A Note on Multilinear Mappings in Modules

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Abstract

The linear compactness of certain linearly topologized modules of multilinear mappings is established, as well as the continuity of multilinear mappings in a known class of linearly topologized modules.

Keywords: modules, multilinear mappings, linearly compact modules.

1. Introduction

In this note a criterion for the linear compactness of modules of multilinear mappings, endowed with the linear module topology of simple convergence, is obtained. It is also shown that, in a certain class of linearly topologized modules, every multilinear mapping is continuous. Other aspects of the study of multilinear mappings in the context of topological modules have been considered, for example, in (Bernardes Jr. N. C. 1996), (Concordido C. F. R. & Pombo Jr. D. P. 2002) and (Concordido C. F. R. & Pombo Jr. D. P. 2003).

In this work we shall adopt the terminology of (Warner S. 1989). R will represent a commutative ring with a non-zero identity element and R-module will always mean unitary R-module. If m is an integer ≥ 2 and $E_1, \ldots, E_m, F$ are R-modules, $\mathcal{F}(E_1, \ldots, E_m; F)$ will denote the R-module of all mappings from $E_1 \times \ldots \times E_m$ into F and $\mathcal{L}_a(E_1, \ldots, E_m; F)$ will denote the submodule of $\mathcal{F}(E_1, \ldots, E_m; F)$ consisting of all m-linear mappings from $E_1 \times \ldots \times E_m$ into F. If F is a topological (resp. linearly topologized) R-module, the topology τ_s of simple convergence on $\mathcal{F}(E_1, \ldots, E_m; F)$ is an R-module (resp. a linear R-module) topology which is a Hausdorff topology if the topology of F has the same property. Thus the topology induced by τ_s on $\mathcal{L}_a(E_1, \ldots, E_m; F)$, also denoted by τ_s, is an R-module (resp. a linear R-module) topology which is a Hausdorff topology if the topology of F has the same property.

Let us recall (Pombo Jr. D. P. & Mauro P. C. G. 2023) that, if R is a complete discrete valuation ring and K is the field of fractions of R, the quotient R-module $R_0 = K/R$ (equipped with the discrete topology) is a linearly compact R-module. Thus the product R-module $(R_0)^I$, equipped with the product topology, is a linearly compact R-module for every non-empty set I; in particular, for each prime natural number p, $\left(\mathbb{Q}_p/\mathbb{Z}_p\right)^I$ is a linearly compact $\mathbb{Z}_p$-module.

2. Multilinear Mappings in Modules

Our first result asserts that, under a special assumption, pointwise boundedness implies uniform boundedness. More precisely:

Proposition 2.1 Let R be a topological ring, m an integer ≥ 2, $E_1, \ldots, E_m$ finitely generated R-modules and F a linearly topologized R-module. If $\mathcal{X}$ is a τ_s-bounded subset of $\mathcal{L}_a(E_1, \ldots, E_m; F)$, then the set

$$L = \{u(x_1, \ldots, x_m); (x_1, \ldots, x_m) \in E_1 \times \ldots \times E_m,\ u \in \mathcal{X}\}$$

is bounded in F.

Proof Let V be a neighborhood of 0 in F which is a submodule of F. For each $i = 1, \ldots, m$ let $x^i_1, \ldots, x^i_{j_i} \in E_i$ be such that $E_i = \left[x^i_1, \ldots, x^i_{j_i}\right]$. By the τ_s-boundedness of $\mathcal{X}$, for all $l_1 \in \{1, \ldots, j_1\}, \ldots, l_m \in \{1, \ldots, j_m\}$, there is a neighborhood $W_{l_1,\ldots,l_m}$ of 0 in R so that

$$W_{l_1,\ldots,l_m} \left\{u(x^1_{l_1}, \ldots, x^m_{l_m}); u \in \mathcal{X}\right\} \subset V.$$

Consequently

$$W = \bigcap_{\substack{1 \le l_1 \le j_1 \\ \dots \\ 1 \le l_m \le j_m}} W_{l_1,\dots,l_m}$$

is a neighborhood of 0 in R such that

$$W\left\{u(x^1_{l_1},\dots,x^m_{l_m}); u \in \mathcal{X}\right\} \subset V$$

for all $l_1 \in \{1,\dots,j_1\},\dots,l_m \in \{1,\dots,j_m\}$. Finally, if $\lambda \in W$, $x_1 = \sum_{l_1=1}^{j_1} \lambda^1_{l_1} x^1_{l_1} \in E_1,\dots,x_m = \sum_{l_m=1}^{j_m} \lambda^m_{l_m} x^m_{l_m} \in E_m$ and $u \in \mathcal{X}$ are arbitrary,

$$\lambda u(x_1,\dots,x_m) = \sum_{\substack{1 \le l_1 \le j_1 \\ \dots \\ 1 \le l_m \le j_m}} \lambda\lambda^1_{l_1}\dots\lambda^m_{l_m} u(x^1_{l_1},\dots,x^m_{l_m}) \in V,$$

because V is a submodule of F. Hence L is a bounded subset of F, thereby concluding the proof.

