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Approximation of the Ultim Ruin Probability by the Finite Difference Method of a Variable-memory Process (HAWKES process) With a Distribution of WEIBULL

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Abstract

In insurance risk management, the probability of ruin is a very important metric to assess. In this article, we give an approximation of the probability of ruin at the infinite horizon, where the inter-arrivals of claims follow the HAWKES process and the amount of claims follows the WEIBULL distribution, with independence between its two processes. This approximation is made using numerical analysis methods, it consists in solving a second-order integro-differential equation of which two cases are considered on the parameter η of WEIBULL: if η is equal to 1, then the distribution of the amounts of claims is exponential, which brings us back to the risk model established in Badini et al. (2024). On the other hand, if η is greater than 1, then the results lead us to a system of linear equations for which we use the finite difference method to obtain a numerical solution. This method is used in both cases ($\eta = 1$ and $\eta > 1$) for u ranging from 0 to 100, so we obtain the analytical solution.

Keywords: numerical method, integro-differential equation, probability of ruin, system of linear equations

1. Introduction

In insurance, the aim of the theory of ruin is to mathematically analyze random fluctuations in insurance companies' calculations. Risk is defined as the probability that a company's reserve will become negative at a certain time. However, there are several risk measures, but the probability of ruin for the moment remains one of the most interesting measures to study.

The risk model we are interested in here is particularly the probability of ruin at the infinite horizon. In this paper, we seek to determine an approximation of the probability of ruin at the infinite horizon of the risk model considered in Badini et al. (2024). This risk model uses the process as the law of inter-arrival claims Hawkes (1971), except that here we will use the WEIBULL distribution as the law of claim amounts which we define in the section (2). We will do this work using the numerical methods described in Ciarlet (1990); Crouzeix and Mignot (1989); Dautray and Lions (1984); Legras (1971); Santana and Rincón (2020).Among which we find the finite difference method which will allow us to solve the second-order integro-differential equation that we will determine from the following integro-differential equation:

$$\phi'(u) = \frac{\delta}{c}\phi(u) + \frac{\lambda\gamma}{2\pi c}\left[e^{\frac{\delta}{c}u}\int_0^{+\infty} e^{-\frac{\delta}{c}y}k(u-y)\sigma(y)dy \right. \tag{1}$$

$$\left. - \int_0^u k(u-y)\sigma(y)dy - \left(1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2}\right)\right] \tag{2}$$

with

$$k(u-y) = \frac{-\alpha\lambda(2\beta-\alpha)\times 2\times\left(-\frac{1}{c}\right)\left(\frac{u-y}{c}\right)}{\left[(\beta-\alpha)^2 + \left(\frac{u-y}{c}\right)^2\right]^2}$$

and

$$\sigma(y) = \int_0^y \phi(y-x)e^{-\gamma x}dx + \int_y^{+\infty} w(y, x-y)e^{-\gamma x}dx$$

One of the results given in Badini et al. (2024). We also draw inspiration from work done in Boots and Shahabuddin (2001); Dufresne and Gerber (1989); Goffard et al. (2016); KAFANDO et al. (2024); Sánchez and Baltazar-Larios (2022); Santana and Rincón (2020) to carry out our work.

First, we will remind you of the elements necessary for the continuation of our work (see section(2)). Then we will give the results obtained in the framework of this article, in particular the result of the numerical resolution in the case $\eta = 1$ and that of $\eta > 1$.

We will certainly end with a conclusion in which we will give the limits and advantages of this approach and also the perspectives.

2. Method

The reserve model $R(t)$ that we use for the purposes of this article is :

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i. \tag{3}$$

In this reserve model, $N(t)$ is a variable memory counting process that represents the number of claims at time $t > 0$.

Inter-arrivals follow a process Hawkes (1971) , whose ruin time τ is defined by:

$$\tau = \inf\{t \geq 0; R(t) < 0\} \tag{4}$$

The probability of ruin at the infinite horizon is therefore defined by:

$$\psi(u) = \mathbb{P}(\tau < \infty | R(0) = u)$$

$\psi(u)$ is the solution of the equation (1) with

$$\lim_{u\to+\infty} \psi(u) = 0$$

The Laplace transform of ψ using the equation (1) is defined as follows:

$$L_\psi(s) = \frac{\left[\left(c\psi(0) + H\psi(z) + \frac{H}{\beta}\right)(s^2 + \beta s) - As - A\beta - Ks^2\right][\gamma + s]}{cs^2(s+\beta)[(Q+1)s + \gamma]} \tag{5}$$

where A, H, K and Q of the constants defined by :

$$A = \frac{\lambda\gamma}{2\pi}\left(1 + \frac{\alpha\lambda(2\beta - \alpha)}{\beta - \alpha}\right); H = \frac{\lambda^2 c^2 \gamma\alpha(2\beta - \alpha)}{2\pi};$$

$$K = \frac{\lambda^2 c\gamma^2\alpha(2\beta - \alpha)}{2\pi(\beta - \alpha)}; Q = \frac{\lambda^2 c^2\gamma\alpha(2\beta - \alpha)}{2\pi(\beta - \alpha)}.$$

For more details (see Badini et al. (2024)).

