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Carleson Type Measures on the Bergman Spaces With Variable Exponent Over the Unit Ball of $\mathbb{C}^n$

Agbor Dieudonne[1], Nzanguim D. Romeo[1]

[1] Department of Mathematics, Faculty of Science, University of Buea,P.O. Box 63 Buea, Cameroon

Correspondence: Nzanguim D. Romeo, Department of Mathematics, Faculty of Science, University of Buea,P.O. Box 63 Buea, Cameroon

Correspondence: Nzanguim D. Romeo; Calgary, AB,T2A 5K9, Canada

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Abstract

We give a characterization of Carleson measures on the variable exponent Bergman spaces using product of holomorphic functions. As a consequence we study the boundedness and compactness of Toeplitz operators between different variable exponent Bergman spaces.

Keywords: Bergman spaces, Carleson measures, variable exponent Bergman spaces, variable exponent Lebesgue spaces, Toeplitz operators

1. Introduction

Let $\mathbb{B}_n$ denote the unit ball of $\mathbb{C}^n$. For $\alpha > -1$ we let $dv_\alpha = c_\alpha(1-|z|^2)^\alpha dv$ where dv is the Lebesgue volume measure on $\mathbb{B}_n$ and c_α is a constant defined such that $v_\alpha(\mathbb{B}_n) = 1$. For $p \in (0, +\infty)$, we let $L^p(\mathbb{B}_n, dv_\alpha) = L^p_\alpha$ be the spaces of measurable functions f on $\mathbb{B}_n$ such that

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z)\right)^{\frac{1}{p}} < \infty.$$

The weighted Bergman space on $\mathbb{B}_n$, denoted A^p_α, is the space of all holomorphic functions in L^p_α. When $p = 2$, A^2_α is a Hilbert space, which is a closed subspace of L^2_α. Let P_α be the (orthogonal) Bergman projection of L^2_α onto A^2_α. For any bounded holomorphic function f on $\mathbb{B}_n$ we define the Toeplitz operator T_f, with symbol f, on A^2_α by

$$T^\alpha_f g(z) = P_\alpha(fg)(z) = \int_{\mathbb{B}_n} \frac{f(w)g(w)}{(1-\langle z, w\rangle)^{n+1+\alpha}} dv_\alpha(w), \quad g \in A^2_\alpha.$$

Moreover, If μ is a positive bounded Borel measure on $\mathbb{B}_n$ we write

$$T^\alpha_\mu g(z) = \int_{\mathbb{B}_n} \frac{g(w)}{(1-\langle z, w\rangle)^{n+1+\alpha}} d\mu(w), \quad g \in A^2_\alpha.$$

For $0 < p, q < \infty$ we say that a positive Borel measure μ on $\mathbb{B}_n$ is a (λ, α)- Carleson measure , with $\lambda = q/p$ if there is a constant $C > 0$ such that

$$\int_{\mathbb{B}_n} |f(z)|^q d\mu(z) \le C\|f\|^q_{p,\alpha},$$

for all $f \in A^p_\alpha$. When $\alpha = 0$ we will write $(\lambda, 0)$- Carleson measure simply by λ- Carleson measure.

Given a set $\Omega \subset \mathbb{R}^n$, by variable exponent we mean a measurable function $p : \Omega \to [c, \infty)$ (where $c > 0$) and we denote by

$$p_+ := ess\sup_{x\in\Omega} p(x), \quad p_- := ess\inf_{x\in\Omega} p(x).$$

For $c > 0$, we will denote by $\mathcal{P}_c(\Omega)$, the set of all variable exponents $p : \Omega \to [c, \infty)$ and we shall set $\mathcal{P}_1(\Omega) = \mathcal{P}(\Omega)$. If $f : \Omega \to \mathbb{C}$ is a measurable function, we define the modular $\rho_{p(\cdot)}$ by

$$\rho_{p(\cdot)}(f) := \int_\Omega |f(x)|^{p(x)} dx$$

Let $\Omega = \mathbb{B}_n$ and $p \in \mathcal{P}_c(\mathbb{B}_n)$ we say that a measurable function f belongs to the variable Lebesgue space of $L^{p(\cdot)}_\alpha$ if

$$\rho_{p(\cdot)}(f) = \int_{\mathbb{B}_n} |f(z)|^{p(z)} d\nu_\alpha(z) < \infty.$$

If there will be no confusion, we will simply write the modular $\rho(f)$ in place of $\rho_{p(\cdot)}(f)$. The norm on the space $L^{p(\cdot)}_\alpha$ is given by

$$\|\phi\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(\frac{\phi}{\lambda}\right) \leq 1\right\}, \quad \phi \in L^{p(\cdot)}_\alpha. \tag{1}$$

Variable exponent Lebesgue spaces appeared in the literature for the first time in an article by Orlicz (1931) and these are somewhat generalizations of the classical Lebesgue spaces, where the constant exponent p is replaced by a variable exponent $p(\cdot) \in \mathcal{P}(\Omega)$. Although the $L^{p(\cdot)}$ spaces have many properties similar to the L^p spaces, they also differ in surprising and subtle ways. This makes the study of variable exponent Lebesgue spaces very interesting and they have very important applications to partial differential equations and variational integrals with non-standard growth conditions.These are related to such physics (and engineering) concepts as image processing, electrorheological and thermorheological fluids. See (Cruz-Uribe & Fiorenza,1989) and (Lars, Petteri, Peter & Michael, 2017) for more details.

Variable exponent Bergman spaces, denoted $A^{p(\cdot)}_\alpha$, are spaces of Holomorphic functions in $L^{p(\cdot)}_\alpha$. These spaces have attracted a lot of research in recent years, for example see (Agbor, 2022), (Agbor, 2018), (Chacon & Rafeiro, 2016) and (Chacon & Rafeiro, 2014). In order to obtain a "fruitful" theory on the variable exponent spaces, (e.g. boundedness of maximal operator) we will impose some regularity conditions on the exponents called log-Hölder continuity. The boundedness of the maximal operator opens up the door for treating a plethora of other operators like the Reisz potential operator and hence to the proofs of Sobolev embeddings.

A function $p : \Omega \to \mathbb{C}$ is said to be log-Hölder continuous on Ω if there exists a positive constant $C_{\log} > 0$ such that

$$|p(x) - p(y)| \leq \frac{C_{\log}}{\log(1/|x - y|)}, \tag{2}$$

for all $x, y \in \Omega$ with $|x - y| < 1/2$. It follows that

$$|p(x) - p(y)| \leq \frac{2lC_{\log}}{\log(2l/|x - y|)}, \tag{3}$$

for all $x, y \in \Omega$ with $|x - y| < l$. We denote by $\mathcal{P}^{\log}_c(\Omega)$ the exponents in $\mathcal{P}_c(\Omega)$ that are log-Hölder continuous on Ω and in a similar manner write $\mathcal{P}^{\log}(\Omega)$ instead of $\mathcal{P}^{\log}_1(\Omega)$. It is known that the condition (2) is equivalent to the condition

$$|B|^{p^-_B - p^+_B} \leq C. \tag{4}$$

for all balls $B \subset \Omega$, where $|\cdot|$ stands for the normalized Lebesgue measure. As a consequence of (4) we have that if $z, w \in B$ then

$$|B|^{p(z)} \approx |B|^{p(w)}, \tag{5}$$

for any ball B. We are going to use this relation several times in our work.

