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Prime Number Theorem and Goldbach Conjecture

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Abstract

Let L_n denote the largest strong Goldbach number generated by the n-th prime P_n, in this paper, we present that there are approximate bounds of L_n such that $2n\log n + 2n\log\log n - 2n < L_n < 2n\log n + 2n\log\log n$ for $n \geq 20542$, based on results about prime number theorem. Let $\xi(n) = L_n/2$ denote the number of strong Goldbach numbers generated by P_n, equivalently, there are approximate bounds of $\xi(n)$ such that $n\log n + n\log\log n - n < \xi(n) < n\log n + n\log\log n$ for $n \geq 20542$. The approximate bounds of L_n have been verified for $20542 \leq n \leq 400000000$, equivalently, the approximate bounds of $\xi(n)$ have also been verified for $20542 \leq n \leq 400000000$. It is obvious that if it can be proven that there is an integer $k > 0$ such that bounds of $2P_n$ can be thought as approximate bounds of L_n for all $n > k$, or equivalently, there is an integer $k > 0$ such that bounds of P_n can be thought as approximate bounds of $\xi(n)$ for all $n > k$, then Goldbach conjecture is true. We also considered another approach to the conjecture, that is, if it can be proven by introducing $\mathrm{Li}(n)$ that there are infinitely many Goldbach steps, then Goldbach conjecture is true.

Keywords: prime, prime number theorem, largest strong Goldbach number, bound, step, Goldbach conjecture

2010 Mathematics Subject Classification: 11A41

1. Introduction

Traditional Goldbach number was defined as an even number to be the sum of two primes and such pair of primes is not subject to any restriction so that there is a tendency that the higher the value of an even number is, the larger the number of prime pairs to form the even number is. Goldbach conjecture states that every even number greater than 2 is the sum of two primes, therefore, there is an approach to Goldbach conjecture based on definition of Goldbach number, that is, researches on the exceptional set of Goldbach numbers may lead to a proof of Goldbach conjecture (Montgomery, & Vaughan, 1975; Chen, & Pan, 1980; Lu, 2010). In our previous works (Zhou, 2017; Zhou, & Ao, 2018; Zhou, 2019), Goldbach number is considered as a product generated by a given prime, that is, an even number is called a Goldbach number generated by the n-th prime P_n if the even number is the sum of two primes not greater than P_n. Further, if some consecutively increased Goldbach numbers, in which 4 is the first such Goldbach number, all are generated by a given prime P_n, then every one among these Goldbach numbers is called a strong Glodbach number generated by P_n, and the largest one among these strong Goldbach numbers is called the largest strong Goldbach number generated by P_n and written as L_n. It is discovered that distribution of largest strong Goldbach numbers generated by primes is a step-type curve based on $L_n \leq L_{n+1}$. A link between prime number theorem and Goldbach conjecture may be established, that is, bounds of P_n imply existence of bounds of $2P_n$ and bounds of $2P_n$ can be thought as approximate bounds of L_n. Another possible approach to the conjecture is to introduce $\mathrm{Li}(n)$ into approximate expression for the number of Goldbach steps, which will lead to a result to be very similar to prime number theorem.

2. Prime Number Theorem and Non-Asymptotic Bounds of P_n

Euclid succinctly proved infinity of primes by using the reduction to absurdity before Christ, and he said that if there is the largest prime then the sum of 1 and product of all primes will be a greater prime than the largest prime since the larger number has not any prime factor, therefore, there is no the largest prime and primes are infinite. But distribution law of infinitely many primes had not been known for long years before prime number theorem was proven in 1896 (Hadamard, 1896; Poussion, 1896).

Let x be positive integer and prime-counting function $\pi(x)$ denote the number of primes not greater than x. The prime number theorem states that there is a limit

$$\lim_{x\to\infty}\frac{\pi(x)}{(\frac{x}{\log x})}=1, \tag{2.1}$$

and corresponding asymptotic expression is

$$\pi(x)\approx\frac{x}{\log x}. \tag{2.2}$$

It is asymptotic distribution law of primes and means that asymptotic average gap between primes is about log*x*. The prime number theorem can also be written as the following approximation for $\pi(x)$,

$$\pi(x)\approx Li(x), \tag{2.3}$$

where Li(x) is the logarithmic integral

$$Li(x)=\int_2^x\frac{dt}{\log t}, \tag{2.4}$$

and there is the asymptotic series for Li(x),

$$Li(x)\approx\frac{x}{\log x}\sum_{k=0}^{\infty}\frac{k!}{(\log x)^k}=\frac{x}{\log x}(1+\frac{1}{\log x}+\frac{2}{(\log x)^2}+\cdots). \tag{2.5}$$

A better expression for $\pi(x)$ than (2.3) is (Koch, 1901)

$$\pi(x)=Li(x)+o(\sqrt{x}\log x), \tag{2.6}$$

where the additional term is equivalent to the Riemann hypothesis.

Of studies on bounds of $\pi(x)$, a pair of weak but sometimes useful bounds is as follows (Rosser, 1941)

$$\frac{x}{\log x+2}<\pi(x)<\frac{x}{\log x-4} \text{ for } x\geq 55. \tag{2.7}$$

An equivalent statement of prime number theorem is that there is a limit

$$\lim_{n\to\infty}\frac{P_n}{n\log n}=1, \tag{2.8}$$

where P_n denotes the n-th prime. The limit means there is an asymptotic expression

$$P_n\approx n\log n. \tag{2.9}$$

It is asymptotic form of prime. There is a better approximation for P_n (Cesaro, 1894),

$$\frac{P_n}{n}=\log n+\log\log n-1+\frac{\log\log n-2}{\log n}-\frac{(\log\log n)^2-6\log\log n+11}{2(\log n)^2}+o(\frac{1}{(\log n)^2}) , \tag{2.10}$$

and it was proven that there are non-asymptotic bounds of P_n such that (Rosser, 1941; Dusart, 1999)

$$n\log n + n\log\log n - n < P_n < n\log n + n\log\log n \text{ for } n\geq 6. \tag{2.11}$$

Note (2.11) may be established as a link between prime number theorem and Goldbah conjecture.

3. Approximate bounds of L_n and Goldbach Conjecture

Definition 3.1. Let P_n denote the n-th prime. Then $P_i + P_k$ is called a *Goldbach number generated by* P_n if $i\leq n$, $k\leq n$. L_n is called *the largest strong Goldbach number generated by* P_n if L_n is an even number such that every even number from 4 to L_n is the sum of two primes not greater than P_n but $L_n + 2$ is not such a sum. Every even number from 4 to L_n is called a *strong Goldbach number generated by* P_n.

By Definition 3.1, there are calculated values of L_n generated by P_n for $1000000 \leq n \leq 100000000$ to be shown in the following figure.

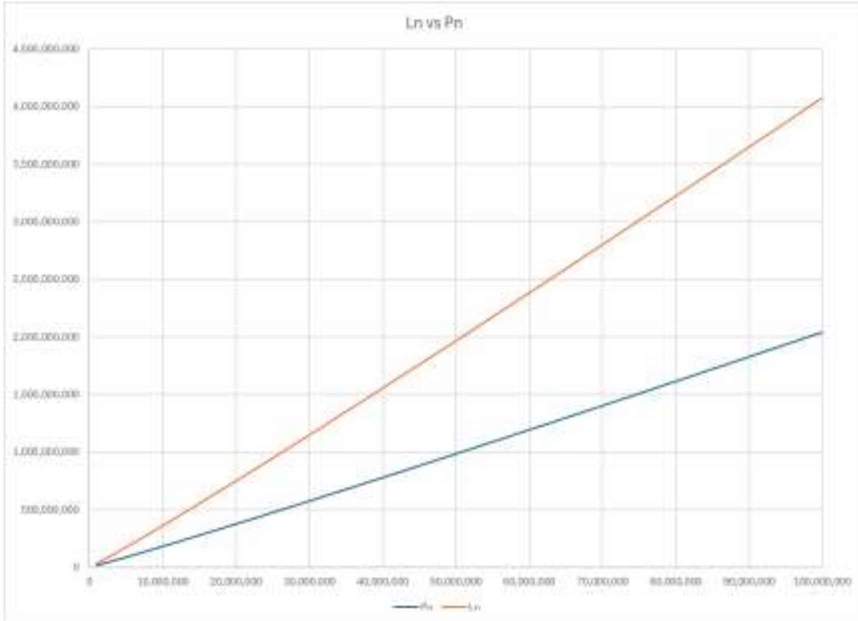

Figure 1. Distribution of Pn and distribution of Ln generated by Pn for 1000000 ≤ n ≤ 100000000

In the figure, the lower curve represents P_n and the upper curve represents L_n. From it we see that $L_n \approx 2P_n$ for $1000000 \le n \le 100000000$. Generally, we have impressive comparison as the following table shows.

Table 1. Relative error between L_n and $2P_n$

n	P_n	L_n	$L_n/2P_n$	$(2P_n–L_n)/2P_n$
100	541	966	0.89279112	0.10720887
1000	7919	15522	0.98004798	0.01995201
10000	104729	208926	0.99746011	0.00253988
100000	1299709	2598332	0.99958221	0.00041778
1000000	15485863	30970934	0.99997442	0.00002557
10000000	179424673	358847082	0.99999369	0.00000630
100000000	2038074743	4076147580	0.99999953	0.00000046

Table 1 emphasizes a fact that relative error between L_n and $2P_n$, which is just relative error between the largest strong Goldbach number generated by P_n and the largest Goldbach number generated by P_n, is coming closer and closer to 0 with growth of n, for example, $0.00002557 \ge (2P_n–L_n)/2P_n \ge 0.00000046$ for $1000000 \le n \le 100000000$. It means that the upper curve representing L_n can be approximately thought as a curve to represent $2P_n$ in Figure 1. Let $\xi(n) = L_n/2$ denote the number of strong Goldbach numbers generated by P_n, where the first even number 2 may be thought as a special or imaginary strong Goldbach number, perhaps, because 2 = 1 + 1 and 1 had been thought as a prime long ago. Then relative error between $\xi(n)$ and P_n is equal to relative error between L_n and $2P_n$, therefore, $0.00002557 \ge (P_n – \xi(n))/P_n \ge 0.00000046$ for $1000000 \le n \le 100000000$ so that the lower curve representing P_n can be approximately thought as a curve to represent $\xi(n)$ in Figure 1.

If it is the first characteristic of L_n that relative error between L_n and $2P_n$ is closer and closer to 0 with growth of n, then the second characteristic of L_n is that distribution of L_n is a step-type curve as Observation 2.6 (Zhou, 2017) and Figure 3 (Zhou, & Ao, 2018) show, and the characteristic arises from an observed fact that $L_n \le L_{n+1}$ for all $n > 0$ though $P_n < P_{n+1}$ for all $n > 0$.

Definition 3.2. Every step in distribution curve of L_n is called a *Goldbach step*.

Definition 3.3. For a given Goldbach step, W is called *width* of the Goldbach step if $W = n2 + 1 – n1$, where $n1$ is n-value at the starting point of the Goldbach step and $n2$ is n-value at the finishing point of the Goldbach step but $n2 + 1$ is n-value at the starting point of next Goldbach step. H is called *height* of the Goldbach step if $H = L_{n1}$ for the Goldbach step, where $n1$ is n-value at the starting point of the Goldbach step.

Remark 3.4. By Definition 3.2 and Definition 3.3, $W \ge 1$ for all $n > 0$, and every L_n for $n > 0$ must be equal to height of a Goldbach step. Considering L_{152} = 1692 < L_{153} = L_{154} = L_{155} = L_{156} = L_{157} = L_{158} = 1722 < L_{159} = 1778, there is a Goldbach step L_{153} = L_{154} = L_{155} = L_{156} = L_{157} = L_{158}, whose height is L_{153} = 1722 and width is $n2$ +1 – $n1$ = 158 +1 – 153 = 6. Considering L_{2571} = 45606 < L_{2572} = 45624 < L_{2573} = 45762, there is a Goldbach step L_{2572}, whose height is L_{2572} = 45624 and width is $n2$ +1 – $n1$ = 2572 +1 – 2572 = 1.

Although there is no a function of n to describe upper or lower bound of L_n, the first characteristic of L_n, relative error between L_n and $2P_n$ being closer and closer to 0 with growth of n, brings us the possibility that bounds of $2P_n$ can be thought as approximate bounds of L_n.

Theorem 3.5. *There is a pair of bounds of $2P_n$ such that $2n\log n + 2n\log\log n - 2n < 2P_n < 2n\log n + 2n\log\log n$ for $n \geq 6$.*

Proof. Let $A_{up}(n)$ denote upper bound of P_n, $A_{low}(n)$ denote lower bound of P_n, $B_{up}(n)$ denote upper bound of $2P_n$ and $B_{low}(n)$ denote lower bound of $2P_n$. By Definition 3.1, $2P_n = P_n + P_n$ is the largest Goldbach number generated by P_n and (P_n, P_n) is the only pair of primes not greater than P_n to form $2P_n$. Hence $B_{up}(n) = A_{up}(n) + A_{up}(n)$ and $B_{low}(n) = A_{low}(n) + A_{low}(n)$. By (2.11) $A_{up}(n) = n\log n + n\log\log n$, we have $B_{up}(n) = (n\log n + n\log\log n) + (n\log n + n\log\log n) = 2n\log n + 2n\log\log n$. By (2.11) $A_{low}(n) = n\log n + n\log\log n - n$, we have $B_{low}(n) = A_{low}(n) + A_{low}(n) = (n\log n + n\log\log n - n) + (n\log n + n\log\log n - n) = 2n\log n + 2n\log\log n - 2n$. Therefore, by (2.11) we have $B_{low}(n) < 2P_{n-}< B_{up}(n)$ for $n \geq 6$, that is, $2n\log n + 2n\log\log n - 2n < 2P_n < 2n\log n + 2n\log\log n$ for $n \geq 6$ and the theorem holds.

Theorem 3.6. *If $L_n = 2P_n$, then upper bound of L_n is equal to upper bound of $2P_n$ and lower bound of L_n is equal to lower bound of $2P_n$.*

Proof. Let $C_{up}(n)$ denote upper bound of L_n and $C_{low}(n)$ denote lower bound of L_n. Since $L_n = 2P_n = P_n + P_n$. Hence $C_{up}(n) = A_{up}(n) + A_{up}(n) = B_{up}(n)$ and $C_{low}(n) = A_{low}(n) + A_{low}(n) = B_{low}(n)$. Thus the theorem holds.

There are seven examples for $L_n = 2P_n$, which have been found among all L_n generated by primes less than 10000000. These examples are listed in the following table.

