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Complete Contraction of Tensor and Its Application

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Abstract

This article presents a mathematical model for tensor complete contraction utilizing permutation group theory. The model helps in identifying the index class that dictates the complete contraction of even-order tensors. Additionally, the concept of a complete contraction closed loop is introduced along with a method for calculating the number of completely contracted images of any even-order tensor under any λ type permutation. The model is applied to investigate the algebraic structure of global conformal invariants on three-dimensional hypersurfaces. Moreover, it is shown that the generalized Willmore functional is the sole global conformal invariant in three-dimensional hypersurfaces when the difference is a constant multiple.

Keywords: complete contraction, geometric scalar, global conformal invariants, permutation group

1. Introduction

The global conformal invariant is a significant topic of interest in mathematics and plays a crucial role in studying conformal anomalies during quantization in physics. One common method to determine the global conformal invariant involves tensor contraction, an operation that reduces the order of tensors. This study focuses on tensor contraction under the standard basis for clarity. In a tensor space (V_q, g) where g is the metric of the tensor space defined as $g_{ij} = \delta_{ij}$, if f is a q-order tensor in V_q with components $f_{i_1 i_2 \cdots i_q}$, then the operation

$$\sum_{i_a, i_b} \delta_{i_a i_b} f_{i_1 i_2 \cdots i_q} = \sum_{i_a = i_b} f_{i_1 i_2 \cdots i_q}, (1 \le a, b \le q), \tag{1}$$

is referred to as the contraction of the tensor. After contracting a tensor function f, it is observed that the index of its components decreases by two, resulting in a reduction of the tensor by two orders. Repeated contractions lead to obtaining a tensor with lower order, making tensor contraction a widely used technique in various fields such as data processing and machine learning. In 2018, Mondino and Nguyen introduced the concept of complete contraction(Mondino & Nguyen, 2018), involving performing k-times contraction on a tensor of order $2k$. This paper simplifies the concept of complete contraction using permutation group theory and establishes a mathematical model for tensor complete contraction (refer to Proposition 1). By utilizing permutation group theory, we classify tensor complete contraction and provide a calculation method for determining the number of different images of tensor complete contraction under each classification. One of the main theorems presented in this paper is:

Theorem 1 *If $f \in V^0_{2k}$ is a $2k$-order tensor and its component $f_{i_1 i_2 \cdots i_{2k}}$ satisfies the index pairwise non-intersection, then for any partition*

$$\lambda = (u_1, u_2, \cdots u_t), \left(\sum_{i=1}^{t} u_i = k, u_1 \ge u_2 \ge \cdots \ge u_t \right),$$

the number Y of λ-type completely contractive different images of f is:
(1) *When $u_1 = u_2 = \cdots = u_t = 1$,*

$$Y = 1,$$

(2)*When at least one of $\{u_1, u_2, \cdots u_t\}$ is greater than 1,*

$$Y = \prod_{p=1}^{l} C^{u_p}_{k-u_1-\cdots-u_{p-1}} 2^{u_p - 1} (u_p - 1)!, (1 \le l \le t, u_p > 1),$$

where the concept of λ-type complete contraction is shown in Section 2 of this paper, $C_n^m = \frac{n!}{m!(n-m)!}$ is a combination number formula, $(u_1, u_2, \cdots, u_t)$ is a sequence of positive integers.

The concept of geometric scalar $\tilde{P}$ is crucial in geometry. Geometric scalars are typically defined by selecting a finite number of tensors for tensor product, followed by complete contraction using the metric of tensor space (refer to section 2 of this article for the definition of complete contraction). Subsequently, a geometric scalar ?$\tilde{P}$ is derived through a linear combination of complete contractions. Specifically, let f and w denote $(0, 2m)$-type and $(0, 2n)$-type tensors with components $f_{i_1\cdots i_{2m}}$ and $w_{i_1\cdots i_{2n}}$, respectively. Initially, we choose r tensors of f and s tensors of w for tensor product, i.e.

$$\underbrace{f\otimes\cdots\otimes f}_{r}\otimes\underbrace{w\otimes\cdots\otimes w}_{s},$$

where $\underbrace{f\otimes\cdots\otimes w}_{r+s} \in (V_{2mr+2ns}, g)$. The tensor product is completely contracted using the complete contraction model. Finally, a geometric scalar $\tilde{P}$ can be obtained by taking the linear combination of complete contractions.

$$\tilde{P}(g, f^r, w^s) = \sum_{\sigma\in\varphi(2mr+2ns)} a_\sigma C^\sigma ontr(f\otimes\cdots\otimes w), \tag{2}$$

where a_σ is a real number, σ is a permutation in the permutation group $\varphi(2mr + 2ns)$ of order $2mr + 2ns$. Due to the various cases resulting from the selection of non-negative integers r and s, as well as the corresponding geometric scalar $\tilde{P}$, we use $P(g, f, w)$ to represent the sum of all geometric scalars $\tilde{P}$ resulting from the complete contraction of f and w,i.e. $P(g, f, w) = \sum\limits_{r,s} \tilde{P}(g, f^r, w^s)$.

In recent years, scholars have explored the algebraic structure of global conformal invariants on manifolds through the use of geometric scalar. This line of inquiry was initiated by the conjecture(Deser & Schwimmer,1993) of Deser and Schwimmer (1993), physicists who were investigating the conformal anomaly during quantization. Alexakis made (Alexakis, 2006,2007,2009)significant progress towards proving this conjecture in 2006, 2007, and 2009, demonstrating that global conformal invariants on even-dimensional Riemannian manifolds can be expressed as a linear combination of the complete contraction of the Wely curvature tensor and a topological invariant. Building on this work, Mondino and Nguyen further extended the research in 2018 to include submanifolds. By focusing on even-dimensional isometric immersion submanifolds and excluding the mixed contraction of curvature R and the second fundamental form h, they were able to determine the algebraic structure of global conformal invariants. For a detailed explanation of the Wely curvature tensor, curvature R, and the second fundamental form h, please refer to Section 2 of this paper.

