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# Local Entropy Solution of a Convection-Diffusion Type Integro-Differential Equation

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**Abstract**

In this work, we prove existence local entropy solution of a convection-diffusion type integro-differential equation

$$\partial_t\Big(k * (j(v) - j(v_0))\Big) - \nabla \cdot \Big(a(x, \nabla\varphi(v)) + F(\varphi(v))\Big) = f$$

in $Q_T := (0, T) \times \Omega$ with Dirichlet boundary condition $v(0, \cdot) = v_0$ in $\Omega$ and $L^1$-data $f \in L^1((0, T) \times \Omega),\ j(v_0) \in L^1(\Omega)$. To that end, regularising the data by $L^\infty$-functions, using the existence result of entropy solution for these more approximate data and a comparison and diagonal principle of the regularised entropy solution, we prove the existence of an local entropy solution.

**Keywords:** fractional time derivative, Nonlinear Volterra equation, Doubly nonlinear, Entropy solution

## 1. Introduction

This paper is devoted to the study of a class of doubly nonlinear history-dependent initial boundary value type probems of the form

$$(EP)^{j,F,k}_{f,\varphi}(v_0)\begin{cases}\partial_t\Big(k * (j(v) - j(v_0))\Big) - \nabla \cdot a(x, \nabla\varphi(v)) & \\ \qquad\qquad +\nabla \cdot F(\varphi(v)) = f & \text{in } Q_T, \\ \qquad\qquad j(v)(0, \cdot) = j(v_0) & \text{in } \Omega, \\ \qquad\qquad v = 0 & \text{on } \Sigma_T. \end{cases}$$

Our aim is to prove existence of an local entropy solutions to the problem $(EP)^{j,F,k}_{f,\varphi}(v_0)$. In our problem, the framework is the following: $\Omega$ is boundary domain of $\mathbb{R}^N$ $(N \geq 2)$, $T$ is positive number, $Q_T := (0, T) \times \Omega$ is the space-time cylinder, $\Sigma_T := (0, T) \times \partial\Omega$ where $\partial\Omega$ denotes the boundary of $\Omega$, $\nabla\varphi(v)$ stands for the gradient of $\varphi(v)$ with respect to the spatial, $p > 1$ is a real number, $p' = \dfrac{p}{p-1}$, $k \in L^1_{loc}([0, \infty))$ is a singular kernel which is type $\mathcal{PC}$, i.e. $k \in L^1_{loc}(\mathbb{R}^+)$, nonnegative, non-increasing, such that there exists a function $l \in L^1_{loc}(\mathbb{R}^+)$ satisfying $(k * l)(t) = 1$ for every $t > 0$ and the expression $(k * u)$ represents the convolution operation over the positive half-line in relation to the time variable

$$(k * u)(t) = \int_0^t k(t - s)u(s)ds,\ t > 0.$$

We further assume that the kernel $k$ satisfies additional assumptions which are introduced in subsection 2.3. Under these assumptions on $k$, our work cover the case of a time-fractional derivative of order $0 < \alpha < 1$, i.e $k(t) = t^{-\alpha}/\Gamma(1 - \alpha),\ t \in (0, \infty)$ where $\Gamma$ denotes the Gamma function.

Here, the partial derivative with respect to time of the product of the functions $k$ and $u$, denoted as $\partial_t(k*u)$, can be expressed as a distributed order derivative. This type of derivative is employed to characterize ultraslow diffusion scenarios, where the mean square displacement exhibits logarithmic growth. Such behaviour of the mean square displacement has been observed in various systems, including polymer physics and signal processing (see Naber, 2004) and others cited therein. The diffusion term, $Av = -\nabla \cdot (a(\cdot, \nabla\varphi(v)))$ is a Leray-Lions operator which is coercive , monotone and which grows like $|\nabla v|^{p-1}$ with respect to $\nabla v$. The $\mathbb{R}^N-$valued function $F$, representing the convection flux term is assumed to be defined and continuous on the whole $\mathbb{R}$. Let us stress that because the convection flux $F$ is assumed merely continuous, existence techniques for $(EP)^{j,F,k}_{f,\varphi}(v_0)$ are those of entropy solutions.

We should note that equations of the form $(EP)^{j,F,k}_{f,\varphi}(v_0)$ finds application in describing nonlinear heat flow in certain dielectric materials at extremely low temperatures. Experimental observations have revealed a finite speed of propagation for thermal disturbances in this situation. Various models have been proposed to explain this phenomenon, including the one presented by Camy, 1977, which introduces constitutive relations for internal energy and heat flux that, unlike Fourier's law, depend on the history of the temperature and the temperature gradient, respectively. As demonstrated by Cleement & Nohel, 1981, this formulation leads to a problem in the form of $(EP)^{j,F,k}_{f,\varphi}(v_0)$ under certain assumptions on the relaxation functions of internal energy and heat flux.

It is worth noting that the assumptions on $k$ are driven by the need to ensure the positivity of solutions, which is a crucial physical requirement in several applications. When modeling nonlinear heat flow in materials with memory, the function $v(t,x)$ in problem $(EP)^{j,F,k}_{f,\varphi}(v_0)$ is considered to represent the absolute temperature at the location $x$ in the domain $\Omega$ at time $t$. Such assumptions were initially introduced by Cleement & Nohel, 1979, giving rise to the concept of complete positivity, as discussed by Cleement & Nohel, 1981.

In the setting of $L^1$-data, we cannot expect weak solutions. As a result, we work with local entropy solutions.

For the doubly-nonlinear history-dependent (degenerated) problem with a time-independent operator existence and uniqueness of entropy solutions (also in the case of $L^1$-data) were shown by Jakubowski 2011, Sapountzoglou 2020, Scholtes and Petra and Soma & Bance 2023. Here the theory about generalized solutions for integro-differential equations (see Gripenberg 1985), using the $m$-accretivity of the time-independent operator, is applied. Note that, in the case $\varphi = id$, an existence result for $L^1$-data has been established by Soma & Bance, 2023.

Note that there are several articles dealing with decay estimates for time-fractional (porous medium type) equations, see, e.g., Bonforte, 2024, Schmitz & Wittbold, 2024 and Soma & Bance, 2024.

The organization of the paper is the following. In the section 2, we prepare assumptions, some tools, namely the adaptation of the regularization method of R. Landes (see Landes, 1981) and some fundamental equality and inequality, which play a crucial role in our proofs.

Afterwards, we formulate in section 3 the definition of local entropy solution and state our main result. Our definition is based on the definition of Soma & Bance 2023 .

In section 4, we formulate the abstract problem corresponding to $(EP)^{j,F,k}_{f,\varphi}(v_0)$, approximate data $v_0,\ f,\ j$ and $k$ by $v_0^{m,n} \in L^\infty(\Omega)$, $f^{m,n} \in L^\infty(Q_T)$, $j_l$ strict increasing and $k_\lambda \in W^{1,1}(0,T)$ and disrupt the problem with the term $\psi^{m,n}$. Setting $z^{m,n}_{\lambda,l} := \varphi(^{m,n}_{\lambda,l})$ and $b_l := j_l \circ \varphi^{-1}$, we know that, there exists an entropy solution $z^{m,n}_{\lambda,l}$ to the regularized and perturbed problem $(EP)^{b_l,F^{m,n},k_\lambda}_{f^{m,n},\psi^{m,n}}(z_0^{m,n})$ and can show by comparison and diagonal principle that $z^{m,n}_{\lambda,l}$ converges (up to subsequences) to a function $z := \varphi(v)$ a.e. in $Q_T$. Finally, we pass to lower limit in the inequality entropy of $(EP)^{b_l,F^{m,n},k_\lambda}_{f^{m,n},\psi^{m,n}}(z_0^{m,n})$. In this end, we use Leray Lions assumptions on the vector field $a$, assumptions on $F,\ j$ and $\varphi$ that the approximate solution $z^{m,n}_{\lambda,l}$ converges (up to subsequences) to a function $z = \varphi(v)$ a.e. in $Q_T$ where $v$ is our local entropy solution of $(EP)^{j,F,k}_{f,\varphi}(v_0)$.

