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2-Distance and 3-Distance Domination Numbers of the Sierpinski Star Graph

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Abstract

The domination set $D(G)$ in graph $G = (V(G), E(G))$ is a subset of the vertex set in graph G such that every vertex in $V(G)\backslash D(G)$ is adjacent to at least one vertex in $D(G)$. The minimum cardinality of a domination set in graph G is called the domination number and is denoted as $\gamma(G)$. The set $S_k(G)$ is called the *k*-distance domination set in graph G if every vertex v in $V(G)\backslash S_k(G)$ has a distance of less than or equal to k from at least one vertex in $S_k(G)$. The minimum cardinality of a *k*-distance domination set in graph G is called the *k*-distance domination number and is denoted as $\gamma_k(G)$. This paper investigated the 2-distance and 3-distance domination sets in the Sierpinski Star graph SS_n and derived the number of 2-distance domination of $\gamma_2(SS_n) = 1$ for $n < 3$ and $\gamma_2(SS_n) = 3.3^{n-3}$ for $n \geq 3$, as well as the 3-distance domination number of $\gamma_3(SS_n) = 1$ for $n < 3$ and $\gamma_3(SS_n) = 3^{n-3}$ for $n \geq 3$.

Keywords: distance domination set, distance domination numbers, Sierpinski star graph

1. Introduction

Domination is one of the topics studied in graph theory. The history of domination in graphs dates back to 1850 when European chess players were enthusiastic about solving the "dominating queens" problem. Dominance was used to determine the number of queens such that each queen could dominate or attack every position with a single move on an 8 x 8 chessboard (Haynes et al., 1998). In graph theory, queens are represented as vertices, while chessboard movements are considered edges. However, domination was first formally defined as a theoretical graph concept in 1958. Domination in graphs has experienced rapid growth since its introduction. By the end of the 1990s, over 1200 papers on domination in graphs had been published (Haynes et al., 2020).

One type of domination in graphs is distance domination. Distance domination in graphs can be applied to various real-life problems and structures that generate graphs, including communication networks, geometric problems, and public facility location problems, for example, in solving the problem of selecting the minimum number of cities as locations for transmission stations so that each city contains a transmitter or can receive messages from at least one transmission station. If communication through paths along k-links is adequate in quality and speed, the problem can be solved by determining the minimum k-distance domination set in the graph (Haynes et al., 2020). In this case, the links represent connections or distances between two cities represented as edges, and the cities are represented as vertices.

The concept of distance domination in graphs has been extensively researched. (Meir & Moon, 1975) combined the concepts of distance and domination in graphs and introduced the concept of distance domination in a graph. Some other articles that discuss distance domination include the study of bounds on 2-distance domination number in graphs by (Sridharan et al., 2002), distance domination in critical graphs discussed by (Tian & Xu, 2008), while (Canales et al., 2015) and (Alvarado et al., 2015) discussed distance domination in outerplanar graphs, and distance domination in graphs with minimum and maximum degrees was explored by (Henning & Lichiardopol, 2017).

This paper investigates the domination numbers of 2-distance and 3-distance in the Sierpinski Star graph SS_n. Geometrically, the Sierpinski Star graph is formed from the Sierpinski triangle. The Sierpinski Star graph $S(n,k)$ was introduced by (Klavžar & Milutinović, 1997), which is a generalization of the Tower of Hanoi problem.

2. Results and Discussion

In this chapter, we will explore the definitions of a k-distance domination set, 2-distance domination set, and 3-distance domination set, and introduce the description of the Sierpinski Star graph, complete with examples and illustrations. Furthermore, we will discuss the research findings regarding 2-distance and 3-distance domination numbers in the Sierpinski Star graph and the labeling of 2-distance and 3-distance domination sets in the Sierpinski Star graph.

The following describes the definition of *k*-distance domination set in a graph.

Definition 1. *A set $S_k(G)$ is called a k-distance domination set in a graph G if every vertex v in $V(G)\backslash S_k(G)$ has a distance of less than or equal to k from at least one vertex in $S_k(G)$. In other words, for every vertex v in $V(G)\backslash S_k(G)$, there is $d(v, S_k(G)) \leq k$, with $d(v, S_k(G))$ being the distance from a vertex v in $V(G)\backslash S_k(G)$ to the set $S_k(G)$. The k-distance domination number of graph G, is the minimum cardinality of a distance-k domination set and is denoted as $\gamma_k(G)$* (Haynes et al., 2020).

When $k = 1$, the domination set is the same as the *k*-distance domination set, such that $D(G) = S_k(G)$. This is because if two vertices in the graph are at a distance of one, that these two vertices are adjacent. This paper refers to the 2-distance and 3-distance domination sets in the graph, along with the 2-distance and 3-distance domination numbers in the graph.

The following is an explanation of the concept of the Sierpinski Star graph, along with an example.