A prominent example of global conformal invariants is the Willmore energy, extensively researched by various scholars (Chen, 1973; Kuwert & Schatzle, 2012; Olanipekun, 2022; Ros & Rosenberg, 2010). In 2007, Professor Guo Zhen identified a category of conformal invariants on hypersurfaces (Guo, 2007), subsequently named the generalized Willmore functional,

$$Wr(M) = \begin{cases} \int_{M^m} Q_r^{\frac{m}{r}} d\mu_g, \ r < m \ and \ \ is \ odd, \\ \int_{M^m} |Q_r|^{\frac{m}{r}} d\mu_g, \ r < m \ and \ \ is \text{ even}, \\ \int_{M^m} Q_r d\mu_g, \ r = m, \end{cases} \tag{3}$$

where $Q_r = \sum\limits_{k=0}^{r} (-1)^{k+1} C_r^k \sigma_1^{r-k} \cdot \sigma_k$ is a conformal invariant of hypersurface. The rth mean curvature of hypersurface at a point is defined by σ_k. This study focuses on analyzing the algebraic structure of global conformal invariants on three-dimensional hypersurfaces (M^3, g) through the mixed contraction of curvature R and the second fundamental form h. It is proven that the generalized Willmore functional is the sole global conformal invariant in three-dimensional hypersurfaces when the difference is a constant multiple, i.e.

Theorem 2 *Let $f : (M^3, g) \to \left(\tilde{M}^4, \tilde{g}\right)$ be a smooth isometric immersion, $g = \tilde{g}|_{f(M^3)}$. If $\int_{M^3} P(\tilde{g}, R, R^\perp, h) d\mu_g$ is a global conformal invariant, then, there exists a real number b such that*

$$P(\tilde{g}, R, R^\perp, h) = bQ_3, \tag{4}$$

where $Q_3 = 2\sigma_1^3 - 2\sigma_1\sigma_2 + \sigma_3$, $R, R^\perp$ and h are Riemannian curvature, normal curvature and the second fundamental form respectively (see Section 2 of this article).

The content of this paper is structured as follows: Section 2 provides background information for the entire article. Section 3 primarily discusses the theory of the complete contraction model of tensor and proves Theorem 1. Section 4 presents the application of the complete contraction model and the proof of Theorem 2.

2. Background Material

2.1 Submanifold Geometry

Let $f : (M^m, g) \to \left(\tilde{M}^n, \tilde{g}\right)(2 \le m \le n)$ be a smooth isometric immersion, where g is the induced metric of f and $g = \tilde{g}|_{f(M^m)}$. The admissible Levi-Civita connection of $\tilde{M}^n$ is denoted by ?$\tilde{\nabla}$. Consider $\{e_A\}(1 \le A \le n)$ as the local standard orthogonal basis of ?$\tilde{M}^n$. When restricted on M^m, $\{e_i\}(1 \le i, j \le m)$ is tangent to M^m, and $\{e_\alpha\}(m+1 \le \alpha \le n)$ is normal to M^m. It is known that for all$p \in M^m$, $\tilde{g}$ induces a decomposition of $T_{f(p)}\tilde{M}^n$. i.e.

$$T_{f(p)}\tilde{M}^n = T_{f(p)}f(M^m) \oplus [T_{f(p)}f(M^m)]^\perp, \tag{5}$$

where $[T_{f(p)}f(M^m)]^\perp$ is the orthogonal complement space of $T_{f(p)}f(M^m) \subset T_{f(p)}\tilde{M}^n$. The Tangent projection and normal projection can be defined as follows:

$$\pi_{\mathrm{T}} : T_{f(p)}\tilde{M}^n \to T_{f(p)}f(M^m), \;\; \pi_N = \mathrm{Id} - \pi_T : T_{f(p)}\tilde{M}^n \to [T_{f(p)}f(M^m)]^\perp.$$

By utilizing these projections, we are able to establish the tangent bundle connection as $\nabla_{e_i}e_j = \pi_T(\tilde{\nabla}_{e_i}e_j)$ and the normal bundle connection as $\nabla^\perp_{e_i}e_\alpha = \pi_T(\tilde{\nabla}_{e_i}e_\alpha)$. Additionally, with the connection in place, we can define the curvature operator.

$$R(e_i, e_j)e_k = -(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i})e_k, \tag{6}$$

$$R^\perp(e_i, e_j)e_\alpha = -({\nabla^\perp}_{e_i}{\nabla^\perp}_{e_j} - {\nabla^\perp}_{e_j}{\nabla^\perp}_{e_i})e_\alpha. \tag{7}$$

The expressions for curvature R_{ijkl} and normal curvature $R^\perp_{ij\alpha\beta}$ are given by

$$R_{ijkl} = g(R(e_i, e_j)e_k, e_l), \;\; R^\perp_{ij\alpha\beta} = g(R^\perp(e_i, e_j)e_\alpha, e_\beta).$$

When considering the standard basis with the metric components $g_{ij} = \delta_{ij}$, the Ricci curvature R_{ik} and scalar curvature R can be defined as

$$R_{ik} = \sum_{j=l} R_{ijkl}$$

and

$$R = \sum_{i=k} R_{ik}.$$

. Curvature R_{ijkl} can be decomposed into two parts with zero trace and nonzero trace, i.e.

$$R_{ijkl} = W_{ijkl} + (S_{ik}\delta_{jl} + S_{jl}\delta_{ik} - S_{il}\delta_{jk} - S_{jk}\delta_{il}), \tag{8}$$

where W_{ijkl} is the Weyl curvature tensor and S_{ij} is the Schouten tensor, the expressions are as follows:

$$W_{ijkl} = R_{ijkl} - \frac{1}{m-2}(R_{ik}\delta_{jl} + R_{jl}\delta_{ik} - R_{il}\delta_{jk} - R_{jk}\delta_{il}) + \frac{R}{(m-1)(m-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{9}$$

$$S_{ij} = \frac{1}{m-2}(R_{ij} - \frac{R}{2(m-1)}\delta_{ij}). \tag{10}$$

Finally, we give the second fundamental form h_{ij} of isometric immersion f, which is defined as follows:

$$h_{ij} = h(e_i, e_j) = \pi_N(\tilde{\nabla}_{e_i}e_j) = \tilde{\nabla}_{e_i}e_j - \nabla_{e_i}e_j. \tag{11}$$