Numerous references are provided at the conclusion of the paper. This list is by no means exhaustive, and additional pertinent references can be found in the cited works.

## 2. Preliminaries

### *2.1 Assumptions*

Throughout the paper, we assume that the following assumptions hold true:

$\Omega$ is boundary domain of $\mathbb{R}^N$ ($N \geq 2$) with boundary $\partial\Omega$, $T > 0$ is given and we set $Q_T := (0,T) \times \Omega$ is the space-time cylinder, $\Sigma_T := (0,T) \times \partial\Omega$, $p > 1$ is a real number, $1 < p < +\infty$ and $p' = \dfrac{p}{p-1}$.

$$a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N \text{ is Carathéodory function,} \tag{H1}$$

i.e. $a(\cdot,\xi) : \Omega \to \mathbb{R}^N$ is measurable for all $\xi \in \mathbb{R}^N$, and $a(x,\cdot) : \mathbb{R}^N \to \mathbb{R}^N$ is continuous vector field a.e. $x \in \Omega$. Moreover, we assume that $a$ satisfies the classical Leray–Lions conditions, i.e., for some real number $1 < p < +\infty$, we assume that $a$ **is monotone:**

$$\text{i.e. } \forall \xi,\zeta \in \mathbb{R}^N \text{ and a.e } x \in \Omega : (a(x,\xi) - a(x,\zeta) \cdot (\xi - \zeta) \geq 0, \tag{H2}$$

**coercive:** i.e.

$$\exists \lambda > 0,\ \forall \xi \in \mathbb{R}^N \text{ and a.e } x \in \Omega : a(x,\xi) \cdot \xi \geq \lambda |\xi|^p, \tag{H3}$$

and satisfies a **growth condition:** i.e.

$$\begin{aligned}&\exists\Lambda>0,\ g\in L^{p'}(\Omega),\ \forall\xi\in\mathbb{R}^N\\ &\text{and a.e } x\in\Omega: |a(x,\xi)|\le\Lambda\left(g(x)+|\xi|^{p-1}\right).\end{aligned}\tag{H4}$$

Thus, assumptions on $a$ are rather general.

Next, we assume that

$$\text{The scalar kernel } k:(0,\infty)\to\mathbb{R} \text{ is of type } \mathcal{PC},\tag{H5}$$

i.e. $k\in L^1_{loc}([0,\infty))$, nonnegative, nonincreasing and such that there exists a function $l\in L^1_{loc}([0,\infty))$ satisfying $(k*l)(t)=1$ for every $t>0$.

$$\begin{aligned}&j \text{ and } \varphi \text{ are two continuous functions on } \mathbb{R} \text{ normalized by,}\\ &j(0)=\varphi(0)=0 \text{ where } j \text{ is non-decreasing}\\ &\text{and } \varphi \text{ is strictly increasing and } \varphi(-r)=-\varphi(r),\ \forall r\in\mathbb{R}.\end{aligned}\tag{H6}$$

$$\begin{aligned}&F \text{ is continuous function defined on } \mathbb{R} \text{ with value in } \mathbb{R}^d\\ &\text{and satisfying } |F(r)|^{p'}\le C\left(1+|r|)^p\right)\end{aligned}\tag{H7}$$

$$\begin{aligned}&v_0 \text{ is a measurable function defined on } \Omega,.\\ &\text{such that } j(v_0)\in L^1(\Omega)\ .\end{aligned}\tag{H8}$$

$$f:Q_T\to\mathbb{R},\ \text{is an element of } L^1(Q_T).\tag{H9}$$

In the following subsection we give some of the notation, functions, definitions and the basic results which will be used later.

*2.2 Notations and Functions*

If $A\subset\Omega$ is a Lebesgue measurable set, we will denote its Lebesgue measure by $|A|$ and by $\chi_A$ its characteristic function. For any real number $K\ge 0$, we denote by $T_K:\mathbb{R}\to\mathbb{R}$, the truncation function at the level $K$, defined by

$$T_K(r)=\min(K,\max(r,-K)).$$

More precisely, for $1\le p<\infty$, the functional space $\mathcal{T}_0^{1,p}(\Omega)$ can be defined by, for every $K>0$

$$\mathcal{T}_0^{1,p}(\Omega):=\left\{v:\Omega\to\mathbb{R}\ :\ v \text{ is a measurable and } T_K(v)\in W_0^{1,p}(\Omega)\right\}.$$

We will frequently use the notation $T_{K,L}:=T_L-T_K$, for $L>K$. For $r\in\mathbb{R}$, $r\mapsto sign_0^+(r)$ is the function defined by $sign_0^+(r)=1$ if $r>0$ and $sign_0^+(r)=0$ if $r\ge 0$.

Throughout the paper, for the sake of simplicity for any $u:Q_T\to\mathbb{R}$ and for $K$ a positive real number, we write $\{|u|\le(<,>,\ge,=)K\}$ for the set $\{(t,x)\in Q_T:|u(t,x)|\le(<,>,\ge,=)K\}$. In addition, we set

$$\begin{aligned}&\mathscr{P}:=\left\{S\in C^1(\mathbb{R})\ :\ S'(t)\ge 0 \text{ for every } t>0,\ \text{Supp } S' \text{ is compact}, S(0)=0\right\};\\ &AC([0,T]):\ \text{Absolutely continuous functions on } [0,T]\end{aligned}$$

and

$$W^{1,q,0}(0,T;X):=\left\{u\in W^{1,q}(0,T;X)\ :\ u(0)=0\right\}.$$

In the sequel, $C$ denotes a constant that may change from line to line.

*2.3 Approximating the Kernel k*

In this subsection, we adapt the regularization method of R-Landes (see Landes 1981) to kernels of type $\mathcal{PC}$. This regularization will be a fundamental tool for the proof of our existence result.

**Definition 1.** A kernel $k \in L^1_{loc}([0,\infty))$ is called to be of type $\mathcal{PC}$ if it is nonnegative, nonincreasing and there exists a kernel $l \in L^1_{loc}([0,\infty))$ such that

$$(k * l)(t) = 1 \quad \text{for all } t \in [0,\infty).$$

In this case, we say that $(k,l)$ is a $\mathcal{PC}$ pair and write $(k,l) \in \mathcal{PC}$.

From $(k,l) \in \mathcal{PC}$ it follows that $l$ is completely positive.

The following corollary gives an approximation result for the kernel $k$ (see Soma & Bance, 2023).

**Corollary 2.** For all kernel $k \in L^1_{loc}(0,T)$ of type $\mathcal{PC}$ and $\lambda > 0$, there exists an approximation $k_\lambda$ of $k$ such that

$$k_\lambda \in W^{1,1}(0,T), \quad k_\lambda(0) = \frac{1}{\lambda} \text{ and } k_\lambda \to k \text{ in } L^1([0,T))$$

as $\lambda \to 0$.

