

# A New Generalization of the Exponentiated Inverted Weibull Distribution With Applications to Tax Revenue and Reliability Data

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## Abstract

The addition of an extra parameter to standard distributions is a common technique in statistical theory. This study introduces a new generalization of the Exponentiated Inverted Weibull distribution named alpha power exponentiated Inverted Weibull distribution (APEIW). The APEIW allows for a significant amount of versatility in modeling various data forms as it accommodates upside-down bathtubs, decreasing, and reversed-J shapes for hazard rate function. Some of the APEIW's mathematical properties are derived in close forms. The maximum likelihood estimation technique was used for the purpose of estimation. Two real life data sets were used to show the applicability of the Alpha Power Exponentiated Inverted Weibull model over all other models considered in this study.

**Keywords:** Alpha Power Exponentiated Inverted Weibull model, upside-down bathtubs, hazard rate function, maximum likelihood estimation

## 1. Introduction

The Addition of an extra shape parameter to a classical distribution is common in statistical distribution theory. Oftentimes introducing an extra shape parameter(s) induce more flexibility to distribution functions mainly for data analysis purposes which improve the modeling potential of the classical distribution. To mention a few, Azzalini (1985) developed the skew-normal distribution by the addition of an extra parameter to the normal distribution to induce more flexibility into the normal distribution. Mudholkar and Srivastava (1993) proposed and studied a method that introduced an extra shape parameter to a two-parameter Weibull distribution and called it exponentiated Weibull model which has two shape parameters and one scale parameter. Marshall and Olkin (1997) developed another method that adds a parameter to any distribution function. The well-known generators are the following: the beta-G family of distribution which was developed by Eugene et al. (2002), Cordeiro and de Castro (2011) developed the Kumaraswamy-G family of distribution, exponentiated generalized-G family of distribution was studied by Cordeiro et al. (2013). Hassan and Eligarhy (2016) developed the Kumaraswamy Weibull generated family of distributions, the Odd generalized exponential family of distribution was proposed and studied Alizadeh et al. (2017), the exponentiated Weibull-H family of distribution was developed by Cordeiro et al. (2017), exponentiated generalized-G Poisson family of distribution was developed and studied by Aryal and Yousof (2017). Nofal et al. (2017) developed the generalized transmuted-G family of distribution, transmuted exponentiated generalized-G family of distribution was developed and studied by Yousof et al. (2015), transmuted geometric-G family of distribution was proposed and studied by Afify et al. (2016). Alizadeh et al. (2018) studied transmuted Weibull-G family of distribution, Marshall-Olkin generalized-G Poisson family of distribution was developed and studied by Korkmaz et al. (2018). Oluyede, et al. (2018) developed the gamma Weibull-G family of distributions by combining the gamma generator with the Weibull-G family of distributions which was defined by Bourguignon et al. (2014) and odd Lomax-G family of distribution was studied by Cordeiro et al. (2019). Reyad et al. (2019) proposed and developed the exponentiated generalized Topp-Leone-G family, Jamal and Chesneau (2020) studied the Sin Kumaraswamy-G family of distribution, the Topp-Leone Marshall-Olkin-G Family of Distributions was developed by Fastel et al. (2020) Recently, the alpha power transformation was proposed and studied by Mahdavi and Kundu (2017).

Let  $J(x)$  be the cumulative distribution function (CDF) of any continuous random variable  $X$ , then CDF of Alpha Power Transformed (APT) family is given by

$$F(x) = \begin{cases} \frac{\alpha^{J(x)} - 1}{\alpha - 1}, & \text{if } \alpha > 0, \alpha \neq 1 \\ J(x), & \alpha = 0 \end{cases} \quad (1)$$

And the associated probability density function (PDF) is

$$f(x) = \begin{cases} \frac{\log \alpha}{\alpha - 1} j(x) \alpha^{J(x)}, & \text{if } \alpha > 0, \alpha \neq 1 \\ J(x), & \text{if } \alpha = 0 \end{cases} \quad (2)$$

The transformation has been widely used by researchers to obtain alpha transformed distributions. Namely, Dey et al. (2017a, 2017b, 2018, 2019) examined the properties of the new extensions of generalized exponential distribution with an application to ozone data, a new extension of Weibull distribution with application to real-life data, extended Weibull distribution with application to real-life data, alpha transformed inverse Lindley distribution which exhibits upside-down bathtub shape failure rate, and alpha power transformed Lindley distribution with applications to earthquake data. Hassan et al. (2018) investigate the properties of alpha power transformed extended exponential distribution, alpha power Weibull distribution was studied by Nasser et al. (2017). Ogunde et al. (2020a, 2020b) studied the properties of alpha power extended Bur II distribution and alpha power extended inverted Weibull distribution respectively.

We derived our motivation from the advantages offered by a generalized distribution which are relevant in modeling lifetime data that are non-monotonic exhibiting different shapes of the hazard function ranges from increasing, decreasing, and bathtub shapes, as well as the versatility of compounding alpha g family of distribution with exponentiated Inverted Weibull distribution in modeling real-life data. Here, we study a new generalization called the Alpha power extended Inverted Weibull (APEIW) distribution which possesses these properties.