Definition 2. *The definition of the Sierpinski Star graph SS_n is as taken after: for SS_1, it comprises as it where one vertex; for $n \geq 2$, the vertex set in SS_n is composed of $(n-1)$-tuples of numbers from the set 0,1,2,3 denoted as $V(SS_n) = \{(0,1,2,3)^{n-1}\}|(n-1) -$ tuples. Two vertices $a = (a_1 a_2 \ldots a_{(n-1)})$ where $a_i \in \{0,1,2,3\}$ and $b = (b_1 b_2 \ldots b_{(n-1)})$ where $b_i \in \{1,2,3\}$ are considered adjacent if and only if there is a value $p \in \{1,2,\ldots,n-1\}$ with the condition that:*

i. $a_i = 0$ *for* $i = p, p+1, \ldots, n-1$

ii. $a_i = b_i$ *for* $i = 1,2,\ldots,p-1$

iii. $b_i \neq b_k$ *for* $i = p+1, p+2, \ldots, n-1$

Moreover, the vertex $(a_1 a_2 \ldots a_{n-1})$ is denoted as $\langle a_1 a_2 \ldots a_{n-1} \rangle$ (Khabibah & Munawwaroh, 2021).

Example 1. The following describes the Sierpinski Star graph SS_3.

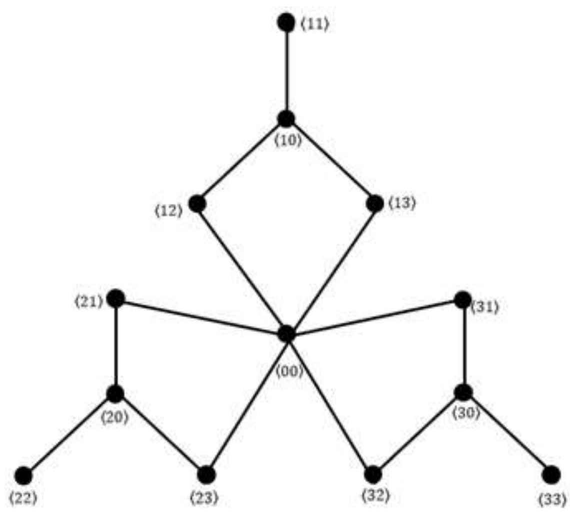

Figure 1. The Sierpinski Star graph SS_3

Figure 1 shows that SS_3 has 13 vertices or $|V(SS_3)| = 13$.

Based on Definition 1 and Definition 2, the 2-distance domination set in the Sierpinski Star graph SS_n will be determined, denoted as $S_2(SS_n)$. The vertices that represent 2-distance domination in the Sierpinski Star graph SS_1 to SS_5 can be seen in Figure 2 to Figure 6, with dominating vertices marked in white color as follows.

Figure 2. The Sierpinski Star Graph SS_1

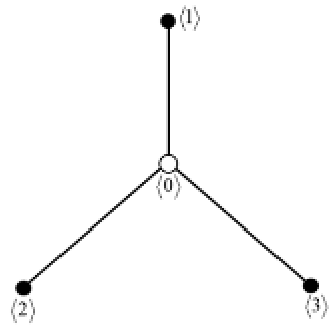

Figure 3. The Sierpinski Star Graph SS_2 with 2-Distance Domination Set $S_2(SS_2) = \{\langle 0 \rangle\}$

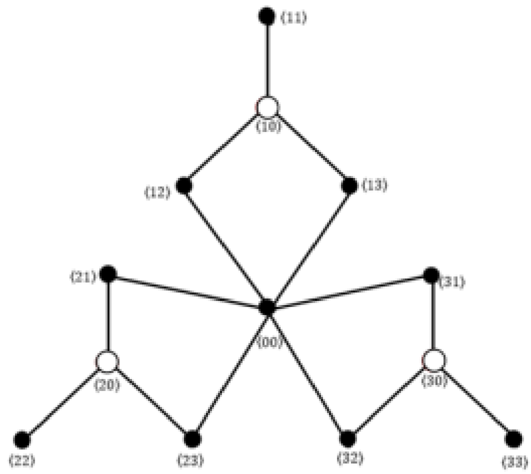

Figure 4. The Sierpinski Star Graph SS_3 with 2-Distance Domination Set $S_2(SS_3) = \{\langle 10 \rangle, \langle 20 \rangle, \langle 30 \rangle\}$