Next, for $1 \leq p < \infty,\ T > 0$ and a real Banach space $X$, we consider the operator $L$ defined by:

$$D(L) = \left\{ u \in L^p(0,T;X) \ : \ k * u \in W^{1,p,0}(0,T;X) \right\}$$

and for $u \in D(L)$

$$L(u)(t) := \partial_t (k * u)(t),$$

and its Yosida approximation

$$L_\lambda(u)(t) := \partial_t (k_\lambda * u)(t), \quad \text{for all } \lambda > 0 \tag{1}$$

with $(k,l) \in \mathcal{PC}$. According to [(Clement & Mitideri, 1988, Theorem 3.1) it is know that the operator $L_\lambda$ is m-accretive in $L^p(0,T;X)$.

We can now to introduce a modification of the regularization in time by R-Landes (see Scoltes & Petra, 2018).

**Definition 3.** Let $X$ be a real Banach space, $X^*$ its dual.

For $v \in L^{p'}(0,T;X^*)$ we define $v_\varepsilon \in L^{p'}(0,T;X^*)$ by

$$v_\varepsilon(t) = \int_t^T r_\varepsilon(\tau - t) v(\tau) d\tau, \quad t \in (0,T), \quad \varepsilon > 0.$$

In the sequel the letter $\mu$ is used in this meaning only. Note that $v_\varepsilon = J^{L^*}_\varepsilon v$, where $L^* : D(L^*) \subset L^{p'}(0,T;X^*) \to L^{p'}(0,T;X^*)$ is the adjoint operator of $L$. Consequently, we have for any $v \in L^{p'}(0,T;X^*)$:

$$v_\varepsilon \to v \quad \text{in } L^{p'}(0,T;X^*) \text{ as } \varepsilon \to 0.$$

To be able to proof to the existence of an entropy solution to $(EP)^{j,F}_{k,f,\varphi}(v_0)$, let us make same further assumptions on $k$ and $k_\lambda$:

$$\begin{aligned} &\text{There exist constants } C_1;\ C_2 > 0 \text{ such that for any } \lambda > 0 \\ &\quad 0 \leq k_\lambda(t) \leq C_1 k(t) + C_2, \quad \text{almost } t \in (0,T) \end{aligned} \tag{K1}$$

$$\begin{aligned} &k \in AC_{loc}((0,T]) \text{ and there exist constants } C_1,\ C_2 > 0 \text{ such that} \\ &\quad 0 \leq -k'_\lambda(t) \leq -C_1 k'(t) + C_2, \quad \text{almost } t \in (0,T) \end{aligned} \tag{K2}$$

for all $\lambda > 0$ and

$$k'_\lambda(t) \to k'(t) \quad \text{a.e. } t \in (0,T) \text{ as } \lambda \to 0. \tag{K3}$$

We next recall a fundamental identity for integro-differential operators of the form $\partial_t(k * u)$ which will be needed for the energy estimate.

**Lemma 4.** . *Let $T > 0$ and $U$ be an open subset of $\mathbb{R}$. Let further $k \in W^{1,1}(0,T)$, $H \in C^1(U)$ and $u \in L^1(0,T)$ with $u(t) \in U$ for almost every $t \in (0,T)$. Suppose that $H(u)$, $H'(u)$, $H'(u)(k' * u) \in L^1(0,T)$. Then, we have for almost every $t \in (0,T)$,*

$$H'(u(t))\partial_t(k * u)(t) = \partial_t(k * H(u))(t) + (H'(u(t))u(t) - H(u(t)))\, k(t)$$
$$+ \int_0^t (H(u(t-s)) - H(u(t)) - H'(u(t))[u(t-s) - u(t)])[-k'(s)]ds$$

Equality (4) is highly important for deriving a priori estimates for problems of the form $(EP)^{j,F}_{k,f,\varphi}(v_0)$.

*2.4 Approach to the Abstract Volterra Equations*

The proof of our an entropy solution existence result presented in this article will be based on the theory of G. Gripenberg (see Gripenberg, 1985) for abstract nonlinear Volterra integro-differentials equations of the form

$$\frac{\partial}{\partial t}(\gamma(u(t) - u_0) + \int_0^t k(t-s)(u(s) - u_0)ds) + A(u(t)) \ni f(t), \quad t \in [0,T) \tag{2}$$

in a real Banach space $X$. Here $\gamma$ is a nonnegative constant, $k$ is a scalar kernel that is assumed to be locally integrable, nonnegative and nonincreasing function on $\mathbb{R}^+$, $A$ is an m-acrretive, possibly multivalued operator in $X$, $u_0 \in X$ and $f \in L^1(0,T;X)$.

In this subsection, we recall the definitions and the main results of the abstract theory. We limit ourselves to the case which will be treated in our purpose, i.e. $\gamma = 0$ and $k$ of type $\mathcal{PC}$. The abstract problem (2) then takes form

$$\partial_t[k * (u - u_0)](t) + A(u(t)) \ni f(t), \quad t \in [0,T) \tag{3}$$

The theory of G. Gripenberg is to consider for $\lambda > 0$ the following approximating problem

$$\partial_t[k_\lambda * (u - u_0)](t) + A(u(t)) \ni f(t), \quad t \in [0,T). \tag{4}$$

Here, $k_\lambda$, $\lambda > 0$ are the kernels associated to the Yosida approximations of the operator given by (1).

**Definition 5.** A measurable function $u : [0,T) \to X$ is called strong solution to the approximating equation (4), if $u \in L^1(0,T;X)$ and there exists $w \in L^1(0,T;X)$ such that $w(t) \in A(u(t))$ and

$$\partial_t[k_\lambda * (u - u_0)](t) + w(t) = f(t), \tag{5}$$

for almost every $t \in [0,T)$.

The abstract approximating problem (4) admits a unique strong solution $u_\lambda$, for every $\lambda > 0$ in the sense of Definition 5 (see Gripenberg, 1985, Theorem 1]). The generalized solution to (3) is defined as follows (see Gripenberg, 1985):

**Definition 6.** Let $(u_\lambda)_\lambda > 0$ be the strong solutions to the approximating problem (4). If there exists a functions $u \in L^1(0,T;X)$ such that $u_\lambda \to u$ in $L^1(0,T;X)$ as $\lambda$ tends to 0, then $u$ is called the generalized solution to (3).

By definition, the generalized solution is unique.

The following theorem is the main existence result of the abstract theory ( for the proof, see Gripenberg, 1985; Theorem 1]).

**Theorem 7.** *Let X be a real Banach space. Assume that A is an m-acrretive operator in X, $u_0 \in \overline{D(A)}$ and $f \in L^1(0,T;X)$, then there exists a generalized solution to* (3).

## 3. Definition of Local Entropy Solution and Main Result

The definition of a local entropy solutions for problem $(EP)^{j,F,k}_{f,\varphi}(v_0)$ can be stated as follows:

**Definition 8.** A measurable function $v : Q_T \to \overline{\mathbb{R}}$ is called an local entropy solution of $(EP)^{b,F,k}_{f,\varphi}(v_0)$ if

(P1) $j(v) \in L^1(Q_T)$ and $T_K(v) \in L^p(0,T;W^{1,p}_0(\Omega))$ for any $K > 0$, and for any functions $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\xi \in \mathscr{D}([0,T))$ with $\xi \geq 0$, $S \in \mathscr{P}$, for all $q \in$ , we have

$$
\begin{aligned}
&\text{(P2)}\\
&-\int_{Q_T}\Big(\int_0^t k_{1,q}(t-\tau)\int_{v_0(x)}^{v(\tau,x)} S(\varphi(\sigma)-\phi(x))dj(\sigma)\Big)d\tau\, \xi_t(t)dxdt\\
&+\int_{Q_T} k_{2,q}(0^+)\left(j(v(t,x))-j(v_0(x))\right)S(\varphi(v(t,x))-\phi(x))\xi(t)dxdt\\
&+\int_{Q_T}\Big(\int_0^t \left(j(v(t-\tau,x))-j(v_0(x))\right)dk_{2,q}(\tau)\Big)S(\varphi(v(t,x))-\phi(x))\xi(t)dxdt\\
&+\int_{Q_T}\left(a(x,\nabla\varphi(v(t,x)))+F(\varphi(v(t,x)))\right)\cdot DS(\varphi(v(t,x))-\phi(x))\xi(t)dxdt\\
&\leq \int_{Q_T} f(t,x)S(\varphi(v(t,x))-\phi(x))\xi(t)dxdt,
\end{aligned}
$$

where $k_{1,q} := (k-q)^+$ and $k_{2,q} := k - k_{1,q}$.

