Using Mathematical Models and Extensions to Solve a Robot’s Equidistant Returning Path

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Abstract
The main purpose of this project is to study the Rusty the Robot problem where a robot travels equidistantly from the starting point back to the starting point. We mainly use mathematical models such as trigonometric functions, recursive sequences, and mathematical induction to explore, deduce and demonstrate findings, while using mathematical computer software for calculation and verification. In the beginning, Geogebra was used to explore the original question and allow us to obtain some properties and results. Then, the topic was extended and the relationship between the number of steps and angles was explored, as well as the recursive relationship between the landing points during the travel process. Next, we altered the starting positions of Rusty to see how this affects our findings. Finally, the two intersecting straight lines are extended into three straight lines intersecting at the same point. Based on the angles between them, we discuss the number of returning paths from the starting point to the starting point in an equidistant traveling method. Based on the mathematical model constructed, this study constructs a return path diagram for the robot to return to the starting point in an equidistant manner according to different starting point positions and obtains the recursive relationship of the relative positions on this return path. During the research process of this study, some interesting mathematical theories were obtained. It is expected that these results can be applied to related fields of AI robot movement patterns in the future.

Keywords: characteristic equation, complex plane, De Moivre’s theorem, recursion formula, returning angle, returning path, trigonometric methods

1. Introduction
1.1 Original Problem
Two lines M and N meet at point O, making an angle \( \theta \leq 90^\circ \) there. (More precisely, \( \theta \) is the measure of that angle in degrees). Rusty the Robot walks in the plane with steps that alternate between those lines M and N, starting from the intersection point O. Rusty is old and has stiff legs so all his steps have the same length (but they may be in any direction). Since Rusty is not allowed to return to a spot he just occupied, he might become stuck with no legal step to make. We can see Rusty’s path because he leaves a trail of red rust behind him as he walks (“Ross Program”, 2016).

Figure 1. Part (a) of original problem

(a) For the angle \( \theta \) pictured above, Rusty steps from O to A to B to C to D to O, returning to O in 5 steps. What is \( \theta \) in this case?
(b) For which angle \( \theta \) will Rusty return to O after 7 steps? What’s the general rule? If Rusty returns to O after k steps, find
(with proof!) the angle $\theta$.

(c) Explore the following: for the $36^\circ$ situation in the picture above, suppose Rusty starts at some point $P$ close to $O$ on line $M$, such as in the figure below. What will his path be? Will Rusty return to his starting point $P$ after some number of steps?

![Figure 2. Part (c) of original problem](image)

1.2 Research Motives

The “Rusty the Robot” problem is problem 4 of the 2016 Ross Summer Program Application. In the problem, Rusty the Robot walks along two lines that meet at an angle $\theta$. All of Rusty’s steps are the same length, and he leaves a trail of red rust behind him, marking his path. His goal is to start at the intersection between the two lines and walk in such a way so that he returns to his original spot. However, he is programmed so that he may not return to a spot on the two lines that he has been to already. For example, if $\theta = 90$ degrees, Rusty can only make one legal step. Parts (a) and (b) of the question ask what $\theta$ is when Rusty returns to his original spot in 5 steps, 7 steps, and $k$ steps. Part (c) of the question presents a scenario where Rusty doesn’t start at the intersection of the two lines; instead, he starts at a point on one of the lines. Although this problem has been solved by many, this problem stood out to us because of its creativity and practicality. The researchers thought this concept could be applied to many things, such as how light reflections can be manipulated to shine at a specific location, or even how robotic systems navigate and map their surroundings (Raheem, Raafat, & Mahdi, 2022).

1.3 Procedure of the Study

First, we solve parts (a), (b), and (c) of the original problem using simple models and basic geometry. We also find mathematical models to explore deep solutions to the problem. For instance, the ratio of $\theta$ in relation to the number of steps.

Second, we extend the problem by finding the ratios of the distances between $O$ and different points on the same line to the distance between $O$ and the first step. We also find the recursive relationships for distance sequences from and to $O$. Meanwhile, we not only find the recursive relationships for scenarios not from and to $O$, but we also find the paths that Rusty takes during the recursive relationship scenarios.

Third, we consider three-line scenarios for certain angles and find the possible returning paths from and to $O$. We do the same for scenarios from and to a point not $O$. 

72
2. Research Methods

2.1 Solution to the Original Problem

2.1.1 Solution to Part (a) of the Original Problem

Figure 3. Solution to original problem’s part (a)

Since $\overline{OD} = \overline{CD}$, $\angle DOC = \angle DCO = \theta$ in figure 3. So, based on the exterior angle theorem, $\angle CDB = 2\theta$. From the angle sum property of $\triangle BCD$, $\theta + 2\theta + 2\theta = 180^\circ$. Therefore, $\theta = 36^\circ$.

We also found that there was no relationship between $\theta$ and the length of one step. This implies that the number of steps Rusty takes to return to his original position is not dependent on the length of one step Rusty takes.

2.1.2 Solution to Part (B) of the Original Problem and Finding the Relation Between $\theta$ and the Number of Steps

Since $\overline{OF} = \overline{FE}$, $\angle FOE = \angle FEO = \theta$. Similarly, since $\overline{EF} = \overline{ED}$, $\angle EFD = \angle EDF = 2\theta$ in figure 4. So, based on the exterior angle theorem, $\angle CDB = 3\theta$. From the angle sum property of $\triangle BCD$, $\theta + 3\theta + 3\theta = 180^\circ$. Therefore, $\theta = 180^\circ / 7$. Similar to the solution to 2.1.1, the implication of 2.1.2 was that the number of steps Rusty takes to return to his original position depends on $\theta$, but not on the length of one of Rusty’s steps.