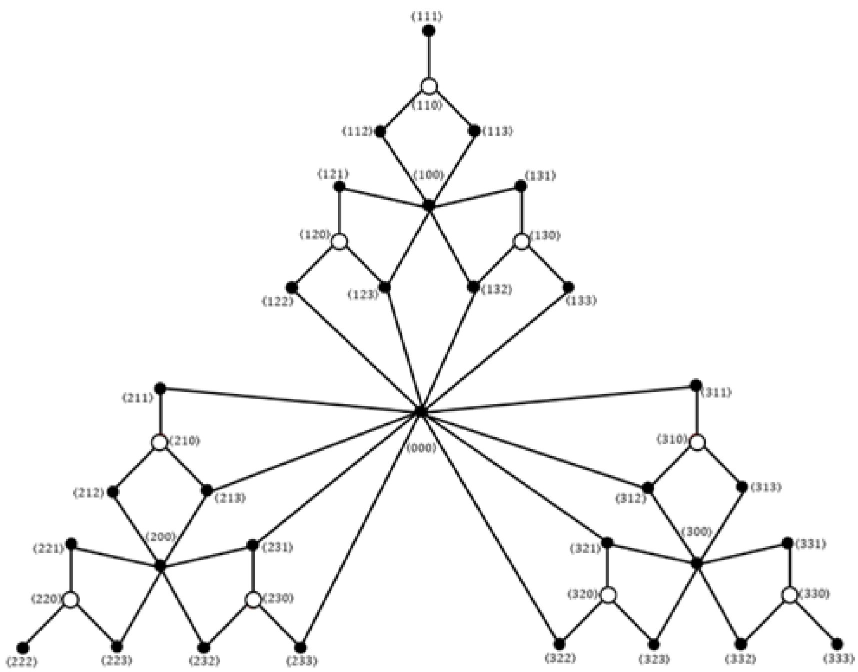

Figure 5. The Sierpinski Star Graph SS_4 with 2-Distance Domination Set
$S_2(SS_4) = \{\langle 110 \rangle, \langle 120 \rangle, \langle 130 \rangle, \langle 210 \rangle, \langle 220 \rangle, \langle 230 \rangle, \langle 310 \rangle, \langle 320 \rangle, \langle 330 \rangle\}$

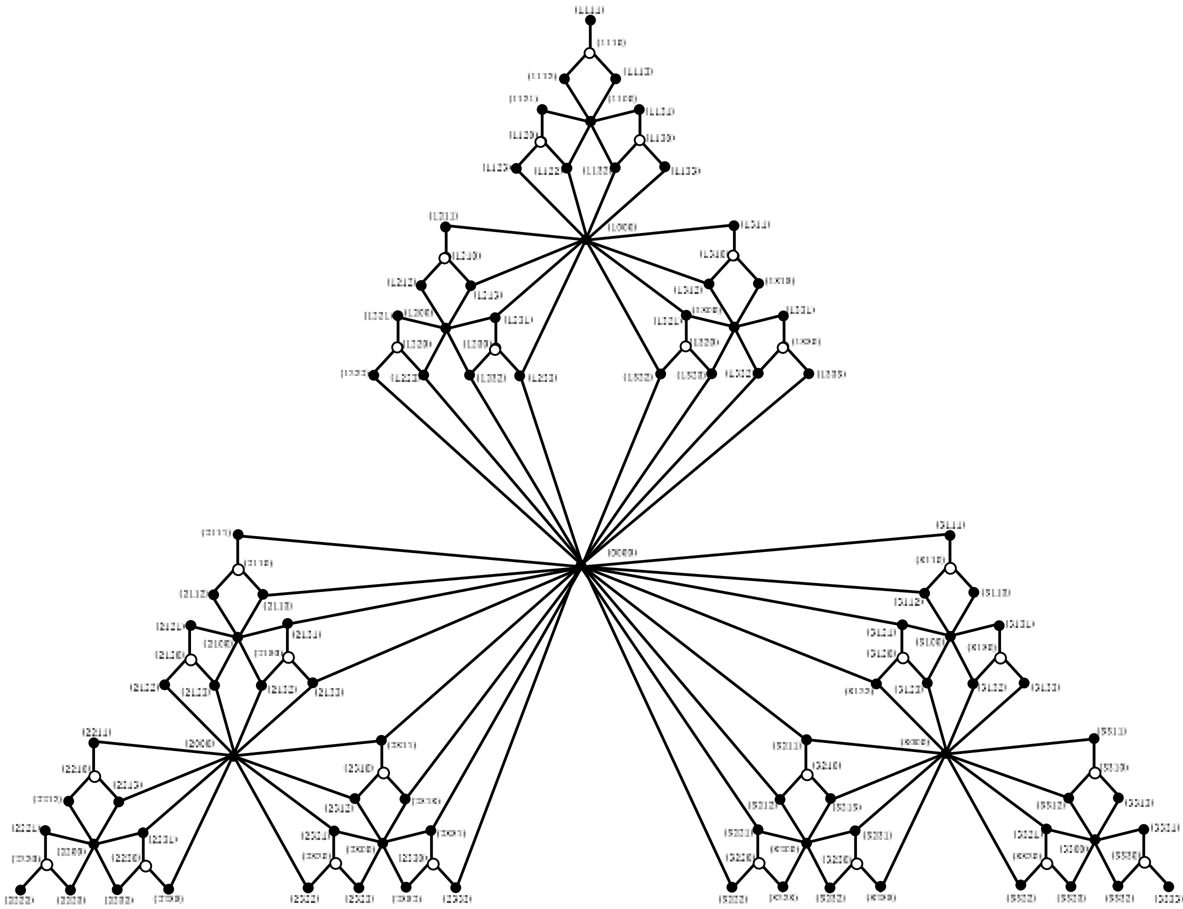

Figure 6. The Sierpinski Star Graph SS_5 with 2-Distance Domination Set
$S_2(SS_5) = \{\langle 1110\rangle, \langle 1120\rangle, \langle 1130\rangle, \langle 1210\rangle, \langle 1220\rangle, \langle 1230\rangle, \langle 1310\rangle, \langle 1320\rangle, \langle 1330\rangle$
$\langle 2110\rangle, \langle 2120\rangle, \langle 2130\rangle, \langle 2210\rangle, \langle 2220\rangle, \langle 2230\rangle, \langle 2310\rangle, \langle 2320\rangle, \langle 2330\rangle$
$\langle 3110\rangle, \langle 3120\rangle, \langle 3130\rangle, \langle 3210\rangle, \langle 3220\rangle, \langle 3230\rangle, \langle 3310\rangle, \langle 3320\rangle, \langle 3330\rangle\}$