We are also motivated to study the APEIW distribution because of its simplicity and extensive usage of Inverted Weibull (IW) distribution in modeling lifetime events. Also, the current generalization promotes a wider application even to complex situations that involve different shapes of the hazard function.

## 2. The Model, Sub-Models, and Properties of Alpha Power Exponentiated Inverted Weibull (APEIW) Distribution

The probability density function (PDF) and the associated distribution function (CDF) of the two-parameter Exponentiated Inverted Weibull (IW) distribution was developed and study by Ahmad Flair (2017) and are given by

$$j(x; \omega, \rho) = \omega \rho x^{-\omega-1} e^{-x^{-\omega-1}} (1 - e^{-x^{-\omega}})^{\rho-1}, \quad x > 0 \quad (3)$$

and

$$J(x; \omega, \rho) = 1 - (1 - e^{-x^{-\omega}})^\rho, \quad x > 0 \quad (4)$$

$\omega$ , and  $\rho$  positive shape parameter ( $\omega > 0$ ). Keller et al. (1982) use the Inverted Weibull (IW) distribution to study mechanical components such as crankshaft and pistons of diesel engines using the IW distribution to. In addition, the IW model has many important applications in reliability engineering, seismography, Insurance, wear-out periods, useful life, service records, and life testing, see Khan and King (2012).

Several generalizations of the Inverted Weibull distribution have been proposed and studied, see, for example, Mashall Olkin Inverted Weibull distribution by Ogunde et al. (2017a, 2017b) studied the transmuted inverted Weibull and exponentiated transmuted inverted Weibull distribution. Oseghale et al. (2023a, 2023b) respectively proposed studied the Marshall-Olkin Extended Generalized Exponential Distribution and Harris Extended Generalized Exponential Distribution among many others.

Given that  $J(x)$  is the CDF of a distribution given in (4), then inserting (4) in (1) gives another distribution called Alpha Power Exponentiated Inverted Weibull distribution (APEIW) which CDF is given by

$$G(x; \alpha, \omega, \rho) = \begin{cases} \frac{\alpha^{1-(1-e^{-x^{-\omega}})^\rho} - 1}{\alpha - 1}, & \text{if } \alpha > 0, \alpha \neq 1 \\ \alpha^{(1-e^{-x^{-\omega}})^\rho}, & \alpha = 0 \end{cases} \quad (5)$$

And the corresponding PDF is given by

$$g(x; \alpha, \omega, \rho) = \begin{cases} \frac{\log \alpha}{\alpha - 1} \omega \rho x^{-\omega-1} e^{-x^{-\omega}} (1 - e^{-x^{-\omega}})^{\rho-1} \alpha^{1-(1-e^{-x^{-\omega}})^\rho}, & \text{if } \alpha > 0, \alpha \neq 1 \\ \omega x^{-\omega-1} e^{-x^{-\omega-1}} (1 - e^{-x^{-\omega}})^{\rho-1}, & \text{if } \alpha = 0 \end{cases} \quad (6)$$

Where  $\omega$ ,  $\rho$  and  $\alpha$  are positive shape parameters respectively. The survival function ( $S(x)$ ), hazard function ( $h(x)$ ), reversed hazard function ( $r(x)$ ), and the cumulative hazard function ( $\zeta(x)$ ) of the APEIW distribution are respectively given by

$$S(x; \alpha, \omega, \rho) = 1 - \frac{\alpha^{1-(1-e^{-x^{-\omega}})^\rho} - 1}{\alpha - 1}, \quad x > 0, \tag{7}$$

$$h(x; \alpha, \omega, \rho) = \frac{\frac{\log \alpha}{\alpha - 1} \omega \rho x^{-\omega-1} e^{-x^{-\omega-1}} (1 - e^{-x^{-\omega}})^{\rho-1} \alpha^{1-(1-e^{-x^{-\omega}})^\rho}}{1 - \frac{\alpha^{1-(1-e^{-x^{-\omega}})^\rho} - 1}{\alpha - 1}}, \quad x > 0, \tag{8}$$

and

$$r(x; \alpha, \omega, \rho) = \frac{\frac{\log \alpha}{\alpha - 1} \omega \rho x^{-\omega-1} e^{-x^{-\omega-1}} (1 - e^{-x^{-\omega}})^{\rho-1} \alpha^{1-(1-e^{-x^{-\omega}})^\rho}}{\alpha^{1-(1-e^{-x^{-\omega}})^\rho} - 1}, \quad x > 0. \tag{9}$$

The graph of the CDF and the PDF of APEIW distribution is given in figure 1 and that of the  $h(x; \alpha, \omega, \rho)$  in figure 2. In particular, figure 2 demonstrate the flexibility of APEIW model in modeling different kinds of data exhibiting different shapes of the hazard function. We observe that the graph of the  $h(x; \alpha, \omega, \rho)$  of APEIW is decreasing, increasing, upside-down bathtub.