Table 2. Examples for $L_n = 2P_n$

n	P_n	L_n	$L_n/2P_n$	$(2P_n–L_n)/2P_n$
1	2	4	1	0
2	3	6	1	0
3	5	10	1	0
4	7	14	1	0
6	13	26	1	0
8	19	38	1	0
29	109	218	1	0

In the table we see the ratio $L_n/2P_n$ = 1 and relative error between L_n and $2P_n$ is 0 for all examples. One may expect that there will not exist example for $L_n = 2P_n$ for n > 29 because density of primes will be smaller and smaller with growth of n. Therefore, we should consider the general case in which $L_n < 2P_n$, for example, there is a Goldbach step $L_{15} = L_{16} = L_{17}$ = 90, whose $n1$ = 15 and $n2$ = 17, because of $2P_{15}$ = 94, $2P_{16}$ = 106, $2P_{17}$ = 118. There is a prime pair (P_{15}, P_{14}) to form L_{15}, that is, $L_{15} = P_{15} + P_{14}$ = 47 + 43, we have $C_{up}(15) = A_{up}(15) + A_{up}(14)$ = 55 + 51 = 106 and $C_{low}(15) = A_{low}(15) + A_{low}(14)$ = 40 + 37 = 77, thus we see 77 < 90 < 106. However, there are two prime pairs (P_{16}, P_{12}) and (P_{15}, P_{14}) to form L_{16}, that is, $L_{16} = P_{16} + P_{12}$ = 53 +37 and $L_{16} = P_{15} + P_{14}$ = 47 + 43. First, we have $C_{up}(16) = A_{up}(16) + A_{up}(12)$ = 60 + 41 = 101 and $C_{low}(16) = A_{low}(16) + A_{low}(12)$ = 44 + 29 = 73, thus we see 73 < 90 < 101. Second, $L_{16} = P_{15} + P_{14}$ = 47 +43, we have $C_{up}(16) = A_{up}(15) + A_{up}(14)$ = 55 + 51 = 106 and $C_{low}(16) = A_{low}(15) + A_{low}(14)$ = 40 + 37 = 77, thus we see 77 < 90 < 106. Further, there are three prime pairs (P_{17}, P_{11}), (P_{16}, P_{12}) and (P_{15}, P_{14}) to form L_{17}, that is, $L_{17} = P_{17} + P_{11}$ = 59 +31, $L_{17} = P_{16} + P_{12}$ = 53 +37 and $L_{17} = P_{15} + P_{14}$ = 47 + 43. First, we have $C_{up}(17) = A_{up}(17) + A_{up}(11)$ = 66 + 36 = 102 and $C_{low}(17) = A_{low}(17) + A_{low}(11)$ = 49 + 25 = 74, thus we see 74 < 90 < 102. Second, we have $C_{up}(17) = A_{up}(16) + A_{up}(12)$ = 60 + 41 = 101 and $C_{low}(17) = A_{low}(16) + A_{low}(12)$ = 44 + 29 = 73, thus we see 73 < 90 < 101. Third, $L_{17} = P_{15} + P_{14}$ = 47 +43, we have $C_{up}(17) = A_{up}(15) + A_{up}(14)$ = 55 + 51 = 106 and $C_{low}(17) = A_{low}(15) + A_{low}(14)$ = 40 + 37 = 77, thus we see 77 < 90 < 106. The example means it is very difficult to establish a function of n to describe upper or lower bound of L_n, because the upper or lower bound of L_n, in fact, is not a number but a set. However, from the example we see there is $C_{low}(n) < L_n < C_{up}(n)$ for every prime pair to form L_n. Based on relative error between L_n and $2P_n$ coming closer and closer to 0 with growth of n, it is reasonable that non-asymptotic bounds of $2P_n$ can be thought as approximate bounds of L_n as the following discussion does.

Theorem 3.7. *If $L_n < 2P_n$, then $C_{low}(n) < L_n < C_{up}(n)$ for a prime pair to form L_n.*

Proof. Suppose $L_n < 2P_n$. Then L_n can be written as $L_n = P_i + P_k$ for $i \le n$ and $k < n$, where (P_i, P_k) is a prime pair to form L_n. By (2.11), we have $C_{low}(n) < L_n = P_i + P_k < C_{up}(n)$ for $i \le n$ and $k < n$, where $C_{low}(n) = A_{low}(i) + A_{low}(k) = (i\log i + i\log\log i) + (k\log k + k\log\log k\ ) - (i + k)$ and $C_{up}(n) = A_{up}(i) + A_{up}(k) = (i\log i + i\log\log i) + (k\log k + k\log\log k)$. Hence $C_{low}(n) < L_n = P_i + P_k < C_{up}(n)$ for $i \le n$ and $k < n$ if $L_n < 2P_n$. The result means that if $L_n < 2P_n$ then $C_{low}(n) < L_n < C_{up}(n)$ for a prime pair to form L_n and the theorem holds.

Corollary 3.8. *If $L_n < 2P_n$, then $C_{low}(n) < L_n < C_{up}(n)$ for every prime pair to form L_n.*

Proof. Since the case for every prime pair to form L_n is same as the case for a prime pair to form L_n described by Theorem 3.7. Hence if $L_n < 2P_n$ then $C_{low}(n) < L_n < C_{up}(n)$ for every prime pair to form L_n and the corollary holds.

Remark 3.9 From previous example $L_{17} = 90$ we see $C_{low}(17) < L_{17} = 90 < C_{up}(17)$ for every prime pair to form L_{17} since $C_{low}(17) = 74$ and $C_{up}(17) = 102$ for $L_{17} = P_{17} + P_{11}$, $C_{low}(17) = 73$ and $C_{up}(17) = 101$ for $L_{17} = P_{16} + P_{12}$, $C_{low}(17) = 77$ and $C_{up}(17) = 106$ for $L_{17} = P_{15} + P_{14}$. It means that although L_n has different upper and lower bounds for different prime pairs to form L_n, there is always $C_{low}(n) < L_n < C_{up}(n)$ for all prime pairs to form L_n.

Theorem 3.10. *If $L_n < 2P_n$, then upper bound of L_n formed by a prime pair is smaller than upper bound of $2P_n$.*

Proof. Considering upper bound of $2P_n$ to be written as $B_{up}(n)$ and $2P_n = P_n + P_n$, by (2.11) we have

$$B_{up}(n) = A_{up}(n) + A_{up}(n) = (n\log n + n\log\log n\ ) + (n\log n + n\log\log n). \tag{3.1}$$

The first case for $L_n < 2P_n$ is that $L_n = P_n + P_i$, $i < n$. In this case we have upper bound of L_n,

$$C_{up}(n) = A_{up}(n) + A_{up}(i) = (n\log n + n\log\log n\ ) + (i\log i + i\log\log i\ ). \tag{3.2}$$

Since $i < n$, we obtain $\log i < \log n$, and get

$$i\log i < n\log n. \tag{3.3}$$

By $i\log\log i < n\log\log n$, from (3.3) we have

$$i\log i + i\log\log i < n\log n + n\log\log n. \tag{3.4}$$

Hence from (3.1), (3.2) and (3.4) we obtain

$$C_{up}(n) < B_{up}(n). \tag{3.5}$$

The second case for $L_n < 2P_n$ is that $L_n = P_i + P_k$, $i < n$, $k < n$. In this case we have upper bound of L_n,

$$C_{up}(n) = A_{up}(i) + A_{up}(k) = (i\log i + i\log\log i\ ) + (k\log k + k\log\log k\ ). \tag{3.6}$$

Since $i < n$, we obtain $\log i < \log n$, and get

$$i\log i < n\log n. \tag{3.7}$$

By $i\log\log i < n\log\log n$, from (3.7) we have

$$i\log i + i\log\log i < n\log n + n\log\log n. \tag{3.8}$$

Since $k < n$, we obtain $\log k < \log n$, and get

$$k\log k < n\log n. \tag{3.9}$$

By $k\log\log k < n\log\log n$, from (3.9) we have

$$k\log k + k\log\log k < n\log n + n\log\log n\ . \tag{3.10}$$

Hence from (3.1), (3.6), (3.8), (3.10) we obtain

$$C_{up}(n) < B_{up}(n). \tag{3.11}$$

Thus the theorem holds.

Corollary 3.11. *If $L_n < 2P_n$, then upper bound of L_n formed by every prime pair is smaller than upper bound of $2P_n$.*

Proof. Since the case for every prime pair to form L_n is same as the case for a prime pair to form L_n described by Theorem 3.10. Hence upper bound of L_n formed by every prime pair is smaller than upper bound of $2P_n$ if $L_n < 2P_n$. Thus the corollary holds.

Corollary 3.12. *Upper bound of $2P_n$ can be thought as approximate upper bound of L_n for $n \ge 6$.*

Proof. By Theorem 3.6 upper bound of L_n is equal to upper bound of $2P_n$ for $L_n = 2P_n$. By Corollary 3.11 upper bound of L_n formed by every prime pair is smaller than upper bound of $2P_n$ for $L_n < 2P_n$. Hence L_n will not be larger than upper bound of $2P_n$ for $n \ge 6$ and upper bound of $2P_n$ can be thought as approximate upper bound of L_n for $n \ge 6$. Thus the corollary holds.

Remark 3.13. By Corollary 3.12, there is no abnormal event such that L_n is larger than upper bound of $2P_n$ for $n \geq 6$, that is, every L_n is smaller than upper bound of $2P_n$ for $n \geq 6$, thus, the ratio of L_n to $2n\log n + 2n\log\log n$, upper bound of $2P_n$, is smaller than 1 and is closer and closer to 1 with growth of n as Figure 2 shows. However, it is not the final scope of application for upper bound of $2P_n$ to be able to be thought as approximate upper bound of L_n, and it needs to be consistent with scope of application for lower bound of $2P_n$ to be able to be thought as approximate lower bound of L_n, that is, the scope of application is $n \geq 20542$.

Theorem 3.14. *If $L_n < 2P_n$, then lower bound of L_n formed by a prime pair is smaller than lower bound of $2P_n$.*

Proof. Considering lower bound of $2P_n$ to be written as $B_{low}(n)$ and $2P_n = P_n + P_n$, by (2.11) we have

$$B_{low}(n) = (n\log n + n\log\log n - n) + (n\log n + n\log\log n - n). \tag{3.12}$$

The first case for $L_n < 2P_n$ is that $L_n = P_n + P_i$, $i < n$. In this case we have lower bound of L_n,

$$C_{low}(n) = A_{low}(n) + A_{low}(i) = (n\log n + n\log\log n - n) + (i\log i + i\log\log i - i). \tag{3.13}$$

Since $i < n$, we obtain $\log i - 1 < \log n - 1$, and get $i(\log i - 1) < n(\log n - 1)$, that is,

$$i\log i - i < n\log n - n. \tag{3.14}$$

By $i\log\log i < n\log\log n$, from (3.14) we have

$$i\log i + i\log\log i - i < n\log n + n\log\log n - n. \tag{3.15}$$

Hence from (3.12), (3.13) and (3.15) we obtain

$$C_{low}(n) < B_{low}(n). \tag{3.16}$$

The second case for $L_n < 2P_n$ is that $L_n = P_i + P_k$, $i < n$, $k < n$. In this case we have lower bound of L_n,

$$C_{low}(n) = (i\log i + i\log\log i - i) + (k\log k + k\log\log k - k). \tag{3.17}$$

Since $i < n$, we obtain $\log i - 1 < \log n - 1$, and get $i(\log i - 1) < n(\log n - 1)$, that is,

$$i\log i - i < n\log n - n. \tag{3.18}$$

By $i\log\log i < n\log\log n$, from (3.18) we have

$$i\log i + i\log\log i - i < n\log n + n\log\log n - n. \tag{3.19}$$

Since $k < n$, we obtain $\log k - 1 < \log n - 1$, and get $k(\log k - 1) < n(\log n - 1)$, that is,

$$k\log k - k < n\log n - n. \tag{3.20}$$

By $k\log\log k < n\log\log n$, from (3.20) we have

$$k\log k + k\log\log k - k < n\log n + n\log\log n - n. \tag{3.21}$$

Hence from (3.12), (3.17), (3.19), (3.21) we obtain

$$C_{low}(n) < B_{low}(n). \tag{3.22}$$

Thus the theorem holds.

Corollary 3.15. *If $L_n < 2P_n$, then lower bound of L_n formed by every prime pair is smaller than lower bound of $2P_n$.*

Proof. Since the case for every prime pair to form L_n is same as the case for a prime pair to form L_n described by Theorem 3.14. Hence lower bound of L_n formed by every prime pair is smaller than lower bound of $2P_n$ if $L_n < 2P_n$. Thus the corollary holds.

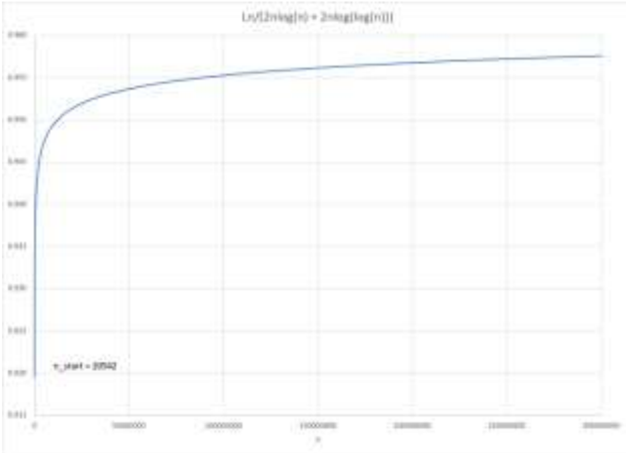

Figure 2. The ratio of L_n to upper bound of $2P_n$ for $20542 \leq n \leq 300000000$

Remark 3.16. Corollary 3.15 means that, generally, lower bound of L_n formed by every prime pair is smaller than lower bound of $2P_n$ if $L_n < 2P_n$, therefore, if lower bound of $2P_n$ is thought as approximate lower bound of L_n though lower bound of L_n is not a number but a set, then there must exist some abnormal events such that L_n is smaller than $B_{low}(n)$ to lead to $L_n/B_{low}(n) < 1$ for $n \geq 6$. Obviously, such abnormal events are not conducive to the effectiveness of using lower bound of $2P_n$ as approximate lower bound of L_n.