The following remarks are concerned with a few comments on Definition 8.

**Remark 9.** 1. Note that in Definition 8, $a(\cdot, \nabla\varphi(v))$ and $F(\varphi(v))$ do note general make sense in the first equation of problem $(EP)^{j,F,k}_{f,\varphi}(v_0)$ , but that to (P1) each term in inequality (P2) has meaning in $\mathcal{D}'(Q_T)$.

2. Since $\varphi$ is continuous stricly increasing, $j$ is continuous non-decreasing and $j(0) = \varphi(0) = 0$, then we can define the function $b := j \circ \varphi^{-1}$ which is continuous non-decreasing and verifies $b(0) = 0$. If we define $z := \varphi(v)$ and $z_0 := \varphi(v_0)$, then $(EP)^{j,F,k}_{f,\varphi}(v_0)$ is equivalent to

$$
(EP)^{b,F}_{k,f}(z_0)\begin{cases}\partial_t\Big(k*(b(z)-b(z_0))\Big)-\nabla\cdot\tilde{a}(x,z,\nabla z)=f & \text{in } Q_T,\\ b(z)(0,\cdot)=b(z_0) & \text{in } \Omega,\\ z=0 & \text{on } \Sigma_T,\end{cases}
$$

where $\tilde{a}(x,z,\nabla z) := a(x,\nabla z) + F(z)$.

Now, we formulate our main existence result of a entropy solution of $(EP)^{j,F,k}_{f,\varphi}(v_0)$ which is given by the following Theorem:

**Theorem 10.** *Assume that $j$ satisfies* (H6)*, the vector fields $a$ and $F$ satisfy* (H1)-(H4) *and* (H7) *and that the scalar kernel $k$ satisfy* (K1)-(K3). *Let $f \in L^1(Q_T)$ and $v_0 : \Omega \to \mathbb{R}$ a measurable function. Then there exists at least one local entropy solution $v$ of problem $(EP)^{j,F,k}_{f,\varphi}(v_0)$.*

To prove Theorem 10, we will use several techniques and approximation procedures. First, we will construct the abstract problem corresponding to our problem $(EP)^{j,F,k}_{f,\varphi}(v_0)$.

## 4. Abstract Problem Corresponding to $(EP)^{j,F,k}_{f,\varphi}(v_0)$ and Approximations

### *4.1 Abstract Problem Corresponding to $(EP)^{j,F,k}_{f,\varphi}(v_0)$*

Since our objective is to apply the abstract theory of G.Gripenberg, let then $b$ a contunuous non-decreasing functions on $\mathbb{R}$ normalized by $b(0) = 0$ and the graph of the possibly multivalued operator $A_b : L^1(\Omega) \to 2^{L^1(\Omega)}$ be defined by

$$
\begin{aligned}
&A_b \subset L^1(\Omega)\times L^1(\Omega),\ (b(z),w)\in A_b\\
&\Leftrightarrow\ b(z),\ w\in L^1(\Omega),\ z\in\mathcal{T}_0^{1,p}(\Omega),\\
&\text{and } \int_\Omega (a(x,\nabla z)+F(z))\cdot\nabla T_K(z-\phi)dx \leq \int_\Omega wT_K(z-\phi)dx
\end{aligned}
\tag{6}
$$

for any $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. This characterization of the operator $A_b$ is based on the results that are shown by Boccardo, 1996.

Thus, using this characterization of the operator $A_b$ and by the same arguments of SCholtes, 2016, we can establish the following Lemma:

**Lemma 11.** *Let $A_b$ the operator defined* (6)*. Then $(b(z), w) \in A_b$ implies that*

$$\int_\Omega (a(x, \nabla z) + F(z)) \cdot \nabla(h(z)\xi)dx = \int_\Omega wh(z)\xi dx \tag{7}$$

*for all $h \in C_c^1(\mathbb{R})$ and all $\xi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Using the result of Lemma 11 we can prove the following result:

**Corollary 12.** *Let $A_b$ the operator defined* (6)*. Then $(b(v), w) \in A_b$ implies that*

$$\int_\Omega (a(x, \nabla z) + F(z)) \cdot DS(z - \phi)dx = \int_\Omega wS(z - \phi)dx \tag{8}$$

*for all $S \in \mathscr{P}$ and all $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

We state some properties of the operator $A_b$ in the following Proposition (for the proof see Zimmermann, 2010).

**Proposition 13.** *The operator $A_b$ satisfies the following properties:*

*i) $A_b$ is m-accretive in $L^1(\Omega)$,*

*ii) $\overline{D(A_b)} = \left\{u \in L^1(\Omega) : u(x) \in \overline{ran(b)} \text{ for almost every } x \in \Omega\right\}$.*

Now, taking the Banach space $X = L^1(\Omega)$ and using the operator $L$ defined in (1), since the operator $A_b$ is m-accretive, then Theorem 7 entails that the abstract Volterra equation

$$L(u - u_0)(t) + A_b(u(t)) \ni f(t), \quad \text{in } L^1(\Omega) \tag{9}$$

admit for almost all $t \in (0, T)$, all $u_0 = b(z_0) \in \overline{D(A_b)}$ and $f \in L^1(0, T; L^1(\Omega)) \equiv L^1(Q_T)$ a unique generalized solution $u \in L^1(Q_T)$. But, it is a priori not clear in which sense the generalized solution satisfies $(EP)_{k,f}^{b,F}(z_0)$ .

In order to show that $u = b(z)$ where $z$ satisfies $(EP)_{k,f}^{b,F}(z_0)$ , we will define approximate and perturbed problems associated to $(EP)_{k,f}^{b,F}(z_0)$ in the next subsection.

*4.2 Approximation of the Data and Entropy Solution of the Regularised Problem Corresponding to $(EP)_{k,f}^{b,F}(z_0)$.*

Note that, for general $j$, we can not expect to find a strong solution which solves the inclusion problem (9).

In order to overcome this difficulty, let us introduce the following regularizations:

(R0) $j_l := j + \frac{1}{l} \cdot id_\mathbb{R}$, for $l > 0$.

(R1) $b_l := j_l \circ \varphi^{-1} = b + \frac{1}{l} \cdot \varphi^{-1}$, for $l > 0$.

(R2) $f^{m,n} := \max(\min(f, m), -n)$, for $m, n \in \mathbb{N}^*$.

(R3) $v_0^{m,n} := \max(\min(v_0, m), -n)$, for $m, n \in \mathbb{N}^*$.

(R4) $\psi^{m,n}(r) := \frac{1}{m}r^+ - \frac{1}{n}r^-$, for all $r \in \mathbb{R}$.

