_\tau}|^2 + k_r|\nabla R_\tau|^2\right) \\
&+ \int_\Omega (\gamma_1+\gamma_2+\mu_e)|\varphi_\tau|^2 + \zeta\int_\Omega (\beta_1 I_{1_\tau} + \beta_2 I_{1_\tau} + \beta_e\lambda^\tau E_\tau + \eta + \mu_s)\,|S_\tau|^2 \\
&+ \int_\Omega (\rho+\delta_2+\mu_2)\,|I_{2_\tau}|^2 + \int_\Omega (\delta_1+\alpha+\mu_1)\,|I_{1_\tau}|^2 + \int_\Omega + \int_\Omega (\delta_r+\mu_r)\,|R_\tau|^2 \\
&\le \int_\Omega (\gamma_1+\gamma_2+\mu_e-\eta-\mu_s)\,S_\tau\varphi_\tau \\
&\quad + \int_\Omega (\Lambda+\delta_r R_\tau)(\varphi_\tau+\zeta S_\tau) + \int_\Omega \gamma_2\lambda^\tau E_\tau I_{2_\tau} + \int_\Omega (\gamma_1\lambda^\tau E_\tau + \rho I_{2_\tau}) I_{1_\tau} \\
&\quad + \int_\Omega (\delta_1 I_{1_\tau} + \delta_2 I_{2_\tau}) R_\tau.
\end{aligned} \tag{3.16}
$$

Thanks to assumptions (2.1)-(2.5) and the positivity of the components of the solution $(S_\tau, E_\tau, I_{1_\tau}, I_{2_\tau}, R_\tau)$, it follows that

$$
\begin{aligned}
\frac{q}{2}\frac{d}{dt}\int_\Omega |\varphi_\tau|^2 &+ |S_\tau|^2 + |I_{1_\tau}|^2 + |I_{2_\tau}|^2 + |R_\tau|^2 \\
&\le \int_\Omega (\gamma_1+\gamma_2+\mu_e-\eta-\mu_s)\,S_\tau\varphi_\tau \\
&\quad + \int_\Omega (\Lambda+\delta_r R_\tau)(\varphi_\tau+\zeta S_\tau) + \int_\Omega \gamma_2\lambda^\tau E_\tau I_{2_\tau} + \int_\Omega (\gamma_1\lambda^\tau E_\tau + \rho I_{2_\tau}) I_{1_\tau} \\
&\quad + \int_\Omega (\delta_1 I_{1_\tau} + \delta_2 I_{2_\tau}) R_\tau
\end{aligned} \tag{3.17}
$$

where $q = \min(1, \zeta)$. We only need to estimate the terms on the right-hand side of inequality (3.17). Thus, using Young's inequality, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\varphi_\tau|^2 &+ |S_\tau|^2 + |I_{1_\tau}|^2 + |I_{2_\tau}|^2 + |R_\tau|^2 \\
&\le c\left(\int_\Omega |\varphi_\tau|^2 + |S_\tau|^2 + |I_{1_\tau}|^2 + |I_{2_\tau}|^2 + |R_\tau|^2 + |\lambda^\tau E_\tau|^2\right) + \frac{\Lambda mes(\Omega)}{2}
\end{aligned} \tag{3.18}
$$

where c is a constant depending on ζ, k^*, β^*, q, δ^*, and certain parameters from (2.3). Integrating (3.18) from 0 to t ($t \in (0, T)$), we obtain

$$
\begin{aligned}
\frac{1}{2}\int_\Omega |\varphi_\tau|^2 &+ |S_\tau|^2 + |I_{1_\tau}|^2 + |I_{2_\tau}|^2 + |R_\tau|^2 \\
&\le c\left(\int_0^t\int_\Omega |\varphi_\tau|^2 + |S_\tau|^2 + |I_{1_\tau}|^2 + |I_{2_\tau}|^2 + |R_\tau|^2 + |\lambda^\tau E_\tau|^2\right) + c_0
\end{aligned} \tag{3.19}
$$

with

$$
c_0 = \frac{\Lambda mes(\Omega) T}{2} + \frac{1}{2}\int_\Omega |\varphi_\tau(0)|^2 + |S_\tau(0)|^2 + |I_{1_\tau}(0)|^2 + |I_{2_\tau}(0)|^2 + |R_\tau(0)|^2.
$$