Similar to the curvature R_{ijkl}, the second fundamental form h_{ij} can also be orthogonally decomposed into two parts with zero trace and non-zero trace, i.e.

$$h_{ij} = h^0_{ij} + H\delta_{ij}, \tag{12}$$

Here, H is the mean curvature vector and h^0_{ij} is the second fundamental form with zero trace. The Gauss-Ricci equation remains valid, represented as

$$\tilde{R}_{ijkl} = R_{ijkl} + h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}, \;\; \tilde{R}_{ij\alpha\beta} = {R^\perp}_{ij\alpha\beta} + h^\alpha_{ik}h^\beta_{kj} - h^\alpha_{ik}h^\beta_{kj}.$$

When combining these definitions and expressions, under the conformal transformation $\hat{\tilde{g}} = e^{2\theta}\tilde{g}$ of the metric $\tilde{g}$, we obtain.

$$\hat{\tilde{g}}_{ij}^{\ -1} = e^{-2\theta}\tilde{g}_{ij}^{-1}, \quad \hat{S}_{ij} = [S_{ij} - \theta_{i,j} + \theta_i\theta_j + \frac{1}{2}|\nabla\theta|^2 g_{ij}], \tag{13}$$

$$\hat{h}_{ij}=e^{\theta}[h_{ij} - g_{ij}\pi_N(\tilde{\nabla}\theta)], \quad \hat{h}^0_{ij} = e^{\theta}h^0_{ij}, \tag{14}$$

$$\hat{H} = e^{-\theta}[H - \pi_N(\tilde{\nabla}\theta)], \quad \hat{W}_{ijkl}=e^{2\theta}W_{ijkl}, \tag{15}$$

where $\theta_i = \nabla_{e_i}\theta, \ \ \theta_{i,j} = \nabla_{e_j}(\theta_i)$.

2.2 Global Conformal Invariants of Submanifolds

In this part, we mainly introduce several important definitions of global conformal invariants. Let $f : (M^m, g) \to \left(\tilde{M}^n, \tilde{g}\right)$ is a smooth isometric immersion, $g = \tilde{g}|_{f(M^m)}$.

Definition 1 (*Weight of geometric scalar*) *Let t be an arbitrary positive real number, $P(\tilde{g}, R, R^{\perp}, h)$ is a geometric scalar. If under the scaling transformation $\hat{\tilde{g}} = t^2\tilde{g}$, there exists an integer k such that*

$$P(t^2\tilde{g}, R_{t^2\tilde{g}}, R^{\perp}_{t^2\tilde{g}}, h_{t^2\tilde{g}}) = t^k P(\tilde{g}, R_{\tilde{g}}, R^{\perp}_{\tilde{g}}, h_{\tilde{g}}), \tag{16}$$

then $P(\tilde{g}, R_{\tilde{g}}, R^{\perp}_{\tilde{g}}, h_{\tilde{g}})$ is called a geometric scalar with weight k.

Since the induced metric g is transformed into $\hat{g} = t^2 g$ under the scaling transformation $\hat{\tilde{g}} = t^2\tilde{g}$ of the metric $\tilde{g}$, the change of the volume form on the submanifold is $d\mu_{\hat{g}} = t^m d\mu_g$. Therefore, for any geometric scalar $P(\tilde{g}, R, R^{\perp}, h)$ with weight $-m$, the integral $\int_{M^m} P(\tilde{g}, R, R^{\perp}, h)d\mu_g$ is invariant under the metric $\tilde{g}$ transformation for any compact orientable submanifold $f(M^m)$. Next, we consider a more general case(the constant scaling condition of the metric is relaxed to the general conformal transformation), then we have the following definition :

Definition 2 (*Global conformal invariants of submanifolds*) *Let $P(\tilde{g}, R, R^{\perp}, h)$ be a geometric scalar with weight $-m$. Consider the conformal transformation $\hat{\tilde{g}} = e^{2\theta}\tilde{g}$ under the metric $\tilde{g}$ of $\tilde{M}^n$. If $P(\tilde{g}, R, R^{\perp}, h)$ for all compact orientable immersed submanifolds M^m in $\tilde{M}^n$ has*

$$\int_{M^m} P(\hat{\tilde{g}}, \hat{R}, \hat{R}^{\perp}, \hat{h})d\mu_{\hat{g}} = \int_{M^m} P(\tilde{g}, R, R^{\perp}, h)d\mu_g.$$

Then $\int_{M^m} P(\tilde{g}, R, R^{\perp}, h)d\mu_g$ is a global conformal invariant of m-dimensional immersed submanifold $f(M^m)$ (Also called $P(\tilde{g}, R, R^{\perp}, h)$ generates a global conformal invariant).

For calculating the algebraic structure of global conformal invariants, Alexakis gives a powerful tool.

Definition 3 (*Hyperdivergence operator $I(\theta)$*) *On m-dimensional submanifolds M^m, where $P(g, R, R^{\perp}, h)$ is a geometric scalar with weight $-m$. If $P(\tilde{g}, R, R^{\perp}, h)$ generates a global conformal invariant, then the operator $I_{\tilde{g},R,R^{\perp},h}(\theta)$ can be defined as*

$$I_{\tilde{g},R,R^{\perp},h}(\theta)=e^{m\theta}P(\hat{\tilde{g}}, \hat{R}, \hat{R}^{\perp}, \hat{h}) - P(\tilde{g}, R, R^{\perp}, h).$$

Alecakis named this formula the superdivergence formula, as the operator ($I(\theta)$) satisfies the equation $\int_{M^m} I_{\tilde{g},R,R^{\perp},h}(\theta)d\mu_g=0$ for any smooth function $\theta \in C^{\infty}{}_{(\tilde{M})}$.