It is clear that although $C_{low}(n) < L_n < C_{up}(n)$ for all prime pairs to form L_n by Corollary 3.8, $C_{low}(n) < B_{low}(n)$ for all prime pairs to form L_n by Corollary 3.15. It may lead to appearing of some abnormal events if lower bound of $2P_n$, $B_{low}(n)$, is thought as approximate lower bound of L_n. For a given L_n, if $L_n/B_{low}(n) > 1$, that is, $L_n > B_{low}(n)$, then the ratio $L_n/B_{low}(n)$ is called a normal event for this approximation, however, if $L_n/B_{low}(n) < 1$, that is, $L_n < B_{low}(n)$, then the ratio $L_n/B_{low}(n)$ is called an abnormal event for this approximation. Still using Goldbach step $L_{15} = L_{16} = L_{17} = 90$ as an example, we discovered that both $L_{15}/B_{low}(15)$ and $L_{16}/B_{low}(16)$ are normal events because $L_{15}/B_{low}(15) = 90/80 = 1.125 > 0$ and $L_{16}/B_{low}(16) = 90/88 = 1.022 > 0$ but $L_{17}/B_{low}(17)$ is an abnormal event because $L_{17}/B_{low}(17) = 90/98 = 0.918 < 0$. This example means some ratios $L_n/B_{low}(n)$ may become abnormal events for $B_{low}(n)$ being thought as approximate lower bound of L_n because of existence of $C_{low}(n) < B_{low}(n)$ for $n \geq 6$ but other ratios $L_n/B_{low}(n)$ may become normal events for $B_{low}(n)$ being thought as approximate lower bound of L_n because of existence of relative error between L_n and $2P_n$ being closer and closer to 0 with growth of n. If the latter has a greater advantage than the former, then it is predictable that density of abnormal events will obviously decrease with growth of n until it reaches 0. In fact, after checking every ratio $L_n/B_{low}(n)$ for $2 \leq n \leq 400000000$, we really found 5225 abnormal events for $2 \leq n \leq 20541$ and $L_{20541}/B_{low}(20541)$ is believed to be the last abnormal event such that $L_{20541}/B_{low}(20541) = 461024/461175 = 0.999672$, however, there is no abnormal event for $20542 \leq n \leq 400000000$, that is, all ratios $L_n/B_{low}(n)$ are normal events for $20542 \leq n \leq 400000000$ and the first normal event is $L_{20542}/B_{low}(20542) = 461882/461200 = 1.001478$ for $20542 \leq n \leq 400000000$ as Figure 3 shows, in the figure, blue curve represents the ratio $L_n/B_{low}(n)$ for $20542 \leq n \leq 300000000$.

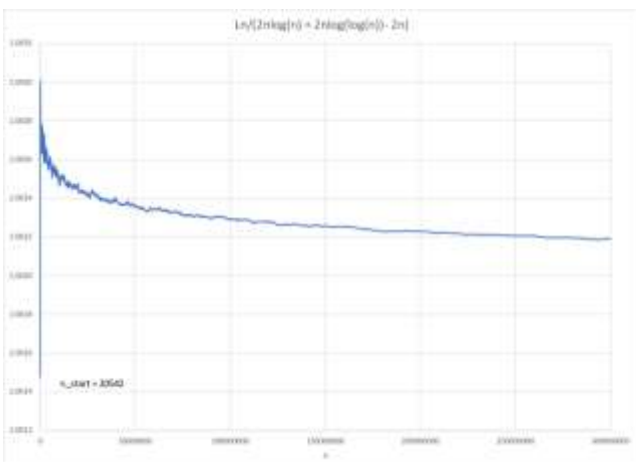

Figure 3. The ratio of L_n to lower bound of $2P_n$ for $20542 \leq n \leq 300000000$

Observation 3.17. *Ratios $L_n/B_{up}(n)$ and $L_n/B_{low}(n)$ at starting point and finishing point for some Goldbach steps.*

Based on checked data, we give ratios $L_n/B_{up}(n)$ and $L_n/B_{low}(n)$ at starting point and finishing point of some Goldbach steps in which the first two Goldbach steps have width being 1 but other Goldbach steps have width greater than 1. Comparing the data with $2P_n/B_{up}(n)$ and $2P_n/B_{low}(n)$, we discovered that $2P_n/B_{up}(n)$ is larger than $L_n/B_{up}(n)$ and $2P_n/B_{low}(n)$ is larger than $L_n/B_{low}(n)$ for every starting or finishing point as Table 3 shows. We also see there are some fluctuations in data caused by existence of complete Goldbach steps.

Table 3. Comparison between $L_n/B_{up}(n)$ or $L_n/B_{low}(n)$ and $2P_n/B_{up}(n)$ or $2P_n/B_{low}(n)$

n	P_n	L_n	$B_{up}(n)$	$B_{low}(n)$	$L_n/B_{up}(n)$	$L_n/B_{low}(n)$	$2P_n/B_{up}(n)$	$2P_n/B_{low}(n)$
20542	231299	461882	502284	461200	0.919563	1.001478	0.920988	1.003031
20543	231317	461918	502311	461225	0.919585	1.001502	0.921011	1.003054
28418	330149	659850	715139	658303	0.922687	1.002349	0.923314	1.003030
28442	330431	659850	715796	658912	0.921840	1.001423	0.923254	1.002959
33637	397211	793746	858915	791641	0.924126	1.002659	0.924913	1.003512
33658	397469	793746	859498	792182	0.923499	1.001974	0.924886	1.003478
38043	454451	908264	981680	905594	0.925213	1.002948	0.925863	1.003652
38072	454859	908264	982492	906348	0.924449	1.002113	0.925929	1.003718
43201	521791	1043010	1126802	1040400	0.925637	1.002508	0.926144	1.003058
43231	522191	1043010	1127650	1041188	0.924941	1.001749	0.926157	1.003067
47512	579011	1157120	1249126	1154102	0.926343	1.002615	0.927065	1.003396
47546	579517	1157120	1250094	1155002	0.925626	1.001833	0.927157	1.003490
52642	647593	1294146	1395790	1290506	0.927178	1.002820	0.927923	1.003626
52660	647789	1294146	1396307	1290987	0.926834	1.002446	0.927860	1.003556
57875	717797	1434450	1546517	1430767	0.927535	1.002574	0.928275	1.003373
57919	718433	1434450	1547789	1431951	0.926773	1.001745	0.928334	1.003432
62381	778777	1556946	1677129	1552367	0.928340	1.002949	0.928702	1.003341
62406	779111	1556946	1677856	1553044	0.927937	1.002512	0.928698	1.003334
67279	845261	1689870	1819901	1685343	0.928550	1.002686	0.928908	1.003072
67325	845981	1689870	1821246	1686596	0.927864	1.001941	0.929013	1.003181
73278	927259	1853690	1995815	1849259	0.928788	1.002396	0.929203	1.002843
73308	927727	1853690	1996697	1850081	0.928378	1.001950	0.929261	1.002904
78302	997247	1993826	2143959	1987355	0.929973	1.003256	0.930285	1.003592
78361	998017	1993826	2145703	1988981	0.929218	1.002435	0.930247	1.003546
84051	1076869	2153040	2314334	2146232	0.930306	1.003172	0.930608	1.003497
84083	1077413	2153040	2315285	2147119	0.929924	1.002757	0.930695	1.003589
89084	1146911	2293004	2464190	2286022	0.930530	1.003054	0.930862	1.003412
89141	1147591	2293004	2465890	2287608	0.929889	1.002358	0.930772	1.003310
94809	1226549	2451950	2635395	2445777	0.930391	1.002523	0.930827	1.002993
94839	1226867	2451950	2636294	2446616	0.930074	1.002180	0.930751	1.002909
98735	1281821	2562962	2753236	2555766	0.930890	1.002815	0.931137	1.003081
98780	1282423	2562962	2754589	2557029	0.930433	1.002320	0.931117	1.003057
103842	1354069	2707116	2907028	2699344	0.931231	1.002879	0.931583	1.003257
103919	1355279	2707116	2909351	2701513	0.930487	1.002074	0.931671	1.003348
109068	1428041	2855210	3064965	2846829	0.931563	1.002943	0.931848	1.003250
109090	1428419	2855210	3065630	2847450	0.931361	1.002725	0.931892	1.003296
113586	1491773	2982710	3201940	2974768	0.931532	1.002669	0.931793	1.002950
113610	1492097	2982710	3202669	2975449	0.931320	1.002440	0.931783	1.002939
118436	1560893	3120656	3349414	3112542	0.931702	1.002606	0.932039	1.002969
118485	1561597	3120656	3350905	3113935	0.931287	1.002158	0.932044	1.002973
122658	1621657	3242634	3478140	3232824	0.932289	1.003034	0.932485	1.003244
122695	1622209	3242634	3479270	3233880	0.931986	1.002706	0.932499	1.003258
128185	1700627	3400380	3647128	3390758	0.932344	1.002837	0.932584	1.003095
128233	1701307	3400380	3648598	3392132	0.931968	1.002431	0.932581	1.003090

In Table 3 we see $L_n/B_{up}(n)$ is very close to $2P_n/B_{up}(n)$ and $L_n/B_{low}(n)$ is also very close to $2P_n/B_{low}(n)$, which means it is reasonable to use bounds of $2P_n$ as approximate bounds of L_n. We also see there is always $B_{low}(n) < L_n < B_{up}(n)$, which means those fluctuations in data caused by existence of complete Goldbach steps, in fact, are not able to interfere the effectiveness of using lower bound of $2P_n$ as approximate lower bound of L_n.

Observation 3.18. *Ratios $L_n/B_{up}(n)$ and $L_n/B_{low}(n)$ at regular watch points.*

Based on checked data, we give the ratios $L_n/B_{up}(n)$ and $L_n/B_{low}(n)$ at regular watch points for $1000000 \le n \le 100000000$ as Table 4 shows. In the table we obviously feel both $L_n/B_{up}(n)$ and $L_n/B_{low}(n)$ are coming closer and closer to 1 with growth of n, which corresponds to the relative error between L_n and $2P_n$ being closer and closer to 0 with growth of n.

Table 4. $L_n/B_{up}(n)$ and $L_n/B_{low}(n)$ at regular watch points for $1000000 \le n \le 100000000$

n	P_n	L_n	$B_{up}(n)$	$B_{low}(n)$	$L_n/B_{up}(n)$	$L_n/B_{low}(n)$	$2P_n/B_{up}(n)$	$2P_n/B_{low}(n)$
1000000	15485863	30970934	32882604	30882604	0.9418637	1.0028601	0.9418878	1.0028858
2000000	32452843	64903652	68733612	64733612	0.9442782	1.0026267	0.9443078	1.0026581
3000000	49979687	99957230	105698588	99698588	0.9456817	1.0025942	0.9457020	1.0026157
4000000	67867967	135733682	143385752	135385752	0.9466329	1,0025699	0.9466486	1.0025865
5000000	86028121	172053716	181609346	171609346	0.9473835	1.0025894	0.9473975	1.0026041
6000000	104395301	208788386	220260081	208260081	0.9479175	1.0025367	0.9479275	1.0025473
7000000	122949823	245897354	259265802	245265802	0.9484372	1.0025749	0.9484461	1.0025843
8000000	141650939	283300394	298575258	282575258	0.9488408	1.0025661	0.9488457	1.0025714
9000000	160481183	320959782	338150150	320150150	0.9491635	1.0025289	0.9491711	1.0025369
10000000	179424673	358847082	377960764	357960764	0.9494294	1.0024760	0.9494354	1.0024823
11000000	198491317	396980480	417983373	395983373	0.9497518	1.0025180	0.9497569	1.0025234
12000000	217645177	435287024	458198588	434198588	0.9499964	1.0025067	0.9500037	1.0025144
13000000	236887691	473773230	498590275	472590275	0.9502255	1.0025031	0.9502298	1.0025076
14000000	256203161	512402706	539144787	511144787	0.9503990	1.0024609	0.9504057	1.0024680
15000000	275604541	551205302	579850440	549850440	0.9505990	1.0024640	0.9506056	1.0024709
16000000	295075147	590147520	620697112	588697112	0.9507818	1.0024637	0.9507862	1.0024684
17000000	314606869	629209844	661675953	627675953	0.9509335	1.0024437	0.9509394	1.0024499
18000000	334214459	668426972	702779155	666779155	0.9511195	1.0024713	0.9511222	1.0024742
19000000	353868013	707734224	743999781	705999781	0.9512559	1.0024567	0.9512583	1.0024592
20000000	373587883	747172802	785331628	745331628	0.9514105	1.0024702	0.9514143	1.0024742
21000000	393342739	786682976	826769113	784769113	0.9515147	1.0024387	0.9515177	1.0024419
22000000	413158511	826312934	868307189	824307189	0.9516366	1.0024332	0.9516413	1.0024382
23000000	433024223	866045744	909941266	863941266	0.9517600	1.0024359	0.9517630	1.0024390
24000000	452930459	905857616	951667155	903667155	0.9518639	1.0024239	0.9518673	1.0024276
25000000	472882027	945760266	993481020	943481020	0.9519661	1.0024157	0.9519699	1.0024197
26000000	492876847	985751742	1035379326	983379326	0.9520682	1.0024125	0.9520700	1.0024144
27000000	512927357	1025851896	1077358819	1023358819	0.9521914	1.0024361	0.9521941	1.0024389
28000000	533000389	1065996186	1119416479	1063416479	0.9522784	1.0024258	0.9522825	1.0024301
29000000	553105243	1106208302	1161549506	1103549506	0.9523557	1.0024093	0.9523575	1.0024112
30000000	573259391	1146516350	1203755294	1143755294	0.9524496	1.0024140	0.9524517	1.0024161
31000000	593441843	1186881338	1246031408	1184031408	0.9525292	1.0024069	0.9525311	1.0024089
32000000	613651349	1227300260	1288375574	1224375574	0.9525951	1.0023887	0.9525969	1.0023907
33000000	633910099	1267817690	1330785657	1264785657	0.9526836	1.0023972	0.9526854	1.0023992
34000000	654188383	1308374094	1373259656	1305259656	0.9527506	1.0023860	0.9527526	1.0023881
35000000	674506081	1349009318	1415795683	1345795683	0.9528276	1.0023879	0.9528296	1.0023900
36000000	694847533	1389691632	1458391961	1386391961	0.9528930	1.0023800	0.9528954	1.0023825
37000000	715225739	1430449692	1501046812	1427046812	0.9529680	1.0023845	0.9529692	1.0023858
38000000	735632791	1471262690	1543758649	1467758649	0.9530393	1.0023873	0.9530411	1.0023893
39000000	756065159	1512127746	1586525965	1508525965	0.9531062	1.0023876	0.9531078	1.0023893
40000000	776531401	1553057954	1629347336	1549347336	0.9531779	1.0023949	0.9531809	1.0023980
41000000	797003413	1594003746	1672221406	1590221406	0.9532252	1.0023784	0.9532271	1.0023804
42000000	817504243	1635005970	1715146885	1631146885	0.9532746	1,0023658	0.9532760	1.0023674
43000000	838041641	1676080584	1758122547	1672122547	0.9533354	1.0023670	0.9533370	1.0023686
44000000	858599503	1717197000	1801147220	1713147220	0.9533906	1.0023639	0.9533918	1.0023651
45000000	879190747	1758378596	1844219788	1754219788	0.9534539	1.0023707	0.9534554	1.0023723
46000000	899809343	1799616438	1887339183	1795339183	0.9535204	1.0023824	0.9535216	1.0023836
47000000	920419813	1840836054	1930504385	1836504385	0.9535518	1.0023586	0.9535537	1.0023605