Thus, the 2-distance domination set and the number of 2-distance domination in the Sierpinski Star graph SS_1 to SS_5 were obtained, as shown in Table 1.

Table 1. 2-Distance Domination Set and 2-Distance Domination Number in The Sierpinski Star Graph SS_n

$\boldsymbol{SS_n}$	$\boldsymbol{S_2(SS_n)}$	$\boldsymbol{\gamma_2(SS_n)}$
$\boldsymbol{SS_1}$	-	0
$\boldsymbol{SS_2}$	$\{\langle 0\rangle\}$	1
$\boldsymbol{SS_3}$	$\{\langle 10\rangle, \langle 20\rangle, \langle 30\rangle\}$	3
$\boldsymbol{SS_4}$	$\{\langle 110\rangle, \langle 120\rangle, \langle 130\rangle, \langle 210\rangle, \langle 220\rangle, \langle 230\rangle, \langle 310\rangle, \langle 320\rangle, \langle 330\rangle\}$	9
$\boldsymbol{SS_5}$	$\{\langle 1110\rangle, \langle 1120\rangle, \langle 1130\rangle, \langle 1210\rangle, \langle 1220\rangle, \langle 1230\rangle, \langle 1310\rangle, \langle 1320\rangle, \langle 1330\rangle, \langle 2110\rangle,$ $\langle 2120\rangle, \langle 2130\rangle, \langle 2210\rangle, \langle 2220\rangle, \langle 2230\rangle, \langle 2310\rangle, \langle 2320\rangle, \langle 2330\rangle, \langle 3110\rangle, \langle 3120\rangle,$ $\langle 3130\rangle, \langle 3210\rangle, \langle 3220\rangle, \langle 3230\rangle, \langle 3310\rangle, \langle 3320\rangle, \langle 3330\rangle\}$	27

Based on Table 1, for $3 \leq n \leq 5$, it is known that

$$\gamma_2(SS_3) = 3$$
$$\gamma_2(SS_4) = 9 = 3.\gamma_2(SS_3)$$
$$\gamma_2(SS_5) = 27 = 3.\gamma_2(SS_4)$$

In this way, it can be seen that the number of 2-distance domination in the Sierpinski Star graph forms a recursive equation that generates a series of numbers that are multiples of 3 because each increment of n in the Sierpinski Star graph SS_n will produce 3 times the previous Sierpinski Star graph. The 2-distance domination numbers in this series start from 3, and each subsequent 2-distance domination number is generated by multiplying the previous 2-distance domination number by 3. Similarly, for $n \geq 6$, the pattern of the number of 2-distance domination is as follows.

$$\gamma_2(SS_6) = 3.\gamma_2(SS_5)$$
$$\vdots$$
$$\gamma_2(SS_n) = 3.\gamma_2(SS_{(n-1)})$$

Since each 2-distance domination number in the Sierpinski Star graph can be obtained by multiplying the previous 2-distance domination number by 3, this also applies to $n+1$ as follows.

$$\gamma_2(SS_{(n+1)}) = 3.\gamma_2(SS_n) \tag{1}$$

Furthermore, the following theorem demonstrates the number of 2-distance domination in the Sierpinski Star graph SS_n.

Theorem 1 *For every Sierpinski Star graph SS_n, for $n \in N$, it holds that*

$$\gamma_2(SS_n) = \begin{cases} 1 & , if\ n < 3 \\ 3.3^{n-3} & , if\ n \geq 3 \end{cases}$$

Proof. Consider a Sierpinski Star graph SS_n. If $n < 3$, then the fewest member in SS_n that fulfills condition $d\big(v, S_2(SS_n)\big) \leq 2$ is one or $\gamma_2(SS_n) = 1$. Furthermore, by using mathematical induction, it can be derived:

i. Base Induction

If $n = 3$, thus based on Figure 4, it is apparent that the minimum member in SS_3 that satisfies $d\big(v, S_2(SS_3)\big) \leq 2$ is three, which is $S_2(SS_3) = \{\langle 10\rangle, \langle 20\rangle, \langle 30\rangle\}$ or can be denoted as $\gamma_2(SS_3) = 3$. Then, it obtained that if $n = 3$, thus $\gamma_2(SS_n) = 3.3^{(n-3)} = 3.3^0 = 3.1 = 3$. This means, the statement $\gamma_2(SS_n) = 3.3^{(n-3)}$ is true when $n = 3$.