□

For the term involving $\lambda^\tau E_\tau$, we can estimate it as follows:

$$
\int_0^t\int_\Omega |\lambda^\tau E_\tau|^2 \le \int_{-\tau}^t\left(\int_\Omega |E_\tau(s)|^2\right) ds \le \tau\,|E_0|^2 + 2\int_0^t\int_\Omega \left(|\varphi_\tau|^2 + |S_\tau|^2\right). \tag{3.20}
$$

Then from (3.19) and (3.20), we obtain

$$\frac{1}{2}\int_\Omega |\varphi_\tau|^2 + |\varphi_\tau|^2 + |S_\tau|^2 + |I_{1_\tau}|^2 + |I_{2_\tau}|^2 + |R_\tau|^2 \leq C\left(\int_0^t \int_\Omega |\varphi_\tau|^2 + |S_\tau|^2 + |I_{1_\tau}|^2 + |I_{2_\tau}|^2 + |R_\tau|^2\right) + C_0 \tag{3.21}$$

for certain positive constants C and C_0. Consequently, applying Gronwall's lemma, we obtain

$$\left\|(\varphi_\tau, S_\tau, I_{1_\tau}, I_{2_\tau}, R_\tau)\right\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq c, \text{ and therefore, } \|E_\tau\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq c. \tag{3.22}$$

Second A Priori Estimate

We aim to prove the estimation

$$\left\|(\partial_t S_\tau, \partial_t E_\tau, \partial_t I_{1_\tau}, \partial_t I_{2_\tau}, \partial_t R_\tau)\right\|_{L^2(0,T;V^*)} \leq c. \tag{3.23}$$

To do this, we first recall the embedding of Sobolev spaces in dimension 3:

$$V \hookrightarrow L^q(\Omega) \quad \text{for} \quad q \in [1, 2^\star] := [1, 6], \tag{3.24}$$

$$\|v\|_p \leq C_\Omega \|v\|_V \quad \text{for all } v \in V \text{ and } p \in [1, 6], \tag{3.25}$$

$$\|v\|_p \leq \delta\|\nabla v\| + C_{\Omega,\delta,p}\|v\| \quad \text{for all } v \in V,\ p \in [1, 6) \text{ and } \delta > 0. \tag{3.26}$$

Consequently, it follows that if $f \in L^2(0, T; V)$ and $g \in L^\infty(0, T; H)$, then the product $fg \in L^2\left(0, T; L^{\frac{3}{2}}(\Omega)\right)$, and we have

$$\begin{aligned}\|fg\|_{L^2(0,T;L^{3/2}(\Omega))} &= \left(\int_0^T \|fg\|^2_{L^{3/2}(\Omega)}\right)^{1/2} \\ &= \left(\int_0^T \left|\int_\Omega |f|^{3/2}\,|g|^{3/2}\right|^{4/3}\right)^{1/2} \\ &\leq \left(\int_0^T \left|\int_\Omega |f|^6\right|^{1/3} \left|\int_\Omega |g|^2\right|\right)^{1/2} \leq c\|f\|_{L^2(0,T;L^6(\Omega))}\|g\|_{L^\infty(0,T;L^2(\Omega))}.\end{aligned} \tag{3.27}$$

Thanks to (3.1) and (3.22), we have

$$\|\lambda^\tau E_\tau\|_{L^\infty(0,T;H)} \leq \max\left\{\|E_0\|, \|E_\tau\|_{L^\infty(0,T;H)}\right\}. \tag{3.28}$$