48000000	941083981	1882164936	1973714415	1877714415	0.9536156	1.0023701	0.9536171	1.0023717
49000000	961748927	1923494912	2016968338	1918968338	0.9536564	1.0023588	0.9536579	1.0023603
50000000	982451653	1964900462	2060265255	1960265255	0.9537123	1.0023645	0.9537137	1.0023660
51000000	1003162753	2006322074	2103604304	2001604304	0.9537545	1.0023569	0.9537561	1.0023587
52000000	1023893771	2047784264	2146984657	2042984657	0.9537954	1.0023493	0.9537970	1.0023509
53000000	1044645379	2089287096	2190405516	2084405516	0.9538357	1.0023419	0.9538374	1.0023437
n	P_n	L_n	$B_{up}(n)$	$B_{low}(n)$	$L_n/B_{up}(n)$	$L_n/B_{low}(n)$	$2P_n/B_{up}(n)$	$2P_n/B_{low}(n)$
54000000	1065433423	2130863682	2233866117	2125866117	0.9538905	1.0023508	0.9538919	1.0023523
55000000	1086218491	2172433442	2277365719	2167365719	0.9539238	1.0023381	0.9539253	1.0023398
56000000	1107029837	2214056762	2320903614	2208903614	0.9539632	1.0023328	0.9539645	1.0023342
57000000	1127870669	2255738460	2364479115	2250479115	0.9540107	1.0023369	0.9540119	1.0023382
58000000	1148739811	2297476292	2408091559	2292091559	0.9540651	1.0023492	0.9540665	1.0023507
59000000	1169604791	2339206464	2451740310	2333740310	0.9541004	1.0023422	0.9541016	1.0023435
60000000	1190494759	2380986204	2495424749	2375424749	0.9541406	1.0023412	0.9541419	1.0023426
61000000	1211405357	2422807086	2539144280	2417144280	0.9541825	1.0023427	0.9541839	1.0023442
62000000	1232332807	2464661234	2582898326	2458898326	0.9542230	1.0023436	0.9542247	1.0023454
63000000	1253270831	2506538544	2626686330	2500686330	0.9542587	1.0023402	0.9542599	1.0023414
64000000	1274224957	2548448312	2670507751	2542507751	0.9542935	1.0023364	0.9542941	1.0023371
65000000	1295202449	2590401626	2714362064	2584362064	0.9543316	1.0023369	0.9543328	1.0023382
66000000	1316196199	2632389182	2758248764	2626248764	0.9543697	1.0023380	0.9543709	1.0023393
67000000	1337195521	2674388664	2802167357	2668167357	0.9544000	1.0023316	0.9544008	1.0023325
68000000	1358208601	2716414334	2846117367	2710117367	0.9544280	1.0023235	0.9544290	1.0023245
69000000	1379256017	2758508666	2890098330	2752098330	0.9544687	1.0023292	0.9544699	1.0023304
70000000	1400305337	2800608066	2934109797	2794109797	0.9545000	1.0023257	0.9545009	1.0023266
71000000	1421376527	2842749122	2978151330	2836151330	0.9545348	1.0023263	0.9545361	1.0023277
72000000	1442467307	2884933790	3022222505	2878222505	0.9545735	1.0023317	0.9545751	1.0023334
73000000	1463554999	2927105882	3066322909	2920322909	0.9545980	1.0023226	0.9545993	1.0023240
74000000	1484670157	2969337290	3110452142	2962452142	0.9546320	1.0023241	0.9546330	1.0023251
75000000	1505776939	3011550096	3154609811	3004609811	0.9546505	1.0023098	0.9546517	1.0023111
76000000	1526922013	3053840274	3198795537	3046795537	0.9546844	1.0023121	0.9546855	1.0023134
77000000	1548074321	3096145686	3243008950	3089008950	0.9547138	1.0023103	0.9547148	1.0023113
78000000	1569250357	3138497756	3287249691	3131249691	0.9547488	1.0023147	0.9547497	1,0023156
79000000	1590425971	3180849132	3331517405	3173517405	0.9547748	1.0023102	0.9547757	1.0023111
80000000	1611623773	3223242996	3375811752	3215811752	0.9548053	1.0023108	0.9548066	1.0023122
81000000	1632828047	3265653684	3420132399	3258132399	0.9548325	1.0023084	0.9548332	1.0023092
82000000	1654054489	3308104874	3464479019	3300479019	0.9548635	1.0023105	0.9548647	1.0023117
83000000	1675293211	3350583830	3508851295	3342851295	0.9548947	1.0023131	0.9548955	1.0023139
84000000	1696528903	3393054734	3553248916	3385248916	0.9549161	1.0023058	0.9549170	1.0023067
85000000	1717783147	3435562872	3597671582	3427671582	0.9549406	1.0023022	0.9549416	1.0023032
86000000	1739062363	3478120850	3642118995	3470118995	0.9549717	1.0023059	0.9549728	1.0023070
87000000	1760341421	3520679450	3686590868	3512590868	0.9549959	1.0023027	0.9549968	1.0023037
88000000	1781636611	3563270910	3731086919	3555086919	0.9550222	1.0023020	0.9550228	1.0023027
89000000	1802933611	3605864532	3775606873	3597606873	0.9550423	1.0022953	0.9550430	1.0022960
90000000	1824261409	3648518604	3820150459	3640150459	0.9550719	1.0022988	0.9550730	1.0023000
91000000	1845587707	3691171752	3864717414	3682717414	0.9550948	1.0022956	0.9550958	1.0022966
92000000	1866941107	3733878180	3909307482	3725307482	0.9551252	1.0023006	0.9551262	1.0023017
93000000	1888303061	3776600942	3953920411	3767920411	0.9551535	1.0023037	0.9551548	1.0023051
94000000	1909662901	3819321876	3998555954	3810555954	0.9551752	1.0023004	0.9551762	1.0023014
95000000	1931045213	3862086372	4043213870	3853213870	0.9552020	1.0023026	0.9552031	1.0023036
96000000	1952429173	3904855992	4087893921	3895893921	0.9552243	1.0023003	0.9552249	1.0023009
97000000	1973828641	3947654816	4132595880	3938595880	0.9552482	1.0023000	0.9552488	1.0023006
98000000	1995230813	3990459452	4177319517	3981319517	0.9552679	1.0022957	0.9552684	1.0022962
99000000	2016634091	4033265634	4222064610	4024064610	0.9552827	1.0022865	0.9552833	1.0022871
100000000	2038074743	4076147580	4266830945	4066830945	0.9553103	1.0022908	0.9553107	1.0022913

It is a surprising result that there is no checked abnormal event for $L_n/B_{low}(n)$ within so large range as $20542 \le n \le 400000000$. Comparing the result with appearing of 5225 abnormal events for $L_n/B_{low}(n)$ for $2 \le n \le 20541$, we discovered that the existence of the relative error between L_n and $2P_n$ coming closer and closer to 0 with growth of n already has an overwhelming advantage than the general existence of $C_{low}(n) < B_{low}(n)$ for all prime pairs to form L_n within the range for $20542 \le n \le 400000000$. Thus we can expect there will not exist abnormal event for $L_n/B_{low}(n)$ for $n > 400000000$ because such overwhelming advantage will be stronger and stronger with growth of n for $n > 400000000$. Therefore, we can conclude that there is no abnormal event for $L_n/B_{low}(n)$ for $n \ge 20542$ and have the following theorem.

Theorem 3.19. *If upper and lower bounds of $2P_n$ can be thought as approximate upper and lower bounds of L_n for $n \ge 20542$, then* $\lim_{n\to\infty} \frac{L_n}{2n\log n} = 1$.

Proof. By Theorem 3.5, upper bound of $2P_n$ is

$$B_{up}(n) = 2n\log n + 2n\log\log n \text{ for } n \ge 6,$$

and lower bound of $2P_n$ is

$$B_{low}(n) = 2n\log n + 2n\log\log n - 2n \text{ for } n \ge 6.$$

Since bounds of $2P_n$ are thought as approximate bounds of L_n for $n \ge 20542$, we have

$$C_{up}(n) = 2n\log n + 2n\log\log n \text{ for } n \ge 20542, \tag{3.23}$$

$$C_{low}(n) = 2n\log n + 2n\log\log n - 2n \text{ for } n \ge 20542. \tag{3.24}$$

By (3.23) we obtain

$$\lim_{n\to\infty} \frac{L_n}{C_{up}(n)} = \lim_{n\to\infty} \frac{L_n}{2n\log n + 2n\log\log n}, \tag{3.25}$$

by (3.24) we obtain

$$\lim_{n\to\infty} \frac{L_n}{C_{low}(n)} = \lim_{n\to\infty} \frac{L_n}{2n\log n + 2n\log\log n - 2n}. \tag{3.26}$$

By Corollary 3.8,

$$\lim_{n\to\infty} \frac{L_n}{C_{up}(n)} = 1,$$

we get

$$\lim_{n\to\infty} \frac{L_n}{2n\log n + 2n\log\log n} = 1. \tag{3.27}$$

By Corollary 3.8,

$$\lim_{n\to\infty} \frac{L_n}{C_{low}(n)} = 1,$$

we get

$$\lim_{n\to\infty} \frac{L_n}{2n\log n + 2n\log\log n - 2n} = 1. \tag{3.28}$$

Since

$$\lim_{n\to\infty} \frac{2n\log\log n}{2n\log n} = \lim_{n\to\infty} \frac{\log\log n}{\log n} = 0, \tag{3.29}$$

by (3.27) we have

$$\lim_{n\to\infty} \frac{L_n}{2n\log n} = 1. \tag{3.30}$$

Since

$$\lim_{n\to\infty}\frac{2n\log\log n-2n}{2n\log n}=\lim_{n\to\infty}\frac{\log\log n}{\log n}-\lim_{n\to\infty}\frac{1}{\log n}=0, \tag{3.31}$$

by (3.28) we have

$$\lim_{n\to\infty}\frac{L_n}{2n\log n}=1. \tag{3.32}$$

From above results we see that by (3.27) we have (3.30) and by (3.28) we have (3.32), and (3.30) is same as (3.32). It means that there is only a limit

$$\lim_{n\to\infty}\frac{L_n}{2n\log n}=1, \tag{3.33}$$

and corresponding asymptotic expression is

$$L_n\approx 2n\log n. \tag{3.34}$$

Thus the theorem holds.

Corollary 3.20. *If upper and lower bounds of $2P_n$ can be thought as approximate upper and lower bounds of L_n for $n \geq 20542$, then Goldbach conjecture is true.*

Proof. Suppose upper and lower bounds of $2P_n$ can be thought as approximate upper and lower bounds of L_n for $n \geq 20542$. Then there is a limit as Theorem 3.19 shows,

$$\lim_{n\to\infty}\frac{L_n}{2n\log n}=1,$$

and corresponding asymptotic expression is

$$L_n\approx 2n\log n.$$

This expression means that, in asymptotic case, if $2n\log n$ approaches infinity as n grows without bound then L_n approaches infinity as n grows without bound. Since it is obvious that $2n\log n$ approaches infinity as n grows without bound. Hence L_n approaches infinity as n grows without bound. The result means that there are infinitely many strong Goldbach numbers so that every even number greater than 2 is the sum of two primes and Glodbach conjecture is true. Thus the corollary holds.

Remark 3.21. Although it seems to be very reasonable that we conclude there is no abnormal event for $L_n/B_{low}(n)$ for $n \geq 20542$, it requires a rigorous proof as a hypothesis used in Theorem 3.19 and Corollary 3.20. The hypothesis can also be understood as a condition in theory, that is, if it can be proven that there is an integer $k > 0$ such that bounds of $2P_n$ can be thought as approximate bounds of L_n for all $n > k$, then Glodbach conjecture is true. In order to get an equivalent result, which is similar to prime number theorem, it is necessary to use $\xi(n) = L_n/2$ as counting function of strong Goldbach numbers and we have the following discussion.

4. Approximate Bounds of $\xi(n)$ and Goldbach Conjecture

Definition 4.1. Suppose the first even number 2 is thought as an imaginary or special strong Goldbach number. Then $\xi(n)$ is called *counting function of strong Goldbach numbers* if $\xi(n) = L_n/2$.

Remark 4.2. By Definition 4.1, $\xi(n) = L_n/2$ denotes the number of strong Goldbach numbers generated by P_n. For example, $\xi(9) = L_9/2 = 42/2 = 21$, which is the number of strong Goldbach numbers generated by $P_9 = 23$ to include (2), 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42.

Considering $\xi(n) = L_n/2$ and $L_n \approx 2P_n$, by (2.10) we have the following approximation for $\xi(n)$ for $n \geq 20542$,

$$\frac{\xi(n)}{n}\approx\log n+\log\log n-1+\frac{\log\log n-2}{\log n}-\frac{(\log\log n)^2-6\log\log n+11}{2(\log n)^2}+o\left(\frac{1}{(\log n)^2}\right). \tag{4.1}$$

By (2.11) we obtain

$$\frac{A_{up}(n)}{n}=\log n+\log\log n, \tag{4.2}$$

$$\frac{A_{low}(n)}{n} = \log n + \log\log n - 1, \tag{4.3}$$

$$\frac{A_{low}(n)}{n} < \frac{\xi(n)}{n} < \frac{A_{up}(n)}{n}, \tag{4.4}$$

where $A_{up}(n)$ denotes upper bound of P_n to be an approximation for upper bound of $\xi(n)$ and $A_{low}(n)$ denotes lower bound of P_n to be an approximation for lower bound of $\xi(n)$ for $n \geq 20542$. Taking $n = 1000000$, we have $\xi(1000000) = L_{1000000}/2 = 15485467$, $\xi(1000000)/1000000 = 15.485467$, $A_{up}(1000000)/1000000 = 16.441302$, and $A_{low}(1000000)/1000000 = 15.441302$. The results mean that (4.2), (4.3) and (4.4) all are reasonable since $15.441302 < 15.485467 < 16.441302$ in the example. On the other hand, $P_n/n = 15485863/1000000 = 15.485863$ for $n = 1000000$. Hence $\xi(n)/n$ is about average gap between primes P_n/n and $\xi(n)$ is about P_n for $n \geq 20542$.