ii. Inductive Induction

Assuming that statement $\gamma_2(SS_t) = 3.3^{(t-3)}$ is true for $n = t$, with t being a specific natural number and $t > 3$. Therefore, it shall be shown that this statement continues to hold for $n = t+1$. It will be proved that $\gamma_2\big(SS_{(t+1)}\big) = 3.3^{(t+1-3)} = 3.3^{(t-2)}$ is according to the induction assumption. Previously, the equation (1) has been derived, which is:

$$\gamma_2(SS_{(n+1)}) = 3.\gamma_2(SS_n)$$

Therefore, because $n = t$, then $\gamma_2\big(SS_{(t+1)}\big) = 3.\gamma_2(SS_t)$.

$$\gamma_2\big(SS_{(t+1)}\big) = 3.\gamma_2(SS_t)$$
$$\gamma_2\big(SS_{(t+1)}\big) = 3.\big(3^{(t-3)}.3\big)$$
$$\gamma_2\big(SS_{(t+1)}\big) = 3.\big(3^{(t-3)}.3^1\big)$$
$$\gamma_2\big(SS_{(t+1)}\big) = 3.3^{(t-3+1)}$$
$$\gamma_2\big(SS_{(t+1)}\big) = 3.3^{(t-2)}$$

Thus, it has been proved that $\gamma_2\big(SS_{(t+1)}\big) = 3.3^{(t+1-3)} = 3.3^{(t-2)}$ with $t \geq 3$. Consequently, this statement holds for $n = t+1$ as well.

According to the mathematical induction provided, it is verified that for each Sierpinski Star graph SS_n, it holds that $\gamma_2(SS_n) = 1$ for $n < 3$ and $\gamma_2(SS_n) = 3.3^{(n-3)}$ for $n \geq 3$. ■

Example 2. The Sierpinski Star Graph SS_4, as depicted in Figure 5, represents the graph of the Sierpinski Star SS_4 where $S_2(SS_4) = \{\langle 110\rangle, \langle 120\rangle, \langle 130\rangle, \langle 210\rangle, \langle 220\rangle, \langle 230\rangle, \langle 310\rangle, \langle 320\rangle, \langle 330\rangle\}$ is the 2-distance domination set from the graph SS_4 because each member in $S_2(SS_4)$ applies $d\big(v, S_2(SS_4)\big) \leq 2$.

Because Sierpinski Star graph SS_4, which means $n = 4$ thus $n > 3$, then based on Theorem 1, obtained

$$\gamma_2(SS_n) = 3.3^{(n-3)}$$
$$\gamma_2(SS_4) = 3.3^{(4-3)}$$

$$\gamma_2(SS_4) = 3.3$$
$$\gamma_2(SS_4) = 9$$

Therefore, the number of 2-distance domination of the Sierpinski Star graph SS_4 is 9 or can be written as $\gamma_2(SS_4) = 9$.

Furthermore, the vertices position of the number of 2-distance domination set in the Sierpinski Star graph SS_n for $n \geq 2$ can be chosen

For $n = 2$, $S_2(SS_2) = \{\langle 0 \rangle\}$

For $n = 3$, $S_2(SS_3) = \{\langle 10 \rangle, \langle 20 \rangle, \langle 30 \rangle\}$

For $n = 4$, $S_2(SS_4) = \{\langle 110 \rangle, \langle 120 \rangle, \langle 130 \rangle, \langle 210 \rangle, \langle 220 \rangle, \langle 230 \rangle, \langle 310 \rangle, \langle 320 \rangle, \langle 330 \rangle\}$

For $n = 5$,

$$S_2(SS_5) = \{\langle 1110 \rangle, \langle 1120 \rangle, \langle 1130 \rangle, \langle 1210 \rangle, \langle 1220 \rangle, \langle 1230 \rangle, \langle 1310 \rangle, \langle 1320 \rangle, \langle 1330 \rangle,$$
$$\langle 2110 \rangle, \langle 2120 \rangle, \langle 2130 \rangle, \langle 2210 \rangle, \langle 2220 \rangle, \langle 2230 \rangle, \langle 2310 \rangle, \langle 2320 \rangle, \langle 2330 \rangle,$$
$$\langle 3110 \rangle, \langle 3120 \rangle, \langle 3130 \rangle, \langle 3210 \rangle, \langle 3220 \rangle, \langle 3230 \rangle, \langle 3310 \rangle, \langle 3320 \rangle, \langle 3330 \rangle\}$$

⋮

Based on that pattern, it is obtained that the general vertices position of the number of 2-distance domination set in the Sierpinski Star graph SS_n is $S_2(SS_2) = \{\langle 0 \rangle\}$ for $n = 2$ and

$S_2(SS_n) = \{\langle u_1 u_2 \ldots u_{(n-2)} 0 \rangle | u_i \in \{1,2,3\} \text{ with } i = 1,2,\ldots,n-2\}$ for $n \geq 3$.