Additionally, we have the following embeddings:

$$V \hookrightarrow L^3(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow (L^3(\Omega))^* := L^{\frac{3}{2}}(\Omega) \hookrightarrow V^\star. \tag{3.29}$$

By using (3.22), (3.27), (3.28), and (3.29), we obtain

$$S_\tau I_{j,\tau}, \quad S_\tau \lambda^\tau E_\tau \in L^2(0, T; V^*), \quad j = 1, 2. \tag{3.30}$$

This leads to the following estimation:

$$\begin{aligned}&\left\|S_\tau I_{1,\tau}\right\|_{L^2(0,T;V^*)} + \left\|S_\tau I_{2,\tau}\right\|_{L^2(0,T;V^*)} + \|S_\tau \lambda^\tau E_\tau\|_{L^2(0,T;V^*)} \\ &\leq c\left(\left\|S_\tau I_{1,\tau}\right\|_{L^2(0,T;L^{3/2}(\Omega))} + \left\|S_\tau I_{2,\tau}\right\|_{L^2(0,T;L^{3/2}(\Omega))} + \|S_\tau \lambda^\tau E_\tau\|_{L^2(0,T;L^{3/2}(\Omega))}\right) \\ &\leq \mathbf{c}\,\|S_\tau\|_{L^2(0,T;L^6(\Omega))}\left(\left\|I_{1,\tau}\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|I_{1,\tau}\right\|_{L^\infty(0,T;L^2(\Omega))} + \|\lambda^\tau E_\tau\|_{L^\infty(0,T;L^2(\Omega))}\right) \leq C.\end{aligned} \tag{3.31}$$

Now, by utilizing (3.2) and the estimates (3.22) and (3.31), we have

$$\int_0^T \langle \partial_t S_\tau(t), v(t)\rangle\, dt \leq c_1\|v\|_{L^2(0,T;V)}. \tag{3.32}$$

Therefore,

$$\|\partial_t S_\tau\|_{L^2(0,T;V^*)} \leq c_2. \tag{3.33}$$

By the same arguments, we obtain similar estimates as (3.32) for the temporal derivatives $\partial_t E_\tau$, $\partial_t I_{1_\tau}$, $\partial_t I_{2_\tau}$, and $\partial_t R_\tau$. We conclude that

$$\left\|(\partial_t S_\tau, \partial_t E_\tau, \partial_t I_{1_\tau}, \partial_t I_{2_\tau}, \partial_t R_\tau)\right\|_{L^2(0,T;V^*)} \leq c \tag{3.34}$$

for some constant $c > 0$.

Third a priori estimate.

Here, we aim to show the following result:

$$\left\|(S_\tau, E_\tau, I_{1_\tau}, I_{2_\tau}, R_\tau)\right\|_\infty \leq c. \tag{3.35}$$

To achieve this, we apply Lemma 3.2, specifically point *ii*). We first establish boundedness for the first component S_τ, using formulation (3.2) where we set

$$f = (\Lambda + \gamma_r) R_\tau \quad \text{and} \quad g = f - (\beta_1 I_{1_\tau} + \beta_2 I_{2_\tau} + \beta_e \lambda^\tau E_\tau + \eta + \mu_s) S_\tau.$$

Recalling that $\|f\|_{L^\infty(0,T;H)} \leq c$ and $g \in L^2(0, T; V^*)$, we can deduce from Lemma 3.2 that

$$S_\tau \in L^\infty(Q) \text{ and } \|S_\tau\|_\infty \leq c. \tag{3.36}$$

We can now apply part *i*) of Lemma 3.2 to (3.3) with

$$f = (\beta_1 I_{1_\tau} + \beta_2 I_{2_\tau} + \beta_e \lambda^\tau E_\tau + \eta + \mu_s) S_\tau - (\gamma_1 + \gamma_2 + \mu_e) E_\tau,$$

and deduce a similar estimate as (3.36) for the component E_τ. Proceeding similarly for the other equations, we conclude that

$$\left\|(S_\tau, E_\tau, I_{1_\tau}, I_{2_\tau}, R_\tau)\right\|_\infty \leq c. \tag{3.37}$$