Figure 4. Solution to original problem’s part (b)

We also noticed that the number of steps was always odd, which could be attributed to symmetry.

Through previous results, we can use GeoGebra to construct a diagram showing Rusty returning to his original point in 9 steps. Since $\overline{OA} = \overline{AB}$, $\angle AOB = \angle ABO = \theta$. Similarly, since $\overline{GH} = \overline{GF}$, $\angle GHF = \angle GFH = 2\theta$ in figure 5. Based on the same logic, $\angle EFD = \angle EDF = 2\theta$. So, from the angle sum property of $\triangle EFD$, $\theta + 4\theta + 4\theta = 180^\circ$. Therefore, $\theta = 20^\circ$.
2.1.3 Define Terms

To make subsequent proofs and research easier, we defined some terms for our use.

Definition 1 – Original point and non-original point: The original point is defined as the intersection of two lines. All other points are non-original points. For example, in figure 5, O is an original point and a start point. In figure 6, P_1 is a non-original point and a start point.

Definition 2 – Step Points: The step points are defined as the locations the robot passes through on his journey. The start point, whether an original point or not, counts as a step point. For example, in figure 6, line N has 5 step points, which are A, C, E, I, and G. Line M also has 5 step points, which are P, D, B, F, and H.

Definition 3 – Returning path: The returning path is defined as the robot’s path from the start point back to the start point. This path will pass many step points. For example, in figure 6, the returning path is P → A → B → C → D → E → F → G → H → I → P.

Definition 4 – Equidistant Step Length: The step length is defined as the equidistant length of each of the robot’s steps between two consecutive forward step points. For example, in figure 6, PA = AB = BC = ... = IP.

Definition 5 – Step point sequence: The step point sequence is defined as the step points on the same line. The order of the sequence depends on the order of the returning path. For example, in figure 5, the step point sequence for line N is P_1, P_2, P_3, and P_4 for line M. In the same way, line M has the step point sequence Q_1, Q_2, Q_3, and Q_4 for line M. This sequence correlates to step points B, D, F, and H.

Definition 6 – Step Distance Sequence: The step distance sequence is defined as the distance between a point in a step point sequence and the original point. For example, step distance sequence a_n is defined as the distance from O to P_n, which is \( a_n = OP_n \). In the same way, \( b_n = OQ_n \).

2.1.4 Initial Findings

Because m is always odd, both line M and line N will have \((m-1)/2\) points, respectively. We can conclude the results as follows:

2.1.4.1 Property of Reflection between \( P_n \) and \( Q_n \)

Lemma 1

If the robot walks \( m \) steps from the original point to itself, both line M and line N will have \((m-1)/2\) points. Then, the point \( P_1 \) is reflected to the point \( Q_{(m+1)/2} \) about the bisector line of M and N. That is, if \( P_i \) and \( Q_j \) are symmetric about the bisector line of M and N, the summation of index of \( P_i \) and \( Q_j \) is \((m+1)/2\).

2.1.4.2 The farthest point from O between \( P_1 \) to \( P_n \) and \( Q_1 \) to \( Q_n \)

Lemma 2

If the robot walks \( m \) steps from the original point to itself, both line M and line N will have \((m-1)/2\) points.

Then:

1. When \( m = 4k + 1, k \in N \), \( P_{k+1} \) and \( Q_k \) are the farthest points from O
2. When \( m = 4k + 3, k \in N \), \( P_{k+1} \) and \( Q_{k+1} \) are the farthest points from O.
2.1.4.3 The relationship between $\theta$ and $m$

**Theorem 1**

When there are two lines that pass through the original point and make the angle $\theta$, and the robot takes $m$ steps from the original point to itself, the relationship between $\theta$ and $m$ is $\theta = \frac{180}{m}$.

**Proof**

By using mathematical induction on $m$:

1. When $m = 3$, there are 3 step distances. Lines M and N have a point that makes an equilateral triangle with point O. Hence, $\theta + \theta + \theta = 180^\circ$, so $\theta = 60^\circ$. Therefore, $m = 3$ is true.

When $m = 5$, there are 5 step distances. Lines M and N have two points as shown in figure 6.

![Figure 6. $m = 3$](image)

Using the external angle theorem of a triangle, the triangle sum theorem, and the properties of an isosceles triangle, $\theta + 2\theta + 2\theta = 180^\circ$, so $\theta = 36^\circ$. Therefore, $m = 5$ is true.

2. Assume that the formula, $\theta = \frac{180}{m}$, is true for $m = 4k - 1$ and $m = 4k + 1$ ($k \in \mathbb{N}$).

(1) Suppose $m = 4k - 1$ and $k \in \mathbb{N}$, lines M and N have $2k - 1$ points, which are $P_1$ to $P_{2k-1}$, and $Q_1$ to $Q_{2k-1}$, as shown in figure 7.

![Figure 7. $m = 4k - 1$](image)

Also, $P_k$ and $Q_k$ are the points that are farthest from O. Hence, $\angle Q_{k+1}P_k = (2k - 1)\theta = \angle Q_kP_j$. So, $\theta + (2k - 1)\theta + (2k - 1)\theta = 180^\circ$. That is, $\theta = \frac{180^\circ}{4k - 1}$ is true and is identical to $\theta = \frac{180^\circ}{m}$ for $2k + 1$.

If $m = 4k + 3$ and $k \in \mathbb{N}$, lines M and N have $2k + 1$ points, which are $P_1$ to $P_{2k+1}$ and $Q_1$ to $Q_{2k+1}$. $P_{k+1}$ and $Q_{k+1}$ are the points farthest from O. By assumption, $\angle Q_{k+1}P_k = 2k\theta$. Using the external angle theorem of a triangle, the triangle sum theorem, and the properties of an isosceles triangle, $\angle Q_{k+1}Q_kP_{k+1} = (2k + 1)\theta = \angle Q_kP_{k+1}P_{k+1}$, so $\theta + (2k + 1)\theta + (2k + 1)\theta = 180^\circ$. That is, $\theta = \frac{180^\circ}{4k + 3}$ is true for $m = 4k + 3$.