Furthermore, 3-distance domination set in the Sierpinski Star graph SS_n will be determined, denoted as $S_3(SS_n)$. The vertices representing 3-distance domination in the Sierpinski Star graph SS_1 to SS_5 can be seen in Figure 8 to 12, with dominating vertices marked in white as follows.

Figure 7. The Sierpinski Star Graph SS_1

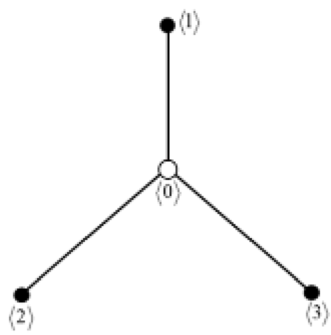

Figure 8. The Sierpinski Star Graph SS_2 with 3-Distance Domination Set $S_3(SS_2) = \{\langle 0 \rangle\}$

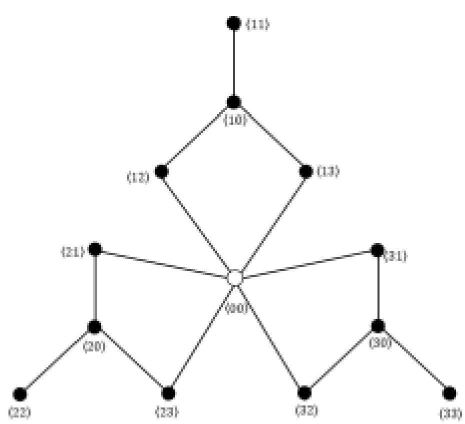

Figure 9. The Sierpinski Star Graph SS_3 with 3-Distance Domination Set $S_3(SS_3) = \{\langle 00 \rangle\}$

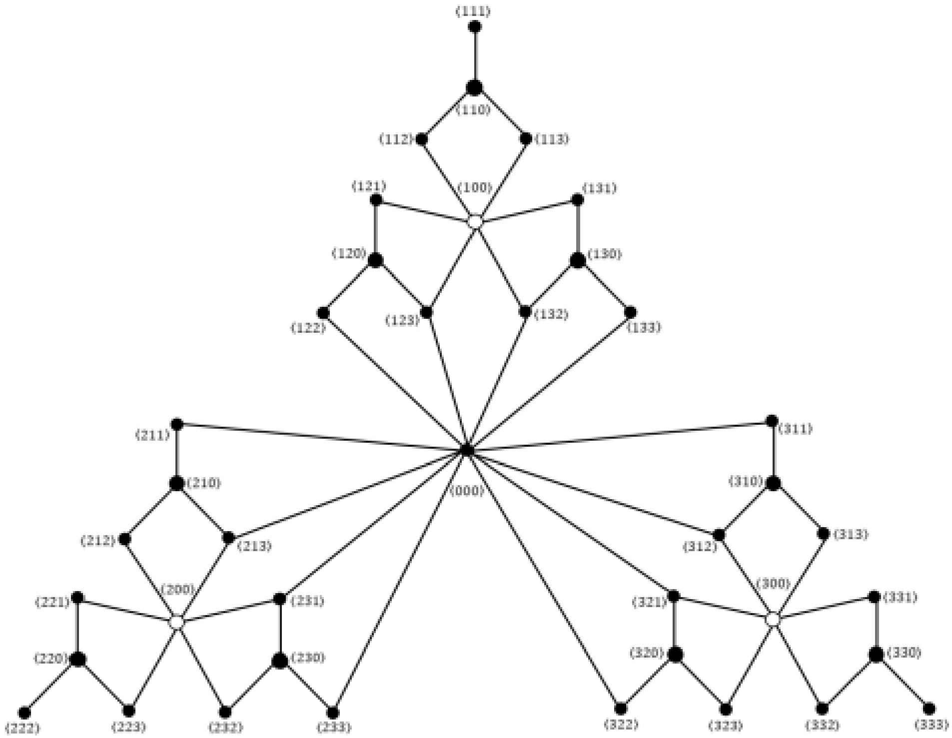

Figure 10. The Sierpinski Star Graph SS_4 with 3-Distance Domination Set $S_3(SS_4) = \{\langle 100\rangle, \langle 200\rangle, \langle 300\rangle\}$