We would also like to recall that the sequences of variables $(X_i)_{i\geq 1}$ represent the amounts of claims that are identically and identically distributed (iid) according to WEIBULL's law. The distribution of WEIBULL Hamzah et al. (2023) is a continuous random variable that is often used to analyze life data, failure time model and access reliability. This distribution was first introduced by Wallodi Weibull in 1951 and has been widely used in reliability engineering, survival analysis, and other fields. It is often applied in insurance companies to model the distribution of claims because of its flexibility. The probability density function f_X of the Weibull random variable is:

$$f_X(x, \gamma, \eta) = \eta\gamma(\gamma x)^{\eta-1} e^{-(\gamma x)^\eta} \tag{6}$$

with $\eta > 0$ and $\gamma > 0$ the parameters.

If $\eta = 1$, the WEIBULL distribution becomes an exponential distribution of parameter γ and probability density f_X defined by :

$$f_X(x) = \gamma e^{-\gamma x} \tag{7}$$

In what follows, we will discuss the probability of ruin at the infinite horizon according to the WEIBULL distribution is divided according to the parameter η. When the parameter $\eta = 1$, the probability of ruin can be determined analytically using Laplace's transform method (5). However, when $\eta > 1$, the Laplace transform method can no longer be used because of a step that requires the computation of an improper integral that is not possible to solve analytically. To meet this challenge (case $\eta > 1$), we will use the finite difference method to solve the equation.

3. Results

In this section, we present the results we have obtained. We determine the probability of ruin at the infinite horizon using the Hawkes process as a process of the inter-arrival of disasters and the distribution of WEIBULL for the amount of claims. However, we first deal with the case $\eta = 1$ and then case $\eta > 1$.

3.1 Case $\eta = 1$

Theorem 3.1.
The probability of ruin at the infinite horizon $\psi(u)$ is defined as a sequence for any $u \geq 0$:

$$M\psi = X \tag{8}$$

with

$$M = \begin{pmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ \frac{c}{(\Delta y)^2} & \frac{-2c}{(\Delta y)^2} + \frac{\delta}{\Delta y} + \delta & \frac{c}{(\Delta y)^2} - \frac{\delta}{\Delta y} & 0 & ... & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & ... & 0 & \frac{c}{(\Delta y)^2} & \frac{-2c}{(\Delta y)^2} + \frac{\delta}{\Delta y} + \delta & \frac{c}{(\Delta y)^2} - \frac{\delta}{\Delta y} \\ 0 & ... & \cdots & ... & ... & 1 \end{pmatrix}$$

$$\psi = \begin{pmatrix} \psi(u_0) \\ \psi(u_1) \\ \psi(u_2) \\ \vdots \\ \psi(u_i) \\ \vdots \\ \psi(u_{N-1}) \\ \psi(u_N) \end{pmatrix}$$

and

$$X = \begin{pmatrix} \frac{\beta\theta_1 - \theta_2}{c(R+\beta)} e^{-\beta u} + \frac{R\theta_1 + \theta_2}{c(R+\beta)} e^{Ru} \\ g(u_1)\left(1 - e^{-\frac{\delta u_1}{c}}\right) \\ g(u_2)\left(1 - e^{-\frac{\delta u_2}{c}}\right) \\ \vdots \\ g(u_i)\left(1 - e^{-\frac{\delta u_i}{c}}\right) \\ \vdots \\ g(u_{N-1})\left(1 - e^{-\frac{\delta u_{N-1}}{c}}\right) \\ g(u_N)\left(1 - e^{-\frac{\delta u_N}{c}}\right) \end{pmatrix}$$

$$g(u_i) = \frac{\alpha\lambda^2(2\beta - \alpha)\left(\frac{1}{c^2}\right)\left[-(\beta - \alpha)^2 + 3\left(\frac{u_i}{c}\right)^2\right]u_i}{2\pi\,(u_i + \beta)\left[(\beta - \alpha)^2 + \left(\frac{u_i}{c}\right)^2\right]^3}$$

The proof of this theorem will be given in the following, because at present we do not have the necessary elements for the proof.