Proposition 2.2 Let R be a topological ring. Let m be an integer ≥ 2, $E_1,\dots,E_m$ R-modules and F a linearly compact R-module. If $\mathcal{X}$ is a τ_s-closed submodule of $\mathcal{L}_a(E_1,\dots,E_m;F)$, then $\mathcal{X}$ is τ_s-linearly compact in $\mathcal{L}_a(E_1,\dots,E_m;F)$.

In order to prove the proposition we shall need an auxiliary result.

Lemma 2.3 Let R be a topological ring, m an integer ≥ 2, $E_1,\dots,E_m$ R-modules and F a Hausdorff topological R-module. Then $\mathcal{L}_a(E_1,\dots,E_m;F)$ is τ_s-closed in $\mathcal{F}(E_1,\dots,E_m;F)$.

Proof. We shall use a known argument (Bourbaki N. 1967). Indeed,

$$\mathcal{L}_a(E_1,\dots,E_m;F) = \bigcap_{\substack{i\in\{1,\dots,m\},\\ \lambda\in R,\ x_1\in E_1,\dots,\\ x_{i-1}\in E_{i-1},\ x_i\in E_i,\ y_i\in E_i,\\ x_{i+1}\in E_{i+1},\dots,\ x_m\in E_m}} \{f \in \mathcal{F}(E_1,\dots,E_m;F);$$

$$f(x_1,\dots,x_{i-1},\lambda x_i + y_i, x_{i+1},\dots,x_m) - \lambda f(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_m) -$$
$$f(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots,x_m) = 0\}.$$

Since, for $i, \lambda, x_1,\dots,x_{i-1},x_i,y_i,x_{i+1},\dots,x_m$ as above, the R-linear mapping

$$f \in (\mathcal{F}(E_1,\dots,E_m;F),\tau_s) \longmapsto f(x_1,\dots,x_{i-1},\lambda x_i + y_i,x_{i+1},\dots,x_m) -$$
$$\lambda f(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_m) - f(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots,x_m) \in F$$

is continuous, its kernel is closed in $(\mathcal{F}(E_1,\dots,E_m;F),\tau_s)$, because $\{0\}$ is closed in F. Therefore, in view of what we have observed at the beginning of the proof, the result is proved.

Now, let us turn to the proof of Proposition 2.2.

Proof. It is easily seen that the topological R-module $(\mathcal{F}(E_1,\dots,E_m;F),\tau_s)$ is isomorphic to the product topological R-module $F^{E_1\times\dots\times E_m}$ (and this does not depend on the linear compactness of F). Thus, by Theorem 31.10 of (Warner S. 1989),

$$(\mathcal{F}(E_1,\dots,E_m;F),\tau_s)$$

is linearly compact. Consequently, by Lemma 2.3 and Theorem 31.9(2) of (Warner S. 1989), one concludes that $\mathcal{X}$ is τ_s-linearly compact in $\mathcal{L}_a(E_1,\dots,E_m;F)$.

Definition 2.4 (Grothendieck A. & Dieudonné J. A. 1971) Let R a linearly topologized ring and let $\mathcal{W}$ be a fundamental system of neighborhoods of 0 in R consisting of ideals of R. Let E be an arbitrary R-module. For each $W \in \mathcal{W}$, let WE be the subset of E formed by the elements of E of the form $\lambda_1 x_1 + \dots + \lambda_n x_n$, where n is an integer ≥ 1, $\lambda_1,\dots,\lambda_n \in W$ and $x_1,\dots,x_n \in E$; WE is a submodule of E. By Theorem 12.3 of (Warner S. 1989), there is a unique R-module topology on E for which $\{WE; W \in \mathcal{W}\}$ is a fundamental system of neighborhoods of 0, called the topology derived from that of R. Thus E, endowed with this topology, is a linearly topologized R-module.

Example 2.5 If the linearly topologized ring R is equipped with the discrete topology, the topology derived from that of R is the discrete topology on E.