Let $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{B}_n)$ and set $\lambda(\cdot) = q(\cdot)/p(\cdot) \in \mathcal{P}_c(\mathbb{B}_n)$. Then a positive Borel measure, μ, on $\mathbb{B}_n$, is said to be a $\lambda(\cdot)$-Bergman Carleson measure if there exists a positive constant $C > 0$ such that

$$\int_{\mathbb{B}_n} |f(z)|^{q(z)} d\mu(z) \leq C\|f\|^s_{p(\cdot),\alpha} \tag{6}$$

for some $s \in \{q_+, q_-\}$. We also denote by

$$\|\mu\|_{\lambda(\cdot),\alpha} = \sup\left\{\int_{\mathbb{B}_n} |f(z)|^{q(z)} d\mu(z) : f \in A^{p(\cdot)}_\alpha, \|f\|_{p(\cdot),\alpha} \leq 1\right\} \tag{7}$$

In the case of constant exponents, it is shown in Theorem 50 of (Zhao & Zhu, 2008) that the condition above depends only on the ratio q/p. We will see that a similar situation arises in the variable exponent setting. Carleson measures are a powerful tool for the study of function spaces and operators acting on them. The Bergman Carleson measures were first studied in (Hastings, 1975) and later on in (Cima & Wogen, 1982), (Lueking, 1983), (Lueking, 1993), (Haiou, 2000), (Michalak, 2001) and (Tchoundja, 2008) just to mention a few. Pau and Zhao (2014) gave characterizations on Bergman Carleson measures with respect to product of functions, for weighted Bergman spaces, with constant exponents, on the unit ball. In (Chacon, Rafeiro & Vallejo, 2016), the authors define and characterize Bergman Carleson measures for the setting of Bergman spaces with the same variable exponent, over the unit disc.

In this paper, we study Bergman Carleson measures and obtain a characterization of these measures with respect to product of functions from Bergman spaces with different variable exponents. Furthermore we characterize boundedness and compactness of Toeplitz operators via Carleson and vanishing Carleson measures respectively, thus extending the results of Pau and Zhao (2014) to the variable exponent Bergman spaces. We use the methods in the paper of Pau and Zhao (2014) and apply variable exponent techniques to obtain our results. The first result characterizes $\lambda(\cdot)$-Bergman Carleson measure in the case when $\lambda(\cdot) \in \mathcal{P}(\mathbb{B}_n)$:

Theorem 1.1. *Let μ be a positive Borel measure on $\mathbb{B}_n$ and $p_i, q_i \in \mathcal{P}^{\log}(\mathbb{B}_n)$, $i = 1, \cdots, k$ with $1 \le p_i(\cdot) \le q_i(\cdot)$ for $i = 1, \cdots, k$. Let*

$$\lambda(\cdot) = \sum_{i=1}^{k} \frac{q_i(\cdot)}{p_i(\cdot)}. \tag{8}$$

Then μ is a $\lambda(\cdot)-$Bergman Carleson measure if and only if there is a constant $C > 0$ such that for any $f_i \in A^{p_i(\cdot)}$, $i = 1, 2, 3, \ldots, k$,

$$\int_{\mathbb{B}_n} \prod_{i=1}^{k} |f_i(z)|^{q_i(z)} d\mu(z) \le C \prod_{i=1}^{k} \|f_i\|_{p_i(\cdot)}^{s_i} \quad s_i \in \{q_{i+}, q_{i-}\}. \tag{9}$$

The next result, which characterizes bounded Toeplitz between two variable Bergman spaces, is also an extention of the result in (Pau & Zhao, 2014).

Theorem 1.2. *Let $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{B}_n)$ with $1 \le p_1(\cdot) \le p_2(\cdot)$, $\lambda(\cdot) = 1 + \frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)}$ and μ be a positive Borel measure on $\mathbb{B}_n$. Then the following statements are equivalent:*

i) T_μ is bounded from $A^{p_1(\cdot)}$ to $A^{p_2(\cdot)}$.

ii) The measure μ is $\lambda(\cdot)-$Bergman Carleson measure.

Let μ be a positive Borel measure on $\mathbb{B}_n$ and $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{B}_n)$ be such that $p(\cdot) \le q(\cdot)$. Then, a positive Borel measure on $\mathbb{B}_n$ is known as a vanishing $\lambda(\cdot) = q(\cdot)/p(\cdot)$-Bergman Carleson measure if for any sequence $\{ f_k \}$ in $A^{p(\cdot)}$ with $\|f_k\|_{p(\cdot)} \le 1$ such that $f_k(z) \to 0$ uniformly on any compact subset of $\mathbb{B}_n$ we have

$$\lim_{k\to\infty} \int_{\mathbb{B}_n} |f_k(z)|^{q(z)} d\mu(z) = 0.$$

We also have the following characterization of vanishing Carleson measures.

Theorem 1.3. *Let μ be a positive Borel measure on $\mathbb{B}_n$. For $i = 1, 2, 3, \ldots, k$ and suppose $p_i, q_i \in \mathcal{P}^{\log}(\mathbb{B}_n)$ be such that $1 \le p_i(\cdot) \le q_i(\cdot)$ and*

$$\lambda(\cdot) = \sum_{i=1}^{k} \frac{q_i(\cdot)}{p_i(\cdot)}.$$

Then the following statements are equivalent:

(i) μ is a vanishing $\lambda(\cdot)$-Bergman Carleson measure.

(ii) For any sequence $\{ f_{1,m} \}$ in the unit ball of $A^{p_1(\cdot)}$ which is convergent to zero uniformly in compact subsets of $\mathbb{B}_n$,

$$\lim_{m\to\infty} \sup_{f_i \in A^{p_i(\cdot)}, \|f_i\|_{p_i(\cdot)} \le 1, i=2,3,\ldots,k} \int_{\mathbb{B}_n} |f_{1,m}(z)|^{q_1(z)} |f_2(z)|^{q_2(z)} \cdots |f_k(z)|^{q_k(z)} d\mu(z) = 0.$$

(iii) For any k sequences, $\{ f_{1,m} \}, \{ f_{2,m} \}, \ldots, \{ f_{k,m} \}$ in the unit balls of $A^{p_1(\cdot)}, A^{p_2(\cdot)}, \ldots, A^{p_k(\cdot)}$, respectively, which are all convergent to zero uniformly in compact subsets of $\mathbb{B}_n$,

$$\lim_{m\to\infty} \int_{\mathbb{B}_n} |f_{1,m}(z)|^{q_1(z)} |f_{2,m}|^{q_2(z)} \cdots |f_{k,m}|^{q_k(z)} d\mu(z) = 0. \tag{10}$$

Finally we characterize compact Toeplitz operators with positive Borel symbols.