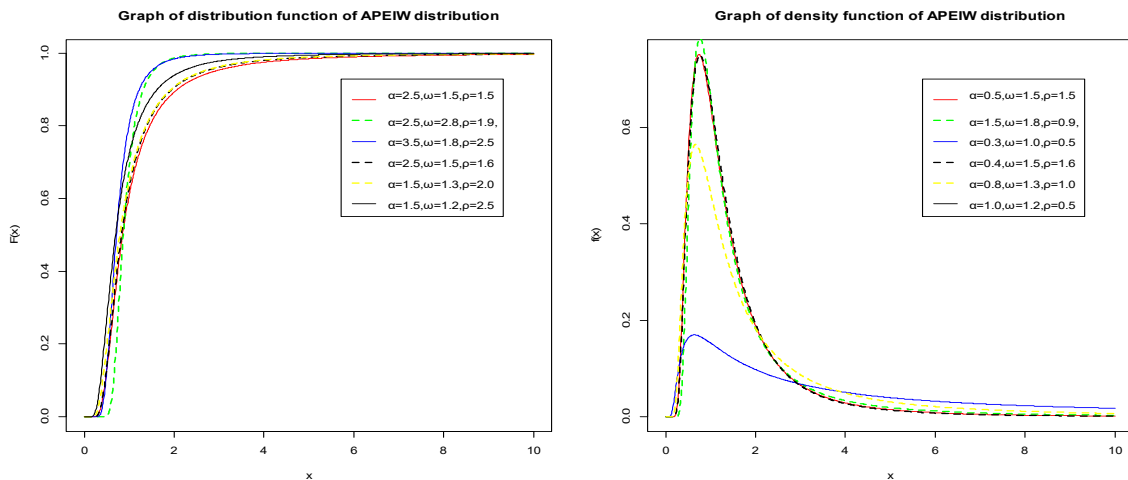


Figure 1. Graph of the CDF and the PDF of APEIW distribution

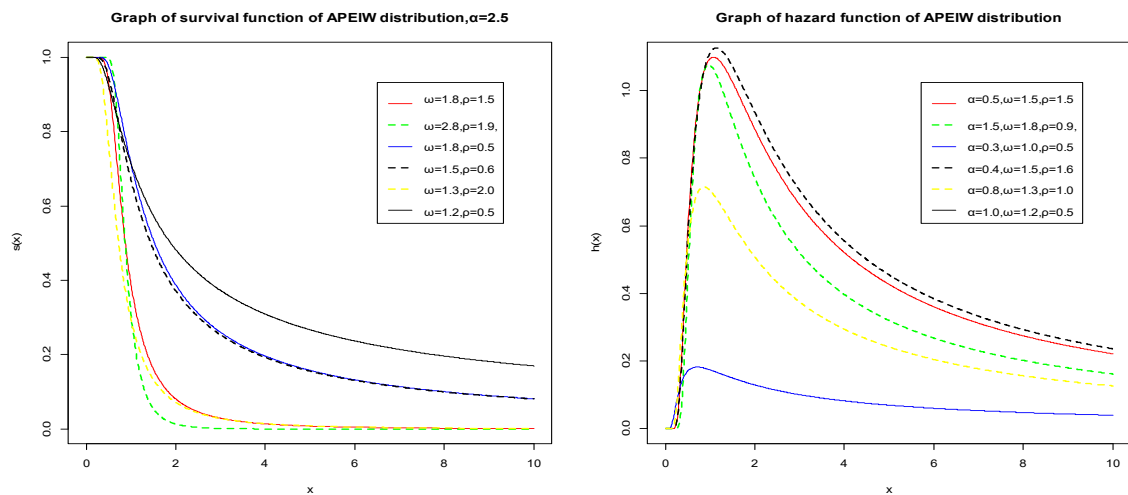


Figure 1. Graph of the survival and the hazard functions of APEIW distribution

### 2.1 Quantile Function

Quantile function can be defined as an inverse of the distribution function. Consider the relation

$$F(X) = U \Rightarrow X = F^{-1}(U)$$

Where  $U$  follows standard Uniform distribution. The  $u^{th}$  quantile of  $APEIW$  distribution is given by

$$X_u = \left\{ - \left[ \log \left( \frac{1}{\log \alpha} [1 - \log(1 + (\alpha - 1)u)] \right)^{1/\rho} \right] \right\}^{-1/\omega} \tag{10}$$

The lower quartile, mean, and the upper quartile  $APEIW$  distribution can be obtained from (10) by setting the value of  $u$  to be 0.25, 0.5, and 0.75 respectively. An expression for the lower quartile, median, and upper quartile is given as

$$X_{0.25} = \left\{ - \left[ \log \left( \frac{1}{\log \alpha} [1 - \log(1 + 0.25(\alpha - 1))] \right)^{1/\rho} \right] \right\}^{-1/\omega} \tag{11}$$

$$X_{0.5} = \left\{ - \left[ \log \left( \frac{1}{\log \alpha} [1 - \log(1 + 0.5(\alpha - 1))] \right)^{1/\rho} \right] \right\}^{-1/\omega} \tag{12}$$

and

$$X_{0.75} = \left\{ - \left[ \log \left( \frac{1}{\log \alpha} [1 - \log(1 + 0.75(\alpha - 1))] \right)^{1/\rho} \right] \right\}^{-1/\omega} \tag{13}$$

