Journal of Mathematics Research; Vol. 16, No. 3; June 2024
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

An Extension to Fermat's Pythagorean Triangle Area Proof, and Further Doubt to His Proof of Fermat's Last Theorem

Richard Kaufman
Correspondence: Office Expander, Andover, Massachusetts, USA

Received: April 17, 2024 Accepted: May 30, 2024 Online Published: June 27, 2024
doi:10.5539/jmr.v16n3p46 URL: https://doi.org/10.5539/jmr.v16n3p46

Abstract

Pierre de Fermat proved that the area of a Pythagorean triangle is not a perfect square. If he also had a proof of Fermat's Last Theorem, then he could have easily extended his triangle area proof to show that primitive Pythagorean triangles could not have an area that is a perfect cube or higher power. Because there is no record of this, we might suppose that he did not ultimately accept that he had a valid proof of Fermat's Last Theorem.

Keywords: Fermat's Pythagorean triangle area proof, Doubt of Fermat's proof of Fermat's Last Theorem

1. Introduction

In the 17th century, Pierre de Fermat proved that the area of a Pythagorean triangle is not a perfect square. A Pythagorean triangle is a right triangle with positive integer sides. The proof that the area is not a perfect square uses the method of infinite descent and is Fermat's only published proof - for which subsequent versions are known (Bogomolny; Weil, 2001, pp. 76-77; Stillwell, 1998, pp. 131-134).

In the next section, we use a basic premise from Fermat's proof to ultimately show that the area of a primitive Pythagorean triangle cannot be a perfect cube or higher power. However, this proof relies on *non-elementary* results, such as Andrew Wiles proof of Fermat's Last Theorem (Wiles, 1995), that were not known in Fermat's time.

Fermat's Last Theorem states that there are no positive integer solutions a, b, and c for the equation, $a^k + b^k = c^k$, with integer $k > 2$. It is well-known that Fermat claimed to have a proof of Fermat's Last Theorem in "the margin of his copy of Bachet's edition of the works of Diophantus," (Ribenboim, 1999, p. 1). However, no elementary proof has ever been found and it is widely believed today that Fermat did not have a valid proof. If Fermat did have a proof of Fermat's Last Theorem, then we could show that he could have easily extended his Pythagorean triangle area results to those shown in this paper. This lays some further doubt to Fermat's claim of a proof of Fermat's Last Theorem.

2. Proof That the Area of a Primitive Pythagorean Triangle Is not a Perfect Cube or Higher Power

A primitive Pythagorean triangle has positive integer sides A and B and hypotenuse C that obeys the Pythagorean theorem $A^2 + B^2 = C^2$. That is, primitive triples A, B, and C, have no two sides with a common factor greater than 1. The Pythagorean theorem follows Euclid's formula:

$$A = m^2 - n^2, \tag{1}$$

$$B = 2mn, \tag{2}$$

$$C = m^2 + n^2. \tag{3}$$

These formulas show that B is even, so A and C are odd. The positive integers m and n have no common factor: for otherwise, A, B, and C would share the common factor (twice) and a contradiction would result (Ogilvy & Anderson, 1988, p. 67; Courant, Robbins, & Stewart, 1996, p. 41). Also, $m > n \geq 1$. Incidentally, m and n have different parity, so one is even, and the other is odd. The area of the Pythagorean triangle is an integer given by:

$$\frac{AB}{2} = mn(m^2 - n^2) = mn(m+n)(m-n) \tag{4}$$

Since m and n have no common factor, then none of $m,\ n,\ m+n,\ m-n$ have a common factor. To see this, note that one of m or n is odd and the other is even (*i.e.*, they have different parity). Since m and n have no common factor, then m, n, and $m+n$ have no common factor. Likewise, m, n, and $m-n$ have no common factor. Furthermore, $(m+n)+(m-n)=2m$, but 2 cannot be a factor of $(m+n)$ or $(m-n)$ since m and n have different parity (*i.e.*, their sum or difference is odd). Thus, $m,\ n,\ m+n$, and $m-n$ have no common factor.

Up to this point, our proof has been similar to Fermat's proof that the area of a Pythagorean triangle cannot be a perfect square. However, we will now consider if the area could be a perfect cube or higher power.

Suppose that the Pythagorean triangle's area is w^k, for positive integer w and integer $k \geq 2$. From Eq. (4), then each of $m,\ n,\ m+n,\ m-n$ would itself be a power of k. Let $m = r^k,\ n = s^k,\ m+n = t^k$, and $m-n = u^k$, for relatively-prime positive integers $r,\ s,\ t$, and u. Then the area is $r^k s^k t^k u^k$.