Lemma 3.1.
The ultim ruin probability $\psi(u)$ satisfies the following integro-differential equation :

$$c\frac{d^2}{du^2}\psi(u)-\delta\frac{d}{du}\psi(u)+\delta\psi(u) = \frac{\lambda\gamma}{2\pi}\left[\int_0^u h(u-y)\sigma(y)dy - \int_0^u k^{'}(u-y)\sigma(y)dy\right] \tag{9}$$

with

$$h(u-y)=e^{-\delta(\frac{u-y}{c})}k^{'}(u-y) \tag{10}$$

and

$$k^{'}(u-y)=\frac{-2\alpha\lambda(2\beta-\alpha)\left(\frac{1}{c^2}\right)\left[-(\beta-\alpha)^2+3\left(\frac{u-y}{c}\right)^2\right]}{\left[(\beta-\alpha)^2+\left(\frac{u-y}{c}\right)^2\right]^3} \tag{11}$$

Proof.
Using the equation (1) with

$$\sigma(y)=\int_0^y\phi(y-x)e^{-\gamma x}dx+\int_y^{+\infty}w(y,x-y)e^{-\gamma x}dx \tag{12}$$

Confers in Badini et al. (2024). We now determine the second derivative of $\psi(u)$.

$$\psi'(u)=\frac{\delta}{c}\psi(u)+\frac{\lambda\gamma}{2\pi c}\left[\int_u^{+\infty}e^{-\delta(\frac{u-y}{c})}k(u-y)\sigma(y)dy-\int_0^u k(u-y)\sigma(y)dy-\left(1+\frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2}\right)\right]$$

$$\begin{aligned}
\psi^{''}(u) &= \frac{\delta}{c}\psi'(u)+\frac{\lambda\gamma}{2\pi c}\left[\frac{d}{du}\left(\int_u^{+\infty}e^{-\delta(\frac{u-y}{c})}k(u-y)\sigma(y)dy-\int_0^u k(u-y)\sigma(y)dy\right)\right]\\
&= \frac{\delta}{c}\psi'(u)+\frac{\lambda\gamma}{2\pi c}\left[\frac{d}{du}\left(\int_u^{+\infty}e^{-\delta(\frac{u-y}{c})}k(u-y)\sigma(y)dy\right)-e^{-\delta\times 0}k(0)\right]\\
&\quad -\frac{\lambda\gamma}{2\pi c}\left[\int_0^u\frac{d}{du}\left(k(u-y)\sigma(y)\right)dy+k(0)\right]\\
&= \frac{\delta}{c}\psi'(u)-\frac{\delta}{c}\psi(u)+\frac{\lambda\gamma}{2\pi c}\left[\int_0^{+\infty}e^{-\delta(\frac{u-y}{c})}k'(u-y)\sigma(y)dy-\int_0^u k'(u-y)\sigma(y)dy\right]
\end{aligned}$$

So it gives us :

$$c\frac{d^2}{du^2}\psi(u)-\delta\frac{d}{du}\psi(u)+\delta\psi(u)=\frac{\lambda\gamma}{2\pi}\left[\int_0^{+\infty}e^{-\delta(\frac{u-y}{c})}k'(u-y)\sigma(y)dy-\int_0^u k'(u-y)\sigma(y)dy\right]$$

We work on the infinite horizon, so $0\le u<+\infty$ and by posing $h(u-y)=e^{-\delta(\frac{u-y}{c})}k^{'}(u-y)$, We get :

$$c\frac{d^2}{du^2}\psi(u)-\delta\frac{d}{du}\psi(u)+\delta\psi(u)=\frac{\lambda\gamma}{2\pi}\left[\int_0^u h(u-y)\sigma(y)dy-\int_0^u k'(u-y)\sigma(y)dy\right]$$

□

Due to the complexity of the analytical solution of the equation (9) in the case where $\psi(u)$ is the probability of ruin at the infinite horizon, we propose a numerical approach for the solution. A numerical method for solving equation (9) (approximation) is given by the following lemma :

Lemma 3.2.
For $0 \leq u < +\infty$ and i ranging from 1 to $N-1$ We have :

$$
\begin{aligned}
& c\left[\frac{\psi(u_{i+1}) - 2\psi(u_i) + \psi(u_{i-1})}{(\Delta y)^2}\right] - \delta\left[\frac{\psi(u_{i+1}) - \psi(u_i)}{\Delta y}\right] + \delta\psi(u_i) \\
= \quad & \frac{\lambda\gamma u_i}{4\pi}\left[h(u_i)\sigma(0) + h(0)\sigma(u_i) - k'(u_i)\sigma(0) - k'(0)\sigma(u_i)\right] \qquad (13)
\end{aligned}
$$

Proof.
Consider the partition of the interval $u \in [0; L]$ which is defined as a sequence :

$$u_0 = 0 < u_1 < u_2 < \cdots < u_N = L$$

with L the size and $\Delta y = \frac{L}{N}$ step in such a way that $u_i = i\Delta y$, $n = 0; 1; \cdots ; N$.
The first and second derivatives of the left-hand side of equation (9) can be approximated by numerical differentiation of the first and second order, gives us :

$$\frac{d}{du}\psi(u) \approx \frac{\psi(u + \Delta y) - \psi(u)}{\Delta y} \qquad (14)$$

$$\frac{d^2}{du^2}\psi(u) \approx \frac{\psi(u + \Delta y) - 2\psi(u) + \psi(u - \Delta y)}{(\Delta y)^2} \qquad (15)$$