### 2.2 Random Numbers Generation

Random numbers can be generated for the  $APEIW$  ( $\alpha, \rho, \omega$ ) distribution, for this let, simulating values of random variable  $X$  with the CDF given in (5) and  $u$  denote a uniform random variable in (0, 1), then the simulated values of  $X$  are obtained by as,

$$X = \left\{ - \left[ \log \left( \frac{1}{\log \alpha} [1 - \log(1 + (\alpha - 1)u)] \right)^{1/\rho} \right] \right\}^{-1/\omega} \tag{14}$$

### 2.3 Mixture Representation for the Density Function

The mixture representation of the density function is a very useful tool used in deriving the statistical properties of generalized distribution. In this section, the mixture representation of the  $APEIW$  density function is obtained. Using the following series representation:

$$\alpha^m = \sum_{t=0}^{\infty} \frac{(\log \alpha)^t}{t!} m^t \tag{15}$$

$$(1 - v)^y = \sum_{t=0}^{\infty} (-1)^t \binom{y}{t} v^t \tag{16}$$

Using the series expansion given in (15) and (8), we obtain a reduce form of the PDF of  $APEIW$  distribution as

$$f(x) = \frac{\omega \rho}{\alpha - 1} \sum_{i=j=k=0}^{\infty} \frac{(\log \alpha)^i}{i!} (-1)^{j+k} \binom{i}{j} \binom{\rho(i+1)}{k} x^{-\omega-1} e^{-(k+1)x^{-\omega}} \tag{17}$$

The above expression is a density of inverted Weibull distribution with scale parameter  $(k + 1)$  and shape parameter  $\omega$ .

### 3. Ordinary and Incomplete Moment

The ordinary moments of distribution play a very important role in statistical applications. The  $r^{th}$  moment of a random variable  $X$  can be obtained using

$$E(X^r) = \mu'_r = \int_{-\infty}^{\infty} x^r f(x) dx \tag{18}$$

Putting (17) in (18), we have

$$\mu'_r = \frac{\omega \rho}{\alpha - 1} \sum_{i=j=k=0}^{\infty} \frac{(\log \alpha)^i}{i!} (-1)^{j+k} \binom{i}{j} \binom{\rho(i+1)}{k} f^\omega \tag{19}$$

where

$$f^\omega = \int_{-\infty}^{\infty} x^{r-\omega-1} e^{-(1+k)x^{-\omega}} dx \tag{20}$$

By letting  $z = (1+k)x^{-\omega}, x = z^{-\frac{1}{\omega}}((1+k))^{\frac{1}{\omega}}$  and putting it in (20), we have

$$f^\omega = \frac{1}{\omega} (k+1)^{\frac{r}{\omega}} \Gamma(1 - r/\omega)$$

Finally  $r^{th}$  moment of APEIW distribution is given by

$$\mu'_r = \frac{\rho}{\alpha - 1} \sum_{i=j=k=0}^{\infty} \frac{(\log \alpha)^i}{i!} (-1)^{j+k} \binom{i}{j} \binom{\rho(i+1)}{k} (k+1)^{\frac{r}{\omega}} \Gamma(1 - r/\omega) \tag{21}$$

$r < \omega$ . For  $r = 1, 2, \dots$   $\Gamma(\cdot)$  is the gamma function. By taking  $r = 1$ , we obtain the mean of  $X$  that is,  $\mu'_1 = \mu$ . The variance of  $X$  obtained by  $\sigma^2 = E[(X - \mu)^2] = \mu'_2 - \mu^2$ . Also, we can determine the  $r^{th}$  central moment and  $r^{th}$  cumulant of  $X$  respectively defined by

$$\mu_r = E[(X - \mu)^r] = \sum_{h=0}^r \binom{r}{h} \mu'_{r-h} (-1)^h \mu^h, \quad k_r = \mu'_r - \sum_{h=1}^{r-1} \binom{r-1}{h-1} k_h \mu'_{r-h},$$

Taking  $k = \mu$ , several measures of skewness and kurtosis based on the central moments (or cumulants) can be obtained.

Table 1 drawn below gives the first six moments and variance ( $\sigma^2$ ), skewness ( $s_k$ ), and kurtosis ( $k_u$ ) of APEIW distributions for various values of the parameter  $\alpha$  and fixed values of  $\omega = \rho = 10.5$

Table 1. first six moments,  $k_u, s_k, \sigma^2$  CV of APEIW

Moments	$\alpha = 0.1$	$\alpha = 3$	$\alpha = 10$	$\alpha = 15$
$\mu_1$	0.8880	0.9199	0.9307	0.9339
$\mu_2$	0.7896	0.8475	0.8673	0.8733
$\mu_3$	0.7030	0.7819	0.8093	0.8176
$\mu_4$	0.6267	0.7225	0.7562	0.7665
$\mu_5$	0.5595	0.6685	0.7075	0.7194
$\mu_6$	0.5000	0.6195	0.6627	0.6760
$\sigma^2$	0.0011	0.0013	0.0011	0.0011
$s_k$	-1.1731	-1.6852	1.6773	-2.2869
$k_u$	40.4745	111.95	-31.3680	178.5223