2.3 Complete Contraction of Tensor

Suppose that $f \in V^0_{2k}$, $f_{i_1 i_2 \cdots i_{2k}}$ is a component of the tensor f for any even number $2k$. Then the complete contraction of tensor f is defined as follows:

Definition 4 (*Complete Contractions of* $(0, 2k)$*-type tensor*) *Any even number $2k$, $f_{i_1 i_2 \cdots i_{2k}}$is a component of the tensor f. We consider that if a set of index pairs group $\{(a_1, b_1), \cdots, (a_k, b_k)\}$ satisfies the property 1) − 3) as follows:*

1) $\{(a_w, b_w)\} \cap \{(a_l, b_l)\} \neq \emptyset, \ for \ \forall \ w, l = 1, 2, \cdots, \ k \ and \ w \neq l$;

2) $a_l \neq b_l, \ for \ \forall \ l = 1, 2, \cdots, k$;

3) $\bigcup_{w=1}^{k} \{(a_w, b_w)\} = \{(i_1, i_2), \cdots, (i_{2k-1}, i_{2k})\}$.

Then the complete contraction of the tensor f can be defined as

$$contr(f) = \sum_{i_1,\cdots,i_{2k}} g^{a_1 b_1} \cdot g^{a_2 b_2} \cdots g^{a_k b_k} \cdot f_{i_1 i_2 \cdots i_{2k}}, \tag{17}$$

where g is a metric on an $2k$-dimensional linear space (V_{2k}, g), $g^{ij} = \frac{G^*_{ij}}{G}$(here, G = det$(g_{ij})$ and G^*_{ij} represents the i row and the j column element of the adjoint matrix of matrix G), which satisfies

$$\sum_{j=1}^{2k} g^{ij} \cdot g_{jl} = \delta^i_l = \begin{cases} 1\ ,\ for\ i = l\ ; \\ 0\ ,\ for\ i \neq l\ . \end{cases} \tag{18}$$

In order to complete the proof of the main theorem, we will give the following definition :

Definition 5 (*r-order contraction closed loop*) *For a given initial index pair group* $A = \{(i_1, i_2), (i_3, i_4), \cdots, (i_{2k-1}, i_{2k})\}$, *we select r index pairs to form a local index pair group* $A_r = \{(a_1, a_2), (a_3, a_4), \cdots, (a_{2r-1}, a_{2r})\}$. *Subsequently, r indexes are chosen from* A_r *to create an ordered array* $\{b_1, b_2, \cdots, b_r\}$, *where* b_i *corresponds to the ith index pair in* A_r. *By performing r-order cyclic permutations on this array, we obtain a new ordered array* $\{c_1, c_2, \cdots c_r\}$. *Replacing* b_i *in* A_r *with* c_i *results in a new local index pair group* $\tilde{A}_r$, *which we refer to as an r-order closed loop of the initial index pair group A.*

From this definition, we can easily find that the essence of r-order contraction closed loop is that r-index pairs in the initial index pair group have changed, while the rest index pairs have not changed.

By the theory of permutation groups (Jacobson, 2012), we know that $\forall \sigma \in \varphi(k)$ can always be expressed as the product of unconnected cyclic permutations (identity permutations also need to be expressed), i.e.

$$\sigma = (a_1 a_2 \cdots a_{u_1})(b_1 b_2 \cdots b_{u_2}) \cdots (c_1 c_2 \cdots c_{u_t}),$$

Where $u_1 \geq u_2 \geq \cdots \geq u_t$, $\sum_{i=1}^{t} u_i = k$, so σ uniquely determines a partition $\lambda = (u_1, u_2, \cdots, u_t)$ of k, $\{u_1, u_2, \cdots u_t\}$ is a sequence of positive integers. It is evident that each positive integer corresponds to a contraction-closed loop of a specific order, denoted as λ being the type of σ. This, in conjunction with the definition of a contraction-closed loop of order r, allows us to decompose the new pair group obtained by A through this permutation into a collection of contraction-closed loop combinations.

Remark 1 *The type of each permutation is unique. However, for a given type, the corresponding permutation may not be unique. This means that a partition determines a class of permutations that share the same type. The complete contraction of the tensor f under this type of permutation is called a* λ*-type complete contraction.*

3. Classification of Complete Contraction of Tensors

Utilizing permutation group theory, we develop a mathematical model for the complete contraction of tensors. Subsequently, we categorize the complete contraction based on this model. Finally, we provide the proof of Theorem 1. The complete contraction of the tensor f is uniquely determined by the index pair group

$$\{(a_1, b_1), (a_2, b_2), \cdots, (a_k, b_k)\},$$

that satisfies properties 1)to3). Assuming the initial index pairs as $A = \{(i_1, i_2), (i_3, i_4), \cdots, (i_{2k-1}, i_{2k})\}$, we can apply the theory of permutation groups to derive:

$$\{(a_1, b_1), \cdots, (a_k, b_k)\} = \{(i_{\sigma(1)}, i_{\sigma(2)}), \cdots, (i_{\sigma(2k-1)}, i_{\sigma(2k)})\},\tag{19}$$

Where $\sigma \in \varphi(2k)$, $\varphi(2k)$ is a $2k$-order permutation group, so we can establish the following complete contraction mathematical model :

Proposition 1 (*Complete contraction model*) *If the tensor f satisfies the conditions mentioned above, then the complete contraction model of f is given by*

$$contr(f) = C^{\sigma}ontr(f) = \sum_{i_1, \cdots, i_{2k}} \delta_{i_{\sigma(1)} i_{\sigma(2)}} \cdots \delta_{i_{\sigma(2k-1)} i_{\sigma(2k)}} f_{i_1 i_2 \cdots i_{2k}},\tag{20}$$

where $contr(f)$ *represents the complete contraction of the tensor f and* $C^{\sigma}ontr(f)$ *represents the complete contraction of f with respect to the permutation* σ.