Convergence Results

In this section, we will need the compactness results, which we present in the following corollary:

Corollary 1. *(see, for example, Simon, J. 1986, Sect. 8, Cor. 4)*
Suppose
$X \subset B \subset Y$ with a compact injection from X to B (X, B, and Y are Banach spaces).
• If F is bounded in $L^p(0, T; X)$ where $1 \leqslant p < \infty$, and $\partial F/\partial t = \{\partial f/\partial t : f \in F\}$ is bounded in $L^1(0, T; Y)$, then F is relatively compact in $L^p(0, T; B)$.
• If F is bounded in $L^\infty(0, T; X)$ and $\partial F/\partial t$ is bounded in $L^r(0, T; Y)$ where $r > 1$, then F is relatively compact in $C(0, T; B)$.

Thanks to estimates (3.22), (3.34), and (3.37), and using results from weak, weak-star, and strong compactness (see, for example, Simon, J. 1986, Sect. 8, Cor. 4), we have that

$$\begin{gathered} S_\tau \to S, \quad E_\tau \to E, \quad I_{1_\tau} \to I_1, \quad I_{2_\tau} \to I_2 \quad \text{and} \quad R_\tau \to R \\ \text{weakly in } H^1(0, T; V^*) \cap L^2(0, T; V) \text{ and weak-star in } L^\infty(Q), \text{ hence weakly} \\ \text{in } C^0([0, T]; H), \text{ strongly in } L^p(Q) \text{ for } 1 \leq p < +\infty, \text{ and a.e in } Q. \end{gathered} \tag{3.38}$$

when $\tau \to 0$. Specifically, the quintuplet (S, E, I_1, I_2, R) satisfies the regularities (2.6)-(2.8), the initial conditions (1.5), and the estimate (2.15) (thanks to the lower semi-continuity of norms). We also demonstrate that (S, E, I_1, I_2, R) solves the variational equations (2.9)-(2.13). To this end, we first establish that

$$\lambda^\tau E_\tau \to E \text{ strongly in } L^2(0, T; H). \tag{3.39}$$

We extend E to $[-\tau, T]$ by setting $E(t) = E_0$ for $t \in [-\tau, 0]$, and we have that

$$\begin{aligned} \|\lambda^\tau E_\tau - E\|^2_{L^2(0,T;H)} &\leq 2\, \|\lambda^\tau E_\tau - \lambda^\tau E\|^2_{L^2(0,T;H)} + 2\, \|\lambda^\tau E - E\|^2_{L^2(0,T;H)} \\ &= 2 \int_0^{T-\tau} \|E_\tau(t) - E(t)\|^2 \, dt + 2 \int_0^T \|\lambda^\tau E(t) - E(t)\|^2 \, dt. \end{aligned} \tag{3.40}$$

As $\tau \to 0$, the integrals on the right-hand side of (3.40) tend to zero, the first due to (3.38) and the second because $E : [-\tau, T] \to H$ is uniformly continuous. Thus, (3.39) is established. At this stage, with the convergence established, our assumptions on $(\beta_1, \beta_2, \beta_e)$ and (3.38) ensure that

$$\beta_1 S_\tau I_{1_\tau} \to \beta_1 S I_1, \quad \beta_2 S_\tau I_{2_\tau} \to \beta_2 S I_2, \quad \text{and} \quad \beta_e S_\tau \lambda^\tau E_\tau \to \beta_e S E \quad \text{strongly in } L^p(Q) \text{ for } p \in [1, 2).$$

Choosing (for example) $p = 3/2$, we can let τ tend to zero in the integrated version of (3.2) with arbitrary test functions $v \in L^2(0, T; V) \cap L^3(Q)$ and obtain the integrated version of (2.9) with the same test functions, which is equivalent to (2.9) itself. The same argument can be applied to derive (2.10), (2.11), (2.12), and (2.13). This concludes the proof of existence of a solution satisfying (2.15).