Table 1 shows that the APEIW model can be used to model both the positively skewed and negatively skewed data and also for various shapes of kurtosis

An expression for an Incomplete moment is given by

$$\varphi_r(t) = \int_0^t x^r f(x) dx \tag{22}$$

Putting (17) in (22), we have

$$\varphi_r(t) = \frac{\omega\rho}{\alpha - 1} \sum_{i=j=k=0}^{\infty} \frac{(\log\alpha)^i}{i!} (-1)^{j+k} \binom{i}{j} \binom{\rho(i+1)}{k} (-1)^k f^*$$

where

$$f^* = \int_0^t x^{r-\omega-1} e^{-(1+k)x^{-\omega}} dx \tag{23}$$

Also, by letting  $z = (1+k)x^{-\omega}$ ,  $x = z^{-\frac{1}{\omega}}((1+k))^{\frac{1}{\omega}}$  and putting it in (23), we have

$$f^\omega = \frac{1}{\omega} ((1+k))^{\frac{r}{\omega}} \Gamma(1 - r/\omega, (1+k)t^{-\omega})$$

Finally the  $r^{th}$  incomplete moment of APEIW distribution is given by

$$\varphi_r(t) = \frac{\rho}{\alpha - 1} \sum_{i=j=k=0}^{\infty} \frac{(\log\alpha)^i}{i!} (-1)^{j+k} \binom{i}{j} \binom{\rho(i+1)}{k} (k+1)^{\frac{r}{\omega}} \Gamma\{1 - r/\omega, (1+k)t^{-\omega}\} \tag{24}$$

Where  $\Gamma(l, n) = \int_n^\infty v^{l-1} e^{-v} dv$  is the complementary incomplete gamma function. The first incomplete moment of APEIW distribution is given as

$$\varphi_1(t) = \frac{\rho}{\alpha - 1} \sum_{i=j=k=0}^{\infty} \frac{(\log\alpha)^i}{i!} (-1)^{j+k} \binom{i}{j} \binom{\rho(i+1)}{k} (k+1)^{\frac{1}{\omega}} \Gamma\{1 - 1/\omega, (1+k)t^{-\omega}\} \tag{25}$$

The mean deviation,  $\gamma_1(x)$  and median deviation,  $\gamma_2(x)$ , can be obtained by using the relation,  $\gamma_1(x) = 2\mu F(\mu) - 2\gamma_1(\mu)$  and  $\gamma_2(x) = \mu - 2\gamma_1(M)$ . Where  $\mu = E(X)$  and  $M$  is the median of the APEIWP random variable. Both the  $\gamma_1(\mu)$  and  $\gamma_1(M)$  are calculated from the first incomplete moment as given in (25)

### 3.1 Inequality Measures

Inequality measures can be applied in biomedical sciences, product quality control economics, insurance and demography, and many more. Here we consider the following inequality measures:

### 3.2 Mean Residual Life (MRL)

Residual life is defined as the expected additional life length for a unit that is alive at age  $t$ , and it is represented mathematically by  $m_x(t) = E(X - t | X > t)$ ,  $t > 0$ .

The MRL of  $X$  can be obtained by using the formula:

$$m_x(t) = \frac{[1 - \varphi_1(t)]}{S(t)} - t, \tag{26}$$

Where  $S(t)$  is the survival function of  $X$  and  $\varphi_1(t)$  as given in (25). Then we have

$$m_x(t) = \frac{1}{S(t)} \left( \frac{\rho}{\alpha - 1} \sum_{i=j=k=0}^{\infty} \frac{(\log\alpha)^i}{i!} (-1)^{j+k} \binom{i}{j} \binom{\rho(i+1)}{k} \Gamma\{1 - 1/\omega, (1+k)t^{-\omega}\} \right) - t,$$

The mean inactivity time (MIT) (mean waiting time) is defined by  $M_x(t) = E(tX | X \leq t)$ ,  $t > 0$ , and it can be obtained by the formula:

$$M_x(t) = t - \left[ \frac{\varphi_1(t)}{F(t)} \right] \tag{27}$$

Also putting (25) in (27), we obtain an expression for MIT for APEIW distribution as

$$M_x(t) = t - \left[ \frac{\frac{\rho}{\alpha - 1} \sum_{i=j=k=0}^{\infty} \frac{(\log\alpha)^i}{i!} (-1)^{j+k} \binom{i}{j} \binom{\rho(i+1)}{k} (k+1)^{\frac{1}{\omega}} \Gamma\{1 - 1/\omega, (1+k)t^{-\omega}\}}{F(t)} \right]$$

### 3.3 Bonferroni And Lorenz Curves

The Bonferroni and Lorenz curve of APEIW distribution are respectively given by

$$\mathfrak{B}_F = \frac{1}{\mu F(t)} \int_0^t x^r f(x) dx \tag{28}$$

Since,

$$\int_0^t x^r f(x) dx = \frac{\rho}{\alpha - 1} \sum_{i=j=k=0}^{\infty} \frac{(\log \alpha)^i}{i!} (-1)^{j+k} \binom{i}{j} \binom{\rho(i+1)}{k} \Gamma\{1 - r/\omega, (1+k)t^{-\omega}\}$$

therefore

$$\begin{aligned} \mathfrak{B}_F(t) &= \frac{1}{\mu F(t)} \frac{\rho}{\alpha - 1} \sum_{i=j=k=0}^{\infty} \frac{(\log \alpha)^i}{i!} (-1)^{j+k} \binom{i}{j} \binom{\rho(i+1)}{k} \\ &\quad \times (k+1)^{\frac{r}{\omega}} \Gamma\{1 - r/\omega, (1+k)t^{-\omega}\} \end{aligned}$$