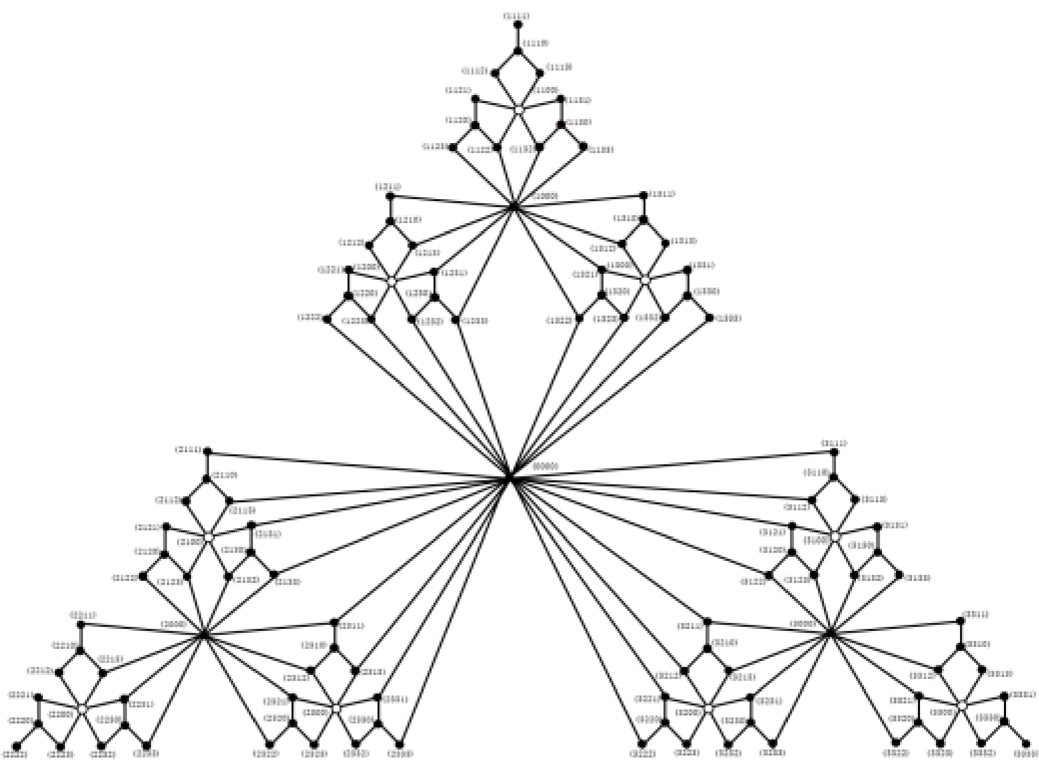

Figure 11. The Sierpinski Star Graph SS_5 with 3-Distance Domination Set
$S_3(SS_5) = \{\langle 1100\rangle, \langle 1200\rangle, \langle 1300\rangle, \langle 2100\rangle, \langle 2200\rangle, \langle 2300\rangle, \langle 3100\rangle, \langle 3200\rangle, \langle 3300\rangle\}$

Thus, obtained the 3-distance domination set and the 3-distance domination number in the Sierpinski Star graph SS_1 to SS_5 as shown in Table 2.

Table 2. 3-Distance Domination Set and 3-Distance Domination Number in The Sierpinski Star Graph SS_n

SS_n	$S_3(SS_n)$	$\gamma_3(SS_n)$
SS_1	-	0
SS_2	$\{\langle 0\rangle\}$	1
SS_3	$\{\langle 00\rangle\}$	1
SS_4	$\{\langle 100\rangle, \langle 200\rangle, \langle 300\rangle\}$	3
SS_5	$\{\langle 1100\rangle, \langle 1200\rangle, \langle 1300\rangle, \langle 2100\rangle, \langle 2200\rangle, \langle 2300\rangle, \langle 3100\rangle, \langle 3200\rangle, \langle 3300\rangle\}$	9

Based on Table 2, for $3 \leq n \leq 5$ it is known that

$$\gamma_3(SS_3) = 1$$
$$\gamma_3(SS_4) = 3 = 3.\gamma_3(SS_3)$$
$$\gamma_3(SS_5) = 9 = 3.\gamma_3(SS_4)$$

Thus, it can be observed that the 3-distance domination numbers in the Sierpinski Star graph form a recursive equation that generates a series of numbers that are multiples of 3, because each increment of *n* in the Sierpinski Star graph SS_n will produce 3 times the previous Sierpinski Star graph. The 3-distance domination numbers in this series start from 1, and each subsequent 3-distance domination number is generated by multiplying the previous 3-distance domination number by 3. Similarly, for $n \geq 6$, the pattern of the 3-distance domination numbers is as follows.

$$\gamma_3(SS_6) = 3.\gamma_3(SS_5)$$
$$\vdots$$
$$\gamma_3(SS_n) = 3.\gamma_3(SS_{(n-1)})$$

Since each 3-distance domination number in the Sierpinski Star graph can be obtained by multiplying the previous 3-distance domination number by 3, this also applies to $n+1$ as follows.

$$\gamma_3(SS_{(n+1)}) = 3.\gamma_3(SS_n) \quad (2)$$

Furthermore, the following theorem demonstrates the 3-distance domination number in the Sierpinski Star graph SS_n.