Ask $\psi(u) = \psi(u_i)$, $\psi(u + \Delta y) = \psi(u_{i+1})$ and $\psi(u - \Delta y) = \psi(u_{i-1})$. So the equations (14) and (15) transforms into :

$$\frac{d}{du}\psi(u_i) \approx \frac{\psi(u_{i+1}) - \psi(u_i)}{\Delta y} \qquad (16)$$

$$\frac{d^2}{du^2}\psi(u_i) \approx \frac{\psi(u_{i+1}) - 2\psi(u_i) + \psi(u_{i-1})}{(\Delta y)^2} \qquad (17)$$

The integral expressions of the right-hand side of the equation (9) can be approximated using the trapezoid method as a sequence :

$$\int_0^{u_i} h(u_i - y)\sigma(y)dy \approx \frac{u_i}{2}\left(h(u_i)\sigma(0) + h(0)\sigma(u_i)\right) \qquad (18)$$

$$\int_0^{u_i} k'(u_i - y)\sigma(y)dy \approx \frac{u_i}{2}\left(k'(u_i)\sigma(0) + k'(0)\sigma(u_i)\right) \qquad (19)$$

Using the equations (9), (16), (17), (18), (19) and starting the calculations from $i = 1$ to $i = N - 1$, we get :

$$
\begin{aligned}
& c\left[\frac{\psi(u_{i+1}) - 2\psi(u_i) + \psi(u_{i-1})}{(\Delta y)^2}\right] - \delta\left[\frac{\psi(u_{i+1}) - \psi(u_i)}{\Delta y}\right] + \delta\psi(u_i) \\
= \quad & \frac{\lambda\gamma u_i}{4\pi}\left[h(u_i)\sigma(0) + h(0)\sigma(u_i) - k'(u_i)\sigma(0) - k'(0)\sigma(u_i)\right]
\end{aligned}
$$

Now we have the necessary elements for the proof of the theorem. □

Proof.
The equation (5) which represents the Laplace transform of $\psi(u)$ gives the following expression (see Badini et al. (2024) for more details) :

$$\psi(u) = \frac{\beta\theta_1 - \theta_2}{c\,(R + \beta)}e^{-\beta u} + \frac{R\theta_1 + \theta_2}{c\,(R + \beta)}e^{Ru} \qquad (20)$$

with

$$R = \frac{-\gamma}{Q + 1} \prec 0$$

The equation (20) implies that :

$$\lim_{u \to +\infty} \psi(u) = 0 \qquad (21)$$

Using Ties (10), (11) and (12), We obtain the following equations :

$$h(u_i) = e^{-\delta(\frac{u_i}{c})} k^{'}(u_i) \tag{22}$$

$$k^{'}(u_i) = \frac{-2\alpha\lambda(2\beta-\alpha)\left(\frac{1}{c^2}\right)\left[-(\beta-\alpha)^2 + 3\left(\frac{u_i}{c}\right)^2\right]}{\left[(\beta-\alpha)^2 + \left(\frac{u_i}{c}\right)^2\right]^3} \tag{23}$$

and

$$\sigma(0) = \frac{1}{\gamma(u_i+\beta)} \tag{24}$$

The equations (13), (22), (23) and (24) lead to :

$$\frac{c\psi(u_{i+1})}{(\Delta y)^2} - \frac{2c\psi(u_i)}{(\Delta y)^2} + \frac{c\psi(u_{i-1})}{(\Delta y)^2} - \frac{\delta\psi(u_{i+1})}{\Delta y} + \frac{\delta\psi(u_i)}{\Delta y} + \delta\psi(u_i) = g(u_i)\left(1 - e^{-\frac{\delta u_i}{c}}\right) \tag{25}$$

with

$$g(u_i) = \frac{\alpha\lambda^2(2\beta-\alpha)\left(\frac{1}{c^2}\right)\left[-(\beta-\alpha)^2 + 3\left(\frac{u_i}{c}\right)^2\right]u_i}{2\pi(u_i+\beta)\left[(\beta-\alpha)^2 + \left(\frac{u_i}{c}\right)^2\right]^3}$$

From the equations (20), (21), and (25), We have the following system of linear equations :

$$M\psi = X$$

with

$$M = \left(m_{i,j}\right)_{1\le i;j\le N+1}$$

defined by :

$$m_{1;1} = m_{N+1;N+1} = 1$$

For i ranging from 2 to N, we have:

$$m_{i;i-1} = \frac{c}{(\Delta y)^2}$$

$$m_{i;i} = \frac{-2c}{(\Delta y)^2} + \frac{\delta}{\Delta y} + \delta$$

and

$$m_{i;i+1} = \frac{c}{(\Delta y)^2} - \frac{\delta}{\Delta y}.$$

For the remaining pairs of even indices $(i;j)$, we have :

$$m_{i;j} = 0.$$

All these elements have allowed us to define the matrix M by :

$$M = \begin{pmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ \frac{c}{(\Delta y)^2} & \frac{-2c}{(\Delta y)^2} + \frac{\delta}{\Delta y} + \delta & \frac{c}{(\Delta y)^2} - \frac{\delta}{\Delta y} & 0 & ... & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & ... & 0 & \frac{c}{(\Delta y)^2} & \frac{-2c}{(\Delta y)^2} + \frac{\delta}{\Delta y} + \delta & \frac{c}{(\Delta y)^2} - \frac{\delta}{\Delta y} \\ 0 & ... & \dots & ... & ... & 1 \end{pmatrix} \tag{26}$$