(R5) $A_b^{\psi^{m,n}}$ the perturbed operator defined as:

$$\begin{aligned} A_b^{\psi^{m,n}} &\subset L^1(\Omega) \times L^1(\Omega),\ (b(z), w) \in A_b^{\psi^{m,n}} \\ &\iff b(z),\ w \in L^1(\Omega),\ z \in \mathcal{T}_0^{1,p}(\Omega), \\ &\text{and} \int_\Omega (a(x, \nabla z) + F(z)) \cdot \nabla T_K(z - \phi)dx \\ &+ \int_\Omega \psi^{m,n}(z)T_K(z - \phi)dx \leq \int_\Omega wT_K(z - \phi)dx \end{aligned} \tag{10}$$

for any $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Furthermore, we set $z^{m,n}_{0,l} = b_l(z^{m,n}_0)$.

By Bénilan & Wittbold, we affirm that operator $A^{\psi^{m,n}}_b$ is m-accretive in $L^1(\Omega)$ and we have

$$\overline{D(A^{\psi^{m,n}}_b)}^{\|\cdot\|_{L^1(\Omega)}} = \left\{u \in L^1(\Omega) : u(x) \in \overline{ran(b)} \text{ for almost every } x \in \Omega\right\} = \overline{D(A_b)}^{\|\cdot\|_{L^1(\Omega)}}.$$

Note that, the function $b_l$ is a strictly increasing approximation of $b$, $f^{m,n} \in L^\infty(Q_T)$ for each $m, n \in \mathbb{N}$, $|f^{m,n}(t, x)| \leq |f(t, x)|$ a.e. in $Q_T$, hence $\lim_{n\to\infty}\lim_{m\to\infty} f^{m,n} = f$ in $L^1(Q_T)$ and almost everywhere in $Q_T$. Similarly, we have $z^{m,n}_0 \in L^\infty(\Omega)$, $z^{m,n}_0 \to z_0$ a.e. in $\Omega$, $b_l(z^{m,n}_0) \to b(z_0)$ in $L^1(\Omega)$ and a.e. in $\Omega$ as $m$ and $n$ tend to infty and $l$ tends to infty.

Let us now consider the following regularized problem:

$$(EP)^{b,F^{m,n},k}_{f^{m,n},\psi^{m,n}}(z^{m,n}_0)\begin{cases}\partial_t\Big(k * (b(z) - b(z^{m,n}_0))\Big) - \nabla \cdot \tilde{a}(x, z, \nabla z) \\ \qquad + \psi^{m,n}(z) = f^{m,n} & \text{in } Q_T,\\ b(z^{m,n})(0, \cdot) = b(z^{m,n}_0) & \text{in } \Omega,\\ z = 0 & \text{on } \Sigma.\end{cases}$$

whose associated abstract problem is

$$L(u - u^{m,n}_0)(t) + A^{\psi^{m,n}}_b(u(t)) \ni f^{m,n}(t), \quad \text{in } L^1(\Omega) \tag{11}$$

We cannot expect to have a strong solution to the abstract problem (11). However, we know by [[**?**], Theorem 1] that the corresponding approximating abstract problem with respect to the Yosida approximating of $L$ defined by:

$$L_\lambda(u - u^{m,n}_{0,l})(t) + A^{\psi^{m,n}}_{b_l}(u(t)) \ni f^{m,n}(t), \quad \text{in } L^1(\Omega) \tag{12}$$

for almost everywhere $t \in (0, T)$, admits a unique strong solution $u^{m,n}_{\lambda,l} = b_l(z^{m,n}_{\lambda,l})$ in $L^1(Q_T)$ in the sense of Definition 5 and through the Theorem 7, there exists a measurable function $u^{m,n}_l$ in $L^1(Q_T)$ such that $u^{m,n}_{\lambda,l} \to u^{m,n}_l$ in $L^1(Q_T)$ as $\lambda$ tends to 0, where $u^{m,n}_l$ is the generalized solution to (11) in the sense of Definition 6.

As the function $b_l$ is bijective, then there exists a unique measurable function $z^{m,n}_{\lambda,l}$ such that $b_l(z^{m,n}_{\lambda,l}) = u^{m,n}_{\lambda,l}$.

**Definition 14.** A measurable function $z : Q_T \to \overline{\mathbb{R}}$ is called an entropy solution of $(EP)^{b,F^{m,n},k}_{f^{m,n},\psi^{m,n}}(z^{m,n}_0)$ if

(A1) $z^{m,n} \in L^1(Q_T)$ and $T_K(z^{m,n}) \in L^p(0, T; W^{1,p}_0(\Omega))$ for any $K > 0$,

and for any functions $\phi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$, $\xi \in \mathscr{D}([0, T))$ with $\xi \geq 0$, $S \in \mathscr{P}$, for all $q \in$ , we have

(A2)

$$\begin{aligned}&- \int_{Q_T}\Big(\int_0^t k_{1,q}(t - \tau)\int_{z^{m,n}_0(x)}^{z^{m,n}(\tau,x)} S(\sigma - \phi(x))db(\sigma)\Big)d\tau\, \xi_t(t)dxdt\\ &+ \int_{Q_T} k_{2,q}(0^+)\Big(b(z^{m,n}(t, x)) - b(z^{m,n}_0(x))\Big)S(z^{m,n}(t, x) - \phi(x))\xi(t)dxdt\\ &+ \int_{Q_T}\Big(\int_0^t\Big(z^{m,n}(t - \tau, x) - z^{m,n}_0(x))\Big)\Big)dk_{2,q}(\tau)\Big)S(z^{m,n}(t, x) - \phi(x))\xi(t)dxdt\\ &+ \int_{Q_T}(a(x, \nabla z^{m,n}(t, x)) + F(z^{m,n}(t, x))) \cdot \nabla S(z^{m,n}(t, x) - \phi(x))\xi(t)dxdt\\ &\leq \int_{Q_T} f(t, x)S(z^{m,n}(t, x) - \phi(x))\xi(t)dxdt,\end{aligned}$$

where $k_{1,q} := (k - q)^+$ and $k_{2,q} := k - k_{1,q}$.

Regularized problems $(EP)^{b_l,F^{m,n},k_\lambda}_{f^{m,n},\psi^{m,n}}(z^{m,n}_0)$ and $(EP)^{b,F^{m,n},k}_{f^{m,n},\psi^{m,n}}(z^{m,n}_0)$ admit at least one solution in the sense of Definition 14 (see Soma & Bance, 2023).

***Step 3:*** *Priori estimates and convergence results.*

The following result gives a priori estimates on the strong solution of abstract problem (12) and the entropy solution of $(EP)^{b,F^{m,n},k}_{f^{m,n},\psi^{m,n}}(z_0^{m,n})$ (see Soma & Bance, 2023).

**Lemma 15.** *Let* $\left\{u^{m,n}_{\lambda,l} = b_l(z^{m,n}_{\lambda,l})\right\}$ *be the sequence of strong solutions of* (12) *with* $z^{m,n}_{\lambda,l} = \varphi(v^{m,n}_{\lambda,l})$*. Then, the sequence* $\left\{v^{m,n}_{\lambda,l}\right\}$ *satisfies,*

*(i)* $\left|\left\{|v^{m,n}_{\lambda,l}| \geq K\right\}\right| \leq \int_0^T \int_\Omega |\nabla T_K(z^{m,n}_{\lambda,l})| dxdt \leq CK$ *for every* $K > 0$*, where* $C = C(f, k, v_0, b) > 0$ *is a constant independant of* $\lambda$*,* $l$*,* $m$ *and* $n$*.*

*(ii)* $\|v^{m,n}_{\lambda,l}\|_{L^\infty(Q_T)} \leq \max(m^2, n^2)$.