(2) Suppose $m = 4k + 1$ and $k \in \mathbb{N}$, lines M and N have $2k$ points, which are $P_1$ to $P_{2k}$, and $Q_1$ to $Q_{2k}$, as shown in figure 8.
Also, $P_{a1}$ and $Q_{k}$ are the points that are farthest from O. Hence, $\angle OP_{a1}Q_{k} = 2k\theta = \angle OP_{i}Q_{i}$. So, $\theta + 2k\theta + 2k\theta = 180^\circ$. That is, $\theta = 180^\circ/(4k+1)$ is true and is identical to $\theta = 180^\circ/m$ for $m = 4k+1$.

If $m = 4k + 5$ and $k \in N$, lines M and N have 2$k + 2$ points, which are $P_i$ to $P_{2k+2}$ and $Q_{i}$ to $Q_{2i+2}$. $P_{i+2}$ and $Q_{i+1}$ are the points farthest from O. By assumption, $\angle OP_{i}Q_{i} = (2k+2)\theta$. Using the external angle theorem of a triangle, the triangle sum theorem, and the properties of an isosceles triangle, $\angle OP_{i+1}Q_{i+1} = (2k+2)\theta = \angle OP_{i+2}Q_{i+3}$, so $\theta + (2k + 2)\theta + (2k + 2)\theta = 180^\circ$. That is, $\theta = 180^\circ/(4k+5)$ is true for $m = 4k+5$.

Thus, using mathematical induction, the formula $\theta = 180^\circ/m$ is true for odd numbers $m \geq 3$.

2.1.4.4 The order of $a_1$ to $a_n$ and $b_1$ to $b_n$

Lemma 3

If the robot walks $m$ steps from the original point to itself, both line M and line N will have $(m-1)/2$ points, which are $P_i$ to $P_{(m-1)/2}$ and $Q_i$ to $Q_{(m-1)/2}$. Their step distance sequences are $\{a_n\}$ and $\{b_n\}$. Then, (1) When $m = 4k + 1, k \in N$, the order of $a_1$ to $a_{(m-1)/2}$ is $a_1 < a_{(m-3)/2} < a_2 < a_{(m-3)/2} < a_3 < a_{(m-5)/2} < \cdots < a_{(m-5)/4} < a_{(m-7)/4} < a_{(m-9)/4} < a_{(m-3)/4}$ and the order of $b_1$ to $b_{(m-1)/2}$ is $b_1 < b_{(m-1)/2} < b_2 < b_{(m-3)/2} < b_3 < \cdots < b_{(m-7)/4} < b_{(m-5)/4} < b_{(m-3)/4} < b_{(m-1)/4}$.

(2) When $m = 4k + 3, k \in N$, the order of $a_1$ to $a_{(m-1)/2}$ is $a_1 < a_{(m-3)/2} < a_2 < a_{(m-3)/2} < a_3 < a_{(m-5)/2} < \cdots < a_{(m-5)/4} < a_{(m-3)/4} < a_{(m+1)/4}$ and the order of $b_1$ to $b_{(m-1)/2}$ is $b_1 < b_{(m-3)/2} < b_2 < b_{(m-1)/2} < b_3 < \cdots < b_{(m-5)/4} < b_{(m-3)/4} < b_{(m+5)/4} < b_{(m+1)/4}$.

2.1.4.5 The size of the exterior angle of $\angle OP_{i}Q_{i}$

Lemma 4

If the robot walks $m$ steps from the original point to itself, both line M and line N will have $(m-1)/2$ points, which are $P_i$ to $P_{(m-1)/2}$ and $Q_i$ to $Q_{(m-1)/2}$.

(1) When $m = 4k + 1, k \in N$, the size of the exterior angle of $\angle OP_{i}Q_{i}$ is $2k\theta$.

(2) When $m = 4k + 3, k \in N$, the size of the exterior angle of $\angle OP_{i}Q_{i}$ is $2k\theta$.

2.1.5 Solution to Part (c) of the Original Problem

According to the results of 2.1.1 and 2.1.2, the number of steps and the length of steps are independent. We explored part (c) through Geogebra software. We first constructed two lines at a 36$^\circ$ angle, then constructed two points P and A randomly on the two lines as shown in figure 9. We constructed many circles using the two adjacent points: one being the center of the circle and one being a point on the circle. We did these operations until a circle passed through point P, the original point, as shown in the figure below. For example, initially, we made a circle with A being the center and AP being the radius, then found the intersection (point B) between the circle and line M. We then used point B as the center of another circle and AB as the radius and found the intersection (point C) between the circle and line N. Finally, we connected all the points together to construct Rusty’s path returning to point P. We found that Rusty took 10 steps. However, we wondered if there was such a case in which Rusty would be unable to return to his original location, like if line PA was perpendicular to line M, or if line AB was perpendicular to line N.
2.2 Extension Problems for Two Lines

2.2.0 Preliminary Theorem

Lemma 5
If \( m \) is an odd number and \( \theta = \frac{180^\circ}{m} \), then
\[
1 + \cos 2\theta + \cos 4\theta + \cos 6\theta + \ldots + \cos(2m-2)\theta = 0.
\]