Analyzing the expression of the complete contraction model allows us to understand that:

$$\delta_{ij} = \delta_{ji},\ \ \delta_{ij}\delta_{kl} = \delta_{kl}\delta_{ij}(1 \leq i, j, k, l \leq 2k),$$

i.e., The same index pair can exchange two indexes inside, and different index pairs can exchange as a whole without affecting the result of complete contraction. Therefore, we can define the equivalence relation "$\sim$" between two permutations $\sigma_1, \sigma_2 \in \varphi(2k)$ as: $\sigma_1 \sim \sigma_2$ if and only if $C^{\sigma_1}ontr(f) = C^{\sigma_2}ontr(f)$. Let σ be a permutation whose equivalence class is $[\sigma]$, and the group of all equivalence classes is $\varphi(2k)/\sim$. It can be easily derived that the number of elements in the group $\varphi(2k)/\sim$ is $\frac{(2k)!}{k!2^k}$. Therefore, we can conclude the following corollary:

Corollary 1 *If f is a $2k$-order tensor and the component $f_{i_1 i_2 \cdots i_{2k}}$ satisfies that the indices are pairwise non-commutable, then the number of different images of f completely contracted is: $\frac{(2k)!}{k!2^k}$.*

Before we begin to prove Theorem 1, we need to prove the following lemmas.

Lemma 1 *Let f be a tensor of order $2k$, where the component $f_{i_1 i_2 \cdots i_{2k}}$ satisfies pairwise non-commutativity of indices. The complete contraction of f is determined by only k indices $(a_1, a_2, \cdots, a_k)$, where a_i, where a_i is the ith initial index pair. Specifically, if the component $f_{i_1 i_2 \cdots i_{2k}}$ satisfies that the index $\{i_{2l-1}\}$ is commutative with $\{i_{2l}\}(1 \leq l \leq k)$, then contr$(f)$ is only determined by all the odd or even indices.*

Proof. Let the components of the tensor f be $f_{i_1 i_2 \cdots i_{2k}}$,, with the initial index pair group denoted as

$$A = \{(i_1, i_2), (i_3, i_4), \cdots, (i_{2k-1}, i_{2k})\}.$$

Assuming a permutation of the initial even pair group by $2k$ orders results in a new even pair group denoted as $\{(a_1, b_1), \cdots, (a_k, b_k)\}$. Due to interior of the same index pair is commutative and the two different index pairs are commutative as a whole, the pair group can be rearranged to obtain $\{(c_1, d_1), \cdots, (c_k, d_k)\}$,, where $\{(c_1, d_1), \cdots, (c_k, d_k)\}$ has at least one index corresponding to the position of the ith pair in the initial pair group $(1 \leqslant i \leqslant k)$. In other words, there is at least one pair of $i_{2l-1} = c_{2l-1}$ and $i_{2l} = c_{2l}$ in the index pair (i_{2l-1}, i_{2l}) and (c_{2l-1}, c_{2l-1}). Representing the pairs in the form of an ordered array, we have

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_{2k} \\ c_1 & c_2 & \cdots & c_{2k} \end{pmatrix}.$$

It is observed that despite a $2k$-order permutation of the initial pair group, only one index in each pair is altered, indicating a k-order permutation of k indexes. When the component $f_{i_1 i_2 \cdots i_{2k}}$ of f satisfies that the indexes $\{i_{2l-1}\}$ and $\{i_{2l}\}$ are commutative, *contr*(f) only relies on all odd indexes or even indexes. This lemma is proven. □

Lemma 2 *For any $2k$-order tensor f, the component is $f_{a_1 a_2 \cdots a_{2k}}$. Then, under the standard basis, the complete contraction expression of the tensor f is :*

$$\text{contr}(f) = \sum_{a_1, \cdots, a_k} f_{\Delta_1 \Delta_2 \cdots \Delta_k}.$$

In particular, if the component $f_{i_1 i_2 \cdots i_{2k}}$ of the tensor f satisfies that the index $\{i_{2l-1}\}$ and $\{i_{2l}\}$ are commutative, the complete contraction expression of the tensor f is

$$contr(f) = \sum_{i_2, i_4, \cdots, i_{2k}} f_{i_{\sigma(2)} i_2 i_{\sigma(4)} i_4 \cdots i_{\sigma(2k)} i_{2k}},$$

or

$$contr(f) = \sum_{i_1, i_3, \cdots i_{2k-1}} f_{i_1 i_{\sigma(1)} i_3 i_{\sigma(3)} \cdots i_{2k-1} i_{\sigma 2k-1}},$$

where the meaning of the symbol $\Delta_i (1 \leq i \leq k)$ is shown in eq(21).

Proof. The contraction of a $2k$-order tensor f is solely dependent on k indexes, as proven by Lemma 1. In the context of the standard basis, we introduce the notation

$$\Delta_i = \begin{cases} a_i a_{\sigma(i)}, & a_i \text{ is odd index} \\ a_{\sigma(i)} a_i, & a_i \text{ is even index} \end{cases}, \tag{21}$$

where $(1 \leq i \leq k)$ (equation 21).The proof is obtained by combining the eq(20), lemma 1 and the eq(21). □

Next, we will categorize the complete contraction of tensor f. Utilizing Lemma 1 and Definition 5, it is established that the complete contraction of a $2k$-order tensor f is determined by k indices. Additionally, for any partition λ of k, there exists at least one λ-type permutation, indicating that a partition determines a class of permutations with the same type. This leads to the following lemma.

Lemma 3 *Let f be a 2k-order tensor, and the component of f is $f_{i_1 i_2 \cdots i_{2k}}$. Then for any partition of k,*

$$\lambda = (u_1, u_2, \cdots u_t), \left(\sum_{i=1}^{t} u_i = k, u_1 \geq u_2 \geq \cdots \geq u_t \right),$$

a class of λ-type complete contractions is determined.

Combining Lemma 3 with the definition of the complete contraction model, it is evident that if the two permutations are of different types, the complete contraction of the tensor under these permutations will also differ. This implies that distinct partitions correspond to different permutations. Hence, the complete contraction of f under these permutations will vary. Additionally, for any permutation σ in $\varphi(k)$, a unique partition is determined, although the same type can correspond to multiple permutations. Consequently, all partitions of k encompass all permutations in $\varphi(k)$. Therefore, it is justifiable to employ partitions of k for classifying the complete contraction of f. By leveraging Lemma 3, it becomes apparent that the classification of the complete contraction of f is based on the number of distinct partitions of k.