We now address the question of solution uniqueness (S, E, I_1, I_2, R) in the following subsection.

3.2 Proof of Solution Uniqueness and Its Continuous Dependence on Transmission Rates

This section discusses the uniqueness of the solution (S, E, I_1, I_2, R) to the problem (2.6) - (2.14) given by Theorem 1, as well as its continuous dependence on the contact rates as given by Theorem 2.

Consider two sets of fixed transmission coefficients $\left(\beta_1^{(j)}, \beta_2^{(j)}, \beta_e^{(j)}\right)$, $j = 1, 2$, whose corresponding solutions are $\left(S^{(j)}, E^{(j)}, I_1^{(j)}, I_2^{(j)}, R^{(j)}\right)$, and let us define

$$\beta_1 = \beta_1^{(1)} - \beta_1^{(2)},\ \beta_2 = \beta_2^{(1)} - \beta_2^{(2)},\ \beta_e = \beta_e^{(1)} - \beta_e^{(2)}$$
$$S = S^{(1)} - S^{(2)},\ E = E^{(1)} - E^{(2)};,\ I_1 = I_1^{(1)} - I_1^{(2)},\ I_2 = I_2^{(1)} - I_2^{(2)},\ \text{et } R = R^{(1)} - R^{(2)}.$$

Now, let's write each of the equations (2.9) to (2.13) for the two solutions and take their difference. Then, by testing the resulting equalities with s, e, i, and r respectively, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |S|^2 + \int_\Omega k_s|\nabla S|^2 + \int_\Omega (\eta + \mu_s)|S|^2 \\
&\qquad = -\int_\Omega \left[(\beta_1^{(1)} I_1^{(1)} S^{(1)} - \beta_1^{(2)} I_1^{(2)} S^{(2)})\right] S - \int_\Omega \left[(\beta_2^{(1)} I_2^{(1)} S^{(1)} - \beta_2^{(2)} I_2^{(2)} S^{(2)})\right] S \\
&\qquad\quad - \int_\Omega \left[(\beta_e^{(1)} E^{(1)} S^{(1)} - \beta_e^{(2)} E^{(2)} S^{(2)})\right] S + \int_\Omega \delta_r R S \qquad (3.41)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |E|^2 + \int_\Omega k_e|\nabla E|^2 + \int_\Omega (\gamma_1 + \gamma_2 + \mu_e)|E|^2 \\
&\qquad = \int_\Omega \left[(\beta_1^{(1)} I_1^{(1)} S^{(1)} - \beta_1^{(2)} I_1^{(2)} S^{(2)})\right] E + \int_\Omega \left[(\beta_2^{(1)} I_2^{(1)} S^{(1)} - \beta_2^{(2)} I_2^{(2)} S^{(2)})\right] E \\
&\qquad\quad + \int_\Omega \left[(\beta_e^{(1)} E^{(1)} S^{(1)} - \beta_e^{(2)} E^{(2)} S^{(2)})\right] E \qquad (3.42)
\end{aligned}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |I_2|^2 + \int_\Omega k_2|\nabla I_2|^2 + \int_\Omega (\rho + \delta_2 + \mu_2)|I_2|^2 = \int_\Omega \gamma_2 E I_2 \qquad (3.43)$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |I_1|^2 + \int_\Omega k_1|\nabla I_1|^2 + \int_\Omega (\delta_1 + \alpha + \mu_1)|I_1|^2 = \int_\Omega \gamma_1 E I_1 + \int_\Omega \rho |I_1|^2 \qquad (3.44)$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |R|^2 + \int_\Omega k_r|\nabla R|^2 + \int_\Omega (\delta_r + \mu_r)|R|^2 = \int_\Omega \delta_1 I_1 R + \delta_2 I_2 R. \qquad (3.45)$$