Theorem 1.4. *Let μ be a positive Borel measure on $\mathbb{B}_n$ and $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{B}_n)$ with $1 \le p_1(\cdot) \le p_2(\cdot)$ and $\lambda(\cdot) = 1 + \frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)}$. Then T_μ is compact from $A^{p_1(\cdot)}$ to $A^{p_2(\cdot)}$ if and only if μ is a vanishing $\lambda(\cdot)$-Bergman Carleson.*

In the following, the notation $A \lesssim B$ means there is a positive constant C such that $A \le CB$ and the notation $A \approx B$ means $A \lesssim B$ and $B \lesssim A$. This paper is organized as follows: In Section 2 we give some preliminaries. Section 3 is devoted to the characterization of $\lambda(\cdot)$-Bergman Carleson measure and the proof of Theorem 1.2. Section 4 deals with the characterization of the vanishing $\lambda(\cdot)$-Bergman Carleson measure and the proof of Theorem 1.4.

2. Preliminaries

In this section we collect some preliminary results that are needed through out the work. For $z = (z_1, \cdots, z_n), w = (w_1, \cdots, w_n) \in \mathbb{C}^n$, we write

$$\langle z, w\rangle = \sum_{i=1}^{n} z_i \overline{w_i}$$

and $|z| = \sqrt{\langle z, z\rangle}$. For $a \in \mathbb{B}_n$ with $a \neq 0$ we denote by φ_a the Möbius transformation on $\mathbb{B}_n$ that interchanges the points 0 and a. It is well known that

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \quad z \in \mathbb{B}_n,$$

where $s_a = 1 - |a|^2$, P_a is the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional space $[a]$ generated by a and Q_a is the orthogonal projection from $\mathbb{C}^n$ onto the complement of $[a]$.

For $z, w \in \mathbb{B}_n$, the Bergman metric distance between z and w is given by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.$$

Let $z \in \mathbb{B}_n$ and $r > 0$. Then the Bergman metric ball with centre z is given by

$$D_r(z) = \{w \in \mathbb{B}_n : \beta(z, w) < r\}.$$

It is known that if $w \in D_r(z)$ then $1 - \langle z, w\rangle \approx (1 - |w|^2) \approx (1 - |z|^2)$,

$$(1 - |z|^2)^{n+1+\alpha} \approx \nu_\alpha(D_r(z)) \approx \nu_\alpha(D_r(w)) \approx (1 - |w|^2)^{n+1+\alpha}. \tag{11}$$

Furthermore, if $\beta(z, w) < r$, then by (5) we have

$$(1 - |z|^2)^{(n+1+\alpha)p(z)} \approx (1 - |z|^2)^{(n+1+\alpha)p(w)}, \quad p(\cdot) \in \mathcal{P}^{\log}(\mathbb{B}_n). \tag{12}$$

The following lemma will be useful, and is Theorem 2.23 of (Zhu, 2005).

Lemma 2.1. *There exists a positive integer N such that for any $r \in (0, 1)$ we can find a sequence $\{a_k\}$ in $\mathbb{B}_n$ satisfying the following properties:*

1. $\mathbb{B}_n = \bigcup_k D_r(a_k)$.
2. *The sets $D_{\frac{r}{4}}(a_k)$ are mutually disjoint.*
3. *Each point $z \in \mathbb{B}_n$ belongs to at most N of the sets $D_r(a_k)$.*

It is known that any sequence $\{a_k\}$ satisfying the properties in Lemma 2.1 is called an **r-lattice** in the Bergman metric. We will need the following well-known estimates, see Theorem 1.12 of (Zhu, 2005):

Lemma 2.2. *Let $t > -1$ and $s > 0$. Then there is a positive constant C such that*

$$\int_{\mathbf{B}} \frac{(1 - |w|^2)^t}{|1 - \langle z, w\rangle|^{n+1+t+s}} d\nu(w) \le C(1 - |z|^2)^{-s} \tag{13}$$

for all $z \in \mathbf{B}$.

We study some properties of a subclass of $\mathcal{P}_c$, $c > 0$, for which $p(\cdot) \in [c, \infty)$. These results are known for the case $p(\cdot) \geq 1$ (see Chapters 2 and 3 of (Lars et al., 2017) for more details).

Lemma 2.3. *Let $c > 0$ and $p : \Omega \subset \mathbb{R}^n \to [c, \infty)$ be a measurable function. Then $\rho_{p(\cdot)}$ and $\|\cdot\|_{p(\cdot)}$ satisfy the following properties:*

1) *For all $\alpha > 0$, the map $\alpha \mapsto \rho_{p(\cdot)}(\alpha\phi)$ is left-continuous;*
2) *$\rho_{p(\cdot)}(\phi) \leq 1$ if and only if $\|\phi\|_{p(\cdot)} \leq 1$;*
3) *$\rho_{p(\cdot)}(\phi) > 1$ if and only if $\|\phi\|_{p(\cdot)} > 1$;*
4) *For all $\alpha \in \mathbb{R}$, $\|\alpha\phi\|_{p(\cdot)} = |\alpha|\|\phi\|_{p(\cdot)}$ (Homogeneity);*
5) $\min\left\{\rho(\phi)^{1/p_-}, \rho(\phi)^{1/p_+}\right\} \leq \|\phi\|_{p(\cdot)} \leq \max\left\{\rho(\phi)^{1/p_-}, \rho(\phi)^{1/p_+}\right\}$.

Proof 1) Follows by the Sandwich theorem for limits. The proof of 2) 3) and 4) is simlar to that for the case when $c = 1$. We are left to proof property 5).

Suppose $\rho(\phi) \leq 1$. We need to show that

$$\rho(\phi)^{1/p_-} < \|\phi\|_{p(\cdot)} \leq \rho(\phi)^{1/p_+} \tag{14}$$

Proving $\|\phi\|_{p(\cdot)} \leq \rho(\phi)^{1/p_+}$ is equivalent (by homogeneity) to proving $\left\|\frac{\phi}{\rho(\phi)^{1/p_+}}\right\|_{p(\cdot)} \leq 1$. By 2) above, this is equivalent to proving that

$$\int_\Omega \left(\frac{|\phi(x)|}{\rho(\phi)^{1/p_+}}\right)^{p(x)} d\nu(x) \leq 1.$$

Since $p(\cdot) \leq p_+$ and $\rho(\phi) \leq 1$ imply $\frac{1}{\rho(\phi)^{p(\cdot)/p_+}} \leq \frac{1}{\rho(\phi)}$, we have

$$\int_\Omega \left(\frac{|\phi(x)|}{\rho(\phi)^{1/p_+}}\right)^{p(x)} d\nu(x) \leq \int_\Omega \frac{|\phi(x)|^{p(x)}}{\rho(\phi)} d\nu(x) = 1.$$

On the other hand, proving $\rho(\phi)^{1/p_-} < \|\phi\|_{p(\cdot)}$ is equivalent (by homogeneity) to proving that $1 < \left\|\frac{\phi}{\rho(\phi)^{1/p_-}}\right\|_{p(\cdot)}$ which by 3) above is equivalent to proving that

$$1 < \int_\Omega \left(\frac{|\phi(x)|}{\rho(\phi)^{1/p_-}}\right)^{p(x)} d\nu(x).$$

But $\rho(\phi) \leq 1$ and $p_- \leq p(\cdot)$ imply

$$1 = \int_\Omega \frac{|\phi(x)|^{p(x)}}{\rho(\phi)} d\nu(x) \leq \int_\Omega \left(\frac{|\phi(x)|}{\rho(\phi)^{1/p_-}}\right)^{p(x)} d\nu(x),$$

from which (14) follows.
For $\rho(\phi) \geq 1$, the expression

$$\rho(\phi)^{1/p_+} < \|\phi\|_{p(\cdot)} \leq \rho(\phi)^{1/p_-} \tag{15}$$

is proven with a similar reasoning. Equations (14) and (15) give the desired result. □

Remark 2.4. *If $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{B}_n)$ then $p(\cdot)/q(\cdot) \in \mathcal{P}_c$ for some $c > 0$.*

We will need the following, which are Corollaries 2.22 and 2.23 of (Cruz-Uribe & Fiorenza, 1989).