Theorem 4.3. *If $L_n < 2P_n$, then $D_{low}(n) < \xi(n) < D_{up}(n)$ for a prime pair to form L_n.*

Proof. Let $D_{up}(n)$ denote upper bound of $\xi(n) = L_n/2$ and $D_{low}(n)$ denote lower bound of $\xi(n) = L_n/2$. Suppose $L_n < 2P_n$. Then L_n can be written as $L_n = P_i + P_k$ for $i \leq n$ and $k < n$, where (P_i, P_k) is a prime pair to form L_n. By (2.11), we have $D_{low}(n) < \xi(n) = (P_i + P_k)/2 < D_{up}(n)$ for $i \leq n$ and $k < n$, where $D_{low}(n) = C_{low}(n)/2 = A_{low}(i)/2 + A_{low}(k)/2 = (i\log i + i\log\log i)/2 + (k\log k + k\log\log k)/2 - (i + k)/2$ and $D_{up}(n) = C_{up}(n)/2 = A_{up}(i)/2 + A_{up}(k)/2 = (i\log i + i\log\log i)/2 + (k\log k + k\log\log k)/2$. Hence $D_{low}(n) < \xi(n) < D_{up}(n)$ for $i \leq n$ and $k < n$ if $L_n < 2P_n$. The result means that if $L_n < 2P_n$ then $D_{low}(n) < \xi(n) < D_{up}(n)$ for a prime pair to form L_n and the theorem holds.

Corollary 4.4. *If $L_n < 2P_n$, then $D_{low}(n) < \xi(n) < D_{up}(n)$ for every prime pair to form L_n.*

Proof. Since the case for every prime pair to form L_n is same as the case for a prime pair to form L_n described by Theorem 4.3. Thus if $L_n < 2P_n$ then $D_{low}(n) < \xi(n) < D_{up}(n)$ for every prime pair to form L_n and the corollary holds.

Remark 4.5. By the example $L_{17} = 90$ to be equivalent to $\xi(17) = 90/2 = 45$, we see $D_{low}(17) < \xi(17) = 45 < D_{up}(17)$ since $D_{low}(17) = 74/2 = 37$ and $D_{up}(17) = 102/2 = 51$ for $L_{17} = P_{17} + P_{11}$, $D_{low}(17) = 73/2 = 36.5$ and $D_{up}(17) = 101/2 = 50.5$ for $L_{17} = P_{16} + P_{12}$, $D_{low}(17) = 77/2 = 38.5$ and $D_{up}(17) = 106/2 = 53$ for $L_{17} = P_{15} + P_{14}$. It means that although $\xi(n)$ has different upper and lower bounds for different prime pairs to form L_n, there is always $D_{low}(n) < \xi(n) < D_{up}(n)$ for all prime pairs to form L_n.

Theorem 4.6. *If $L_n < 2P_n$, then $D_{up}(n)$ for a prime pair to form L_n is smaller than $A_{up}(n)$.*

Proof. Considering upper bound of $2P_n$ to be written as $B_{up}(n)$ and $2P_n = P_n + P_n$, by (2.11) we have

$$A_{up}(n) = B_{up}(n)/2 = n\log n + n\log\log n. \tag{4.5}$$

The first case for $L_n < 2P_n$ is that $L_n = P_n + P_i$, $i < n$. In this case we have

$$D_{up}(n) = C_{up}(n)/2 = A_{up}(n)/2 + A_{up}(i)/2 = (n\log n + n\log\log n)/2 + (i\log i + i\log\log i)/2. \tag{4.6}$$

Since $i < n$, we obtain $(\log i)/2 < (\log n)/2$, and get

$$(i\log i)/2 < (n\log n)/2. \tag{4.7}$$

By $(i\log\log i)/2 < (n\log\log n)/2$, from (4.7) we have

$$(i\log i + i\log\log i)/2 < (n\log n + n\log\log n)/2, \tag{4.8}$$

and

$$(i\log i + i\log\log i)/2 + (n\log n + n\log\log n)/2 < n\log n + n\log\log n. \tag{4.9}$$

By (4.5), (4.6) and (4.9) we get

$$D_{up}(n) < A_{up}(n). \tag{4.10}$$

The second case for $L_n < 2P_n$ is that $L_n = P_i + P_k$, $i < n$, $k < n$. In this case we have

$$D_{up}(n) = C_{up}(n)/2 = A_{up}(i)/2 + A_{up}(k)/2 = (i\log i + i\log\log i)/2 + (k\log k + k\log\log k)/2. \tag{4.11}$$

Since $i < n$, we obtain $(\log i)/2 < (\log n)/2$, and get

$$(i\log i)/2 < (n\log n)/2. \tag{4.12}$$

By $(i\log\log i)/2 < (n\log\log n)/2$, from (4.12) we have

$$(i\log i + i\log\log i)/2 < (n\log n + n\log\log n)/2. \tag{4.13}$$

Since $k < n$, we obtain $(\log k)/2 < (\log n)/2$, and get

$$(k\log k)/2 < (n\log n)/2. \tag{4.14}$$

By $(k\log\log k)/2 < (n\log\log n)/2$, from (4.14) we have

$$(k\log k + k\log\log k)/2 < (n\log n + n\log\log n)/2 . \tag{4.15}$$

Therefore, from (4.13) and (4.15) we obtain

$$(i\log i + i\log\log i)/2 + (k\log k + k\log\log k)/2 < n\log n + n\log\log n. \tag{4.16}$$

By (4.5), (4.11) and (4.16) we get

$$D_{up}(n) < A_{up}(n). \tag{4.17}$$

Thus the theorem holds.

Corollary 4.7. *If $L_n < 2P_n$, then $D_{up}(n)$ for every prime pair to form L_n is smaller than $A_{up}(n)$.*

Proof. Since the case for every prime pair to form L_n is same as the case for a prime pair to form L_n described by Theorem 4.6. Hence $D_{up}(n)$ for every prime pair to form L_n is smaller than $A_{up}(n)$ if $L_n < 2P_n$. Thus the corollary holds.

Corollary 4.8. *$A_{up}(n)$ can be thought as approximate $D_{up}(n)$ for $n \geq 6$.*

Proof. By Theorem 3.6, equivalently, $D_{up}(n) = A_{up}(n)$ for $L_n = 2P_n$. By Corollary 4.7 $D_{up}(n)$ for every prime pair to form L_n is smaller than $A_{up}(n)$ for $L_n < 2P_n$. Hence $\xi(n)$ will not be larger than $A_{up}(n)$ for $n \geq 6$ and $A_{up}(n)$ can be thought as approximate $D_{up}(n)$ for $n \geq 6$. Thus the corollary holds.

Remark 4.9. By Corollary 4.8, there is no any abnormal event such that $\xi(n)$ is larger than $A_{up}(n)$ for $n \geq 6$, therefore, every $\xi(n)$ is smaller than $A_{up}(n)$ for $n \geq 6$ so that the ratio of $\xi(n)$ to $A_{up}(n) = n\log n + n\log\log n$ is smaller than 1 and is closer and closer to 1 with growth of n as Figure 2 shows. In the figure, since $\xi(n) = L_n/2$ and $A_{up}(n) = B_{up}(n)/2$, $\xi(n)/A_{up}(n) = L_n/B_{up}(n)$, the curve also represents $\xi(n)/A_{up}(n)$. However, it is not the final scope of application for upper bound of P_n to be able to be thought as approximate upper bound of $\xi(n)$, and it needs to be consistent with scope of application for lower bound of P_n to be able to be thought as approximate lower bound of $\xi(n)$, that is, the scope of application is also $n \geq 20542$.

Theorem 4.10. *If $L_n < 2P_n$, then $D_{low}(n)$ for a prime pair to form L_n is smaller than $A_{low}(n)$.*

Proof. Considering lower bound of $2P_n$ to be written as $B_{low}(n)$ and $2P_n = P_n + P_n$, by (2.11) we have

$$A_{low}(n) = B_{low}(n)/2 = n\log n + n\log\log n - n . \tag{4.18}$$

The first case for $L_n < 2P_n$ is that $L_n = P_n + P_i$, $i < n$. In this case we have lower bound of $\xi(n)$,

$$D_{low}(n) = C_{low}(n)/2 = A_{low}(n)/2 + A_{low}(i)/2 = (n\log n + n\log\log n - n)/2 + (i\log i + i\log\log i - i)/2. \tag{4.19}$$

Since $i < n$, we obtain $(\log i - 1)/2 < (\log n - 1)/2$, and get $i(\log i - 1)/2 < n(\log n - 1)/2$, that is,

$$(i\log i - i)/2 < (n\log n - n)/2. \tag{4.20}$$

By $(i\log\log i)/2 < (n\log\log n)/2$, from (4.20) we have

$$(i\log i + i\log\log i - i)/2 < (n\log n + n\log\log n - n)/2. \tag{4.21}$$

Hence from (4.18), (4.19) and (4.21) we obtain

$$D_{low}(n) < A_{low}(n). \tag{4.22}$$

The second case for $L_n < 2P_n$ is that $L_n = P_i + P_k$, $i < n$, $k < n$. In this case we have lower bound of $\xi(n)$,

$$D_{low}(n) = C_{low}(n)/2 = (i\log i + i\log\log i - i)/2 + (k\log k + k\log\log k - k)/2. \quad (4.23)$$

Since $i < n$, we obtain $(\log i - 1)/2 < (\log n - 1)/2$, and get $i(\log i - 1)/2 < n(\log n - 1)/2$, that is,

$$(i\log i - i)/2 < (n\log n - n)/2. \quad (4.24)$$

By $(i\log\log i)/2 < (n\log\log n)/2$, from (4.24) we have

$$(i\log i + i\log\log i - i)/2 < (n\log n + n\log\log n - n)/2. \quad (4.25)$$

Since $k < n$, we obtain $(\log k - 1)/2 < (\log n - 1)/2$, and get $k(\log k - 1)/2 < n(\log n - 1)/2$, that is,

$$(k\log k - k)/2 < (n\log n - n)/2. \quad (4.26)$$

By $(k\log\log k)/2 < (n\log\log n)/2$, from (4.26) we have

$$(k\log k + k\log\log k - k)/2 < (n\log n + n\log\log n - n)/2. \quad (4.27)$$

Hence from (4.18), (4.23), (4.25), (4.27) we obtain

$$D_{low}(n) < A_{low}(n). \quad (4.28)$$

Thus the theorem holds.

Corollary 4.11. *If $L_n < 2P_n$, then $D_{low}(n)$ for every prime pair to form L_n is smaller than $A_{low}(n)$.*

Proof. Since the case for every prime pair to form L_n is same as the case for a prime pair to form L_n described by Theorem 4.10. Hence $D_{low}(n)$ for every prime pair to form L_n is smaller than $A_{low}(n)$ if $L_n < 2P_n$. Thus the corollary holds.

Remark 4.12. Corollary 4.11 means that, generally, $D_{low}(n)$ for every prime pair to form L_n is smaller than $A_{low}(n)$ if $L_n < 2P_n$. Therefore, if lower bound of P_n is thought as approximate lower bound of $\xi(n)$ though lower bound of $\xi(n)$ is not a number but a set, then there must exist some abnormal events such that $\xi(n)$ is smaller than $A_{low}(n)$ to lead to $\xi(n)/A_{low}(n) < 1$ for $n \geq 6$. Obviously, such abnormal events are not conducive to the effectiveness of using $A_{low}(n)$ as approximate $D_{low}(n)$.

For a given $\xi(n)$, if $\xi(n)/A_{low}(n) > 1$, that is, $\xi(n) > A_{low}(n)$, then the ratio $\xi(n)/A_{low}(n)$ is called a normal event for using $A_{low}(n)$ as approximate $D_{low}(n)$, however, if $\xi(n)/A_{low}(n) < 1$, that is, $\xi(n) < A_{low}(n)$, then the ratio $\xi(n)/A_{low}(n)$ is called an abnormal event for this approximation. Taking Goldbach step $L_{15} = L_{16} = L_{17} = 90$ to be equivalent to $\xi(15) = \xi(16) = \xi(17) = 90/2 = 45$ as an example, we discovered that both $\xi(15)/A_{low}(15)$ and $\xi(16)/A_{low}(16)$ are normal events because $\xi(15)/A_{low}(15) = 45/40 = 1.125 > 0$ and $\xi(16)/A_{low}(16) = 45/44 = 1.022 > 0$ but $\xi(17)/A_{low}(17)$ is an abnormal event because $\xi(17)/A_{low}(17) = 45/49 = 0.918 < 0$. This example means some ratios $\xi(n)/A_{low}(n)$ may become abnormal events for $A_{low}(n)$ being thought as approximate $D_{low}(n)$ because of existence of $D_{low}(n) < A_{low}(n)$ for $n \geq 6$ but other ratios $\xi(n)/A_{low}(n)$ may become normal events for $A_{low}(n)$ being thought as approximate $D_{low}(n)$ because of existence of relative error between $\xi(n)$ and P_n being closer and closer to 0 with growth of n. If the latter has a greater advantage than the former, then it is predictable that density of abnormal events will obviously decrease with growth of n until it reaches 0. In fact, after checking every ratio $L_n/B_{low}(n)$ for $2 \leq n \leq 400000000$, equivalently, after checking every ratio $\xi(n)/A_{low}(n)$ for $2 \leq n \leq 400000000$, 5225 abnormal events have been found for $2 \leq n \leq 20541$ and $\xi(20541)/A_{low}(20541)$ is believed to be the last abnormal event such that $\xi(20541)/A_{low}(20541) = 230512/230587.5 = 0.999672$, however, there is no abnormal event for $20542 \leq n \leq 400000000$, that is, all ratios $\xi(n)/A_{low}(n)$ are normal events for $20542 \leq n \leq 400000000$ and the first normal event is $\xi(20542)/A_{low}(20542) = 230941/230600 = 1.001478$ for $20542 \leq n \leq 400000000$. In Figure 3, the blue curve also equivalently represents the ratio $\xi(n)/A_{low}(n)$ for $20542 \leq n \leq 300000000$.

Observation 4.13. *Ratios $\xi(n)/A_{up}(n)$ and $\xi(n)/A_{low}(n)$ at the starting point and the finishing point for some Goldbach steps.*

By Definition 4.1 $\xi(n)$ is equal to half-height of Goldbach step, therefore, it would not change status of Goldbach steps to introduce $\xi(n) = L_n/2$. So, we can give ratios $\xi(n)/A_{up}(n)$ and $\xi(n)/A_{low}(n)$ at starting point and finishing point of some Goldbach steps in which the first two Goldbach steps have width being 1 but other Goldbach steps have width greater than 1. Comparing the data with $P_n/A_{up}(n)$ and $P_n/A_{low}(n)$, we discovered that $P_n/A_{up}(n)$ is larger than $\xi(n)/A_{up}(n)$ and $P_n/A_{low}(n)$ is larger than $\xi(n)/A_{low}(n)$ for every starting or finishing point as Table 5 shows, and there are some fluctuations in data caused by existence of complete Goldbach steps.