Example 2.6 If R is a discrete valuation ring (Serre J. -P. 1968) and π is a generator of the maximal ideal of R, $\{\pi^n E; n = 1, 2, \ldots\}$ is a fundamental system of neighborhoods of 0 in E with respect to the topology derived from that of R.

It has been observed in (Grothendieck A. & Dieudonné J. A. 1971) that, if R is a linearly topologized ring and E, F are R-modules endowed with the respective topologies derived from that of R, then the R-module $\mathcal{L}_a(E; F)$ of all R-linear mappings from E into F is equicontinuous. In fact, for each neighborhood W of 0 in R which is an ideal of R, $u(WE) \subset WF$ for all $u \in \mathcal{L}_a(E; F)$. The next result is an extension of the just mentioned fact to the multilinear case.

Proposition 2.7 Let R be a linearly topologized ring, m an integer ≥ 2, and let $E_1, \ldots, E_m, F$ be R-modules, each one being endowed with the topology derived from that of R. Then $\mathcal{L}_a(E_1, \ldots, E_m; F)$ is equicontinuous. In particular, every $u \in \mathcal{L}_a(E_1, \ldots, E_m; F)$ is continuous.

Proof. We shall establish the equicontinuity of $\mathcal{L}_a(E_1, \ldots, E_m; F)$ at an arbitrary element $(x_1, \ldots, x_m)$ of $E_1 \times \ldots \times E_m$. For this purpose let us first observe that, if $(h_1, \ldots, h_m) \in E_1 \times \ldots \times E_m$ and $u \in \mathcal{L}_a(E_1, \ldots, E_m; F)$ are arbitrary, then

$$u(x_1 + h_1, \ldots, x_m + h_m) - u(x_1, \ldots, x_m) = \sum_{\substack{H \subset \{1,\ldots,m\} \\ H \neq \{1,\ldots,m\}}} u_H,$$

where $u_H = u(y_1, \ldots, y_m)$, with $y_i = x_i$ if $i \in H$ and $y_i = h_i$ if $i \in \{1, \ldots, m\} \setminus H$.

Let $\mathcal{W}$ be as in Definition 2.4. Let $W \in \mathcal{W}$, $h_1 = \sum_{j_1=1}^{n_1} \mu^1_{j_1} x^1_{j_1} \in WE_1$, …, $h_m = \sum_{j_m=1}^{n_m} \mu^m_{j_m} x^m_{j_m} \in WE_m$ and $u \in \mathcal{L}_a(E_1, \ldots, E_m; F)$ be arbitrary, and let $H \subset \{1, \ldots, m\}$ with $H \neq \{1, \ldots, m\}$ be arbitrary. If $H = \emptyset$,

$$u_H = u(h_1, \ldots, h_m) = \sum_{\substack{1 \leq j_1 \leq n_1 \\ \cdots \\ 1 \leq j_m \leq n_m}} \mu^1_{j_1} \ldots \mu^m_{j_m} u(x^1_{j_1}, \ldots, x^m_{j_m}) \in WF$$

(incidentally this guarantees the equicontinuity of $\mathcal{L}_a(E_1, \ldots, E_m; F)$ at $(0, \ldots, 0) \in E_1 \times \ldots \times E_m$). On the other hand, if $H \neq \emptyset$, $\{1, \ldots, m\} \setminus H = \{s_1, \ldots, s_k\}$, where $1 \leq k < m$ and $s_1 \leq \ldots \leq s_k$. Thus u_H may be expressed in the form

$$u_H = \sum_{\substack{1 \leq j_{s_1} \leq n_{s_1} \\ \cdots \\ 1 \leq j_{s_k} \leq n_{s_k}}} \mu^{s_1}_{j_{s_1}} \ldots \mu^{s_k}_{j_{s_k}} z_{j_{s_1}, \ldots, j_{s_k}},$$

where each $z_{j_{s_1}, \ldots, j_{s_k}} \in F$, and hence $u_H \in WF$. Therefore

$$u(x_1 + h_1, \ldots, x_m + h_m) - u(x_1, \ldots, x_m) = \sum_{\substack{H \subset \{1,\ldots,m\} \\ H \neq \{1,\ldots,m\}}} u_H \in WF,$$

proving the equicontinuity of $\mathcal{L}_a(E_1, \ldots, E_m; F)$ at $(x_1, \ldots, x_m)$. This completes the proof.

A general criterion for the linear compactness of modules of continuous multilinear mappings, equipped with the linear module topology of simple convergence, may be found in (Pombo Jr. D. P. 2006).

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