Using a priori estimate of the preceeding Lemma, we have the following convergence and a priori estimate (see Soma & Bance, 2023):

**Corollary 16.** By the diagonal principle, there exist subsequences $(\lambda(l))_l$, $(z^{m,n}_{\lambda(l)})_l$ and $(v^{m,n}_{\lambda(l)})_l$ such that, as $l \to \infty$

$$\begin{aligned}
&\lambda(l) \to 0,\\
&z^{m,n}_{\lambda(l)} \to z^{m,n} \quad \text{in } L^1(Q_T) \text{ and } a.e \text{ in } Q_T\\
&v^{m,n}_{\lambda(l)} \to v^{m,n} \quad \text{in } L^1(Q_T) \text{ and } a.e \text{ in } Q_T.
\end{aligned}$$

Here, $z^{m,n}$ is entropy solution of $(EP)^{b,F^{m,n},k}_{f^{m,n},\psi^{m,n}}(z_0^{m,n})$ and $z^{m,n} = \varphi(v^{m,n})$. An additional, we have the following a priori estimate

$$\|z^{m,n}\|_{L^\infty(Q_T)} \leq \max(m^2\ ,\ n^2), \quad \|v^{m,n}\|_{L^\infty(Q_T)} \leq \max(m^2\ ,\ n^2).$$

Now, for $m, n \in \mathbb{N}^*$, $f \in L^1(Q_T)$ and $b(z_0) \in \overline{D(A_b)}^{\|\cdot\|}_{L^1(\Omega)}$, let $z^{m,n} = \varphi(v^{m,n})$ be the entropy solution to $(EP)^{b,F^{m,n},k}_{f^{m,n},\psi^{m,n}}(z_0^{m,n})$. Then, by Lemma 15 and Corollary 16, there exists subsequences $(m(n))_n$, $(l(n))_n$ and $(\lambda(l(n)))_n$ and such that setting $\psi^n := \psi^{m(n),n}$; $f^n := f^{m(n),n}$; $j_n := j_{l(n)}$; $v_0^n := v_0^{m(n),n}$, $v^n := v^{m(n),n}_{\lambda(l(n)),l(n)}$, $\tilde{v}^n := v^{m(n),n}$ and $k_n := k_{\lambda(l(n))}$, we get convergence results in the following three lemma:

**Lemma 17.** *As* $n \to \infty$*, the following basic convergences hold true:*

$$\begin{aligned}
&\text{(i)} \quad f^n \longrightarrow f \quad \textit{in } L^1(Q_T),\\
&\text{(ii)} \quad k_n \longrightarrow k \quad \textit{in } L^1([0,T]),\\
&\text{(iii)} \quad \psi^n \longrightarrow \quad \textit{everywhere in } \mathbb{R},\\
&\text{(iv)} \quad v_0^n \longrightarrow v_0 \quad \textit{a.e. in } \Omega,\\
&\text{(v)} \quad \tilde{v}^n \longrightarrow v \quad \textit{a.e. in } Q_T,\\
&\text{(vi)} \quad v^n \longrightarrow v \quad \textit{a.e. in } Q_T.
\end{aligned}$$

*Proof:* Applying the diagonal principle, we can construct sub-sequences $(m(n))_n$, $(l(n))_n$ all converging towards infinity and a subsequence $(\lambda(l(n)))_n$ converging to zero as $n \to \infty$. Thus, setting $f^n := f^{m(n),n}$, $k_n := k_{\lambda(l(n))}$, $\psi^n := \psi^{m(n),n}$, $v_0^n := v_0^{m(n),n}$, we get that (i) − (iv) are direct consequences of approximation procedure. Next, by comparison principle (see Soma & Bance, 2023, Lemma 4.14), strictly increasing and continuity of $\varphi$ and Lemma 15, there exist a measurable function $v : Q_T \to \overline{\mathbb{R}}$ verifying $|v| < \infty$ a.e. in $Q_T$ such that (v) − (vi) hold true. □

Using Lemma 17, we get the following Lemma:

**Lemma 18.** *As* $n \to \infty$*, the following convergences hold true:*

$$\begin{aligned}
&\text{(i)} \quad j_n(v_0^n) \longrightarrow j(v_0) \textit{ in } L^1(\Omega),\\
&\text{(ii)} \quad j(\tilde{v}^n) \longrightarrow j(v) \textit{ in } L^1(Q_T),\\
&\text{(iii)} \quad j_n(v^n) \longrightarrow j(v) \textit{ in } L^1(Q_T).
\end{aligned}$$

*Proof:* (i): Taking $l(n)$ such that $l(n) \geq m(n)$ for any $n \in \mathbb{N}^*$, we obtain

$$|b_n(z_0^n) - b(z_0)| \leq |b_n(z_0^n) - b(z_0^n)| + |b(z_0^n) - b(z_0)| = \frac{1}{l(n)}|v_0^n| + |b(z_0^n) - b(z_0)|$$

Since $|v_0^n| \leq m(n) \leq l(n)$ a.e. in $\Omega$ and for every $n \in \mathbb{N}$, then we obtain that $|b_n(v_0^n) - b(v_0)| \to 0$ a.e. in $\Omega$ as $n \to \infty$. In addition, we also have $|b_n(z_0^n) - b(z_0)| \leq 1 + 2|b(z_0)|$. So, by Lebesgue's theorem, we get $b_n(z_0^n) \longrightarrow b(z_0)$ in $L^1(\Omega)$ as $n \to \infty$. By the definitions of $z_0,\ z_0^n,\ b$ and $b_n$, it follows that (i) holds true. (ii) : We know that by Gripenberg, 1985, the abstract problem

$$L\left(u - b(z_0^{m,n})\right)(t) + A_b^{\psi^{m,n}}(u(t)) \ni f^{m,n}(t), \quad t \in [0, T)$$

admits a unique generalized solution $u^{m,n} = b(z^{m,n})$ belongs to $L^1(Q_T)$ and by diagonal principle, there exists some function $u \in L^1(Q_T)$ such that $b(z^{m(n),n}) \to u$ in $L^1(Q_T)$ as $n \to \infty$. Since, $z^{m(n),n} \to z$ a.e. in $Q_T$ and $b$ is continuous on $\mathbb{R}$, then it follows that $u = b(z)$ i.e. $u = j(v)$. (iii): It's the same approach as in (vii), since $v^n \to v$ a.e. in $Q_T$ and $|v| < \infty$ a.e. on $Q_T$, it also comes that $j_n(v^n) \to j(v)$ in $L^1(Q_T)$. $\square$

The third lemma below gives us the convergence of truncated energies.