Table 5. Comparison between $\xi(n)/A_{up}(n)$ or $\xi(n)/A_{low}(n)$ and $P_n/A_{up}(n)$ or $P_n/A_{low}(n)$

n	P_n	$\xi(n)$	$A_{up}(n)$	$A_{low}(n)$	$\xi(n)/A_{up}(n)$	$\xi(n)/A_{low}(n)$	$P_n/A_{up}(n)$	$P_n/A_{low}(n)$
20542	231299	230941	251142	230600	0.919563	1.001478	0.920988	1.003031
20543	231317	230959	251155	230612	0.919585	1.001502	0.921011	1.003054
28418	330149	329925	357569	329151	0.922687	1.002349	0.923314	1.003030
28442	330431	329925	357898	329456	0.921840	1.001423	0.923254	1.002959
33637	397211	396873	429457	395820	0.924126	1.002659	0.924913	1.003512
33658	397469	396873	429749	396091	0.923499	1.001974	0.924886	1.003478
38043	454451	454132	490840	452797	0.925213	1.002948	0.925863	1.003652
38072	454859	454132	491246	453174	0.924449	1.002113	0.925929	1.003718
43201	521791	521505	563401	520200	0.925637	1.002508	0.926144	1.003058
43231	522191	521505	563825	520594	0.924941	1.001749	0.926157	1.003067
47512	579011	578560	624563	577051	0.926343	1.002615	0.927065	1.003396
47546	579517	578560	625047	577501	0.925626	1.001833	0.927157	1.003490
52642	647593	647073	697895	645253	0.927178	1.002820	0.927923	1.003626
52660	647789	647073	698153	645493	0.926834	1.002446	0.927860	1.003556
57875	717797	717225	773258	715383	0.927535	1.002574	0.928275	1.003373
57919	718433	717225	773894	715975	0.926773	1.001745	0.928334	1.003432
62381	778777	778473	838564	776183	0.928340	1.002949	0.928702	1.003341
62406	779111	778473	838928	776522	0.927937	1.002512	0.928698	1.003334
67279	845261	844935	909950	842671	0.928550	1.002686	0.928908	1.003072
67325	845981	844935	910623	843298	0.927864	1.001941	0.929013	1.003181
73278	927259	926845	997907	924629	0.928788	1.002396	0.929203	1.002843
73308	927727	926845	998348	925040	0.928378	1.001950	0.929261	1.002904
78302	997247	996913	1071979	993677	0.929973	1.003256	0.930285	1.003592
78361	998017	996913	1072851	994490	0.929218	1.002435	0.930247	1.003546
84051	1076869	1076520	1157167	1073116	0.930306	1.003172	0.930608	1.003497
84083	1077413	1076520	1157642	1073559	0.929924	1.002757	0.930695	1.003589
89084	1146911	1146502	1232095	1143011	0.930530	1.003054	0.930862	1.003412
89141	1147591	1146502	1232954	1143804	0.929889	1.002358	0.930772	1.003310
94809	1226549	1225975	1317697	1222888	0.930391	1.002523	0.930827	1.002993
94839	1226867	1225975	1318147	1223308	0.930074	1.002180	0.930751	1.002909
98735	1281821	1281481	1376618	1277883	0.930890	1.002815	0.931137	1.003081
98780	1282423	1281481	1377294	1278514	0.930433	1.002320	0.931117	1.003057
103842	1354069	1353558	1453514	1349672	0.931231	1.002879	0.931583	1.003257
103919	1355279	1353558	1454675	1350756	0.930487	1.002074	0.931671	1.003348
109068	1428041	1427605	1532482	1423414	0.931563	1.002943	0.931848	1.003250
109090	1428419	1427605	1532815	1423725	0.931361	1.002725	0.931892	1.003296
113586	1491773	1491355	1600970	1487384	0.931532	1.002669	0.931793	1.002950
113610	1492097	1491355	1601334	1487724	0.931320	1.002440	0.931783	1.002939
118436	1560893	1560328	1674707	1556271	0.931702	1.002606	0.932039	1.002969
118485	1561597	1560328	1675452	1556967	0.931287	1.002158	0.932044	1.002973
122658	1621657	1621317	1739070	1616412	0.932289	1.003034	0.932485	1.003244
122695	1622209	1621317	1739635	1616940	0.931986	1.002706	0.932499	1.003258
128185	1700627	1700190	1823564	1695379	0.932344	1.002837	0.932584	1.003095
128233	1701307	1700190	1824299	1696066	0.931968	1.002431	0.932581	1.003090

Observation 4.14. *Ratios $\xi(n)/A_{up}(n)$ and $\xi(n)/A_{low}(n)$ at regular watch points.*

Because $\xi(n) = L_n/2$ leads to an equivalent statement for using bounds of $2P_n$ as approximate bounds of L_n for $n \geq 20542$, that is, bounds of P_n can be thought as approximate bounds of $\xi(n)$ for $n \geq 20542$, we can give the ratios $\xi(n)/A_{up}(n)$ and $\xi(n)/A_{low}(n)$ at regular watch points for $1000000 \leq n \leq 100000000$ as Table 6 shows.

Table 6. $\xi(n)/A_{up}(n)$ and $\xi(n)/A_{low}(n)$ at regular watch points for $1000000 \le n \le 100000000$.

n	P_n	$\xi(n)$	$A_{up}(n)$	$A_{low}(n)$	$\xi(n)/A_{up}(n)$	$\xi(n)/A_{low}(n)$	$P_n/A_{up}(n)$	$P_n/A_{low}(n)$
1000000	15485863	15485467	16441302	15441302	0.9418637	1.0028601	0.9418878	1.0028858
2000000	32452843	32451826	34366806	32366806	0.9442782	1.0026267	0.9443078	1.0026581
3000000	49979687	49978615	52849294	49849294	0.9456817	1.0025942	0.9457020	1.0026157
4000000	67867967	67866841	71692876	67692876	0.9466329	1,0025699	0.9466486	1.0025865
5000000	86028121	86026858	90804673	85804673	0.9473835	1.0025894	0.9473975	1.0026041
6000000	104395301	104394193	110130040	104130040	0.9479175	1.0025367	0.9479275	1.0025473
7000000	122949823	122948677	129632901	122632901	0.9484372	1.0025749	0.9484461	1.0025843
8000000	141650939	141650197	149287629	141287629	0.9488408	1.0025661	0.9488457	1.0025714
9000000	160481183	160479891	169075075	160075075	0.9491635	1.0025289	0.9491711	1.0025369
10000000	179424673	179423541	188980382	178980382	0.9494294	1.0024760	0.9494354	1.0024823
11000000	198491317	198490240	208991686	197991686	0.9497518	1.0025180	0.9497569	1.0025234
12000000	217645177	217643512	229099294	217099294	0.9499964	1.0025067	0.9500037	1.0025144
13000000	236887691	236886615	249295137	236295137	0.9502255	1.0025031	0.9502298	1.0025076
14000000	256203161	256201353	269572393	255572393	0.9503990	1.0024609	0.9504057	1.0024680
15000000	275604541	275602651	289925220	274925220	0.9505990	1.0024640	0.9506056	1.0024709
16000000	295075147	295073760	310348556	294348556	0.9507818	1.0024637	0.9507862	1.0024684
17000000	314606869	314604922	330837976	313837976	0.9509335	1.0024437	0.9509394	1.0024499
18000000	334214459	334213486	351389577	333389577	0.9511195	1.0024713	0.9511222	1.0024742
19000000	353868013	353867112	371999890	352999890	0.9512559	1.0024567	0.9512583	1.0024592
20000000	373587883	373586401	392665814	372665814	0.9514105	1.0024702	0.9514143	1.0024742
21000000	393342739	393341488	413384556	392384556	0.9515147	1.0024387	0.9515177	1.0024419
22000000	413158511	413156467	434153594	412153594	0.9516366	1.0024332	0.9516413	1.0024382
23000000	433024223	433022872	454970633	431970633	0.9517600	1.0024359	0.9517630	1.0024390
24000000	452930459	452928808	475833577	451833577	0.9518639	1.0024239	0.9518673	1.0024276
25000000	472882027	472880133	496740510	471740510	0.9519661	1.0024157	0.9519699	1.0024197
26000000	492876847	492875871	517689663	491689663	0.9520682	1.0024125	0.9520700	1.0024144
27000000	512927357	512925948	538679409	511679409	0.9521914	1.0024361	0.9521941	1.0024389
28000000	533000389	532998093	559708239	531708239	0.9522784	1.0024258	0.9522825	1.0024301
29000000	553105243	553104151	580774753	551774753	0.9523557	1.0024093	0.9523575	1.0024112
30000000	573259391	573258175	601877647	571877647	0.9524496	1.0024140	0.9524517	1.0024161
31000000	593441843	593440669	623015704	592015704	0.9525292	1.0024069	0.9525311	1.0024089
32000000	613651349	613650130	644187787	612187787	0.9525951	1.0023887	0.9525969	1.0023907
33000000	633910099	633908845	665392828	632392828	0.9526836	1.0023972	0.9526854	1.0023992
34000000	654188383	654187047	686629828	652629828	0.9527506	1.0023860	0.9527526	1.0023881
35000000	674506081	674504659	707897841	672897841	0.9528276	1.0023879	0.9528296	1.0023900
36000000	694847533	694845816	729195980	693195980	0.9528930	1.0023800	0.9528954	1.0023825
37000000	715225739	715224846	750523406	713523406	0.9529680	1.0023845	0.9529692	1.0023858
38000000	735632791	735631345	771879324	733879324	0.9530393	1.0023873	0.9530411	1.0023893
39000000	756065159	756063873	793262982	754262982	0.9531062	1.0023876	0.9531078	1.0023893
40000000	776531401	776528977	814673668	774673668	0.9531779	1.0023949	0.9531809	1.0023980
41000000	797003413	797001873	836110703	795110703	0.9532252	1.0023784	0.9532271	1.0023804
42000000	817504243	817502985	857573442	815573442	0.9532746	1,0023658	0.9532760	1.0023674
43000000	838041641	838040292	879061273	836061273	0.9533354	1.0023670	0.9533370	1.0023686
44000000	858599503	858598500	900573610	856573610	0.9533906	1.0023639	0.9533918	1.0023651
45000000	879190747	879189298	922109894	877109894	0.9534539	1.0023707	0.9534554	1.0023723
46000000	899809343	899808219	943669591	897669591	0.9535204	1.0023824	0.9535216	1.0023836
47000000	920419813	920418027	965252192	918252192	0.9535518	1.0023586	0.9535537	1.0023605
48000000	941083981	941082468	986857207	938857207	0.9536156	1.0023701	0.9536171	1.0023717
49000000	961748927	961747456	1008484169	959484169	0.9536564	1.0023588	0.9536579	1.0023603
50000000	982451653	982450231	1030132627	980132627	0.9537123	1.0023645	0.9537137	1.0023660
51000000	1003162753	1003161037	1051802152	1000802152	0.9537545	1.0023569	0.9537561	1.0023587
52000000	1023893771	1023892132	1073492328	1021492328	0.9537954	1.0023493	0.9537970	1.0023509
n	P_n	$\xi(n)$	$A_{up}(n)$	$A_{low}(n)$	$\xi(n)/A_{up}(n)$	$\xi(n)/A_{low}(n)$	$P_n/A_{up}(n)$	$P_n/A_{low}(n)$

53000000	1044645379	1044643548	1095202758	1042202758	0.9538357	1.0023419	0.9538374	1.0023437
54000000	1065433423	1065431841	1116933058	1062933058	0.9538905	1.0023508	0.9538919	1.0023523
55000000	1086218491	1086216721	1138682859	1083682859	0.9539238	1.0023381	0.9539253	1.0023398
56000000	1107029837	1107028381	1160451807	1104451807	0.9539632	1.0023328	0.9539645	1.0023342
57000000	1127870669	1127869230	1182239557	1125239557	0.9540107	1.0023369	0.9540119	1.0023382
58000000	1148739811	1148738146	1204045779	1146045779	0.9540651	1.0023492	0.9540665	1.0023507
59000000	1169604791	1169603232	1225870155	1166870155	0.9541004	1.0023422	0.9541016	1.0023435
60000000	1190494759	1190493102	1247712374	1187712374	0.9541406	1.0023412	0.9541419	1.0023426
61000000	1211405357	1211403543	1269572140	1208572140	0.9541825	1.0023427	0.9541839	1.0023442
62000000	1232332807	1232330617	1291449163	1229449163	0.9542230	1.0023436	0.9542247	1.0023454
63000000	1253270831	1253269272	1313343165	1250343165	0.9542587	1.0023402	0.9542599	1.0023414
64000000	1274224957	1274224156	1335253875	1271253875	0.9542935	1.0023364	0.9542941	1.0023371
65000000	1295202449	1295200813	1357181032	1292181032	0.9543316	1.0023369	0.9543328	1.0023382
66000000	1316196199	1316194591	1379124382	1313124382	0.9543697	1.0023380	0.9543709	1.0023393
67000000	1337195521	1337194332	1401083678	1334083678	0.9544000	1.0023316	0.9544008	1.0023325
68000000	1358208601	1358207167	1423058683	1355058683	0.9544280	1.0023235	0.9544290	1.0023245
69000000	1379256017	1379254333	1445049165	1376049165	0.9544687	1.0023292	0.9544699	1.0023304
70000000	1400305337	1400304033	1467054898	1397054898	0.9545000	1.0023257	0.9545009	1.0023266
71000000	1421376527	1421374561	1489075665	1418075665	0.9545348	1.0023263	0.9545361	1.0023277
72000000	1442467307	1442466895	1511111252	1439111252	0.9545735	1.0023317	0.9545751	1.0023334
73000000	1463554999	1463552941	1533161454	1460161454	0.9545980	1.0023226	0.9545993	1.0023240
74000000	1484670157	1484668645	1555226071	1481226071	0.9546320	1.0023241	0.9546330	1.0023251
75000000	1505776939	1505775048	1577304905	1502304905	0.9546505	1.0023098	0.9546517	1.0023111
76000000	1526922013	1526920137	1599397768	1523397768	0.9546844	1.0023121	0.9546855	1.0023134
77000000	1548074321	1548072843	1621504475	1544504475	0.9547138	1.0023103	0.9547148	1.0023113
78000000	1569250357	1569248878	1643624845	1565624845	0.9547488	1.0023147	0.9547497	1,0023156
79000000	1590425971	1590424566	1665758702	1586758702	0.9547748	1.0023102	0.9547757	1.0023111
80000000	1611623773	1611621498	1687905876	1607905876	0.9548053	1.0023108	0.9548066	1.0023122
81000000	1632828047	1632826842	1710066199	1629066199	0.9548325	1.0023084	0.9548332	1.0023092
82000000	1654054489	1654052437	1732239509	1650239509	0.9548635	1.0023105	0.9548647	1.0023117
83000000	1675293211	1675291915	1754425647	1671425647	0.9548947	1.0023131	0.9548955	1.0023139
84000000	1696528903	1696527367	1776624458	1692624458	0.9549161	1.0023058	0.9549170	1.0023067
85000000	1717783147	1717781436	1798835791	1713835791	0.9549406	1.0023022	0.9549416	1.0023032
86000000	1739062363	1739060425	1821059497	1735059497	0.9549717	1.0023059	0.9549728	1.0023070
87000000	1760341421	1760339725	1843295434	1756295434	0.9549959	1.0023027	0.9549968	1.0023037
88000000	1781636611	1781635455	1865543459	1777543459	0.9550222	1.0023020	0.9550228	1.0023027
89000000	1802933611	1802932266	1887803436	1798803436	0.9550423	1.0022953	0.9550430	1.0022960
90000000	1824261409	1824259302	1910075229	1820075229	0.9550719	1.0022988	0.9550730	1.0023000
91000000	1845587707	1845585876	1932358707	1841358707	0.9550948	1.0022956	0.9550958	1.0022966
92000000	1866941107	1866939090	1954653741	1862653741	0.9551252	1.0023006	0.9551262	1.0023017
93000000	1888303061	1888300471	1976960205	1883960205	0.9551535	1.0023037	0.9551548	1.0023051
94000000	1909662901	1909660938	1999277977	1905277977	0.9551752	1.0023004	0.9551762	1.0023014
95000000	1931045213	1931043186	2021606935	1926606935	0.9552020	1.0023026	0.9552031	1.0023036
96000000	1952429173	1952427996	2043946960	1947946960	0.9552243	1.0023003	0.9552249	1.0023009
97000000	1973828641	1973827408	2066297940	1969297940	0.9552482	1.0023000	0.9552488	1.0023006
98000000	1995230813	1995229726	2088659758	1990659758	0.9552679	1.0022957	0.9552684	1.0022962
99000000	2016634091	2016632817	2111032305	2012032305	0.9552827	1.0022865	0.9552833	1.0022871
100000000	2038074743	2038073790	2133415472	2033415472	0.9553103	1.0022908	0.9553107	1.0022913