We know from the above that different types of complete contractions must be different. It's natural to ask if two permutations of the same type must belong to the same equivalence class? The following lemma will answer this question. Let f be a tensor of order $2k$, and its component $f_{i_1 i_2 \cdots i_{2k}}$ be pairwise noncommutative. The initial index pairs are :

$$A = \{(i_1, i_2), (i_3, i_4), \cdots, (i_{2k-1}, i_{2k})\}.$$

From Lemma 1, it is possible to assume that the k indices determining the complete contraction of f are $N_1 = \{a_1, a_2, \cdots a_k\}$, and the remaining k indices are $N_2 = \{b_1, b_2, \cdots b_k\}$, where $a_i, b_i (1 \leq i \leq k)$ belong to the i-th index pair in A.

Lemma 4 *Let f be a tensor of order $2k$ satisfying the above conditions, then the complete contraction of f under the index set N_1 and N_2 is the same.*

Proof. The complete contraction model demonstrates that

$$\delta_{ij} = \delta_{ji}, \delta_{ij}\delta_{kl} = \delta_{kl}\delta_{ij} (1 \leq i, j, k, l \leq 2k).$$

This means that indexes within the same pair can be interchanged, and different pairs of indexes can be exchanged as a whole without altering the outcome of the complete contraction. Consequently, for any index set $N_1 = \{a_1, a_2, \cdots a_k\}$ and the remaining index set $N_2 = \{b_1, b_2, \cdots b_k\}$, a_i and b_i are commutative, making N_1 and N_2 equivalent. Hence, the complete contraction of f under the index set N_1 and N_2 remains the same. □

Based on this lemma and the proof process, we have established that when two permutations of the same type $\sigma_1 \in \varphi_1(k), \sigma_2 \in \varphi_2(k)$, where the elements in the permutation group $\varphi_1(k)$ belong to the index set N_1, and the elements in$\varphi_2(k)$ belong to the index set N_2, then σ_1 and σ_2 are equivalent. This means that the complete contraction of f under permutations σ_1 and σ_2 is identical. Next, we will proceed to prove Theorem 1.

Proof. For any partition λ of k, a class of λ-type complete contractions of f is determined. Under this partition, the corresponding permutations can be expressed as the product of unconnected cyclic permutations, i.e.,

$$\sigma = (a_1 a_2 \cdots a_{u_1})(b_1 b_2 \cdots b_{u_2}) \cdots (c_1 c_2 \cdots c_{u_t}),$$

and each u_i-order cyclic permutation corresponds to a u_i-order contraction closed loop. For each initial index pair A, it is posit to set the new index pair group obtained after the replacement σ effect as $\tilde{A}$. Combined with the definition of complete contraction closed-loop, $\tilde{A}$ can be expressed as a combination of multiple contraction closed-loop $\tilde{A}_{u_i}$, i.e.

$$\tilde{A} = (\tilde{A}_{u_1}, \tilde{A}_{u_2}, \cdots, \tilde{A}_{u_t}),$$

so the number of λ-type complete contraction different images of tensor f is the number of different combinations of $\tilde{A}$, i.e. to calculate the number of different cases of $\tilde{A}$, we need to calculate the number of closed-loop contraction $\tilde{A}_{u_i}$ of order u_i, and then multiply them. Let B_{u_i} be the number of contraction closed-loops of order u_i, and then we will discuss the classification. (1)When $u_1 = u_2 = \cdots = u_t = 1, \tilde{A} = A$, then $Y = 1$. (2)When at least one of $\{u_1, u_2, \cdots u_t\}$ is greater than 1. For the 1-order contraction closed loop, we can easily know that $B_1 = 1$. Next, we consider the elements greater than 1. Let the elements greater than 1 be $\{u_1, u_2, \cdots u_l\} (1 \leq l \leq t)$. From the definition of complete contraction closed loop, we first need to select u_1 pairs from k index pairs, and there are a total of $C_k^{u_1}$ cases. Then, we need to select one index from these u_1 pairs to form an index set N_1, and the remaining index set is N_2.Combined with Lemma 4, we know that N_1 and

N_2 are equivalent in the sense of complete contraction. Therefore, the selection of indicators has a total of 2^{u_1-1} different cases. Finally, we have $(u_1-1)!$ different cases for u_1-order cyclic permutation of N_1.Therefore,

$$B_{u_1} = C_k^{u_1} 2^{u_1-1}(u_1-1)!.$$

In the same way, we can get

$$B_{u_j} = C_{k-u_1-\cdots-u_{j-1}}^{u_j} 2^{u_j-1}(u_j-1)!, 1 < j \le l.$$

In summary,

$$Y = B_{u_1} B_{u_2} \cdots B_{u_l} \cdot 1^{t-l} = \prod_{p=1}^{l} C_{k-u_1-\cdots-u_{p-1}}^{u_p} 2^{u_p-1}(u_p-1)!, 1 \le p \le l.$$

□

Finally, we give the following corollary.

Corollary 1 *If f is a $2k$-order tensor and its component $f_{i_1 i_2 \cdots i_{2k}}$ satisfies that the index $\{i_{2l-1}\}$ is commutative with $\{i_{2l}\}$, then for any partition of k, $\lambda = (u_1, u_2, \cdots, u_t)$, $\left(\sum_{i=1}^{t} u_i = k, u_1 \ge u_2 \ge \cdots \ge u_t\right)$, the number Y of λ-type completely contractive images of f is :*

(1) *When $k \le 2$ or $u_1 = u_2 = \cdots = u_t = 1$:*

$$Y = 1,$$

(2) *When $k > 2$ and $\{u_1, u_2, \cdots u_t\}$ has at least one greater than 1:*

$$Y = \prod_{i=1}^{l} C_{k-u_1-\cdots-u_{i-1}}^{u_i} \frac{(u_i-1)!}{2}, (u_l > 1, t > l \ge 1).$$

Proof. By combining the definitions of complete contraction closed-loop and lemma 1-4, it is evident that the complete contraction of f is dependent solely on k odd or k even indices. This implies that when $k \le 2$, there exists only one case of the 1st-order contraction closed-loop and the 2nd-order contraction closed-loop. Furthermore, in conjunction with theorem 1, it can be deduced that $Y = 1$. On the other hand, when $k > 2$ and the set $\{u_1, u_2, \cdots u_t\}$ contains at least one element greater than 1, the theorem can be proven by combining Theorem 1 and the commutativity of the f-component index. □

4. Application of Complete Contraction Model

In this section, we mainly give the application of the complete contraction model and the proof of the second main theorem of this paper.