And the Lorenz curve

$$\begin{aligned} L_F(t) &= \frac{1}{\mu} \int_0^t x^r f(x) dx \tag{29} \\ &= \frac{1}{\mu} \frac{\rho}{\alpha - 1} \sum_{i=j=k=0}^{\infty} \frac{(\log \alpha)^i}{i!} (-1)^{j+k} \binom{i}{j} \binom{\rho(i+1)}{k} (k+1)^{\frac{r}{\omega}} \Gamma\{1 - r/\omega, (1+k)t^{-\omega}\} \end{aligned}$$

### 3.4 Probability Weighted Moments (PWMs)

The  $(q, r)^{th}$  PWM of X, denoted by  $\mathfrak{E}_{q,r}$  is given by

$$\mathfrak{E}_{q,r} = E[x^q F(X)^r] = \int_{-\infty}^{\infty} x^q F(X)^r f(x) dx \tag{30}$$

Putting equation (5) and (6) in (30), then followed by algebraic manipulation we obtain an expression for the PWMs of APEIW distribution as

$$\begin{aligned} \mathfrak{E}_{q,r} &= \frac{\rho}{(\alpha - 1)^{q+1}} \sum_{i=j=k=0}^{\infty} \sum_{p=0}^{\rho(j+1)-1} \frac{(-1)^{i+k+p+q}}{j!} \binom{q}{i} \binom{j}{k} \binom{\rho(j+1)-1}{p} \\ &\quad \times (i+1)^j (\log \alpha)^j (p+1)^{\frac{q-\omega}{\omega}} \Gamma(1 - q/\omega) \\ &\quad \left[ \frac{\alpha^{1-(1-e^{-x^{-\omega}})^{\rho}} - 1}{\alpha - 1} \right]^r = \sum_{i=0}^{\infty} (-1)^{i+q} \binom{q}{i} \left( \alpha^{1-(1-e^{-x^{-\omega}})^{\rho}} \right)^i \end{aligned}$$

## 4. Order Statistics

Order statistics is widely applied in statistical theory. Suppose  $X_1, X_2, \dots, X_n$  be a random sample having CDF  $F(x)$ . Let  $x_{1:1}, x_{2:n}, \dots, x_{n:n}$  be the ordered sample of size  $n$ , then the density of  $r^{th}$  order statistics is given as

$$f_{r:n}(x) = P \sum_{l=0}^{n-r} (-1)^l \binom{n-r}{l} f(x) F(x)^{l+r-1} \tag{31}$$

Where  $P = \frac{n!}{(n-r)!r!}$ . The PDF of the  $r^{th}$  order statistics of APEIW distribution is obtained by putting (5) and (6) in ( ), changing  $s$  with  $l + r - 1$  followed by simple algebraic manipulation, we have

$$f_{r:n}(x) = P\alpha\omega\theta \sum_{l=0}^{n-r} \sum_{i,j=0}^{\infty} Q_p x^{-(\omega+1)} e^{-(p+1)x^{-\omega}} \tag{32}$$

where

$$Q_p = \frac{\log\alpha}{(\alpha - 1)^{s+1}} \sum_{k=0}^{\infty} \sum_{p=0}^{\rho(j+1)-1} (-1)^{i+k+p+s} \binom{s}{i} \binom{j}{k} \binom{\rho(j+1)-1}{p} (i+1)^j \frac{(\log\alpha)^j}{j!}$$

Consequently, the  $s^{th}$  moment of the  $r^{th}$  order statistics for APEIW distribution is given by

#### 4.1 Maximum Likelihood Estimation

Given a random sample of  $x_1, x_2, \dots, x_n$  from the APEIW distribution, the likelihood function for  $\xi = (\alpha, \omega, \rho)$  is

$$L(\underline{x}, \xi) = \prod_{i=1}^n \frac{\log\alpha}{\alpha - 1} \omega \rho x_i^{-\omega-1} e^{-x_i^{-\omega-1}} (1 - e^{-x_i^{-\omega}})^{\rho-1} \alpha^{1-(1-e^{-x_i^{-\omega}})^{\rho}} \tag{33}$$

And the log-likelihood function  $\log L(\underline{x}, \xi) = l$  is presented as

$$l = n\log(\rho) + n\log(\omega) + n\log\left(\frac{\log\alpha}{\alpha - 1}\right) - (\omega + 1) \sum_{i=1}^n \log(x_i) - \sum_{i=1}^n x_i^{-\omega-1} - (\rho - 1) \sum_{i=1}^n \log(1 + e^{-x_i^{-\omega}}) + \log\alpha \sum_{i=1}^n 1 - (1 - e^{-x_i^{-\omega}})^{\rho} \tag{34}$$

We differentiate (34) with respect  $\alpha, \rho$  and  $\omega$ , to obtain the element of the score vector  $(V_{\alpha} = \frac{\partial l}{\partial \alpha}, V_{\rho} = \frac{\partial l}{\partial \rho}, V_{\omega} = \frac{\partial l}{\partial \omega})^T$ .