ψ is represented by :

$$\psi = \begin{pmatrix} \psi(u_0) \\ \psi(u_1) \\ \psi(u_2) \\ \vdots \\ \psi(u_i) \\ \vdots \\ \psi(u_{N-1}) \\ \psi(u_N) \end{pmatrix} \tag{27}$$

The column vector X is also defined by :

$$X = \begin{pmatrix} \frac{\beta\theta_1-\theta_2}{c(R+\beta)}e^{-\beta u} + \frac{R\theta_1+\theta_2}{c(R+\beta)}e^{Ru} \\ g(u_1)\left(1-e^{-\frac{\delta u_1}{c}}\right) \\ g(u_2)\left(1-e^{-\frac{\delta u_2}{c}}\right) \\ \vdots \\ g(u_i)\left(1-e^{-\frac{\delta u_i}{c}}\right) \\ \vdots \\ g(u_{N-1})\left(1-e^{-\frac{\delta u_{N-1}}{c}}\right) \\ g(u_N)\left(1-e^{-\frac{\delta u_N}{c}}\right) \end{pmatrix}$$

□

Application 1

Here we use MATLAB software to solve the system of linear equations $M\psi = X$ in the case $\eta = 1$. And by varying the reserve u from 0 to 100 in millions of euros as units, then using the parameters $\beta = 0.7$, $\alpha = 0.5$, $\lambda = 0.2$, $\gamma = 0.3$, $\pi = 3.14$, $\delta = 0$ and $c = 6$. The result of this simulation is given in Figure 1, its results show that the exact values of ψ are substantially at the numerical (approximate) values of ψ. Figure 1 also gives an indication of the curve corresponding to both the exact and the numerical solution of the probability of ruin at the finite horizon. It is clear that when the reserve u varies from 0 to 100, the exact and numerical solutions coincide and approach 0.

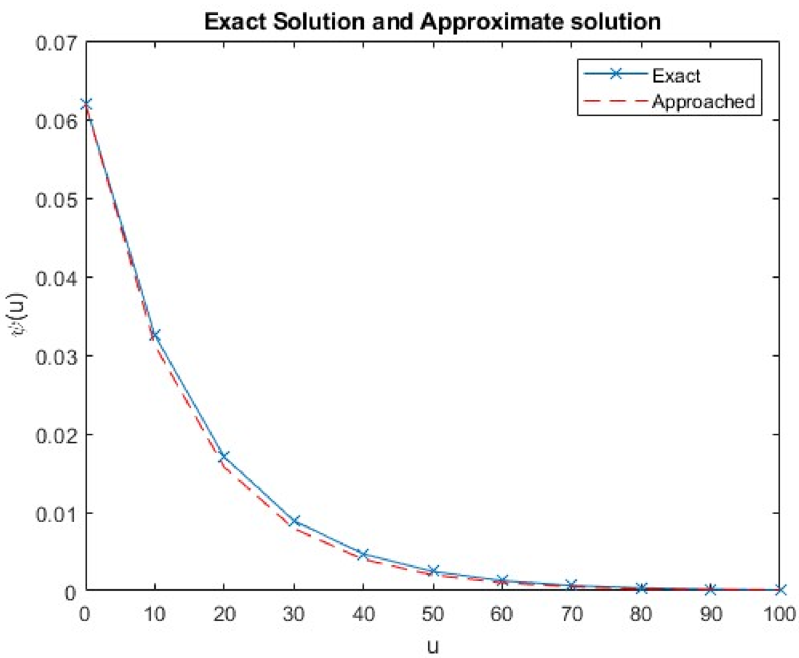

Figure 1. Curve of the exact solution and the approximate solution u ranging from 0 to 100

In figure 1, the results are between 0 and 1.

3.2 Case $\eta > 1$

For this hypothesis $\eta > 1$, we work more precisely with $\eta = 2$ by restoring the expression of the second-order integro-differential equation, as well as the system of linear equations $M\psi = Z$. Except that here the colone vector X defined in the equation (8) changes into another colone vector Z, on the other hand the tridiagonal matrix M does not change. For $\eta = 2$ the density function of the claim amounts (WEIBULL distribution) is defined by:

$$f_X(x) = 2\gamma^2 x e^{-(\gamma x)^2} \tag{28}$$

whose distribution function is defined as follows:

$$F_X(x) = 1 - e^{-(\gamma x)^2} \tag{29}$$

Moreover, the Laplace transform of the equation (28) is :

$$L_f(s) = \int_0^{+\infty} 2\gamma^2 x e^{-\left((\gamma x)^2 + sx\right)} dx \tag{30}$$