**Lemma 19.** *Let $z^n := z_{\lambda(l(n)),l(n)}^{m(n),n}$ be a entropy solution of $(EP)_{f^n,\psi^n}^{b_n,F^n,k_n}(z_0^n)$. Then for every $K > 0$ and up to subsequences, as $n \to \infty$, the following result hold true:*

$$\text{(i)} \quad T_K(z^n) \rightharpoonup T_K(z) \quad \textit{in } L^p(0, T; W_0^{1,p}(\Omega)),$$
$$\text{(ii)} \quad a(\cdot, \nabla T_K(z^n)) \rightharpoonup a(\cdot, \nabla T_K(z)) \quad \textit{in } \left(L^{p'}(Q_T)\right)^N.$$

*Proof:* (i) can be deduced from Lemma 15 and the convergence $z_n \to z$ a.e. in $Q_T$. Concerning (ii), by Lemma 15 and growth condition (H4), it follows that for every $K > 0$, exists $\tilde{\Phi}_K \in \left(L^{p'}(Q_T)\right)^N$ such that

$$a(\cdot, \nabla T_K(z^n)) \rightharpoonup \tilde{\Phi}_K \quad \text{weakly in } \left(L^{p'}(Q_T)\right)^N$$

as $n \to \infty$. To prove that $\tilde{\Phi}_K = a(\cdot, \nabla T_K(z))$, we proceed as Soma & Bance, 2023 . Just replace $\tilde{z}_l^{m,n}$ by $z^n$; $z^{m,n}$ by $z$; $b_l$ by $b_n$; $k_l$ by $k_n$; $z_0^{m,n}$ by $z_0$ and $\psi^{m,n}$ by $\psi^n$. Note that, as $z^n \to z$ a.e. in $Q_T$, $\psi^n \to 0$ uniformly on compact sets, then by Fatou's lemma it follows that

$$\begin{aligned}\liminf_{n\to\infty} \int_{Q_\tau} &\psi^n(z^n)\,(T_K(z^n) - h_i(z^n) T_K(z)_\varepsilon)\,dxdt \\ &\geq \int_{Q_\tau} \liminf_{n\to\infty} \psi^n(z^n)\,(T_K(z^n) - h_i(z^n) T_K(z)_\varepsilon)\,dxdt = 0.\end{aligned}$$

Thus, we obtain that

$$\liminf_{i\to\infty} \liminf_{\varepsilon\to 0} \liminf_{n\to\infty} \int_{Q_\tau} \psi^n(z^n)\,(T_K(z^n) - h_i(z^n) T_K(z)_\varepsilon)\,dxdt \geq 0.$$

So, by the same argument like those of Soma & Bance, 2023, we can deduce that

$$a(\cdot, \nabla T_K(z^n)) \rightharpoonup a(\cdot, \nabla T_K(z)) \quad \text{weakly in } \left(L^{p'}(Q_T)\right)^N$$

for every $K > 0$ and as $n \to \infty$. $\square$

*4.3 Conclusion of the Proof of the Theorem 10: Passage to the Lower Limit*

Let $z^n := \varphi(v^n)$ be an entropy solution of $(EP)_{f,\psi^n}^{b_n,F^n,k_n}(z_0^n)$. On the basis of the convergence results above, we can state the following proposition

**Proposition 20.** *As $n \to \infty$, the following convergences hold true:*

$$\begin{cases} k_{1,n,q} * \int_{z_0^n}^{z^n} S(\sigma-\phi)db_n(\sigma) \\ \to k_{1,q} * \int_{\varphi(v_0)}^{\varphi(v)} S(\varphi(\sigma)-\phi)dj(\sigma), \text{ stronly in } L^1(Q_T), \end{cases} \tag{13}$$

$$\begin{cases} \partial_t[k_{2,n,q} * (b_n(z^n) - b_n(z_0^n))](t)S(z^n(t,\cdot)-\phi) \\ \rightharpoonup \partial_t[k_{2,q} * (j(v) - j(v_0))](t)S(\varphi(v(t,\cdot))-\phi) \\ \text{weakly in } L^1(Q_T), \end{cases} \tag{14}$$

$$\begin{cases} F^n(z^n) \cdot \nabla S(z^n(t,\cdot)-\phi) \\ \rightharpoonup F(\varphi(v)) \cdot \nabla S(\varphi(v)(t,\cdot)-\phi), \text{ weakly in } L^1(Q_T), \end{cases} \tag{15}$$

$$\left\{ f^n S(z^n-\phi)\xi \longrightarrow fS(\varphi(v)-\phi), \text{ stronly in } L^1(Q_T). \right. \tag{16}$$

*Proof:* Thanks to the convergence result of the Lemma 17 with the continuity of $\varphi$ and by Lebesgue's convergence theorem, we note that

$$\int_{z_0^n}^{z^n} S(\sigma-\phi)db_n(\sigma) \overset{n\to\infty}{\longrightarrow} \int_{\varphi(v_0)}^{\varphi(v)} S(\varphi(\sigma)-\phi)dj(\sigma) \quad \text{in } L^1(Q_T).$$

Since $k_{1,n,q} \to k_{1,q}$ in $L^1([0,T))$, it follows by Young's inequality that

$$k_{1,q} * \int_{\varphi(v_0)}^{\varphi(v)} S(\varphi(\sigma)-\phi)dj(\sigma) \in L^1(Q_T)$$

for a subsequence if necessary a.e. in $Q_T$. Hence, (13) holds true and

$$\begin{aligned} &-\int_0^T \int_\Omega \Big[k_{1,n,q} * \int_{z_0^n}^{z^n} S(\sigma-\phi)db_n(\sigma)\Big]\xi_t(t)dxdt \\ &\overset{n\to\infty}{\longrightarrow} -\int_0^T \int_\Omega \Big[k_{1,q} * \int_{\varphi(v_0)}^{\varphi(v)} S(\varphi(\sigma)-\phi)dj(\sigma)\Big](t)\xi_t dxdt. \end{aligned} \tag{17}$$

By the triangle ineguality, we have the following estimate:

$$\begin{aligned} &\|\partial_t[k_{2,n,q} * (b_n(z^n) - b_n(z_0^n))] - \partial_t[k_{2,q} * (b(z) - b(z_0))]\|_{L^1(Q_T)} \\ &\le \|\partial_t[k_{2,n,q} * ((b_n(z^n) - b_n(z_0^n)) - (b(z) - b(z_0)))]\|_{L^1(Q_T)} \\ &\quad + \|\partial_t[k_{2,n,q} * (b(z) - b(z_0))] - \partial_t[k_{2,q} * (b(z) - b(z))]\|_{L^1(Q_T)} \\ &:= D_q^n + E_q^n. \end{aligned}$$

Since $k_{2,n,q} \in W^{1,1}(0,T)$ verifies $0 \le k_{2,n,q}(0) \le q$ for any $n$ and $q \in \mathbb{N}$. So, from Young's inequality, we have

$$\begin{aligned} D_q^n &\le q\|(b_n(z^n) - b_n(v_0^{nn})) - (b(z) - b(z_0))\|_{L^1(Q_T)} \\ &\quad + \|k'_{2,n,q}\|_{L^1(0,T)}\|(b_n(z^n) - b_n(z_0)) - (b(z) - b(z))\|_{L^1(Q_T)} \end{aligned}$$

for any $n > 0$ and $q \in \mathbb{N}$. As $k'_{2,n,j}$ is non-negative and non-increasing, then

$$Var_{[0,T]}k_{2,n,q} = \|k'_{2,n,q}\|_{L^1(0,T)} \le k_{2,n,q}(0) \le q$$

for any $n > 0$ and $q \in \mathbb{N}$, where $Var_{[0,T]}$ denotes the variation on the interval $[0,T]$. Thus, we have just shown that $A_q^n \longrightarrow 0$ as $n \to \infty$.