By summing (3.41) to (3.45), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |S|^2 + |E|^2 + |I_1|^2 + |I_2|^2 + |R|^2 + \int_\Omega k_s|\nabla S|^2 + k_e|\nabla E|^2 + k_1|\nabla I_1|^2 + k_2|\nabla I_2|^2 + k_r|\nabla R|^2 \\
&\quad + \int_\Omega (\eta + \mu_s)|S|^2 + (\gamma_1 + \gamma_2 + \mu_e)|E|^2 + (\delta_1 + \alpha + \mu_1)|I_1|^2 + (\rho + \delta_2 + \mu_2)|I_2|^2 + (\delta_r + \mu_r)|R|^2 \\
&\quad = -\int_\Omega \left[(\beta_1^{(1)} I_1^{(1)} S^{(1)} - \beta_1^{(2)} I_1^{(2)} S^{(2)})\right](S - E) - \int_\Omega \left[(\beta_2^{(1)} I_2^{(1)} S^{(1)} - \beta_2^{(2)} I_2^{(2)} S^{(2)})\right](S - E) \\
&\quad\quad - \int_\Omega \left[(\beta_e^{(1)} E^{(1)} S^{(1)} - \beta_e^{(2)} E^{(2)} S^{(2)})\right](S - E) \\
&\quad\quad + \int_\Omega \delta_r R S + \gamma_1 E I_1 + \gamma_2 E I_2 + \delta_1 I_1 R + \delta_2 I_2 R + \int_\Omega \rho |I_1|^2 \qquad (3.46)
\end{aligned}$$

which implies that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |S|^2 + |E|^2 + |I_1|^2 + |I_2|^2 + |R|^2 \\ &\leq -\int_\Omega \left[(\beta_1^{(1)} I_1^{(1)} S^{(1)} - \beta_1^{(2)} I_1^{(2)} S^{(2)})\right](S-E) - \int_\Omega \left[(\beta_2^{(1)} I_2^{(1)} S^{(1)} - \beta_2^{(2)} I_2^{(2)} S^{(2)})\right](S-E) \\ &\quad - \int_\Omega \left[(\beta_e^{(1)} E^{(1)} S^{(1)} - \beta_e^{(2)} E^{(2)} S^{(2)})\right](S-E) \\ &\quad + \int_\Omega \delta_r RS + \gamma_1 E I_1 + \gamma_2 E I_2 + \delta_1 I_1 R + \delta_2 I_2 R + \int_\Omega \rho |I_1|^2. \end{aligned} \tag{3.47}$$

Using Young's inequality, we can estimate each term on the right-hand side, the first of which can be estimated as follows:

$$-\int_\Omega \left[(\beta_1^{(1)} I_1^{(1)} S^{(1)} - \beta_1^{(2)} I_1^{(2)} S^{(2)})\right](S-E) \tag{3.48}$$

$$\begin{aligned} &= -\int_\Omega \left(\beta_1^{(1)} S I_1^{(1)} + \beta_1^{(1)} S^{(2)} I_1 + \beta_1 S^{(2)} I_1^{(2)}\right)(S-E) \\ &\leq \beta^* (\|I^{(1)}\|_\infty + \|S^{(2)}\|_\infty + \|I^{(1)}\|_\infty \|S^{(2)}\|_\infty) \int_\Omega \left(|\beta_1|^2 + |S|^2 + |E|^2 + |I_1|^2\right), \end{aligned} \tag{3.49}$$

where we estimate the other terms on the right-hand side of (3.47) similarly using the same arguments and integrating over $(0, t)$, we obtain

$$\frac{1}{2}\int_\Omega |S|^2 + |E|^2 + |I_1|^2 + |I_2|^2 + |R|^2$$

$$\leq c\int_0^t \int_\Omega \left(|\beta_1|^2 + \beta_2|^2 + \beta_e|^2 + |S|^2 + |E|^2 + |I_1|^2 + |I_2|^2 + |R|^2\right) \tag{3.50}$$