Proposition 2.5. *Suppose $p(\cdot) \in \mathcal{P}^{\log}(\Omega)$ and $|\Omega| < +\infty$. If $\|f\|_{p(\cdot)} > 1$ then $\|f\|_{p(\cdot)} \leq \rho(f)$ and*

$$\rho(f)^{1/p_+} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_-}.$$

If $0 < \|f\|_{p(\cdot)} \leq 1$ then $\rho(f) \leq \|f\|_{p(\cdot)}$ and

$$\rho(f)^{1/p_-} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_+}.$$

Lemma 2.6. *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{B}_n)$. Then for every $z \in \mathbb{B}_n$ we have*

$$||K_z^{2/p(z)}||_{p(\cdot)} \lesssim \frac{C}{(1-|z|^2)^{(n+1+\alpha)/p(z)}}.$$

Lemma 2.7. *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{B}_n)$. Then for every $a \in \mathbb{B}_n$ we have*

$$|f(a)| \approx \frac{||f||_{p(\cdot)}}{(1-|a|^2)^{(n+1+\alpha)/p(a)}}.$$

3. Carleson Type Measures

Proposition 3.1. *Suppose $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{B}_n)$ with $1 \le p(\cdot) \le q(\cdot)$. Then the following are equivalent:*

(a) There is an $s \in \{q_-, q_+\}$ such that

$$\int_{\mathbb{B}_n} |f(z)|^{q(z)} d\mu(z) \lesssim ||f||_{p(\cdot)}^s,$$

for every $f \in A_\alpha^{p(\cdot)}$.

(b) There exists a constant C such that for all $t \ge n+1+\alpha$

$$\sup_{w\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{\frac{q(z)}{p(w)}t}}{|1-\langle z,w\rangle|^{(n+1+\alpha)q(z)/p(w)+\frac{q(z)}{p(w)}t}} d\mu(z) < C \tag{16}$$

(c) For any $r > 0$

$$\mu(D_r(z)) \lesssim (1-|z|^2)^{(n+1+\alpha)q(z)/p(z)}. \tag{17}$$

Proof Suppose (a) holds, and let $f_w(z) = \frac{(1-|w|^2)^{\frac{t}{p(w)}}}{(1-\langle z,w\rangle)^{(n+1+\alpha)/p(w)+\frac{t}{p(w)}}}$. Then by Lemma 3.2 of (Agbor, 2022), we have that

$$\sup_{w\in\mathbb{B}_n} ||f_w||_{p(\cdot)} < \infty, \tag{18}$$

whenever $t = n+1+\alpha$. When $t > n+1+\alpha$ we can see easily that

$$|f_w(z)| \le \frac{(1-|w|^2)^{\frac{n+1+\alpha}{p(w)}}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)/p(w)}}$$

and this shows that (18) hold for all $t \ge n+1+\alpha$. By assertion (a) we have that $\int_{\mathbb{B}_n} |f_w(z)|^{q(z)} d\mu(z) < \infty$ which is (b).
Suppose (b) holds.

$$\begin{aligned}
&\int_{\mathbb{B}_n} \frac{(1-|w|^2)^{\frac{q(z)}{p(w)}t}}{|1-\langle z,w\rangle|^{(n+1+\alpha)q(z)/p(w)+\frac{q(z)}{p(w)}t}} d\mu(z) \\
\ge\ & \int_{D_r(w)} \frac{(1-|w|^2)^{\frac{q(z)}{p(w)}t}}{|1-\langle z,w\rangle|^{(n+1+\alpha)q(z)/p(w)+\frac{q(z)}{p(w)}t}} d\mu(z) \\
\approx\ & \int_{D_r(w)} \frac{(1-|w|^2)^{\frac{q(w)}{p(w)}t}}{(1-|w|^2)^{(n+1+\alpha)q(w)/p(w)+\frac{q(w)}{p(w)}t}} d\mu(z) \\
=\ & \frac{\mu(D_r(w))}{(1-|w|^2)^{(n+1+\alpha)q(w)/p(w)}},
\end{aligned}$$

which gives (c).
Suppose (c) holds. We will show that for all $f \in A_\alpha^{p(\cdot)}$

$$||f||_{A^{q(\cdot)}(\mu)} \lesssim ||f||_{p(\cdot),\alpha}. \tag{19}$$

Let $f \in A^{p(\cdot)}_\alpha$ be such that $\|f\|_{p(\cdot)} = 1$ and set $d\mu_{f,\alpha}(w) = |f(w)|^{p(w)}d\nu_\alpha(w)$. Then by applying the Diening inequality, for variable exponents (see for example (Rafeiro & Samko, 2011)) we have that

$$|f(z)|^{p(z)} \lesssim \frac{1}{\nu_\alpha(D_{2r}(a))}\int_{D_{2r}(a))} d\mu_{f,\alpha}(w) + 1$$

whenever $z, w \in D_r(a)$, $a \in \mathbb{B}_n$. Now, if $p(z) < q(z)$ we write $q(z) = s(z)p(z)$, for some $s(\cdot) \in \mathcal{P}$, then

$$\begin{aligned} |f(z)|^{q(z)} &= |f(z)|^{p(z)s(z)} \lesssim \left(\frac{1}{\nu_\alpha D_{2r}(a))}\int_{D_{2r}(a)} |f(w)|^{p(w)}d\nu_\alpha(w) + 1\right)^{s(z)} \\ &\lesssim \left(\left(\frac{1}{\nu_\alpha(D_{2r}(a))}\int_{D_{2r}(a)} |f(w)|^{p(w)}d\nu_\alpha(w)\right)^{s(z)} + 1\right) \\ &\leq \left(\frac{1}{(\nu_\alpha(D_{2r}(a))^{s(z)}}\int_{D_{2r}(a)} |f(w)|^{p(w)}d\nu_\alpha(w) + 2\right) \\ &\approx \left(\frac{1}{\nu_\alpha(D_{2r}(a))^{s(a)}}\int_{D_{2r}(a)} |f(w)|^{p(w)}d\nu_\alpha(w) + 2\right) \end{aligned} \tag{20}$$

where we have used the identity (12) to obtain the last equation. It follows that

$$\int_{D_r(a)} |f(z)|^{q(z)}d\mu(z) \lesssim \frac{\mu(D_r(a))}{(1-|a|^2)^{(n+1+\alpha)q(a)/p(a)}}\int_{D_{2r}(a)} |f(w)|^{p(w)}d\nu_\alpha(w) + \mu(D_r(a)).$$