By simple substitution, $m+n = r^k + s^k = t^k$ and $m-n = r^k - s^k = u^k$, and we have the following two equations, respectively:

$$r^k + s^k = t^k, \tag{5}$$

$$r^k - s^k = u^k. \tag{6}$$

Each resembles a Fermat equation for integer $k > 2$, where $r,\ s,\ t$, and u are relatively-prime positive integers. Yet another Fermat equation arises using $A = m^2 - n^2 = (m+n)(m-n) = t^k u^k$, whereby $A = m^2 - n^2 = (r^k)^2 - (s^k)^2 = (r^k + s^k)(r^k - s^k) = (tu)^k$. Note that since A is odd per Eq. (1), then tu is odd. Here the Fermat equation is:

$$(tu)^k + (s^2)^k = (r^2)^k \tag{7}$$

which is not a multiple of Eq. (5) or (6).

But none of these last 3 equations are possible because of Andrew Wiles' non-elementary proof of Fermat's Last Theorem. This completes the proof. (See Appendix for other non-elementary proofs).

As an observation, a Pythagorean triangle can have an area that is a perfect cube, such as for $18^2 + 24^2 = 30^2$, which has area $216 = 6^3$. However, this is not a primitive Pythagorean triangle that is used in this paper.

3. Final Remarks

If Fermat did have a proof of Fermat's Last Theorem, it would have been simple for him to expand upon his Pythagorean triangle proof that the area is not a perfect square. Using the same outline of his triangle area proof, Fermat could have used Fermat's Last Theorem to show that no primitive Pythagorean triangles can have an area that is a perfect cube or higher power. So, this leaves further doubt to his claim for a proof of Fermat's Last Theorem.

For further research, it might be possible to consider a proof by contradiction that starts with the premise of a Pythagorean triangle with an area that is a perfect cube or higher power. The Fermat equations shown in Eq. (5) (6) and (7) could then be used in an infinite descent argument, if one can be found. This would then show that the area of a Pythagorean triangle could not actually be a perfect cube or higher power. Fermat himself invented the method of infinite descent (Bussey, 1918), which is not considered to be modern mathematics (it is considered to be elementary mathematics, although that does not mean that it is easy).

Appendix

This appendix shows other non-elementary proofs (besides Andrew Wiles proof of Fermat's Last Theorem) that could have been used for the proof in this paper. Darmon and Merel (1997) proved the following two main results for arbitrary positive integer n:

1. The equation $x^n + y^n = 2z^n$ has no non-trivial primitive solution when $n \geq 3$.
2. The equation $x^n + y^n = z^2$ has no non-trivial primitive solution when $n \geq 4$.

Adding Eq. (5) and (6), we have: $t^k + u^k = 2r^k$. The first result from Darmon and Merel shows that this is impossible, completing the proof.

Also, Eq. (7) can be rewritten as $(tu)^k + (s^2)^k = (r^k)^2$. The second result from Darmon and Merel shows that this is impossible for $k \geq 4$ (and the case of $k = 3$ is known by elementary means), also completing the proof.

References

Bogomolny, A. (n.d.). Fermat's only published proof. [Online] Available: https://www.cut-the-knot.org/arithmetic/FermatsInfiniteDescent.shtml (Accessed 13 April 2024)

Bussey, W. H. (1918). Fermat's method of infinite descent. *The American Mathematical Monthly*, *25*(8), 333. https://doi.org/10.2307/2974148

Courant, R., Robbins, H., & Stewart, I. (1996). *What is Mathematics?: An elementary approach to ideas and methods*. Oxford University Press. https://doi.org/10.1093/oso/9780195105193.001.0001

Darmen H., & Merel, L. (1997) Winding quotients and some variants of Fermat's Last Theorem. *J. Reine Angew. Math. 490*, 81–10. https://doi.org/10.1515/crll.1997.490.81

Ogilvy, C. S., & Anderson, J. T. (1988). *Excursions in number theory*. Dover.

Ribenboim, P. (1999). *Fermat's last theorem for amateurs*. Springer.

Stillwell, J. (1998). *Numbers and geometry*. Springer. https://doi.org/10.1007/978-1-4612-0687-3

Weil, A. (2001). *Number theory: An approach through history from Hammurapi to Legendre*. Springer.

Wiles, A. (1995). Modular elliptic curves and Fermat's last theorem. *The Annals of Mathematics*, *141*(3), 443. https://doi.org/10.2307/2118559

Acknowledgements

The author wishes to thank Ken Ribet for providing the example $18^2 + 24^2 = 30^2$ used in this paper. He pointed out a key difference between Fermat's Proof that the area of a Pythagorean triangle is not a square, and its application to higher powers, which ultimately requires consideration of primitive Pythagorean triangles.

The author also wishes to thank Kara Kaufman for her help in editing a final draft of this paper.

Authors contributions

Richard Kaufman is responsible for the entire article.

Funding

Not applicable.

Competing interests

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Informed consent

Not applicable.

Ethics approval

The Publication Ethics Committee of the Canadian Center of Science and Education.

The journal's policies adhere to the Core Practices established by the Committee on Publication Ethics (COPE).

Provenance and peer review

Not commissioned; externally double-blind peer reviewed.

Data availability statement

Not applicable.

Data sharing statement

No additional data are available.

Open access

Copyrights