The idea of how to construct geometric scalars has been explained in the previous article. The next problem is that we will select tensors from geometric objects immersed in submanifolds for tensor product, and then construct geometric scalars. Let $f : (M^3, g) \to \left(\tilde{M}^4, \tilde{g}\right)$ be smooth isometric immersion, under the standard basis, $g = \tilde{g}|_{f(M^3)} = \delta_{ij}$. The basic concepts of Riemannian manifolds and submanifolds have been described in the background material, and are not repeated here. By equation(8) we have

$$R_{ijkl} = \mathrm{W}_{ijkl} + (S_{ik}\delta_{jl} + S_{jl}\delta_{ik} - S_{il}\delta_{jk} - S_{jk}\delta_{il}).$$

On the three-dimensional submanifold, the Weyl curvature tensor disappears, so the decomposition formula can be written as:

$$R_{ijkl} = S_{ik}\delta_{jl} + S_{jl}\delta_{ik} - S_{il}\delta_{jk} - S_{jk}\delta_{il}. \tag{22}$$

Among the geometric objects of submanifolds, the tensors are:

$$g_{ij}, \ R_{ijkl}, \ h_{ij}, \ R^{\perp}_{\alpha\beta ij}.$$

The quantity $C^{\sigma}ontr(f \otimes g)$ is only different from $C^{\sigma}ontr(f)$ by a constant multiple. Therefore, the metric g_{ij} does not need to take into account the selection for tensor product. On the hypersurface, $R^{\perp}_{\alpha\beta ij}$ disappears, eliminating the need for consideration. To summarize, we only need to focus on how R_{ijkl} and h_{ij} together form the geometric scalar. This leads to the following propositions.

Proposition 2 *Let* $f : (M^3, g) \to \left(\tilde{M}^4, \tilde{g}\right)$ *be a smooth isometric immersion,* $g = \tilde{g}|_{f(M^3)}$*. If the geometric scalar* $P(\tilde{g}, R, R^{\perp}, h)$ *produces a global conformal invariant, then*

$$P(\tilde{g}, R, R^{\perp}, h) = P(\tilde{g}, R, h) = \tilde{P}(g, (h^0)^3), \tag{23}$$

where $\tilde{P}(g, (h^0)^3)$ *is a pointwise conformal invariant and defined by eq(2).*

Proof. Since the dimension of the hypersurface is 3, $P(\tilde{g}, R, R^{\perp}, h)$ generates a global conformal invariant, so the weight of $P(\tilde{g}, R, R^{\perp}, h)$ can only be -3. According to the definition 1, the weights of $g^{-1} \otimes h$ and $g^{-1} \otimes g^{-1} \otimes R$ are -1 and -2 respectively, and then the geometric scalar $P(\tilde{g}, R, h)$ with a weight of -3 has only one decomposition :

$$\begin{aligned} P(\tilde{g}, R, h) =& \tilde{P}(g, h^3) + \tilde{P}(g, R, h) \\ =& \sum_{\sigma_1 \in \varphi(6)} a_{\sigma_1} C^{\sigma_1} ontr(h \otimes h \otimes h) + \sum_{\sigma_2 \in \varphi(6)} b_{\sigma_2} C^{\sigma_2} ontr(h \otimes R), \end{aligned} \tag{24}$$

where a_{σ_1} and b_{σ_2} are real numbers. By the equation(12), equation(22) and direct calculation we have:

$$\tilde{P}(g, h^3) = \tilde{P}(g, (h^0)^3) + a'_1 \left|h^0\right|^2 \cdot H + a'_2 H^3, \tag{25}$$

where a'_1 and a'_2 are real numbers. Combining Definition 1 we have

$$\tilde{P}(g, h, R) = b'_1 \left\langle h^0, S \right\rangle_g + b'_2 HtrS,$$

where b'_1 and b'_2 are real numbers. Thus, we can decompose $P(\tilde{g}, R, h)$ as follows:

$$P(\tilde{g}, R, h) = \tilde{P}(g, (h^0)^3) + a'_1 \left|h^0\right|^2 \cdot H + a'_2 H^3 + b'_1 \left\langle h^0, S \right\rangle_g + b'_2 HtrS. \tag{26}$$

Let

$$G(\tilde{g}, h^0, H, S) = P(\tilde{g}, R, h) - \tilde{P}(g, (h^0)^3) = a'_1 \left|h^0\right|^2 \cdot H + a'_2 H^3 + b'_1 \left\langle h^0, S \right\rangle_g + b'_2 HtrS.$$

Since $P(\tilde{g}, R, , h)$ produces a global conformal invariant, $\tilde{P}(g, (h^0)^3)$ is a pointwise conformal invariant, then $G(g, h^0, H, S)$ also produces a global conformal invariant. Taking into account definition 2 and definition 3, we can define the hyperdivergence formula of $G(\tilde{g}, h^0, H, S)$ as follows:

$$I_{(\tilde{g}, h^0, H, S)}(\theta) = I_{(g, h^0, H, S)} = (\theta) e^{3\theta} G(\hat{g}, \hat{h}^0, \hat{H}, \hat{S}) - G(g, h^0, H, S), \tag{27}$$

$$\int_{M^m} I_{(g, h^0, H, S)}(\theta) d\mu_g = 0.$$

Under the conformal transformation $\hat{\tilde{g}} = e^{2\theta}\tilde{g}$ of the metric $\tilde{g}$. Then, By eq(13), eq(14), eq(15) and direct calculation we have:

$$\begin{aligned} I_{(g, h^0, H, S)}(\theta) =& -[a'_1 \left|h^0\right|_g^2 \pi_N(\tilde{\nabla}\theta) + a'_2 H^2 \pi_N(\tilde{\nabla}\theta) + b'_1 \left\langle h^0_{ij}, \theta_{k,l} \right\rangle_g + b'_2 H\Delta\theta + b'_2 \pi_N(\tilde{\nabla}\theta) trS] \\ &+ \{a'_2 H[\pi_N(\tilde{\nabla}\theta)]^2 + b'_1 \left\langle h^0_{ij}, \theta_k\theta_l \right\rangle_g - b'_2 \frac{n-2}{2} H \cdot \left|\tilde{\nabla}\theta\right|^2 + b'_2 \Delta\theta \pi_N(\tilde{\nabla}\theta)\} \\ &+ \{-a'_2 H[\pi_N(\tilde{\nabla}\theta)]^3 + b'_2 \frac{n-2}{2} |\nabla\theta|^2 \cdot \pi_N(\tilde{\nabla}\theta)\}. \end{aligned} \tag{28}$$