$$\leq c\int_0^T \int_\Omega \left(|\beta_1|^2 + \beta_2|^2 + \beta_e|^2\right) + c\int_0^t \int_\Omega \left(|S|^2 + |E|^2 + |I_1|^2 + |I_2|^2 + |R|^2\right) \tag{3.51}$$

where the value of c also depends on the L^∞ norms of certain components of solutions and certain parameters from (2.3). Applying the Gronwall's lemma, we obtain

$$\|(S, E, I_1, I_2, R)\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq c\,\|(\beta_1, \beta_2, \beta_e)\|_{L^2(0,T;H)}\,. \tag{3.52}$$

This proves Theorem 2.

We observe that applying (3.52) with $\beta_1^{(1)} = \beta_1^{(2)}, \beta_2^{(1)} = \beta_2^{(2)}$, and $\beta_e^{(1)} = \beta_e^{(2)}$, i.e., with $\beta_1 = 0, \beta_2 = 0$, and $\beta_e = 0$, we obtain $(S, E, I_1, I_2, R) = (0, 0, 0, 0, 0)$, meaning $\left(S^{(1)}, E^{(1)}, I_1^{(1)}, I_2^{(1)}, R^{(1)}\right) = \left(S^{(2)}, E^{(2)}, I_1^{(2)}, I_2^{(2)}, R^{(2)}\right)$. Hence, the uniqueness of the solution is established. □

References

Auricchio, F., Colli, P., Gilardi, G., Reali, A., & Rocca, E. (2023). Well?posedness for a diffusion-reaction compartmental model simulating the spread of COVID-19. *Mathematical Methods in the Applied Sciences, 46*(12), 12529-12548.

Ayoola, T. A., Edogbanya, H. O., Peter, O. J., Oguntolu, F. A., Oshinubi, K., & Olaosebikan, M. L. (2021). *Modelling and optimal control analysis of typhoid fever*.

Bellomo, N., Brezzi, F., & Chaplain, M. A. J. (2021). Special Issue on "Mathematics Towards COVID19 and Pandemic". *Math. Models Methods Appl. Sci, 31*, 2391-2397.

De Mottoni, P., Orlandi, E., & Tesei, A. (1979). Asymptotic behavior for a system describing epidemics with migration and spatial spread of infection. *Nonlinear Anal, 3*(5), 663-675.

Deressa, C. T., & Duressa, G. F. (2021). Modeling and optimal control analysis of transmission dynamics of COVID-19: The case of Ethiopia. *Alexandria Engineering Journal, 60*(1), 719-732. https://doi.org/10.1016/j.aej.2020.10.004

Fitzgibbon, W. E., Morgan, J. J., Webb, G. F., & Wu, Y. (2018). A vector-host epidemic model with spatial structure and age of infection. *Nonlinear Analysis: Real World Applications, 41*, 692-705. https://doi.org/10.1016/j.nonrwa.2017.11.005

Garad, A., Budiyanto, G. U. N. A. W. A. N., & Ansi, A. M. A. L. (2021). Impact of covid-19 pandemic on the global economy and future prospects: A systematic review of global reports. *Journal of Theoretical and Applied Information Technology, 99*(4), 1-15.

Gatto, M., Bertuzzo, E., Mari, L., Miccoli, S., Carraro, L., Casagrandi, R., & Rinaldo, A. (2020). Spread and dynamics of the COVID-19 epidemic in Italy: Effects of emergency containment measures. *Proceedings of the National Academy of Sciences, 117*(19), 10484-10491. https://doi.org/10.1073/pnas.2004978117

Ibn-Mohammed, T., Mustapha, K. B., Godsell, J., Adamu, Z., Babatunde, K. A., Akintade, D. D., ... & Koh, S. C. L. (2021). A critical analysis of the impacts of COVID-19 on the global economy and ecosystems and opportunities for circular economy strategies. *Resources, Conservation and Recycling, 164*, 105169. https://doi.org/10.1016/j.resconrec.2020.105169

Ladyzhenskaia, O. A., Solonnikov, V. A., & Ural'tseva, N. N. (1968). *Linear and quasi-linear equations of parabolic type* (Vol. 23). American Mathematical Soc..