Lemma 6
If \( m \) is an odd number and \( \theta = \frac{180^\circ}{m} \), then
1. When \( m \equiv 1 \pmod{4} \), then
\[
1 + 2\cos 2\theta + 2\cos 4\theta + 2\cos 6\theta + \ldots + 2\cos(m-1)\theta / 2 = 2\cos\theta + 2\cos 3\theta + 2\cos 5\theta + \ldots + 2\cos(m-3)\theta / 2.
\]
2. When \( m \equiv 3 \pmod{4} \), then
\[
1 + 2\cos 2\theta + 2\cos 4\theta + 2\cos 6\theta + \ldots + 2\cos(m-3)\theta / 2 = 2\cos\theta + 2\cos 3\theta + 2\cos 5\theta + \ldots + 2\cos(m-1)\theta / 2.
\]

Lemma 7
If \( k \in \mathbb{N} \), then
1. \( 1 + 2\cos 2\theta + 2\cos 4\theta + 2\cos 6\theta + \ldots + 2\cos 2k\theta = \frac{\sin(2k+1)\theta}{\sin \theta} \).
2. \( 2\cos\theta + 2\cos 3\theta + 2\cos 5\theta + \ldots + 2\cos(2k-1)\theta = \frac{\sin 2k\theta}{\sin \theta} \).

2.2.1 Finding the Ratio of Step Length and Step Point Sequence on the Same Line

Result of \( m=7 \) case

Rusty’s path returning from \( O \) to \( O \) is \( O \rightarrow A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow O \), as shown in figure 10. Using basic trigonometric principles, we found that the ratios of \( \overline{OB} / \overline{OA} \), \( \overline{OC} / \overline{OD} / \overline{OA} \), and \( \overline{OF} / \overline{OA} \) are \( 2\cos \theta \), \( 1 + 2\cos 2\theta \), and \( 1 \), respectively.
Figure 10. Ratios of $n=7$ case

By the definition of step distance sequence, these ratios are described as $a_i/a_1$, $a_2/a_1$, and $a_3/a_1$. These ratios are also described as $b_i/a_1$, $b_2/a_1$, and $b_3/a_1$.

Result of $m=9$ case

Rusty’s path from O to O is $O\rightarrow A\rightarrow B\rightarrow C\rightarrow D\rightarrow E\rightarrow F\rightarrow G\rightarrow H\rightarrow O$, as shown in figure 11. Using basic trigonometric principles, we found that the ratios of $\frac{OB}{OA}$, $\frac{OC}{OA}$, $\frac{OD}{OA}$, and $\frac{OH}{OA}$ are $2\cos \theta$, $1 + 2\cos 2\theta$, $2\cos \theta + 2\cos 3\theta$, and 1, respectively.

Figure 11. Ratios of $n=9$ case

By the definition of step distance sequence, these ratios are described as $a_i/a_1$, $a_2/a_1$, $a_3/a_1$, and $a_4/a_1$. These ratios are also described as $b_i/a_1$, $b_2/a_1$, $b_3/a_1$, and $b_4/a_1$.

Result of $m=15$ case

Rusty’s path from O to O is $O\rightarrow A\rightarrow B\rightarrow C\rightarrow D\rightarrow E\rightarrow F\rightarrow G\rightarrow H\rightarrow I\rightarrow J\rightarrow K\rightarrow L\rightarrow M\rightarrow O$, as shown in figure 12. Using basic trigonometric principles, we found that the ratios of $\frac{OB}{OA}$, $\frac{OC}{OA}$, $\frac{OD}{OA}$, $\frac{OE}{OA}$, $\frac{OF}{OA}$, $\frac{OG}{OA}$, $\frac{OH}{OA}$, $\frac{OI}{OA}$, $\frac{OJ}{OA}$, $\frac{OK}{OA}$, $\frac{OL}{OA}$, $\frac{OM}{OA}$, $\frac{ON}{OA}$, $\frac{OP}{OA}$, $\frac{OQ}{OA}$, $\frac{OR}{OA}$, $\frac{OS}{OA}$, $\frac{OT}{OA}$, $\frac{OU}{OA}$, $\frac{OV}{OA}$, $\frac{OW}{OA}$, $\frac{OX}{OA}$, $\frac{OY}{OA}$, $\frac{OZ}{OA}$, $\frac{O\ell}{OA}$, $\frac{O\ell'}{OA}$, $\frac{O\ell''}{OA}$, $\frac{O\ell'''}{OA}$, $\frac{O\ell''''}{OA}$, $\frac{O\ell'''''}{OA}$, $\frac{O\ell''''''}{OA}$, and 1, respectively.

Figure 12. Ratios of $n=15$ case

Based off the deduction and conjecture of these three examples, the contents of the conjecture are described and proved as follows in Theorem 2.

**Theorem 2**

If the robot walks $m$ steps from the original point to itself, both line M and line N will have $(m-1)/2$ points, which are $P_i$ to $P_{(m-1)/2}$ and $Q_i$ to $Q_{(m-1)/2}$, then

1. the ratios of $a_{(m-1)/2}$ to $b_{(m-1)/2}$ are $a_i/a_1 = \frac{\sin (2i-1)\theta}{\sin \theta}$ for $i = 1$ to $(m-1)/2$.

2. the ratios of $b_{(m-1)/2}$ to $b_{(m-1)/2}$ are $b_i/a_i = \frac{\sin 2\theta}{\sin \theta}$ for $i = 1$ to $(m-1)/2$. 