Because the equation (30) cannot be solved analytically, we cannot obtain an expression for the Laplace transform L_ψ of the probability of ruin as well as its analytic expression $\psi(u)$. For all these reasons, a numerical method is chosen using a reasoning similar to the case $\eta = 1$, the calculations of which are detailed below:

$$\begin{aligned}
\psi(u) &= \mathbb{E}[e^{-\delta\tau} w\left(U\left(\tau^-\right), |U(\tau)|\right) \mathbb{1}_{\{\tau<\infty\}} \mid U(0) = u] \\
&= \int_0^{+\infty}\int_0^{u+ct} e^{-\delta t}\phi(u+ct-x) dF(x,t) \\
&\quad + \int_0^{+\infty}\int_{u+ct}^{+\infty} e^{-\delta t} w(u+ct, x-u-ct) dF(x,t) \\
&= \int_0^{+\infty}\int_0^{u+ct} e^{-\delta t}\phi(u+ct-x) f_X(x) f_w(t) dx dt \\
&\quad + \int_0^{+\infty}\int_{u+ct}^{+\infty} e^{-\delta t} w(u+ct, x-u-ct) f_X(x) f_w(t) dx dt.
\end{aligned}$$

but

$$f_w(t) = \frac{\lambda}{2\pi}\left[1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2 + t^2}\right],$$

we deduce

$$\begin{aligned}
\psi(u) &= \int_0^{+\infty}\int_0^{u+ct} e^{-\delta t}\phi(u+ct-x) f_X(x) \frac{\lambda}{2\pi}\left[1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right] dx dt \\
&\quad + \int_0^{+\infty}\int_{u+ct}^{+\infty} e^{-\delta t} w(u+ct, x-u-ct) f_X(x) \frac{\lambda}{2\pi}\left[1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right] dx dt \\
&= \frac{\lambda}{2\pi}\int_0^{+\infty} e^{-\delta t}\left(\int_0^{u+ct}\phi(u+ct-x) f_X(x)\left[1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right] dx\right) dt \\
&\quad + \frac{\lambda}{2\pi}\int_0^{+\infty} e^{-\delta t}\left(\int_{u+ct}^{+\infty} w(u+ct, x-u-ct) f_X(x)\left[1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right] dx\right) dt. \\
\psi(u) &= \frac{\lambda}{2\pi}\int_0^{+\infty} e^{-\delta t}\left[1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]\left(\int_0^{u+ct}\phi(u+ct-x) f_X(x) dx\right. \\
&\quad \left. + \int_{u+ct}^{+\infty} w(u+ct, x-u-ct) f_X(x) dx\right) dt
\end{aligned}$$

We obtain that

$$\psi(u) = \frac{\lambda}{2\pi}\int_0^{+\infty} e^{-\delta t}\left[1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2+t^2}\right]\sigma(u+ct) dt$$

with

$$\sigma(y) = \int_0^y \phi(y-x) f_X(x) dx + \int_y^{+\infty} w(y, x-y) f_X(x) dx$$

Ask $y = u + ct$, then $t = \frac{y-u}{c}$ And this implies that $dt = \frac{1}{c}dy$.
if $t = 0$, then $y = u$ and if $t = +\infty$,then $y = +\infty$,So we have:

$$\psi(u) = \frac{\lambda}{2\pi c}\int_u^{+\infty} e^{-\delta\left(\frac{y-u}{c}\right)}\left[1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2 + \left(\frac{y-u}{c}\right)^2}\right]\sigma(y) dy$$

The first derivative of $\psi(u)$ with respect to u gives :

$$\psi'(u) = \frac{\delta}{c}\psi(u) + \frac{\lambda}{2\pi c}\left[e^{\frac{\delta}{c}u}\int_0^{+\infty} e^{-\frac{\delta}{c}y}k(y-u)\sigma(y)dy - \int_0^u k(y-u)\sigma(y)dy - \left(1 + \frac{\alpha\lambda(2\beta-\alpha)}{(\beta-\alpha)^2}\right)\right]$$

with

$$k(u-y) = \frac{-\alpha\lambda(2\beta-\alpha)\times 2\times\left(-\frac{1}{c}\right)\left(\frac{u-y}{c}\right)}{\left[(\beta-\alpha)^2 + \left(\frac{u-y}{c}\right)^2\right]^2}$$

which implies

$$\psi''(u) = \frac{\delta}{c}\psi'(u) - \frac{\delta}{c}\psi'(u) + \frac{\lambda}{2\pi c}\int_0^{+\infty} e^{-\frac{\delta(u-y)}{c}y}k'(u-y)\sigma(y)dy - \frac{\lambda}{2\pi c}\int_0^u k'(u-y)\sigma(y)dy$$

by laying $h(u-y) = e^{-\delta\left(\frac{u-y}{c}\right)}k^{'}(u-y)$ and the fact that $0 \le u < +\infty$, We get :

$$c\frac{d^2}{du^2}\psi(u) - \delta\frac{d}{du}\psi(u) + \delta\psi(u) = \frac{\lambda}{2\pi}\left[\int_0^u h(u-y)\sigma(y)dy - \int_0^u k'(u-y)\sigma(y)dy\right]$$