Manfredi, P., & D'Onofrio, A. (Eds.). (2013). *Modeling the interplay between human behavior and the spread of infectious diseases*. Springer Science & Business Media.

Omame, A., Sene, N., Nometa, I., Nwakanma, C. I., Nwafor, E. U., Iheonu, N. O., & Okuonghae, D. (2021). Analysis of COVID-19 and comorbidity co-infection model with optimal control. *Optimal Control Applications and Methods, 42*(6), 1568-1590. https://doi.org/10.1002/oca.2748

Simon, J. (1986). Compact sets in the space L p (O, T; B). *Annali di Matematica pura ed applicata, 146,* 65-96.

Soma, S., Kambele, S., & Guiro, A. (2024). Global existence of a weak solution for a reaction-diffusion system in a porous media with membrane conditions and mass control. *Canad. J. Math. 2024*, pp. 1–23. http://dx.doi.org/10.4153/S0008414X24000737.

Soma, S., Kambele, S., & Guiro, A. (2024). Spatio-temporal mathematical modeling of infectious diseases with cross diffusion effects. *JP Journal of Mathematical Sciences, 35*(1), 1–35.

Song, P., & Xiao, Y. (2022). Analysis of a diffusive epidemic system with spatial heterogeneity and lag effect of media impact. *Journal of Mathematical Biology, 85*(2), 17.

Stone, L., Shulgin, B., & Agur, Z. (2000). Theoretical examination of the pulse vaccination policy in the SIR epidemic model. *Mathematical and computer modelling, 31*(4-5), 207-215. https://doi.org/10.1016/S0895-7177(00)00040-6

Taboe, H. B., Salako, K. V., Tison, J. M., Ngonghala, C. N., & Kakaï, R. G. (2020). Predicting COVID-19 spread in the face of control measures in West Africa. *Mathematical biosciences, 328*, 108431. https://doi.org/10.1016/j.mbs.2020.108431

Viguerie, A., Lorenzo, G., Auricchio, F., Baroli, D., Hughes, T. J., Patton, A., ... & Veneziani, A. (2021). Simulating the spread of COVID-19 via a spatially-resolved susceptible-exposed-infected-recovered-deceased (SEIRD) model with heterogeneous diffusion. *Applied mathematics letters, 111,* 106617. https://doi.org/10.1016/j.aml.2020.106617

Viguerie, A., Veneziani, A., Lorenzo, G., Baroli, D., Aretz-Nellesen, N., Patton, A., ... & Auricchio, F. (2020). *Diffusion-reaction compartmental models formulated in a continuum mechanics framework: application to COVID-19*, mathematical analysis, and numerical study. *Computational Mechanics, 66,* 1131-1152. https://doi.org/10.1007/s00466-020-01888-0

Wang, Z., Zhang, X., Teichert, G. H., Carrasco-Teja, M., & Garikipati, K. (2020). System inference for the spatio-temporal evolution of infectious diseases: Michigan in the time of COVID-19. *Computational Mechanics, 66*, 1153-1176. https://doi.org/10.1007/s00466-020-01894-2

Webb, G. F. (1981). A reaction-diffusion model for a deterministic diffusive epidemic. *Journal of Mathematical Analysis and Applications, 84*(1), 150-161.

Zohdi, T. (2020). An agent-based computational framework for simulation of global pandemic and social response on planet X. *Computational Mechanics, 66*(5), 1195-1209. https://doi.org/10.1007/s00466-020-01886-2

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