78
Proof:

(1) When \( m = 4k + 1 \) and \( k \in N \). That is, line M and line N both have \( 2k \) points, which are \( P_i \) to \( P_{2k} \) and \( Q_i \) to \( Q_{2k} \), where \( P_{4i+1} \) and \( Q_i \) are the farthest points from O. By lemma 4, the exterior angle of \( \angle OPQ \) is \( 2\theta \) for \( i = 1 \) to \( 2k \), and \( \angle OPQ = (2i-1)\theta \) for \( i = 1 \) to \( 2k \).
In every \( \Delta OPQ \) for \( i = 1 \) to \( 2k \), we use law of sin to show that \( \frac{OP}{\sin \angle OPQ} = \frac{PQ}{\sin \angle POQ} \), which means \( a_i = \sin(2\theta)/\sin \theta \). From the first and second equations, we can get \( a_i \) for \( i = 1 \) to \( 2k \). For example, the step point sequence can also be shown with \( \Delta OPQ \) and \( \Delta OPQ_k \). The second point on line N and M were \( 2\cos 36\) and \( 2\cos 36 \), respectively, and the points before returning to point O were \( 2\cos 36 \). We use law of sin to show that \( \theta = \angle OPQ \). The series from \( a_1 \) to \( a_2 \) are also known as \( a_1 \) to \( a_2 \), and \( b_1 \) to \( b_2 \), respectively. The step point sequence can also be shown with \( P_k \) and \( Q_k \) in figure 19 below.

(2) When \( m = 4k + 3 \) and \( k \in N \). That is, line M and line N both have \( 2k + 1 \) points, which are \( P_i \) to \( P_{2k+1} \) and \( Q_i \) to \( Q_{2k+1} \), where \( P_{4i+1} \) and \( Q_{4i+1} \) are the farthest points from O. By lemma 4, the exterior angle of \( \angle OPQ \) is \( 2\theta \) for \( i = 1 \) to \( 2k+1 \), and \( \angle OPQ = (2i-1)\theta \) for \( i = 1 \) to \( 2k+1 \).
In every \( \Delta OPQ \) for \( i = 1 \) to \( 2k+1 \), we use law of sin to show that \( \frac{OP}{\sin \angle OPQ} = \frac{PQ}{\sin \angle POQ} \), which means \( a_i = \sin(2\theta)/\sin \theta \). That is, \( a_i = \sin(2\theta)/\sin \theta \) for \( i = 1 \) to \( 2k+1 \).
On the other hand, by lemma 1 and lemma 6, \( b_i = a_{2k+2i} \) and \( a_i = \sin(2\theta)/\sin \theta \) for \( i = 1 \) to \( 2k+1 \).
That is, \( b_i = a_{2k+2i} \) and \( a_i = \sin(2\theta)/\sin \theta \) for \( i = 1 \) to \( 2k+1 \).
And the ratios of \( b_{(m-1)/2} \) to \( b_1 \) are \( b_i = a_{2k+2i} \) for \( i = 1 \) to \( (m-1)/2 \). QED

2.2.2 Finding the Recursive Relationship of the Points From and to O on Different Lines

According to the results of 2.2.1, we discovered that these points’ locations had a relationship, which we explored further in 2.2.2. We assumed that the starting location was point O, the first point on line N was \( P_1 \), and the first point on line M was \( Q_1 \). The second point on line N and M were \( P_2 \) and \( Q_2 \), respectively, and the points before returning to point O were \( P_1 \) and \( Q_1 \), respectively. The series from \( P_i \) to \( P_{2k} \) and the series from \( Q_i \) to \( Q_{2k} \) were also known as \( a_i \) to \( a_{2k} \) and \( b_i \) to \( b_{2k} \), respectively. For example, the step point sequence can also be shown with \( P_n \) and \( Q_n \) in figure 19 below.

![Figure 19. Relationship of 15 steps](image-url)
same logic, from the second and third equations, we get \( b_1 + b_2 = 2a_1 \cos 36^\circ \). We can conclude that \( a_1 + a_2 = 2b_2 \cos 36^\circ \), and \( b_2 + b_3 = 2a_2 \cos 36^\circ \).

Figure 20. Relationship of \( \theta = 36^\circ \) case

Therefore, the recursive formulas for the \( <a_n> \) sequence and \( <b_n> \) sequence are \( a_{n+1} + a_n = 2b_n \cos 36^\circ \), and \( b_{n+1} + b_n = 2a_n \cos 36^\circ \), respectively (“Recursive Formula”, 2021). Coincidentally, \( 2 \cos 36^\circ = \phi \), and \( \phi^2 = \phi + 1 \). Thus, we can conclude:

\[
a_2 = \phi b_1 - a_1
\]
\[
b_2 = \phi a_2 - b_1 = \phi (\phi b_1 - a_1) - b_1 = (\phi^2 - 1)b_1 - \phi a_1 = \phi (b_1 - a_1)
\]
\[
a_3 = \phi b_2 - a_2 = \phi^2 (b_1 - a_1) - \phi b_1 + a_1 = (\phi^2 - \phi)b_1 - (1 - \phi^2)a_1 = b_1 - \phi a_1 = 0
\]

This means that the robot’s path is \( O(0) \rightarrow P_1(a_1) \rightarrow Q_1(h_1) \rightarrow P_2(a_2) \rightarrow Q_2(h_2) \rightarrow O(a_3 = 0) \), as shown in figure 21.

Figure 21. Returning path for \( \theta = 36^\circ \) case

The meaning of \( a_3 = 0 \) means that the robot has returned to the original point because \( a_3 \) is the step distance from the origin to \( P_2 \). We also discovered that \( P_1 \) and \( P_2 \) on line \( N \) were symmetric to \( Q_1 \) and \( Q_2 \) on line \( M \), respectively, with respect to the angle bisector of \( \theta \).