Now let the sequence $\{a_k\}$ be an r-lattice on $\mathbb{B}_n$ then

$$\begin{aligned} \int_{\mathbb{B}_n} |f(z)|^{q(z)}d\mu(z) &\leq \sum_{k=1}^{\infty}\int_{D_r(a_k)} |f(z)|^{q(z)}d\mu(z) \\ &\lesssim \sum_{k=1}^{\infty}\int_{D_{2r}(a_k)} |f(w)|^{p(w)}d\nu_\alpha(w) + \sum_{k=1}^{\infty}\mu(D(a_k, r)) \\ &\leq N\int_{\mathbb{B}_n} |f(w)|^{p(w)}d\nu_\alpha(w) + \mu(\mathbb{B}_n) \leq N(1 + \mu(\mathbb{B}_n)). \end{aligned}$$

Thus,

$$\int_{\mathbf{B}} |f(z)|^{q(z)}d\mu(z) \lesssim 1 = \|f\|^s_{p(\cdot),\alpha},$$

for any $s > 0$, since $\|f\|_{p(\cdot),\alpha} = 1$. Now, by Proposition 2.5 there is an $s \in \{q_-, q_+\}$ such that

$$\|f\|_{A^{q(\cdot)}(\mu)} \lesssim \left(\int_{\mathbf{B}} |f(z)|^{q(z)}d\mu(z)\right)^{\frac{1}{s}} \lesssim \|f\|_{p(\cdot),\alpha},$$

which gives (19). If $\|f\|_{p(\cdot),\alpha} \neq 1$ we set $g = f/\|f\|_{p(\cdot),\alpha}$ and use the Homogeneity of the norms to obtain equation (19) for all $f \in A^{p(\cdot)}_\alpha$. Finally, we apply Proposition 2.5 to the left of equation (19) to obtain the required result. □

The next two results will be needed to prove the main results in this section.

Proposition 3.2. *For $i = 1, 2 \ldots, k$, $p_i(\cdot), q_i(\cdot) \in \mathcal{P}^{\log}(\mathbb{B}_n)$. Let*

$$\lambda(\cdot) = \sum_{i=1}^{k}\frac{q_i(\cdot)}{p_i(\cdot)}$$

If $f_i \in A^{p_i(\cdot)/q_i(\cdot)}$, for each $i = 1, 2 \ldots, k$, then $\prod_{i=1}^k f_i \in A^{\frac{1}{\lambda}(\cdot)}$ and

$$\left\|\prod_{i=1}^{k} f_i\right\|_{\frac{1}{\lambda(\cdot)}} \lesssim \prod_{i=1}^{k}\|f_i\|_{p_i(\cdot)/q_i(\cdot)}.$$

Proof Let $f_i \in A^{p_i(\cdot)/q_i(\cdot)}$ for each $i = 1, 2 \ldots, k$. Since $\frac{p_i(\cdot)\lambda(\cdot)}{q_i(\cdot)} > 1$ and $\sum_{i=1}^{k} \frac{q_i(\cdot)}{p_i(\cdot)\lambda(\cdot)} = 1$ we apply Hölders' inequality to obtain

$$\int_{\mathbb{B}_n} \prod_{i=1}^{k} |f_i(z)|^{\frac{1}{\lambda(z)}} d\nu(z) \lesssim \prod_{i=1}^{k} \|f_i\|_{p_i(\cdot)/q_i(\cdot)} . \tag{21}$$

Now, suppose $\|f_i\|_{p_i(\cdot)/q_i(\cdot)} = 1$. Then (21) implies that there is an $s \in \{\lambda_-^{-1}, \lambda_+^{-1}\}$, such that

$$\left\| \prod_{i=1}^{k} f_i \right\|_{\frac{1}{\lambda(\cdot)}} \lesssim \left(\prod_{i=1}^{k} \|f_i\|_{p_i(\cdot)/q_i(\cdot)} \right)^s = \prod_{i=1}^{k} \|f_i\|_{p_i(\cdot)/q_i(\cdot)} .$$

Now, if $\|f_i\|_{p_i(\cdot)/q_i(\cdot)} \neq 1$ for each $i = 1, 2, \ldots, k$ we set $g_i = f_i/\|f_i\|_{p_i(\cdot)/q_i(\cdot)}$ and conclude by the homogeneity of norms. □

Proposition 3.3. *Suppose μ is a positive Borel measure on $\mathbb{B}_n$, $p_i(\cdot), q_i(\cdot) \in \mathcal{P}^{\log}(\mathbb{B}_n)$, $i = 1, 2 \ldots, k$, and let*

$$\lambda(\cdot) = \sum_{i=1}^{k} \frac{q_i(\cdot)}{p_i(\cdot)}.$$

If μ is a $\lambda(\cdot)$−Bergman Carleson measure then for any $f_i \in A^{p_i(\cdot)/q_i(\cdot)}$, $i = 1, 2 \ldots, k$,

$$\int_{\mathbb{B}_n} \prod_{i=1}^{k} |f_i(z)|^{q_i(z)} d\mu(z) \lesssim \prod_{i=1}^{k} \|f_i\|_{p_i(\cdot)}^{s_i}, \tag{22}$$

$s_i \in \{q_{i+}, q_{i-}\}$, $i = 1, 2 \ldots, k$.

Proof Note that the case $k = 1$ is just the definition of a $\lambda(\cdot)$−Bergman Carleson.
Suppose $k \geq 2$ and let $h_i \in A^{p_i(\cdot)/q_i(\cdot)}$ for $i = 1, 2, 3, \ldots, k$. By proposition 3.2,

$$\prod_{i=1}^{k} h_i \in A^{\frac{1}{\lambda}(\cdot)} \quad \text{and} \quad \left\| \prod_{i=1}^{k} h_i \right\|_{\frac{1}{\lambda}(\cdot)} \lesssim \prod_{i=1}^{k} \|h_i\|_{p_i(\cdot)/q_i(\cdot)} .$$

Since μ is a $\lambda(\cdot)$−Bergman Carleson measure, we have

$$\begin{aligned} \int_{\mathbb{B}_n} \left| \prod_{i=1}^{k} h_i(z) \right| d\mu(z) &\lesssim \left\| \prod_{i=1}^{k} h_i \right\|_{\frac{1}{\lambda}(\cdot)} \\ &\lesssim \prod_{i=1}^{k} \|h_i\|_{p_i(\cdot)/q_i(\cdot)} . \end{aligned} \tag{23}$$

Let

$$d\mu_1(z) = \left(\prod_{i=2}^{k} |h_i| d\mu(z) \right) \Big/ \left(\prod_{i=2}^{k} \|h_i\|_{p_i(\cdot)/q_i(\cdot)} \right)$$

then by (23) we get $\int_{\mathbb{B}_n} |h_1(z)| d\mu_1(z) \lesssim \|h_1\|_{p_1(\cdot)/q_1(\cdot)}$. Hence, μ_1 is a $\left(\frac{q_1(\cdot)}{p_1(\cdot)}\right)$-Bergman Carleson measure and so for any $f_1 \in A^{p_1(\cdot)}$,

$$\int_{\mathbb{B}_n} |f_1(z)|^{q_1(z)} d\mu_1(z) \lesssim \|f_1\|_{p_1(\cdot)}^{s_1},$$

for some $s_1 \in \{q_{1+}, q_{1-}\}$. Thus

$$\int_{\mathbb{B}_n} |f_1(z)|^{q_1(z)} \left(\prod_{i=2}^{k} |h_i| \right) d\mu(z) \lesssim \|f_1\|_{p_1(\cdot)}^{s_1} \times \left(\prod_{i=2}^{k} \|h_i\|_{p_i(\cdot)/q_i(\cdot)} \right). \tag{24}$$