The elements of the score vector are given by

$$V_{\rho} = \frac{n}{\rho} - \sum_{i=1}^n \log(1 + e^{-x_i^{-\omega}}) + \sum_{i=1}^n (1 - e^{-x_i^{-\omega}})^{\rho} \log(1 - e^{-x_i^{-\omega}}) \tag{35}$$

$$V_{\omega} = \frac{n}{\omega} - \sum_{i=1}^n \log(x_i) + (\omega + 1) \sum_{i=1}^n x_i^{-\omega-1} + (\rho - 1) \sum_{i=1}^n \frac{x_i^{-\omega-1} e^{-x_i^{-\omega}}}{(1 + e^{-x_i^{-\omega}})} \tag{36}$$

$$V_{\alpha} = \frac{n}{\alpha - 1} + \frac{n(\alpha - 1 - \alpha \log(\alpha))}{\alpha(\alpha - 1)\log(\alpha)} + \frac{1}{\alpha} \sum_{i=1}^n (1 - e^{-x_i^{-\omega}})^{\rho} \tag{37}$$

The maximum likelihood estimator  $\hat{\xi} = [(\hat{\alpha}, \hat{\omega}, \hat{\rho})]^T$  can be derived by solving the following simultaneous equations

$$\frac{\partial l(\xi)}{\partial \alpha} \Big|_{\xi=\hat{\xi}} = 0, \quad \frac{\partial l(\xi)}{\partial \omega} \Big|_{\xi=\hat{\xi}} = 0, \quad \frac{\partial l(\xi)}{\partial \rho} \Big|_{\xi=\hat{\xi}} = 0$$

However, since there is no closed-form expression for the maximum likelihood estimator, and computation can only be done numerically using nonlinear optimization techniques. We can adopt the use of iterative techniques such as a Newton–Raphson-type algorithm to obtain the numerical value of  $\hat{\xi}$ . The observed information matrix used for computing confidence intervals for the model parameters can be computed from standard maximization routines numerically, such as the SAS procedure NL Mixed, the Ox function Max BFGS, R, and many others, to estimate the



values of the observed information matrix numerically.

#### 4.2 Practical Applications

In this subsection, we evaluate the performance of the APEIW distributions with the other four competing models to two reliability data sets. The data sets are described as follows:

##### Applications

In this section, the APEIW distribution is compared with Alpha Power Inverse Exponential (APIE), Alpha Power Exponentiated Inverse Exponential (APEIE), Alpha Power Inverted Weibull (APIW), and IW distributions. Different goodness of fit measures like Cramer-von Mises (W), Anderson Darling (A), Kolmogorov- Smirnov (K-S) statistics with p-values, Akaike Information Criterion (AIC), consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC) and likelihood ratio statistics are computed using R-package for real data sets: fracture toughness, taxes revenue’s data and coal mining disasters data. The better fit corresponds to smaller, AIC, CAIC, BIC, HQIC and  $-l$  value. The Maximum Likelihood Estimates (MLEs) of the unknown parameters and values of goodness of fit measures are computed for APEIW distribution and its sub-models.

Dat set I: (Fracture Toughness) The values fracture toughness MPa m<sup>1/2</sup> data from the material Alumina ( 23 Al O ) are: 5.5, 5, 4.9, 6.4, 5.1, 5.2, 5.2, 5, 4.7, 4, 4.5, 4.2, 4.1, 4.56, 5.01, 4.7, 3.13, 3.12, 2.68, 2.77, 2.7, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.8, 3.73, 3.71, 3.28, 3.9, 4, 3.8, 4.1, 3.9, 4.05, 4, 3.95, 4, 4.5, 4.5, 4.2, 4.55, 4.65, 4.1, 4.25, 4.3, 4.5, 4.7, 5.15, 4.3, 4.5, 4.9, 5, 5.35, 5.15, 5.25, 5.8, 5.85, 5.9, 5.75, 6.25, 6.05, 5.9, 3.6, 4.1, 4.5, 5.3, 4.85, 5.3, 5.45, 5.1, 5.3, 5.2, 5.3, 5.25, 4.75, 4.5, 4.2, 4, 4.15, 4.25, 4.3, 3.75, 3.95, 3.51, 4.13, 5.4, 5, 2.1, 4.6, 3.2, 2.5, 4.1, 3.5, 3.2, 3.3, 4.6, 4.3, 4.3, 4.5, 5.5, 4.6, 4.9, 4.3, 3, 3.4, 3.7, 4.4, 4.9, 4.9, 5.

Data set II: (Taxes Revenue’s Data): The values of tax revenue data are 3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65. Figure 3 gives the diagrammatic representation for the Box plot for the two data sets. Table 2 provides the exploratory data analysis (EDA) for the data sets which shown that data I negatively skewed, under-dispersed with excess kurtosis of 0.94 which further indicates that the data is leptokurtic, data II is positively skewed, under-dispersed with excess kurtosis of 0.177 which also show that data II is leptokurtic. Tables 3 and 4 gives details on the values of the measures of goodness of fit for the two data set.