2.2.2.2 Case of \( \theta = 180^\circ / 7 \).

Using figure 22 below, and the law of cosine, in \( \Delta OP_1Q \), \( \overrightarrow{P_1Q} = a_1^2 + b_1^2 - 2a_1b_1 \cos(180^\circ / 7) \). In \( \Delta OP_2Q \), \( \overrightarrow{P_2Q} = a_2^2 + b_2^2 - 2a_2b_2 \cos(180^\circ / 7) \). Finally, in \( \Delta OP_1P_2 \), \( \overrightarrow{P_1P_2} = a_1^2 + b_1^2 - 2a_1b_1 \cos(180^\circ / 7) \). From the first and second equations, we can get \( a_1 + a_2 = 2b_2 \cos(180^\circ / 7) \). With the same logic, from the second and third equations, we get \( b_1 + b_2 = 2a_2 \cos(180^\circ / 7) \). We can conclude that \( a_1 + a_2 = 2b_2 \cos(180^\circ / 7) \), and \( b_2 + b_3 = 2a_2 \cos(180^\circ / 7) \).
Suppose that \(2\cos(180/7) = \lambda\). Then \(a_{n+1} + a_n = \lambda b_{n+1}\), and \(b_{n+1} + b_n = \lambda a_n\). While \(\lambda\) satisfies \(\lambda^3 - \lambda^2 - 2\lambda + 1 = 0\) and \(\lambda a_1 = b_1\), we can conclude:

\[
a_2 = \lambda b_1 - a_1 \\
b_2 = \lambda a_1 - b_1 = \lambda(\lambda b_1 - a_1) - b_1 = (\lambda^2 - 1)b_1 - \lambda a_1 \\
a_3 = \lambda b_2 - a_2 = (\lambda^3 - 2\lambda)b_1 - (\lambda^2 - 1)a_1 \\
b_3 = \lambda a_1 - b_2 = (-\lambda^3 + \lambda^2 + 3\lambda - 1)b_1 + (-\lambda^2 + 2\lambda)a_1 = \lambda b_1 + (-\lambda^2 + 1)a_1 \\
a_4 = \lambda b_3 - a_2 = \lambda(\lambda b_1 + \lambda(-\lambda^2 + 1)a_1 - (\lambda^2 - 1)b_1 + (\lambda^2 - 1)a_1) = b_1 - \lambda a_1 = 0
\]

This means that the robot’s path is \(O(0) \rightarrow P_1(a_1) \rightarrow Q_1(b_1) \rightarrow P_2(a_1) \rightarrow Q_2(b_1) \rightarrow P_3(a_1) \rightarrow Q_3(b_1) \rightarrow O(a_1 = 0)\), as shown in figure 23.

The meaning of \(a_1 = 0\) means that the robot has returned to the original point because \(a_1\) is the step distance from the origin to \(P_1\). We also discovered that \(P_1, P_2,\) and \(P_3\) on line N were symmetric to \(Q_1, Q_2,\) and \(Q_3\) on line M, respectively, with respect to the angle bisector of \(\theta\).

Finally, we consider the general recursive formula, where \(\theta = 180^\circ / m\), where \(m\) is an odd number. The distance from \(O\) to \(P_n\) is \(a_n\), and the distance from \(O\) to \(Q_n\) is \(b_n\). According to the cosine law of triangles, we can determine the recursive relationship between \(a_n\) and \(b_n\):

\[
a_{n+1} = 2b_{n+1} \cos(180^\circ / m) - b_n + b_{n+1} = 2a_n \cos(180^\circ / m)
\]

where \(2\cos(180^\circ / m)\) satisfies the equation of \((m-1)/2\) power and a cosine form. For example, \(2\cos 36^\circ\) satisfies \((2\cos 36^\circ)^2 = 2\cos 36^\circ + 1\). Also, \(2\cos(180^\circ / 7)\) satisfies \([2\cos(180^\circ / 7)]^3 - [2\cos(180^\circ / 7)]^2 - 2[2\cos(180^\circ / 7)] + 1 = 0\).

Therefore, the path of Rusty is \(O(0) \rightarrow P_1(a_1) \rightarrow Q_1(b_1) \rightarrow P_2(a_1) \rightarrow Q_2(b_1) \rightarrow P_3(a_1) \rightarrow Q_3(b_1) \rightarrow \ldots \rightarrow P_{(m-1)/2}(a_{(m-1)/2}) \rightarrow Q_{(m-1)/2}(b_{(m-1)/2}) \rightarrow P_{(m+1)/2}(a_{(m+1)/2} = 0) = O\), which means Rusty returned to the original point.
Proof:
From the cosine law of $\Delta OP_0Q_0$, $\Delta OP_1Q_1$, and $\Delta OP_2Q_2$, we can get $a_1 + a_2 = 2b_1 \cos \theta$ and $b_1 + b_2 = 2a_1 \cos \theta$. From this, we can get $a_{n+1} + a_n = 2b_n \cos \theta$ and $b_{n+1} + b_n = 2a_n \cos \theta$. Next, we can get $a_{n+1} = (4 \cos^2 \theta - 2)a_n - a_{n-1}$. Using its characteristic equation, we can get $x^2 = (4 \cos^2 \theta - 2)x - 1$. The two roots are $\cos 2\theta + i \sin 2\theta$ and $\cos 2\theta - i \sin 2\theta$. So, $a_n$ can be expressed with a combination of $(\cos 2\theta + i \sin 2\theta)^n$ and $(\cos 2\theta - i \sin 2\theta)^n$. $a_n = c_1(\cos 2\theta + i \sin 2\theta)^n + c_2(\cos 2\theta - i \sin 2\theta)^n$.
Here, the values of $c_1$ and $c_2$ will be determined by $a_1$ and $a_2$.

We also have to find the relationship between $a_i$ and $a_i$.
$a_1 \cdot \cos 2\theta = a_2 - a_1^2$, which implies that $a_2 = (3 - 4 \sin^2 \theta)a_1$.

Finally, we have to prove $a_{(m+1)/2} = 0$, where $m$ is odd.
$a_{(m+1)/2} = c_1(\cos 2\theta + i \sin 2\theta)^{m+1/2} + c_2(\cos 2\theta - i \sin 2\theta)^{m+1/2}$.