Theorem 2 *For every Sierpinski Star graph* SS_n, *for* $n \in N$, *it holds that*

$$\gamma_3(SS_n) = \begin{cases} 1 & , if\ n < 3 \\ 3^{n-3} & , if\ n \geq 3 \end{cases}$$

Proof. Consider a Sierpinski Star graph SS_n. In case $n < 3$, then the fewest member in SS_n that fulfills condition $d\big(v, S_3(SS_n)\big) \leq 3$ is one or $\gamma_3(SS_n) = 1$. Furthermore, by using mathematical induction, it can be derived:

i. Base Induction

If $n = 3$, thus according to Figure 10, it can be observed that the minimum member in SS_3 that satisfies $d\big(v, S_3(SS_n)\big) \leq 3$, is one, which is or can be denoted as $\gamma_3(SS_3) = 1$. Then, obtained that if $n = 3$, thus $\gamma_3(SS_3) = 3^{(n-3)} = 3^{(3-3)} = 3^0 = 1$. $\gamma_2(SS_n) = 3.3^{(n-3)} = 3.3^0 = 3.1 = 3$. In other words, statement $\gamma_3(SS_n) = 3^{(n-3)}$ is true when $n = 3$.

ii. Inductive Induction

Assuming that statement $\gamma_3(SS_t) = 3^{(t-3)}$ is true for $n = t$, where t is a specific natural number and $t > 3$. Then, it shall be shown that this statement continues to hold for $n = t + 1$. It will be proved that $\gamma_3\big(SS_{(t+1)}\big) = 3^{(t+1-3)} = 3^{(t-2)}$ is according to the induction assumption. Previously, the equation (2) has been derived, which is:

$$\gamma_3(SS_{(n+1)}) = 3.\gamma_3(SS_n)$$

Therefore, because $n = t$, then $\gamma_3\big(SS_{(t+1)}\big) = 3.\gamma_3(SS_t)$.

$$\gamma_3\big(SS_{(t+1)}\big) = 3.\gamma_3(SS_t)$$
$$\gamma_3\big(SS_{(t+1)}\big) = 3.\big(3^{(t-3)}\big)$$
$$\gamma_3\big(SS_{(t+1)}\big) = 3^1.3^{(t-3)}$$
$$\gamma_3\big(SS_{(t+1)}\big) = 3^{(t-3+1)}$$

$$\gamma_3\left(SS_{(t+1)}\right) = 3^{(t-2)}$$

Thus, it has been proved that $\gamma_3\left(SS_{(t+1)}\right) = 3^{(t+1-3)} = 3^{(t-2)}$ with $t \geq 3$. Consequently, this statement holds for $n = t + 1$ as well.

According to the mathematical induction provided, it is verified that for every Sierpinski Star graph SS_n, it holds that $\gamma_3(SS_n) = 1$ for $n < 3$ and $\gamma_3(SS_n) = 3^{(n-3)}$ for $n \geq 3$. ■

Example 3. The Sierpinski Star Graph SS_4, as depicted in Figure 10, represents the graph of the Sierpinski Star SS_4 where $S_3(SS_4) = \{\langle 100\rangle, \langle 200\rangle, \langle 300\rangle\}$ is the 3-distance domination set from the graph SS_4 because each member in $S_3(SS_4)$ applies $d\left(v, S_3(SS_4)\right) \leq 3$.

Because Sierpinski Star graph SS_4, which means $n = 4$ thus $n > 3$, then based on Theorem 2, obtained

$$\gamma_3(SS_n) = 3^{(n-3)}$$
$$\gamma_3(SS_4) = 3^{(4-3)}$$
$$\gamma_3(SS_4) = 3$$

Thus, the number of 3-distance domination of the Sierpinski Star graph SS_4 is 3 or can be written as $\gamma_3(SS_4) = 3$.

Furthermore, the vertices position of the 3-distance domination number set in the Sierpinski Star graph SS_n for $n \geq 2$ can be chosen

For $n = 2$, $S_3(SS_2) = \{\langle 0\rangle\}$

For $n = 3$, $S_3(SS_3) = \{\langle 00\rangle\}$

For $n = 4$, $S_3(SS_4) = \{\langle 100\rangle, \langle 200\rangle, \langle 300\rangle\}$

For $n = 5$, $S_3(SS_5) = \{\langle 1100\rangle, \langle 1200\rangle, \langle 1300\rangle, \langle 2100\rangle, \langle 2200\rangle, \langle 2300\rangle, \langle 3100\rangle, \langle 3200\rangle, \langle 3300\rangle\}$

⋮

Based on that pattern, it is obtained that the general vertices position of the number of 3-distance domination set in the Sierpinski Star graph SS_n is

$S_3(SS_n) = \left\{\left\langle u_1 u_2 \ldots u_{(n-3)} 00\right\rangle \middle| u_i \in \{1,2,3\} \ with \ i = 1,2,\ldots,n-3\right\}$ for $n \geq 2$.

3. Conclusion

From the analysis of the set and the number of 2-distance and 3-distance domination in the Sierpinski Star graph SS_n it may be concluded that the number of 2-distance and 3-distance domination in the Sierpinski Star graph SS_n is 1 if *n* is less than three. Furthermore, for the Sierpinski Star graph SS_n with *n* is greater than or equal to three, the number of 2-distance domination is $3.3^{(n-3)}$ and the 3-distance domination number is $3^{(n-3)}$.

Therefore, in general, vertices position of the number of 2-distance domination set in the Sierpinski Star graph SS_n is at vertices with $(n-1)$-tuple being 0, except the central vertex. Meanwhile, the vertices position of the 3-distance domination number set in the Sierpinski Star graph SS_n is at vertices with $(n-1)$-tuple and $(n-2)$-tuple being 0, except the central vertex.

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Author Contributions

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Competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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