Using the above reasoning, we have :

$$\begin{aligned} & c\left[\frac{\psi(u_{i+1}) - 2\psi(u_i) + \psi(u_{i-1})}{(\Delta y)^2}\right] - \delta\left[\frac{\psi(u_{i+1}) - \psi(u_i)}{\Delta y}\right] + \delta\psi(u_i) \\ = \; & \frac{\lambda u_i}{4\pi}\left[h(u_i)\sigma(0) + h(0)\sigma(u_i) - k'(u_i)\sigma(0) - k'(0)\sigma(u_i)\right] \end{aligned}$$

as $h(u_i) = e^{-\delta\left(\frac{u_i}{c}\right)}k^{'}(u_i)$, then $h(0) = k^{'}(0)$ which gives us :

$$\begin{aligned} \frac{c\psi(u_{i+1})}{(\Delta y)^2} - \frac{2c\psi(u_i)}{(\Delta y)^2} + \frac{c\psi(u_{i-1})}{(\Delta y)^2} - \frac{\delta\psi(u_{i+1})}{\Delta y} + \frac{\delta\psi(u_i)}{\Delta y} + \delta\psi(u_i) &= \frac{\lambda u_i}{4\pi}\left[h(u_i)\sigma(0) - k'(u_i)\sigma(0)\right] \\ &= \frac{-\lambda u_i k^{'}(u_i)\sigma(0)}{4\pi}\left(1 - e^{-\delta\left(\frac{u_i}{c}\right)}\right) \end{aligned}$$

with

$$k^{'}(u_i) = \frac{-2\alpha\lambda(2\beta-\alpha)\left(\frac{1}{c^2}\right)\left[-(\beta-\alpha)^2 + 3\left(\frac{u_i}{c}\right)^2\right]}{\left[(\beta-\alpha)^2 + \left(\frac{u_i}{c}\right)^2\right]^3}$$

and

$$\sigma(0) = \frac{1}{\gamma}\left(1 - F_X(u_i)\right)$$

We finally get :

$$\begin{aligned} & \frac{c\psi(u_{i+1})}{(\Delta y)^2} - \frac{2c\psi(u_i)}{(\Delta y)^2} + \frac{c\psi(u_{i-1})}{(\Delta y)^2} - \frac{\delta\psi(u_{i+1})}{\Delta y} + \frac{\delta\psi(u_i)}{\Delta y} + \delta\psi(u_i) = \\ & \frac{\alpha\lambda^2(2\beta-\alpha)\left(\frac{1}{c^2}\right)\left[-(\beta-\alpha)^2 + 3\left(\frac{u_i}{c}\right)^2\right]\left[1 - F_X(u_i)\right]u_i}{2\pi\gamma\left[(\beta-\alpha)^2 + \left(\frac{u_i}{c}\right)^2\right]^3}\left(1 - e^{-\delta\left(\frac{u_i}{c}\right)}\right) \end{aligned}$$

with

$$F_X(u_i) = 1 - e^{-(\gamma u_i)^2}$$

Using the same approach as before, we obtain a system of linear equations of the form $M\psi = Z$, with M and ψ respectively described by the equations (26) and (27). As for the column vector Z it is different from the previous vector colone X in that this time $\eta = 2$, and is defined by :

$$Z = \begin{pmatrix} \frac{\alpha\lambda^2(2\beta-\alpha)\left(\frac{1}{c^2}\right)\left[-(\beta-\alpha)^2+3\left(\frac{u_1}{c}\right)^2\right][1-F_X(u_1)]u_1}{2\pi\gamma\left[(\beta-\alpha)^2+\left(\frac{u_i}{c}\right)^2\right]^3}\left(1-e^{-\delta\left(\frac{u_1}{c}\right)}\right) \\ \frac{\alpha\lambda^2(2\beta-\alpha)\left(\frac{1}{c^2}\right)\left[-(\beta-\alpha)^2+3\left(\frac{u_2}{c}\right)^2\right][1-F_X(u_2)]u_2}{2\pi\gamma\left[(\beta-\alpha)^2+\left(\frac{u_2}{c}\right)^2\right]^3}\left(1-e^{-\delta\left(\frac{u_2}{c}\right)}\right) \\ \frac{\alpha\lambda^2(2\beta-\alpha)\left(\frac{1}{c^2}\right)\left[-(\beta-\alpha)^2+3\left(\frac{u_3}{c}\right)^2\right][1-F_X(u_3)]u_3}{2\pi\gamma\left[(\beta-\alpha)^2+\left(\frac{u_3}{c}\right)^2\right]^3}\left(1-e^{-\delta\left(\frac{u_3}{c}\right)}\right) \\ \vdots \\ \frac{\alpha\lambda^2(2\beta-\alpha)\left(\frac{1}{c^2}\right)\left[-(\beta-\alpha)^2+3\left(\frac{u_i}{c}\right)^2\right][1-F_X(u_i)]u_i}{2\pi\gamma\left[(\beta-\alpha)^2+\left(\frac{u_i}{c}\right)^2\right]^3}\left(1-e^{-\delta\left(\frac{u_i}{c}\right)}\right) \\ \vdots \\ \frac{\alpha\lambda^2(2\beta-\alpha)\left(\frac{1}{c^2}\right)\left[-(\beta-\alpha)^2+3\left(\frac{u_{N-1}}{c}\right)^2\right][1-F_X(u_{N-1})]u_{N-1}}{2\pi\gamma\left[(\beta-\alpha)^2+\left(\frac{u_{N-1}}{c}\right)^2\right]^3}\left(1-e^{-\delta\left(\frac{u_{N-1}}{c}\right)}\right) \\ \frac{\alpha\lambda^2(2\beta-\alpha)\left(\frac{1}{c^2}\right)\left[-(\beta-\alpha)^2+3\left(\frac{u_N}{c}\right)^2\right][1-F_X(u_N)]u_N}{2\pi\gamma\left[(\beta-\alpha)^2+\left(\frac{u_N}{c}\right)^2\right]^3}\left(1-e^{-\delta\left(\frac{u_N}{c}\right)}\right) \end{pmatrix}$$