By De Moivre’s Theorem (“De Moivre’s Theorem”, 2023), this can be written as
$c_1[(\cos(m+1)\theta + i \sin(m+1)\theta)] + c_2[(\cos(m+1)\theta - i \sin(m+1)\theta)]$. Here $c_1 = \frac{a_1 - a_2(\cos 2\theta - i \sin 2\theta)}{2 \sin 2\theta(\cos 90^\circ + 2\theta + i \sin 90^\circ + 2\theta)}$.
$c_2 = \frac{a_1 - a_2(\cos 2\theta + i \sin 2\theta)}{-2 \sin 2\theta(\cos 90^\circ - 2\theta + i \sin 90^\circ - 2\theta)}$. $m \theta = 180^\circ$, and $a_{(m+1)/2} = c_1(-\cos \theta - i \sin \theta) + c_2(-\cos \theta + i \sin \theta)$

$$a_{n+1} = 2 \sin \theta(a_n - a_{n-1}(-3 - 4 \sin^2 \theta))$$
$$= \frac{2 \sin \theta(a_n - a_{n-1})}{2 \sin 2\theta} = 0.$$ Q.E.D

This result can be described in Theorem 3.

Theorem 3
Suppose that two lines M and N intersect at a point O at an unknown angle, such that points $P_1$ to $P_n$ are on line N and points $Q_1$ to $Q_n$ are on line M. The distances from $P_1$ to O is $a_1$, the distance from $P_2$ to O is $a_2$, and so on. Similarly, the distances from $Q_1$ to O is $b_1$, the distances from $Q_2$ to O is $b_2$, and so on. If and only if the unknown angle is $180^\circ / m$, Rusty takes m steps from O to O to return, and vice versa.

Also, when $\theta = 180^\circ / m$ and $m$ is an odd number, according to the cosine law of triangles, we can determine the recursive relationship between $a_n$ and $b_n$:
$a_{n+1} + a_n = 2b_n \cos(180^\circ / m)$, $b_{n+1} + b_n = 2a_n \cos(180^\circ / m)$.

Therefore, the path of Rusty is $O \rightarrow P_1(a_1) \rightarrow Q_1(b_1) \rightarrow P_2(a_2) \rightarrow Q_2(b_2) \rightarrow P_3(a_3) \rightarrow Q_3(b_3) \rightarrow \cdots \rightarrow P_{(m-1)/2}(a_{(m-1)/2}) \rightarrow Q_{(m-1)/2}(b_{(m-1)/2}) \rightarrow P_{(m-1)/2}(a_{(m-1)/2}) = 0 \rightarrow O$, which means Rusty returned to the starting point.

The meaning of $a_{(m+1)/2} = 0$ means that the robot has returned to the original point because $a_{(m+1)/2}$ is the step distance from the origin to $P_{(m+1)/2}$. We also discovered that $P_1$ to $P_{(m+1)/2}$ on line N were symmetric to $Q_{(m+1)/2}$ to $Q_1$ on line M, respectively, with respect to the angle bisector of theta.

2.2.3 Finding the Recursive Relationship of the Points Not From and to O on Different Lines
First, we consider case 1, where $\theta = 36^\circ$ in figure 24. According to cosine law of $\Delta OP_0Q_0$, $\Delta OP_1Q_1$, and $\Delta OP_2Q_2$, we can conclude:

$l^1 = a_1^2 + b_1^2 - 2a_1b_1 \cos 36^\circ$
$l^2 = a_2^2 + b_2^2 - 2a_2b_2 \cos 36^\circ$
$l^3 = a_3^2 + b_3^2 - 2a_3b_3 \cos 36^\circ$

Therefore, $a_1 + a_2 = 2b_1 \cos 36^\circ$ and $b_1 + b_2 = 2a_1 \cos 36^\circ$.

We can conclude that $a_2 + a_3 = 2b_2 \cos 36^\circ$, and $b_2 + b_3 = 2a_2 \cos 36^\circ$.

Therefore, the recursive formulas for the $<a_1>$ sequence and $<b_2>$ sequence are $a_{n+1} + a_n = 2b_{n+1} \cos 36^\circ$, and $b_{n+1} + b_n = 2a_n \cos 36^\circ$, respectively. We can conclude that $a_{n+1} + a_{n+1} = 2b_{n+1} \cos 36^\circ$ is also true. Adding the two recursive formulas for the $<a_1>$ sequence together, we get $a_{n+1} + 2a_n + a_{n+1} = 2 \cos 36^\circ(b_{n+1} + b_n) = 4 \cos^2 36^\circ \cdot a_n$. Simplified, this is $a_{n+1} = 2(2 \cos^2 36^\circ - 1) \cdot a_n - a_{n-1}$.
Suppose that \(2(2 \cos^2 36° - 1) = \alpha + \beta\), and that \(\alpha \beta = 1\). Then, \(a_{n+1} = 2(2 \cos^2 36° - 1) \cdot a_n - a_{n-1}\) becomes

\[a_{n+1} = (\alpha + \beta)a_n - \alpha \beta a_{n-1}.\]

We can conclude that \(a_n - \alpha a_n = \beta^{n-2}(a_2 - \alpha a_1)\), and

\[a_n - \beta a_n = \alpha^{n-2}(a_2 - \beta a_1).\]

Simplifying even more, we get the formula of \(a_n\), which is

\[a_n = a\alpha^n(a_2 - \beta a_1) + \beta^n(a_2 - \alpha a_1)\]

for all natural numbers \(n\), and \(\alpha + \beta = 2(2 \cos^2 36° - 1) = \frac{\sqrt{9} - 1}{2}\), and \(\alpha \beta = 1\).