Because the existence of the relative error between $\xi(n)$ and P_n coming closer and closer to 0 with growth of n already has an overwhelming advantage than the general existence of $D_{low}(n) < A_{low}(n)$ for all prime pairs to form L_n within scope for $20542 \le n \le 400000000$, we can expect that there will not exist abnormal event for $\xi(n)/A_{low}(n)$ for $n > 400000000$ because such overwhelming advantage will be stronger and stronger with growth of n for $n > 400000000$. Therefore, we can conclude that there is no abnormal event for $\xi(n)/A_{low}(n)$ for $n \ge 20542$ and have the following theorem.

Theorem 4.15. *If upper and lower bounds of P_n can be thought as approximate upper and lower bounds of $\xi(n)$ for $n \geq$ 20542, then* $\lim_{n\to\infty}\frac{\xi(n)}{n\log n}=1$.

Proof. By (2.11), upper bound of P_n is

$$A_{up}(n) = n\log n + n\log\log n \text{ for } n \geq 6,$$

and lower bound of P_n is

$$A_{low}(n) = n\log n + n\log\log n - n \text{ for } n \geq 6.$$

Since bounds of P_n are thought as approximate bounds of $\xi(n)$ for $n \geq 20542$, we have

$$D_{up}(n) = n\log n + n\log\log n \text{ for } n \geq 20542, \tag{4.29}$$

$$D_{low}(n) = n\log n + n\log\log n - n \text{ for } n \geq 20542. \tag{4.30}$$

By (4.29) we obtain

$$\lim_{n\to\infty}\frac{\xi(n)}{D_{up}(n)} = \lim_{n\to\infty}\frac{\xi(n)}{n\log n + n\log\log n}, \tag{4.31}$$

by (4.30) we obtain

$$\lim_{n\to\infty}\frac{\xi(n)}{D_{low}(n)} = \lim_{n\to\infty}\frac{\xi(n)}{n\log n + n\log\log n - n}. \tag{4.32}$$

By Corollary 4.4,

$$\lim_{n\to\infty}\frac{\xi(n)}{D_{up}(n)} = 1,$$

we get

$$\lim_{n\to\infty}\frac{\xi(n)}{n\log n + n\log\log n} = 1. \tag{4.33}$$

By Corollary 4.4,

$$\lim_{n\to\infty}\frac{\xi(n)}{D_{low}(n)} = 1,$$

we get

$$\lim_{n\to\infty}\frac{\xi(n)}{n\log n + n\log\log n - n} = 1. \tag{4.34}$$

Since

$$\lim_{n\to\infty}\frac{n\log\log n}{n\log n} = \lim_{n\to\infty}\frac{\log\log n}{\log n} = 0, \tag{4.35}$$

by (4.33) we have

$$\lim_{n\to\infty}\frac{\xi(n)}{n\log n} = 1. \tag{4.36}$$

Since

$$\lim_{n\to\infty}\frac{n\log\log n - n}{n\log n} = \lim_{n\to\infty}\frac{\log\log n}{\log n} - \lim_{n\to\infty}\frac{1}{\log n} = 0, \tag{4.37}$$

by (4.34) we have

$$\lim_{n\to\infty}\frac{\xi(n)}{n\log n} = 1. \tag{4.38}$$

From above results we see that, in fact, (4.36) arising from (4.33) is same as (4.38) arising from (4.34), thus, there is only a limit

$$\lim_{n\to\infty}\frac{\xi(n)}{n\log n}=1, \tag{4.39}$$

and corresponding asymptotic expression is

$$\xi(n)\approx n\log n. \tag{4.40}$$

Thus the theorem holds.

Corollary 4.16. *If upper and lower bounds of P_n can be thought as approximate upper and lower bounds of $\xi(n)$ for $n \geq$ 20542, then Goldbach conjecture is true.*

Proof. Suppose upper and lower bounds of P_n can be thought as approximate upper and lower bounds of $\xi(n)$ for $n \geq$ 20542. Then there is a limit as Theorem 4.15 shows,

$$\lim_{n\to\infty}\frac{\xi(n)}{n\log n}=1,$$

and corresponding asymptotic expression is

$$\xi(n)\approx n\log n.$$

This expression means that, in asymptotic case, if $n\log n$ approaches infinity as n grows without bound then $\xi(n)$ approaches infinity as n grows without bound. Since it is obvious that $n\log n$ approaches infinity as n grows without bound. Hence $\xi(n)$ approaches infinity as n grows without bound. The result means there are infinitely many strong Goldbach numbers so that every even number greater than 2 is the sum of two primes and Glodbach conjecture is true. Thus the corollary holds.

Remark 4.17. It seems to be reasonable that we conclude there is no abnormal event for $\xi(n)/A_{low}(n)$ for $n \geq 20542$, but it requires a rigorous proof as a hypothesis used in Theorem 4.15 and Corollary 4.16. The hypothesis can also be understood as a condition in theory, that is, if it can be proven that there is an integer $k > 0$ such that bounds of P_n can be thought as approximate bounds of $\xi(n)$ for all $n > k$, then Glodbach conjecture is true. In fact, all results based on definition of $\xi(n)$ are symmetrical with prime number theorem.

5. Asymptotic Distribution Law of Goldbach Steps to be Similar to Prime Number Theorem

Theorem 5.1 *If there are infinitely many Goldbach steps, then Goldbach conjecture is true.*

Proof. Let H_i denote height of the i-th Goldbach step and suppose there are infinitely many Goldbach steps. Then $H_i < H_{i+1}$ for $i > 0$. By Definition 3.3 $H_i = L_{n1}$, where $n1$ is n-value at the starting point of the i-th Goldbach step. Since $H_i < H_{i+1}$ for $i > 0$, H_i approaches infinity as i grows without bound. Considering $L_{n1} = H_i$, L_{n1} of the i-th Goldbach step is smaller than L_{n1} of the $(i+1)$-th Goldbach step. Therefore, L_{n1} approaches infinity as i grows without bound. Since i approaches infinity as n grows without bound. Hence L_{n1} approaches infinity as n grows without bound. Considering L_{n1} to be special L_n, that is, $\{L_{n1}\}$ to be a subset of $\{L_n\}$, L_n approaches infinity as n grows without bound, therefore, there are infinitely many strong Goldbach numbers. It means that every even number greater than 2 is the sum of two primes and Goldbach conjecture is true. Thus the theorem holds.

Remark 5.2 In order to prove Goldbach conjecture, by Theorem 5.1 we should study it will become an expected result under what condition that there are infinitely many Goldbach steps.

Let $Q(n)$ denote the number of Goldbach steps formed by the first n largest strong Goldbach numbers L_n, which is the number of $n1$ not greater than n and can be called Goldbach step-counting function. By observation for distribution of Goldbach steps formed by the first 4000000 largest strong Goldbach numbers we assumed that there is the following approximation for $Q(n)$,

$$Q(n)\approx\frac{n}{\log n}\left(1+\frac{1}{\log\log n}\right), \tag{5.1}$$

and got a distribution curve of the ratio

$$\frac{Q(n)}{\frac{n}{\log n}\left(1+\frac{1}{\log\log n}\right)}$$

for $2 \leq n \leq 4000000$ as Figure 9 shows (Zhou & Ao, 2018). The curve is closer and closer to 1 with growth of n. Considering the first term in (5.1) to be just the first term in asymptotic series for $\mathrm{Li}(n)$ and $Q(n)$ to be greater than calculated result , we can improve (5.1) to the following form

$$Q(n) \approx Li(n) + \frac{n}{\log n \log\log n}, \tag{5.2}$$

where

$$Li(n) = \int_2^n \frac{dt}{\log t},$$

and the asymptotic series for Li(n) is that

$$Li(n) \approx \frac{n}{\log n}\sum_{k=0}^{\infty}\frac{k!}{(\log n)^k} = \frac{n}{\log n}\left(1+\frac{1}{\log n}+\frac{2}{(\log n)^2}+\cdots\right). \tag{5.3}$$

Taking the first three terms in (5.3) (Derbyshire, 2004), we obtain

$$Q(n) \approx \frac{n}{\log n}\left(1+\frac{1}{\log n}+\frac{2}{(\log n)^2}+\frac{1}{\log\log n}\right)$$

$$\approx Li(n) + \frac{n}{\log n \log\log n}. \tag{5.4}$$

In Figure 4, the upper curve represents $Q(n)$ and the lower curve represents Li(n) + n/log$n$loglog$n$, where Li(n) ≈ $\frac{n}{\log n}(1+\frac{1}{\log n}+\frac{2}{(\log n)^2})$. In Figure 5, Li($n$) ≈ $\frac{n}{\log n}(1+\frac{1}{\log n}+\frac{2}{(\log n)^2})$, and the ratio curve indicates that there is a tendency such that the ratio is closer and closer to 1 with growth of n.

We can verify (5.4) and get other further results as Table 7 shows. In the table, $\pi(n)$ denotes practical number of primes not greater than n, $Q(n)$ is practical number of Goldbach steps formed by largest strong Goldbach numbers L_n generated by the first n primes, $\frac{Q(n)}{\pi(n)}$ is the ratio of $Q(n)$ to $\pi(n)$, $Li(n) + \frac{n}{\log n \log\log n}$ is calculated number of Goldbach steps in theory, there is also the ratio of $Q(n)$ to $Li(n) + \frac{n}{\log n \log\log n}$. From the table we see $Q(n)$ is greater than $\pi(n)$ but there is a tendency such that the ratio of $Q(n)$ to $\pi(n)$ is closer and closer to 1 with growth of n, and we also see there is another tendency such that the ratio of practical number to calculated number for Goldbach steps is closer and closer to 1 with growth of n. These tendencies make us feel that, in asymptotic case, $Q(n) \approx \pi(n)$ and $Q(n) \approx Li(n) + \frac{n}{\log n \log\log n}$, that is, there are two limits as follows

$$\lim_{n\to\infty}\frac{Q(n)}{\pi(n)} = 1, \tag{5.5}$$

$$\lim_{n\to\infty}\frac{Q(n)}{Li(n)+\frac{n}{\log n \log\log n}} = 1. \tag{5.6}$$

Table 7. Observation and calculation for the number of Goldbach steps

n	$\pi(n)$	$Q(n)$	$\frac{Q(n)}{\pi(n)}$	$Li(n)+\frac{n}{\log n \log\log n}$	$\frac{Q(n)}{Li(n)+\frac{n}{\log n \log\log n}}$
100	25	55	2.2000	43	1.279069
1000	168	277	1.6488	247	1.121457
10000	1229	1868	1.5199	1718	1.087310
100000	9592	13693	1.4275	13125	1.043276
1000000	78498	109565	1.3957	105932	1.034295
10000000	664579	912224	1.3726	886767	1.028707
100000000	5761455	7819295	1.3571	7617525	1.026487

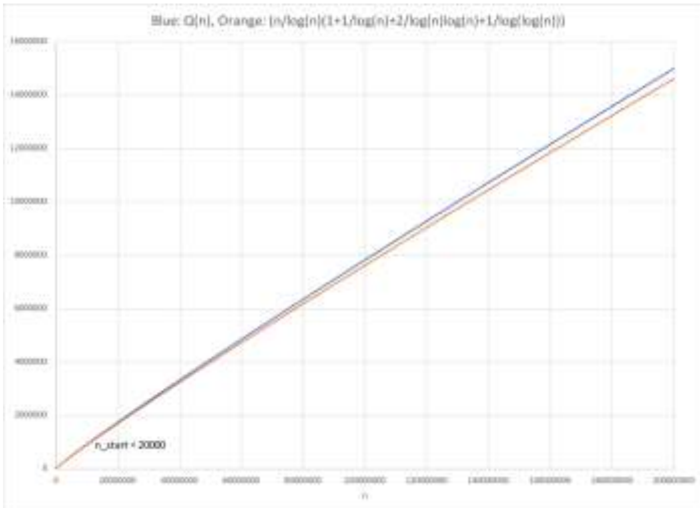

Figure 4. Distribution of Q(n) and distribution of Li(n) + n/lognloglogn for 20000 ≤ n ≤ 200000000

In Table 8, $Li(n)+\frac{n}{\log n \log\log n}$ is written as $Q'(n)$ to express involved ratios and all results are calculated results in theory to discuss possible asymptotic results.