Moreover, we may conclude by Gripenberg, 1985 that $B_q^n \longrightarrow 0$ as $n \to \infty$. Its follows that

$$\partial_t[k_{2,n,q} * (b_n(z^n) - b_n(z_0^n))] \to \partial_t[k_{2,q} * (b(z) - b(z_0))]$$

in $L^1(Q_T)$ as $n \to \infty$. Since, $S$ is bounded and continuous, the convergence $z^n \to z$ a.e. in $Q_T$ implies that $S(z^n - \phi)$ converges to $S(\varphi(v) - \phi)$ a.e. in $Q_T$ and $L^\infty(Q_T)$ weak-$\star$. So, (14) holds true and

$$\begin{aligned} &\int_0^T \int_\Omega \partial_t[k_{2,n,q} * (b_n(z^n) - b_n(z_0^n))](t)S(z^n(t,\cdot)-\phi)\xi(t)dxdt \\ &\overset{n\to\infty}{\longrightarrow} \int_0^T \int_\Omega \partial_t[k_{2,q} * (j(v) - j(v_0))](t)S(\varphi(v(t,\cdot))-\phi)\xi(t)dxdt. \end{aligned} \tag{18}$$

Next, suppose that $Supp(S) \subset [-R, R]$ and $K := R + \|\phi\|_{L^\infty(\Omega)}$ where $R > 0$, then

$$F^n(z^n) \cdot \nabla S(z^n - \phi) = S'(T_K(z^n) - \phi) F^n(T_K(z^n)) \cdot \nabla(T_K(z^n) - \phi),$$

a.e. in $Q_T$. Due to $\phi \in W_0^{1,p}(\Omega)$, the weak convergence $T_K(z^n) \rightharpoonup T_K(z^n)$ in $L^p(0,T; W_0^{1,p}(\Omega))$ and a.e. in $Q_T$, we deduce that

$$D(T_K(z^n) - \phi) \rightharpoonup \nabla(T_K(z) - \phi) \quad \text{weakly in } (L^p(Q_T))^N$$

as $n$ tends to infinity, while $S'(T_K(z^n) - \phi) F^n(T_K(z^n))$ is uniformly bounded with respect to $n$ and converges a.e. in $Q_T$ to $S'(T_K(z) - \phi) F(T_K(z))$. As a consequence, it follows that for $1 \le q < \infty$

$$S'(T_K(z^n) - \phi) F^n(T_K(z^n)) \longrightarrow S'(T_K(z) - \phi) F(T_K(z))$$

strongly in $L^q(Q_T)$ and

$$F^n(T_K(z^n)) \cdot \nabla S(T_K(z^n) - \phi) \overset{n\to\infty}{\longrightarrow} F(T_K(z)) \cdot \nabla S(T_K(z) - \phi)$$

weakly in $L^1(Q_T)$. Thus, (15) holds true and

$$\begin{aligned} &\int_0^T \int_\Omega F^n(z^n) \cdot \nabla S(z^n - \phi) \xi dx dt \\ &\overset{n\to\infty}{\longrightarrow} \int_0^T \int_\Omega F(\varphi(v)) \cdot \nabla S(\varphi(v) - \phi) \xi dx dt. \end{aligned} \tag{19}$$

To the end, recalling that $f^n$ belongs to $L^1(Q_T)$, $z^n \longrightarrow z$ a.e. in $Q_T$ and that $S(z^n - \phi)$ is bounded and convergences to $S(\varphi(v) - \phi)$ *a.e.* $Q_T$. as $n$ tends to infinity. Then, it possible to obtain

$$f^n S(z^n - \phi)\xi \longrightarrow f S(\varphi(v) - \varphi)\xi \quad \text{in } L^1(Q_T) \tag{20}$$

as $n$ tends infinity. □

**Proposition 21.** *Let $z^n := \varphi(v^n)$ be an entropy solution of $(EP)_{f,\psi^n}^{b_n,F^n,k_n}(z_0^n)$. The following inequalities hold:*

$$\begin{cases} \liminf\limits_{n\to\infty} \displaystyle\int_0^T \int_\Omega a(x, \nabla z^n) \cdot \nabla S(z^n(t,\cdot) - \phi)\xi(t) dx dt \\ \qquad \ge \displaystyle\int_0^T \int_\Omega a(x, \nabla\varphi(v)) \cdot \nabla S(\varphi(v(t,\cdot)) - \phi)\xi(t) dx dt, \end{cases} \tag{21}$$

$$\begin{cases} \liminf\limits_{n\to\infty} \displaystyle\int_{Q_T} \psi^n(z^n) S(z^n - \phi)\xi dx dt \ge 0. \end{cases} \tag{22}$$

*Proof:* Recall that it has been assumed that $Supp(S) \subset [-R, R]$ and $K := R + \|\phi\|_{L^\infty(\Omega)}$ where $R > 0$. Furthermore,

$$S'(z^n(t,\cdot) - \phi) a(\cdot, \nabla z^{m,n}) \cdot \nabla(z^n(t,\cdot) - \phi)$$

is identified with the term

$$S'(T_K(z^n(t,\cdot)) - \phi) a(\cdot, DT_K(z^n)) \cdot D(T_K(z^n(t,\cdot)) - \phi).$$

Thus, from the monotonicity assumption (H2), weak convergences in Lemma 19 and the same argument like those of Scoltes, 2016, we obtain

$$\begin{aligned} &\underline{\lim}_{n\to\infty} \int_0^T \int_\Omega a(x, \nabla z^n) \cdot \nabla S(z^n(t,\cdot) - \phi)\xi(t) dx dt \\ &\quad \ge \int_0^T \int_\Omega a(x, \nabla\varphi(v)) \cdot \nabla S(\varphi(v(t,\cdot)) - \phi)\xi(t) dx dt \end{aligned}$$

for any $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $S \in \mathscr{P}$ and $\xi \in \mathcal{D}$, $\xi \ge 0$. So, (21) holds true.

Now, observe that for any $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $S \in \mathcal{P}$ and $\xi \in D([0,T))$ with $\xi \geq 0$, there exists a constant $C > 0$ such that

$$\psi^n(z^n)S(z^n - \phi)\xi \geq -C.$$

From Fatou's Lemma, we get that

$$\liminf_{n\to\infty} \int_{Q_T} \psi^n(v^n)S(z^n - \phi)\xi dxdt \geq 0. \qquad \square$$

To show that $v$ is local entropy solution of $(EP)^{j,F,k}_{f,\varphi}(v_0)$, we only have to prove the inequality (P2). In (12), taking $S(z_n - \phi(x))$ as a test function with $z^n := z^{m,n}_{\lambda,l}$ an entropy solution of $(EP)^{b_l,F^{m,n},k_\lambda}_{f^{m,n},\psi^{m,n}}(z_0^{m,n})$, we have

$$\begin{aligned}
&-\int_{Q_T}\Big(\int_0^t k_{1,n,q}(t-\tau)\int_{z_0^n(x)}^{z^n(\tau,x)} S(\sigma-\phi(x))db_n(\sigma)\Big)d\tau\, \xi_t(t)dxdt\\
&\quad+\int_{Q_T} k_{2,n,q}(0^+)\Big(b_n(z^n(t,x))-b_n(z_0^n(x)))\Big)S(z^n(t,x)-\phi(x))\xi(t)dxdt\\
&\quad+\int_{Q_T}\Big(\int_0^t\Big(z^n(t-\tau,x)-z_0^n(x)))\Big)dk_{2,q}(\tau)\Big)S(z^n(t,x)-\phi(x))\xi(t)dxdt\\
&\quad+\int_{Q_T}(a(x,\nabla z^n(t,x))+F^n(z^n(t,x)))\cdot\nabla S(z^n(t,x)-\phi(x))\xi(t)dxdt\\
&\quad+\int_{Q_T}\psi_n(z^n(t,x))S(z^n(t,x)-\phi(x))\xi(t)dxdt\\
&\leq \int_{Q_T} f(t,x)S(z^n(t,x)-\phi(x))\xi(t)dxdt. \qquad (23)
\end{aligned}$$

Now, the idea is to calculate the lower limit of both sides of the inequality (23) as $n \to \infty$.

Thanks to the convergence results (17)-(20) and the inequalities (21)-(22), it follows that $v$ satisfies the condition (P2).

This shows that $v$ is an local entropy solution of $(EP)^{j,F,k}_{f,\varphi}(v_0)$.

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