Let

$$d\mu_2(z) = \left(|f_1|^{q_1(z)} \times \prod_{i=3}^{k} |h_i| d\mu(z) \right) \Big/ \left(\|f_1\|_{p_1(\cdot)}^{s_1} \times \prod_{i=3}^{k} \|h_i\|_{p_i(\cdot)/q_i(\cdot)} \right)$$

then by (24) we get $\int_{\mathbb{B}_n} |h_2(z)|d\mu_2(z) \lesssim \|h_2\|_{p_2(\cdot)/q_2(\cdot)}$. Hence, μ_2 is a $\left(\frac{q_2(\cdot)}{p_2(\cdot)}\right)$-Bergman Carleson measure and so for any $f_2 \in A^{p_2(\cdot)}$, there is an $s_2 \in \{q_{2+}, q_{2-}\}$ such that

$$\int_{\mathbb{B}_n} |f_2(z)|^{q_2(z)} d\mu_2(z) \lesssim \|f_2\|^{s_2}_{p_2(\cdot)}.$$

Thus

$$\int_{\mathbb{B}_n} |f_1(z)|^{q_1(z)}|f_2(z)|^{q_2(z)} \left(\prod_{i=3}^{k} |h_i|\right) d\mu(z) \lesssim \|f_1\|^{s_1}_{p_1(\cdot)}\|f_2\|^{s_2}_{p_2(\cdot)} \times \left(\prod_{i=3}^{k} \|h_i\|_{p_i(\cdot)/q_i(\cdot)}\right). \tag{25}$$

It follows that

$$\int_{\mathbb{B}_n} \prod_{i=1}^{2} |f_i(z)|^{q_i(z)} \left(\prod_{i=3}^{k} |h_i|\right) d\mu(z) \lesssim \prod_{i=1}^{2} \|f_i\|^{s_i}_{p_i(\cdot)} \times \left(\prod_{i=3}^{k} \|h_i\|_{p_i(\cdot)/q_i(\cdot)}\right). \tag{26}$$

Let

$$d\mu_3(z) = \left(\prod_{i=1}^{2} |f_i|^{q_i(z)} \times \prod_{i=4}^{k} |h_i| d\mu(z)\right) \Big/ \left(\prod_{i=1}^{2} \|f_i\|^{s_i}_{p_i(\cdot)} \times \prod_{i=4}^{k} \|h_i\|_{p_i(\cdot)/q_i(\cdot)}\right)$$

then by (26) we get

$$\int_{\mathbb{B}_n} |h_3(z)| d\mu_3(z) \lesssim \|h_3\|_{p_3(\cdot)/q_3(\cdot)}.$$

Hence, μ_3 is a $\left(\frac{q_3(\cdot)}{p_3(\cdot)}\right)$-Bergman Carleson measure and so for any $f_3 \in A^{p_3(\cdot)}$, there exists an $s_3 \in \{q_{3+}, q_{3-}\}$ such that

$$\int_{\mathbb{B}_n} |f_3(z)|^{q_3(z)} d\mu_3 \lesssim \|f_3\|^{s_3}_{p_3(\cdot)}.$$

Thus

$$\int_{\mathbb{B}_n} \prod_{i=1}^{3} |f_i(z)|^{q_i(z)} \left(\prod_{i=4}^{k} |h_i|\right) d\mu(z) \lesssim \prod_{i=1}^{3} \|f_i\|^{s_i}_{p_i(\cdot)} \times \left(\prod_{i=4}^{k} \|h_i\|_{p_i(\cdot)/q_i(\cdot)}\right). \tag{27}$$

We continue this process to obtain the required result. □

Proposition 3.4. *Let μ be a positive Borel measure on $\mathbb{B}_n$. For $i = 1, 2 \ldots, k$, let $p_i(\cdot), q_i(\cdot) \in \mathcal{P}^{\log}(\mathbb{B}_n)$ be such that $1 \le p_i(\cdot) \le q_i(\cdot)$ and $\lambda(\cdot) = \sum_{i=1}^{k} \frac{q_i(\cdot)}{p_i(\cdot)}$. If there is an $s_i \in \{q_{i+}, q_{i-}\}$ such that*

$$\int_{\mathbb{B}_n} \prod_{i=1}^{k} |f_i(z)|^{q_i(\cdot)} d\mu(z) \lesssim \prod_{i=1}^{k} \|f_i\|^{s_i}_{p_i(\cdot)}, \quad f_i \in A^{p_i(\cdot)/q_i(\cdot)}, \tag{28}$$

$i = 1, 2 \ldots, k$ then μ is a $\lambda(\cdot)-$Bergman Carleson measure.

Proof. Consider the functions

$$f_{i,a}(z) = \frac{(1 - |a|^2)^{(n+1)/p_i(a)}}{(1 - \langle z, a\rangle)^{2(n+1)/p_i(a)}} = (1 - |a|^2)^{(n+1)/p_i(a)} K_a^{2/p_i(a)}(z).$$

By Lemma 2.6,

$$\|f_{i,a}\|_{p_i(\cdot)} = (1 - |a|^2)^{(n+1)/p_i(a)} \|K_a^{2/p_i(a)}\|_{p(\cdot)} \lesssim 1$$

Therefore $f_{i,a} \in A^{p_i(\cdot)}$ for $i = 1, 2, \ldots, k$. It follows by equation (28)

$$\int_{\mathbb{B}_n} \prod_{i=1}^{k} |f_{i,a}(z)|^{q_i(z)} d\mu(z) \lesssim \prod_{i=1}^{k} \|f_{i,a}\|^{s_i}_{p_i(\cdot)} \lesssim 1.$$

That is, using equation (12) we have

$$
\begin{aligned}
1 &\gtrsim \int_{\mathbb{B}_n} \prod_{i=1}^{k} |f_{i,a}(z)|^{q_i(z)} d\mu(z) = \int_{\mathbb{B}_n} \prod_{i=1}^{k} \left| \frac{(1-|a|^2)^{(n+1)/p_i(a)}}{(1-\langle z,a\rangle)^{2(n+1)/p_i(a)}} \right|^{q_i(z)} d\mu(z) \\
&= \int_{\mathbb{B}_n} \prod_{i=1}^{k} \left(\frac{(1-|a|^2)}{|1-\langle z,a\rangle|^2} \right)^{(n+1)q_i(z)/p_i(a)} d\mu(z) \\
&\gtrsim \int_{D_r(a)} \prod_{i=1}^{k} \left(\frac{(1-|a|^2)}{|1-\langle z,a\rangle|^2} \right)^{(n+1)q_i(z)/p_i(a)} d\mu(z) \\
&\approx \int_{D_r(a)} \prod_{i=1}^{k} \left(\frac{(1-|a|^2)}{(1-|a|^2)^2} \right)^{(n+1)q_i(a)/p_i(a)} d\mu(z).
\end{aligned}
$$

Thus,

$$
\frac{\mu(D_r(a))}{(1-|a|^2)^{(n+1)\lambda(a)}} \lesssim 1.
$$

Hence, by Proposition 3.1 and Proposition 2.5, μ is a $\lambda(\cdot)$-Bergman Carleson measure. □

The proof of Theorem 1.1 follows from Propositions 3.3 and 3.4 .