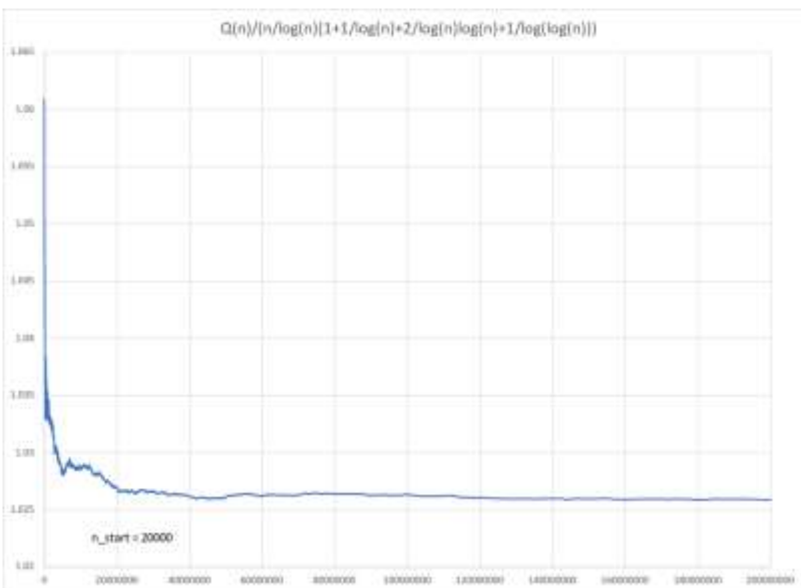

Figure 5. The ratio of $Q(n)$ to Li(n) + n/log$n$loglog$n$ for 20000 ≤ n ≤ 200000000

Table 8. The ratio of $Q'(n) = Li(n) + \frac{n}{\log n \log\log n}$ to $\frac{n}{\log n}$.

n	$\frac{n}{\log n}$	$Q'(n) = Li(n) + \frac{n}{\log n \log\log n}$	$\frac{Q'(n)}{(\frac{n}{\log n})}$	$\frac{Li(n)}{Q'(n)}$	$\frac{(\frac{n}{\log n \log\log n})}{Q'(n)}$
10^2	22	43	1.9545	0.6669	0.3330
10^3	145	247	1.7034	0.6963	0.3036
10^4	1086	1718	1.5819	0.7154	0.2845
10^5	8686	13125	1.5110	0.7291	0.2708
10^6	72382	105932	1.4635	0.7398	0.2601
10^7	620421	886767	1.4293	0.7483	0.2516
10^8	5428681	7617525	1.4032	0.7554	0.2445
10^9	48254942	66722108	1.3827	0.7614	0.2385
10^{10}	434294482	593202832	1.3659	0.7666	0.2333
10^{11}	3948131654	5337479183	1.3519	0.7711	0.2288
10^{12}	36191206825	48496217145	1.3400	0.7751	0.2248
10^{13}	334072678387	444249847719	1.3298	0.7787	0.2212
10^{14}	3102103442166	4097258226412	1.3208	0.7820	0.2179
10^{15}	28952965460217	38009453056172	1.3128	0.7849	0.2150
10^{16}	271434051189532	354438584043290	1.3058	0.7876	0.2123

From Table 8 we see the ratio of $Q'(n)$ to $n/\log n$ is closer and closer to 1 with growth of n, and the proportion of Li(n) in $Q'(n)$ is increasing but the proportion of $n/\log n\log\log n$ in $Q'(n)$ is reducing with growth of n. These tendencies mean it is possible that there are two limits such that

$$\lim_{n\to\infty} \frac{Li(n) + \frac{n}{\log n \log\log n}}{\frac{n}{\log n}} = 1, \tag{5.7}$$

$$\lim_{n\to\infty} \frac{Li(n)}{Li(n) + \frac{n}{\log n \log\log n}} = 1. \tag{5.8}$$

By (5.6) and (5.7) we obtain

$$\lim_{n\to\infty} \frac{Q(n)}{(\frac{n}{\log n})} = 1. \tag{5.9}$$

If the limit (5.9) is proven, then there is an asymptotic expression such that

$$Q(n) \approx \frac{n}{\log n}, \tag{5.10}$$

which is completely similar to asymptotic expression for $\pi(n)$, that is,

$$\pi(n) \approx \frac{n}{\log n}. \tag{5.11}$$

Formulae (5.9) and (5.10) mean that there are infinitely many Goldbach steps since $n/\log n$ approaches infinity as n grows without bound, therefore, by Theorem 5.1 Goldbach conjecture is true.

It is obvious that (5.11) arises from statement of prime number theorem, that is, there is a limit as follows

$$\lim_{n\to\infty}\frac{\pi(n)}{(\frac{n}{\log n})}=1. \tag{5.12}$$

By (5.12) and (5.9) we have (5.5), and (5.5) implies the following asymptotic expression

$$Q(n)\approx\pi(n). \tag{5.13}$$

From (5.13) we see, in asymptotic case, the number of Goldbach steps is about the number of primes. In fact, by (5.2) we have $\lim_{n\to\infty}\frac{Q(n)}{Li(n)+\frac{n}{\log n\log\log n}}=1$ and have the following theorem.

Theorem 5.3. *If* $\lim_{n\to\infty}\frac{Q(n)}{Li(n)+\frac{n}{\log n\log\log n}}=1$, *then* $\lim_{n\to\infty}\frac{Q(n)}{(\frac{n}{\log n})}=1$.

Proof. Using (5.3) to express Li(*n*), that is,

$$Li(n)\approx\frac{n}{\log n}\sum_{k=0}^{\infty}\frac{k!}{(\log n)^k}=\frac{n}{\log n}+\frac{n}{(\log n)^2}+\frac{2n}{(\log n)^3}+\cdots, \tag{5.14}$$

it is clear that, in the expression, the first term approaches higher order infinity than every term among infinitely many terms after the first term as *n* grows without bound. For example, for the second term we have

$$\lim_{n\to\infty}\frac{(\frac{n}{(\log n)^2})}{(\frac{n}{\log n})}=\lim_{n\to\infty}\frac{1}{\log n}=0,$$

and for the third term we have

$$\lim_{n\to\infty}\frac{(\frac{2n}{(\log n)^3})}{(\frac{n}{\log n})}=\lim_{n\to\infty}\frac{2}{(\log n)^2}=0,$$

further, for every $k > 2$ we have

$$\lim_{n\to\infty}\frac{(\frac{k!n}{(\log n)^{k+1}})}{\frac{n}{\log n}}=\lim_{n\to\infty}\frac{k!}{(\log n)^k}=0.$$

On the other hand, for additional term $n/\log n\log\log n$ in condition of the theorem we have

$$\lim_{n\to\infty}\frac{(\frac{n}{\log n\log\log n})}{(\frac{n}{\log n})}=\lim_{n\to\infty}\frac{1}{\log\log n}=0,$$

which means that *n*/log*n* approaches higher order infinity than *n*/log*n*loglog*n* as *n* grows without bound. Based on above results, by assumed $\lim_{n\to\infty}\frac{Q(n)}{Li(n)+\frac{n}{\log n\log\log n}}=1$ we have

$$\lim_{n\to\infty}\frac{Q(n)}{(\frac{n}{\log n})}=1. \tag{5.15}$$

Thus the theorem holds.

Further, (5.10) means that, in asymptotic case, average width of Goldbach steps is about log*n*, which is similar to average gap between primes in asymptotic case. However, in non-asymptotic case, average width of Goldbach steps is as follows

$$\frac{n}{Q(n)}\approx\frac{n}{Li(n)+\frac{n}{\log n\log\log n}}, \tag{5.16}$$

and we can get the following verifications.

Table 9. Observation and calculation for average width of Goldbach steps

n	$\log n$	$Q(n)$	$\frac{n}{Q(n)}$	$Li(n)+\frac{n}{\log n\log\log n}$	$\frac{n}{Li(n)+\frac{n}{\log n\log\log n}}$
100	4.6051	55	1.8181	43	2.3255
1000	6.9077	277	3.6101	247	4.0485
10000	9.2103	1868	5.3533	1718	5.8207
100000	11.5129	13693	7.3030	13125	7.6190
1000000	13.8155	109565	9.1270	105932	9.4400
10000000	16.1180	912224	10.9622	886767	11.2769
100000000	18.4206	7819295	12.7888	7617525	13.1276

In Table 9, log*n* is approximate average gap between primes, *n*/*Q*(*n*) is practical average width of Goldbach steps, and the last term is calculated average width of Goldbach steps by (5.16).

Taking the first three terms of the asymptotic series for Li(*n*), by (5.3) we get approximation for *Q*(*n*) as follows

$$Q(n)\approx\frac{n}{\log n}\left(1+\frac{1}{\log n}+\frac{2}{(\log n)^2}+\frac{1}{\log\log n}\right).$$

Referring to (2.7), we suppose there is a pair of weak bounds of *Q*(*n*), for $n\ge 55$,

$$\frac{n}{\log n+2}\left(1+\frac{1}{\log n}+\frac{2}{(\log n)^2}+\frac{1}{\log\log n}\right)<Q(n)<\frac{n}{\log n-4}\left(1+\frac{1}{\log n}+\frac{2}{(\log n)^2}+\frac{1}{\log\log n}\right). \tag{5.17}$$

We can verify (5.17) in Table 10, where $Q_{up}(n)$ and $Q_{low}(n)$ denote upper and lower bounds of *Q*(*n*).

Table 10. Verification for bounds of Q(n)

n	$Q(n)$	$Q_{up}(n)$	$Q_{low}(n)$	$Q(n)/Q_{up}(n)$	$Q(n)/Q_{low}(n)$
100	55	327	30	0.168195	1.833333
1000	277	587	191	0.471890	1.450261
10000	1868	3023	1410	0.617929	1.324822
100000	13693	20107	11181	0.681006	1.224666
1000000	109565	149120	92531	0.734743	1.184089
10000000	912224	1179400	789222	0.773464	1.155852
100000000	7819295	9727579	6871007	0.803827	1.138012

6. Additional Discussion

Above researches on Goldbach conjecture allow us to see the possibility that if the conjecture is true then there are some results which express compelling similarity between the conjecture and the prime number theorem.

Asymptotic distribution law of primes is expressed by the prime number theorem as

$$\lim_{x\to\infty}\frac{\pi(x)}{\left(\frac{x}{\log x}\right)}=1, \tag{6.1}$$

$$\pi(x)\approx\frac{x}{\log x}. \tag{6.2}$$

However, asymptotic distribution law of Goldbach steps is expressed in this paper as

$$\lim_{n\to\infty}\frac{Q(n)}{\left(\frac{n}{\log n}\right)}=1, \tag{6.3}$$

$$Q(n)\approx\frac{n}{\log n}. \tag{6.4}$$

An equivalent statement of the prime number theorem is that there is a limit

$$\lim_{n\to\infty}\frac{P_n}{n\log n}=1, \tag{6.5}$$

and there is an asymptotic expression

$$P_n\approx n\log n. \tag{6.6}$$

However, asymptotic distribution law of the number of strong Goldbach numbers is expressed in this paper as

$$\lim_{n\to\infty}\frac{\xi(n)}{n\log n}=1, \tag{6.7}$$

and an asymptotic expression

$$\xi(n)\approx n\log n. \tag{6.8}$$

It is very interesting to discover similarity between (6.1) and (6.3), similarity between (6.2) and (6.4), similarity between (6.5) and (6.7), and similarity between (6.6) and (6.8). It can be thought as the result arising from a link between the prime number theorem and Goldbach conjecture, which is just known bounds of P_n. Further, we also found other similarities, that is, in asymptotic case, we see there are some relations as follows

$$\lim_{n\to\infty}\frac{\pi(n)}{\left(\frac{n}{\log n}\right)}=1, \tag{6.9}$$

from (6.3) and (6.9) we obtain

$$\lim_{n\to\infty}\frac{Q(n)}{\pi(n)}=1, \tag{6.10}$$

$$Q(n)\approx\pi(n), \tag{6.11}$$

and from (6.5) and (6.7) we get

$$\lim_{n\to\infty}\frac{\xi(n)}{P_n}=1, \tag{6.12}$$

$$\xi(n)\approx P_n. \tag{6.13}$$

It is a noteworthy result that asymptotic expression (6.13) means value of the n-th prime P_n is about the number of strong Goldbach numbers generated by the prime.

7. Conclusion

Based on proved bounds of P_n and checked numerical evidences, we discovered that if bounds of $2P_n$ are thought as approximate bounds of L_n, or equivalently, bounds of P_n are thought as approximate bounds of $\xi(n) = L_n/2$ then all 5225 abnormal events for $L_n/B_{low}(n)$ or $\xi(n)/A_{low}(n)$, $L_n/B_{low}(n) < 1$ or $\xi(n)/A_{low}(n) < 1$, only appear within the range for $2 \le n \le 20541$ but there are no such abnormal events within the range for $20542 \le n \le 400000000$. Thus we can expect there will not exist abnormal events for $L_n/B_{low}(n)$ or $\xi(n)/A_{low}(n)$ for $n \ge 20542$ because the relative error between L_n and $2P_n$, or equivalently, the relative error between $\xi(n)$ and P_n coming closer and closer to 0 with growth of n already has an overwhelming advantage than the general existence of $C_{low}(n) < B_{low}(n)$ or $D_{low}(n) < A_{low}(n)$ within so large range as $20542 \le n \le 400000000$ and such advantage will be stronger and stronger with growth of n for $n > 400000000$. If the expectation can be rigorously proven, then Goldbach conjecture is true. It also can be understood that, in theory, if there is an integer $k > 0$ such that bounds of $2P_n$ can be thought as approximate bounds of L_n, equivalently, bounds of P_n can be thought as approximate bounds of $\xi(n) = L_n/2$ for all $n > k$, then Goldbach conjecture is true. Using $\xi(n) = L_n/2$, we got a limit and corresponding asymptotic expression to be completely similar to prime number theorem. On the other hand, researches on asymptotic distribution law of Goldbach steps by introducing Li(n) also presented some results to be similar to prime number theosrem. From these results we see that if Goldbach conjecture is true then the conjecture, in asymptotic case, will be symmetrical with prime number theorem, and value of the n-th prime P_n is just about the number of strong Goldbach numbers generated by the prime for $n \ge 20542$.

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