Application 2

For this simulation, we set the values of the parameters $\eta = 2, \beta = 0.7, \alpha = 0.5$, $\lambda = 0.2$, $\gamma = 0.3$, $\pi = 3.14, \delta = 0$ and $c = 10$; Then we vary the value of the reserve u from 0 to 100 in a cross-sectional manner to observe the behavior of the probability of ruin at the infinite horizon. The results of this simulation are shown in Table.

Table 1. Result of the simulation of the approximate solution

u	0	5	10	15	20	25	30	35	40
$\psi(u)$	0.0611	0.0467	0.0362	0.0275	0.0203	0.0134	0.0107	0.0096	0.0051

Figure (2) gives a simulation of the numerical solution of the probability of ruin at the infinite horizon for $\eta = 2$. It is clearly visible (Figure (2)) that an increase in u leads to a decrease in the probability of ruin ψ. When the initial reserve varies and tends to more infinity, the probability of ruin ($\psi(u)$) tends to 0.

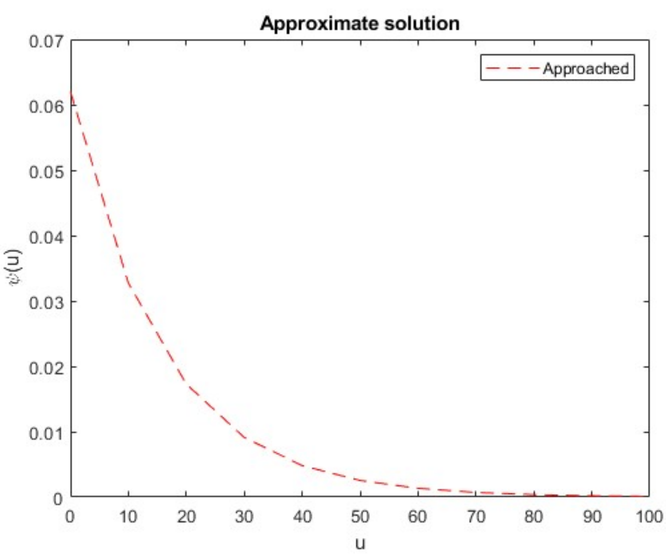

Figure 2. Curve of the approximate solution u ranging from 0 to 100

4. Discussion

The numerical analysis methods allowed us to transform the equation (9) into (13) without resorting to the solvency conditions of the integral encountered during the Laplace transform made in Badini et al. (2024), especially the case $\eta > 1$. In this paper, we have presented the main results of the ruin theory: exact expressions, approximations of the probability of ruin at the infinite horizon when the inter-arrivals of the losses follow the HAWKES process and the amount of the losses is from the WEIBULL distribution . The search for approximations for the probability of ruin in risk models has been one of the main points in this work. On the other hand, numerical methods, such as finite difference methods, are becoming increasingly important and produce excellent results for the case of approximation of the probability of ruin. Nevertheless, the distribution of WEIBULL used as a distribution of the amount of the claims is still of interest. In the case $\eta = 1$, we obtained an exact solution and an apprehended solution, this approximation seems to be in agreement with the exact solution,(1), but on the other hand when $\eta > 1$ we get a solution that we have(1) due to the complexity of the probability density of the amount of claims (WEIBULL distribution). On the other hand, numerical methods, such as finite difference methods, are becoming increasingly important and produce excellent results for the case of approximation of probability of ruin. Nevertheless, the WEIBULL distribution used as the distribution of the amount of claims is still of interest, but the most interesting thing is that the numerical solutions are between 0 and 1, as this effectively shows that it is a probability. In the near future, we intend to determine the probability of ruin on the infinite horizon.

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