3.1 Proof of Theorem 1.2

Suppose (i) holds, that is T_μ is bounded from $A^{p_1(\cdot)}$ to $A^{p_2(\cdot)}$, with $p_1, p_2 \in \mathcal{P}^{\log}(\mathbb{B}_n)$ and $p_1(\cdot) \le p_2(\cdot)$. For $z \in \mathbb{B}_n$, we know that $K_z \in A^{p_1(\cdot)}(\mathbb{B}_n)$ where $K_z(w) = \frac{1}{(1-\langle w,z\rangle)^{n+1}}$. Thus,

$$
T_\mu K_z(\xi) = \int_{\mathbb{B}_n} \frac{K_z(w)}{(1-\langle \xi, w\rangle)^{n+1}} d\mu(w),
$$

we have that

$$
\begin{aligned}
T_\mu K_z(z) &= \int_{\mathbb{B}_n} \frac{1}{|1-\langle z,w\rangle|^{2(n+1)}} d\mu(w) \\
&\gtrsim \int_{D_r(z)} \frac{1}{(1-|z|^2)^{2(n+1)}} d\mu(w) \approx \frac{\mu(D_r(z))}{(1-|z|^2)^{2(n+1)}}.
\end{aligned} \tag{29}
$$

Now, $K_z \in A^{p_1(\cdot)}(\mathbb{B}_n)$ implies $T_\mu K_z \in A^{p_2(\cdot)}(\mathbb{B}_n)$. Thus applying Lemma 3.3 of (Agbor, 2022) we have

$$
\begin{aligned}
|T_\mu K_z(z)| &\lesssim \frac{\|T_\mu K_z\|_{p_2(\cdot)}}{(1-|z|^2)^{(n+1)/p_2(z)}} \le \frac{\|T_\mu\|\|K_z\|_{p_1(\cdot)}}{(1-|z|^2)^{(n+1)/p_2(z)}} \\
&\approx \frac{\|T_\mu\|(1-|z|^2)^{(n+1)\left(\frac{1}{p_1(z)}-1\right)}}{(1-|z|^2)^{(n+1)/p_2(z)}}.
\end{aligned} \tag{30}
$$

It follows that

$$
T_\mu K_z(z) \lesssim \|T_\mu\|(1-|z|^2)^{(n+1)/p_1(z)-(n+1)/p_2(z)-(n+1)}.
$$

From this and (29), we get

$$
\frac{\mu(D_r(z))}{(1-|z|^2)^{2(n+1)}} \lesssim T_\mu K_z(z) \lesssim \|T_\mu\|(1-|z|^2)^{(n+1)/p_1(z)-(n+1)/p_2(z)-(n+1)}.
$$

This shows that

$$
\mu(D_r(z)) \lesssim \|T_\mu\|(1-|z|^2)^{(n+1)/p_1(z)-(n+1)/p_2(z)+(n+1)} = \|T_\mu\|(1-|z|^2)^{(n+1)\lambda(z)},
$$

and hence

$$
\frac{\mu(D_r(z))}{(1-|z|^2)^{(n+1)\lambda(z)}} \lesssim \|T_\mu\|.
$$

Therefore, T_μ bounded implies $\frac{\mu(D_r(z))}{(1-|z|^2)^{(n+1)\lambda(z)}}$ is bounded and so, by Proposition 3.1 μ is a $\lambda(\cdot)$-Bergman Carleson measure.

Conversely, suppose ii) holds, that is, μ is $\lambda(\cdot)$-Bergman Carleson measure. We need to prove that T_μ is bounded. To this end, let $p_2'(\cdot)$ be the conjugate exponent of $p_2(\cdot)$. Then the dual of $A^{p_2(\cdot)}$ is $(A^{p_2(\cdot)})^* = A^{p_2'(\cdot)}$. So, for $h \in A^{p_2'(\cdot)}$ and $f \in A^{p_1(\cdot)}$, we have by Fubini's theorem that

$$\langle h, T_\mu f\rangle = \int_{\mathbb{B}_n} h(z)\overline{T_\mu f(z)}d\nu(z) = \int_{\mathbb{B}_n} h(z)\overline{f(z)}d\mu(z). \tag{31}$$

Now recall that

$$\lambda(\cdot) = 1 + \frac{1}{p_1(\cdot)} - \frac{1}{p_2(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2'(\cdot)}.$$

Hence, using the assumption that μ is a $\lambda(\cdot)$-Bergman Carleson measure, we have by Propositions 3.2 and 3.3, that

$$\begin{aligned}|\langle h, T_\mu f\rangle| &\leq \int_{\mathbb{B}_n} |h(z)f(z)|d\mu(z)\\ &\lesssim \|fh\|_{\frac{1}{\lambda(\cdot)}} \lesssim \|f\|_{p_1(\cdot)}\|h\|_{p_2'(\cdot)}.\end{aligned}$$

Thus T_μ is bounded, which completes the proof of the theorem. □

4. Vanishing Carleson Measures

The following proposition is Proposition 3.7 of (Agbor, 2022).

Proposition 4.1. *Let μ be a Borel measure on $\mathbb{B}_n$. Also let the function $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{B}_n)$ be such that $p(\cdot) \leq q(\cdot)$. Then, the following statements are equivalent:*

(a) For any sequence $\{f_k\}$ in $A^{p(\cdot)}$ with $\|f_k\|_{p(\cdot)} \leq 1$ and $f_k(z) \to 0$ uniformly on any compact subset of $\mathbb{B}_n$,

$$\lim_{k\to\infty}\int_{\mathbb{B}_n} |f_k(z)|^{q(z)}d\mu(z) = 0.$$

(b) For every (or some) $r > 0$ we have

$$\lim_{|a|\to 1^-}\frac{\mu(D_r(a))}{(1-|a|^2)^{(n+1)q(a)/p(a)}} = 0,$$

for $z \in \mathbb{B}_n$.

4.1 Proof of Theorem 1.3

Assume (i) holds. Let $\{f_{1,l}\}$ be a sequence in the unit ball of $A^{p_1(\cdot)}$ which converges uniformly on compact subsets of $\mathbb{B}_n$ and suppose $\{f_i\}$ are arbitrary functions in the unit ball of $A^{p_i(\cdot)}$, $i = 2, 3, \ldots, k$. Let $\mu_r = \mu|_{\mathbb{B}_n\setminus\overline{B_r}}$ where $B_r = \{z \in \mathbb{B}_n : |z| < r\}$. Then μ is a $\lambda(\cdot)-$Bergman Carleson measure and $\lim_{r\to 1}\|\mu_r\|_{\lambda(\cdot)} = 0$ (see pages 130-131 of (Cowen & MacCluer, 1995)). Let

$$I_1 := \int_{\mathbb{B}_n\setminus\overline{B_r}} |f_{1,l}(z)|^{q_1(z)}|f_2(z)|^{q_2(z)}\cdots|f_k(z)|^{q_k(z)}d\mu(z)$$

and

$$I_2 := \int_{\overline{B_r}} |f_{1,l}(z)|^{q_1(z)}|f_2(z)|^{q_2(z)}\cdots|f_k(z)|^{q_k(z)}d\mu(z).$$