For any real number t, $t\theta \in C^{\infty}_{\tilde{M}}$. Take both sides of the above integral and take $1, 2, 3$ derivatives at $t = 0$ to obtain :

$$\int_{M^m} [a'_1 \left|h^0\right|_g^2 \pi_N(\tilde{\nabla}\theta) + a'_2 H^2 \pi_N(\tilde{\nabla}\theta) + b'_1 \left\langle h^0_{ij}, \theta_{k,l} \right\rangle_g + b'_2 H\Delta\theta + b'_2 \pi_N(\tilde{\nabla}\theta) trS] d\mu_g = 0,$$

$$\int_{M^m} \{a'_2 H[\pi_N(\tilde{\nabla}\theta)]^2 + b'_1 \left\langle h^0_{ij}, \theta_k\theta_l \right\rangle_g - b'_2 \frac{n-2}{2} H \cdot \left|\tilde{\nabla}\theta\right|^2 + b'_2 \Delta\theta \pi_N(\tilde{\nabla}\theta)\} d\mu_g = 0,$$

$$\int_{M^m} \{-a'_2 H[\pi_N(\tilde{\nabla}\theta)]^3 + b'_2 \frac{n-2}{2} \left|\tilde{\nabla}\theta\right|^2 \cdot \pi_N(\tilde{\nabla}\theta)\} d\mu_g = 0.$$

For any $x \in M^3$, choose the local coordinate domain of $f(M^3) \in \tilde{M}^4$, for $\forall \psi \in C^{\infty}_{\tilde{M}}$, $\exists \theta \in C^{\infty}_{\tilde{M}}$ such that :

$$\psi = \frac{\partial\theta}{\partial x^4} \circ f = \pi_N(\tilde{\nabla}\theta), \quad \nabla_M(\theta \circ f) = 0.$$

Substituting it into the above equation, we get :

$$\begin{cases}\int_{M^m}[a'_1\left|h^0\right|_g^2+a'_2H^2+b'_2trS]\cdot\psi d\mu_g=0\\ \int_{M^m}a'_2H\cdot\psi^2d\mu_g=0\\ \int_{M^m}a'_2\cdot\psi^3d\mu_g=0.\end{cases}\tag{29}$$

Then we get :

$$G(g,h^0,H,S)=b'_1\left\langle h^0,S\right\rangle_g.$$

Next, we will discuss the h^0_{ij} classification. The equation $G(g,h^0,H,S)=0$ holds when $h^0_{ij}=0$, indicating that $f(M^3)$ represents the total umbilical epidemic in $\tilde{M}^4$, thus confirming the conclusion. If $h^0_{ij}\neq 0$, then $G(g,h^0,H,S)$ yields a global conformal invariant, leading to

$$\int_{M^m}b'_1\left\langle h^0_{ij},\theta_{k,l}-\theta_k\theta_l\right\rangle_g d\mu_g=0,\ \forall\theta\in C^\infty_{(\tilde{M})}.$$

Due to the arbitrariness of θ and $h^0_{ij}\neq 0$, we obtain $b'_1=0$. This completes the proof. □

Now, we will prove Theorem 2.

Proof. On the one hand, it is known from Proposition 2 that if $P(\tilde{g},R,h)$ produces a global conformal invariant, then

$$P(\tilde{g},R,h)=\tilde{P}(g,(h^0)^3).$$

Combining $tr(h^0_{ij})=0$, then $P(\tilde{g},R,h)=0$ or $G_3(h^0,h^0,h^0)$, where

$$G_3((h^0)^3)=\sum_{1\le i,j,k\le 3}h^0_{ij}\cdot h^0_{ik}\cdot h^0_{jk}.$$

So there exists a real number k such that

$$P(\tilde{g},R,h)=k\sum_{1\le i,j,k\le 3}h^0_{ij}\cdot h^0_{ik}\cdot h^0_{jk}.$$

We diagonalize the second fundamental form, i.e. $h_{ij}=\lambda_i\delta_{ij}$, and directly calculate:

$$P(g,R,h)=k[\frac{2}{9}(\lambda_1^3+\lambda_2^3+\lambda_3^3)-\frac{1}{3}(\lambda_1^2\lambda_2+\lambda_1^2\lambda_3+\lambda_2^2\lambda_1+\lambda_2^2\lambda_3+\lambda_3^2\lambda_1+\lambda_3^2\lambda_2)+\frac{4}{3}\lambda_1\lambda_2\lambda_3].\tag{30}$$

On the other hand, it is known from the content of the generalized Willmore functional[**?**],

$$\sigma_1=\frac{1}{3}(\lambda_1+\lambda_2+\lambda_3),\ \sigma_2=\frac{1}{3}(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3),\ \sigma_3=\lambda_1\lambda_2\lambda_3\tag{31}$$

Combining the formula $Q_3=2\sigma_1^3-2\sigma_1\sigma_2+\sigma_3$, we have

$$Q_3=\frac{2}{27}(\lambda_1^3+\lambda_2^3+\lambda_3^3)-\frac{1}{9}(\lambda_1^2\lambda_2+\lambda_1^2\lambda_3+\lambda_2^2\lambda_1+\lambda_2^2\lambda_3+\lambda_3^2\lambda_1+\lambda_3^2\lambda_2)+\frac{4}{9}\lambda_1\lambda_2\lambda_3.\tag{32}$$

From eq (30) and eq (32), it can be seen that the global conformal invariant on the three-dimensional hypersurface is unique in the case of a constant difference. The proof is completed. □

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