Table 2. Exploratory data analysis for the two data sets

	Min	$q_1$	Median	mean	$q_3$	Max.	variance	skewness	Kurtosis
Data I	1.680	3.850	4.380	4.325	5.0	6.810	1.037	-0.417	3.094
Data II	0.390	1.840	2.675	2.611	3.197	5.560	1.017	0.393	3.177

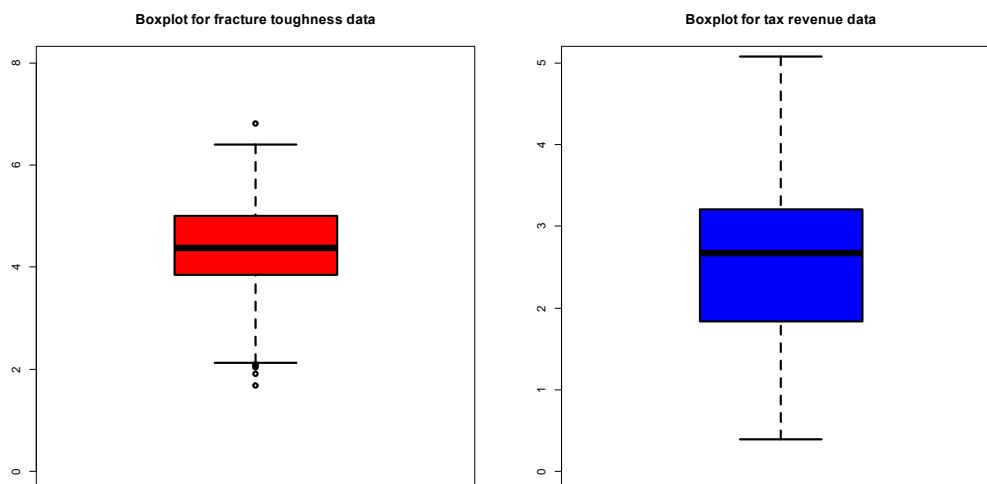


Figure 3. Boxplot for the two data sets

Table 3. Analytical results of the APEIW model and other competing models for fracture toughness data

<i>Model</i>	$\alpha$	$\omega$	$\rho$	$-l$	<i>AIC</i>	<i>CAIC</i>	BIC	HQIC
<i>APEIW</i>	28.353 (4.641)	0.118 (0.032)	5.697 (1.518)	184.453	374.905	375.114	383.243	378.291
<i>APIW</i>	37.554 (8.786)	1.450 (0.081)	– (–)	287.457	578.914	579.018	584.472	581.171
<i>EIW</i>	– (–)	0.135 (0.069)	5.169 (2.602)	332.302	668.604	668.707	674.162	670.861
<i>APEIE</i>	32.265 (6.818)	0.418 (0.041)	0.886 (0.139)	254.458	514.915	515.124	523.253	518.301
<i>IW</i>	– (–)	0.938 (0.071)	– (–)	199.624)	401.248	401.282	404.027	402.376

Table 4. Analytical results of the APEIW model and other competing models for Tax revenue’s data

<i>Model</i>	$\alpha$	$\omega$	$\rho$	$-l$	<i>AIC</i>	<i>CAIC</i>	BIC	HQIC
<i>APEIW</i>	26.133 (5.104)	0.306 (0.034)	3.041 (0.189)	97.331	200.661	200.911	208.477	203.824
<i>APIW</i>	32.351 (10.547)	2.048 (0.131)	– (–)	165.829	335.658	335.782	340.869	337.767
<i>EIW</i>	– (–)	0.451 (0.061)	2.397 (0.242)	197.163	398.326	398.450	403.537	400.435
<i>IW</i>	– (–)	1.426 (0.117)	– (–)	126.284	254.570	254.610	257.175	255.624

The new APEIW model is much better than other three important competitive models with smallest value of AIC, CAIC, BIC, and HQIC value in modeling the second data set.

**5. Concluding Remarks**

We have developed and studied the APEIW distribution along with its properties such as: descriptive measures based on the quantiles, moments, incomplete moments, Lorenz and Bonferroni curves, stress-strength reliability, weighted moment, entropy, and order statistics. Maximum Likelihood estimates are computed. The simulation study is performed using the quantile function of APEIW distribution. to illustrate the performance of the MLEs two different data sets were used. Applications of the APEIW model to fracture toughness, and taxes revenue’s data are presented to demonstrate its tractability and flexibility in modeling real life data. We have shown that the APEIW distribution is empirically better for fracture toughness, and taxes revenue’s data.

### Authors contributions

Dr. Oseghale O. I and Dr. Ayoola J.F were responsible for study design and revising. Dr. Oluwole A. Nuga was responsible for data collection and properties investigation. Dr. Oseghale O. I. drafted the manuscript and Dr. Ayoola J. F and Dr. Oluwole A. Nuga revised it. All authors read and approved the final manuscript and contributed equally towards the development of the manuscript.

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