Then

$$\begin{aligned}I_1 &\leq \int_{\mathbb{B}_n} |f_{1,l}(z)|^{q_1(z)}|f_2(z)|^{q_2(z)}\cdots|f_k(z)|^{q_k(z)}d\mu_r(z)\\ &\lesssim \|\mu_r\|_{\lambda(\cdot)} < \epsilon.\end{aligned}$$

as r gets sufficiently close to 1. On the other hand, if we fix such an r, then since $\{f_{1,l}\}$ converges to zero uniformly in compact subsets of $\mathbb{B}_n$, there exists a constant $K > 0$ such that for any $l > K$, $|f_{1,l}| < \epsilon$ for any $z \in \overline{B_r}$. Hence, applying Theorem 1.1 we get

$$\begin{aligned}I_2 &\leq \epsilon\int_{\mathbb{B}_n} |f_2(z)|^{q_2(z)}\cdots|f_k(z)|^{q_k(z)}d\mu(z)\\ &= \epsilon\int_{\mathbb{B}_n} |1|^{q_1(z)}|f_2(z)|^{q_2(z)}\cdots|f_k(z)|^{q_k(z)}d\mu(z)\\ &\lesssim \epsilon\|1\|^{s_1}_{p_1(\cdot)}\prod_{i=2}^{k}\|f_i\|^{s_i}_{p_i(\cdot)}.\end{aligned}$$

for some $s_i \in \{q_{i_+}, q_{i_-}\}$, $i = 1, 2, \ldots, k$. Thus $I_2 \lesssim \epsilon$, for any $z \in \overline{B_r}$. The assertion ii) follows from the fact that

$$\int_{\mathbb{B}_n} |f_{1,l}(z)|^{q_1(z)}|f_2(z)|^{q_2(z)} \cdots |f_k(z)|^{q_k(z)} d\mu(z) = I_1 + I_2.$$

Clearly, (*ii*) implies (*iii*). Now, assume (*iii*) holds. We need to show that μ is a vanishing $\lambda(\cdot)-$Bergman Carleson measure. For $a \in \mathbb{B}_n$, consider the functions

$$f_{i,a}(z) = \frac{(1 - |a|^2)^{(n+1)/p_i(a)}}{(1 - \langle z, a\rangle)^{2(n+1)/p_i(a)}}.$$

Then $\|f_{i,a}\|_{p_i(\cdot)} \lesssim 1$. Let K be a compact subset of $\mathbb{B}_n$. Then obviously there exists $C_K > 0$ such that

$$|f_{i,a}(z)| = C_K(1 - |a|^2)^{(n+1)/p_i^-} \to 0$$

as $|a| \to 1^-$. Therefore

$$\lim_{|a|\to 1} |f_{i,a}| = 0,$$

uniformly on any compact subset of $\mathbb{B}_n$. Now using equation (12) together with the assertion (iii) implies

$$\begin{aligned}
0 &= \lim_{|a|\to 1} \int_{\mathbb{B}_n} \prod_{i=1}^{k} \frac{(1 - |a|^2)^{(n+1)q_i(z)/p_i(a)}}{|1 - \langle z, a\rangle|^{2(n+1)q_i(z)/p_i(a)}} d\mu(z) \\
&\geq \lim_{|a|\to 1} \int_{D_r(a)} \prod_{i=1}^{k} \frac{(1 - |a|^2)^{(n+1)\frac{q_i(a)}{p_i(a)}}}{|1 - \langle z, a\rangle|^{2(n+1)\frac{q_i(a)}{p_i(a)}}} d\mu(z) \\
&= \lim_{|a|\to 1} \int_{D_r(a)} \frac{(1 - |a|^2)^{(n+1)\lambda(a)}}{|1 - \langle z, a\rangle|^{2(n+1)\lambda(a)}} d\mu(z) \\
&\gtrsim \lim_{|a|\to 1} \frac{\mu(D_r(a))}{(1 - |a|^2)^{(n+1)\lambda(a)}}
\end{aligned}$$

Thus μ is a vanishing $\lambda(\cdot)$-Bergman Carleson measure. □

We now use vanishing $\lambda(\cdot)$-Bergman Carleson measure to establish the compactness of Toeplitz operators, T_μ between Bergman spaces.

4.2 Proof of Theorem 1.4

Suppose that the operator T_μ is compact. Let $\{ a_j \}$ be an r-lattice in $\mathbb{B}_n$ such that $|a_j| \to 1^-$ as $j \to \infty$, and set

$$f_j(z) = \frac{(1 - |a_j|^2)^{(n+1)-(n+1)/p_1(a_j)}}{(1 - \langle z, a_j\rangle)^{n+1}} = (1 - |a_j|^2)^{(n+1)(1-1/p_1(a_j))} K_{a_j}(z)$$

for $z \in \mathbb{B}_n$. Then, $\|f\|_{p_1(\cdot)} \lesssim 1$ and $\sup_{z\in K} |f_j(z)| \to 0$ for each compact subset K of $\mathbb{B}_n$. Since T_μ is compact we have $\|T_\mu f_j\|_{p_2(\cdot)} \longrightarrow 0$ as $j \to \infty$. Also, just as in the proof of Theorem 1.2, we have that

$$T_\mu f_j(a_j) \gtrsim \frac{\mu(D_r(a_j))}{(1 - |a_j|^2)^{(n+1)+(n+1)/p_1(a_j)}},$$

and that

$$\frac{\mu(D_r(a_j))}{(1 - |a_j|^2)^{(n+1)\lambda(a_j)}} \lesssim \|T_\mu f_j\|_{p_2(\cdot)}.$$

Thus as $|a_j| \to 1^-$, $\frac{\mu(D_r(a_j))}{(1-|a_j|^2)^{(n+1)\lambda(a_j)}} \to 0$ which shows by Proposition 4.1 that μ is a vanishing $\lambda(\cdot)$-Bergman Carleson measure.

Conversely, suppose that μ is a vanishing $\lambda(\cdot)$-Bergman Carleson measure. We need to prove that T_μ is compact. Let $\{ f_j \}$ be a bounded sequence in $A^{p_1(\cdot)}$ converging uniformly to 0 on compact subsets of $\mathbb{B}_n$, then by Theorem 4.3 of (Rudin, 1991), we write

$$\begin{aligned}
\|T_\mu f_j\|_{p_2(\cdot)} &\approx \sup\{ |\langle h, T_\mu f_j\rangle| : \|h\|_{p_2'(\cdot)} \leq 1 \} \\
&\lesssim \sup_{\|h\|_{p_2'(\cdot)}\leq 1} \int_{\mathbb{B}_n} |f_j(z)||h(z)| d\mu(z) \qquad (32)
\end{aligned}$$

Thus applying Theorem 1.3 with $q_1(z) = q_2(z) = 1$, we get

$$\lim_{j\to\infty} \sup_{\|h\|_{p_2'(\cdot)}\leq 1} \int_{\mathbb{B}_n} |f_j(z)||h(z)|d\mu(z) = 0.$$

This, together with equation (32) gives

$$\|T_\mu f_j\|_{p_2(\cdot)} \lesssim \sup_{\|h\|_{p_2'(\cdot)}\leq 1} \int_{\mathbb{B}_n} |f_j(z)||h(z)|d\mu(z) \to 0,$$

as $j \to \infty$. □

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