According to \(a_{n+1} = (\alpha + \beta)a_n - \alpha \beta a_{n-1}\), let \(\alpha + \beta = \frac{\sqrt{9} - 1}{2} = k\). Then, \(k^2 + k = 1\), and \(a_{n+1} = ka_n - a_{n-1}\).

For \(n=2, n=3, n=4\), and \(n=5\), we get:

\(a_1 = ka_2 - a_1, a_2 = ka_3 - ka_1, a_3 = ka_4 - a_3 = -a_2 + ka_1\)

\(a_4 = ka_5 - a_4 = k(-a_2 + ka_1) - (-ka_2 - ka_1) = a_1\)

This means that the robot’s path is \(P_1(a_1) \rightarrow Q_1(b_1) \rightarrow P_2(a_2) \rightarrow Q_2(b_2) \rightarrow P_3(a_3) \rightarrow Q_3(b_3) \rightarrow P_4(a_4) \rightarrow Q_4(b_4) \rightarrow P_5(a_5) \rightarrow Q_5(b_5) \rightarrow P_1(a_1)\), meaning when the robot starts at a point \(P_1\) not \(O\), he returns to \(P_1\) in ten steps.

Figure 24. Returning path for \(\theta = 36°\) case

Second, we consider case 2, where \(\theta = 180°/7\) in figure 25. According to the cosine law of \(\Delta OP_1Q_1\), \(\Delta OP_2Q_1\), and \(\Delta OP_3Q_2\), we can conclude:

\[l^2 = a_1^2 + b_1^2 - 2a_1b_1 \cos(180°/7)\]

\[l^2 = a_2^2 + b_2^2 - 2a_2b_2 \cos(180°/7)\]

\[l^2 = a_3^2 + b_3^2 - 2a_3b_3 \cos(180°/7)\]

Therefore, \(a_1 + a_2 = 2b_1 \cos(180°/7)\) and \(b_1 + b_2 = 2a_2 \cos(180°/7)\)

We can conclude that \(a_2 + a_3 = 2b_2 \cos(180°/7)\), and \(b_2 + b_3 = 2a_3 \cos(180°/7)\).

Therefore, the recursive formulas for the \(<a_n>\) sequence and \(<b_n>\) sequence are \(a_{n+1} + a_n = 2b_{n-1} \cos(180°/7)\), and \(b_{n+1} + b_n = 2a_n \cos(180°/7)\), respectively. We can conclude that \(a_{n+1} + a_{n+2} = 2b_{n} \cos(180°/7)\) is also true. Adding the two recursive formulas for the \(<a_n>\) sequence together, we get

\[a_{n+1} + 2a_n + a_{n+1} = 2(b_{n-1} + b_n) \cos(180°/7) = 4 \cos^2(180°/7) \cdot a_n\]

Simplified, this is

\[a_{n+1} = 2(2 \cos^2(180°/7) - 1) \cdot a_n - a_{n-1}.

83
Suppose that $2[2\cos^2(\cos 180^\circ/7) - 1] = \alpha + \beta$, and that $\alpha \beta = 1$. Then, $a_{n+1} = 2[2\cos^2(\cos 180^\circ/7) - 1] \cdot a_n - a_{n-1}$ becomes $a_{n+1} = (\alpha + \beta)a_n - \alpha \beta a_{n-1}$. We can conclude that $a_n - \alpha a_n = \beta^{n-2} (a_2 - \alpha a_1)$, and $a_n - \beta a_n = \alpha^{n-2} (a_2 - \beta a_1)$.

Simplifying even more, we get the formula of $a_n$, which is $a_n = \alpha^{(\alpha - \beta)} + \beta^{(\alpha - \alpha)}$, for all natural numbers $n$.

According to $\alpha + \beta = 2[2\cos^2(\cos 180^\circ/7) - 1]$, and $\alpha \beta = 1$. According to $a_{n+1} = (\alpha + \beta)a_n - \alpha \beta a_{n-1}$, let $2[2\cos^2(\cos 180^\circ/7) - 1] = k$. Then, $k^3 + k^2 - 2k = 1$, and $a_{n+1} = k a_n - a_{n-1}$.

For $n=2$, $n=3$, $n=4$, $n=5$, $n=6$, and $n=7$, we get:

- $a_2 = k a_1 - a_0 = (k^2 + 1) a_2 - k a_1$, $a_3 = k a_2 - a_1 = -(k^2 + 1) a_2 - (k^2 - 1) a_1$
- $a_4 = k a_3 - a_2 = -k a_2 + (k^2 - 1) a_1$, $a_5 = k a_4 - a_3 = -a_2 + k a_1$
- $a_6 = k a_5 - a_4 = -k a_2 + k^2 a_1 + k a_2 - (k^2 - 1) a_1 = a_1$

This means that the robot’s path is $P(a_1) \rightarrow Q_1(b_1) \rightarrow P_1(a_2) \rightarrow Q_2(b_2) \rightarrow P_2(a_3) \rightarrow Q_3(b_3) \rightarrow P_3(a_4) \rightarrow Q_4(b_4) \rightarrow P_4(a_5) \rightarrow Q_5(b_5) \rightarrow P_5(a_6) \rightarrow Q_6(b_6) \rightarrow P_6(a_7) \rightarrow Q_7(b_7) \rightarrow P_7(a_8) = P_1(a_1)$, meaning when the robot starts at a point $P_1$ not O, he returns to $P_1$ in 14 steps.

Finally, we consider the general recursive formula, where $\theta = 180^\circ/7$, where $m$ is an odd number. The distance from O to $P_n$ is $a_n$, and the distance from O to $Q_n$ is $b_n$. According to the cosine law of triangles, we can determine the recursive relationship between $a_n$ and $b_n$:

$$a_{n+1} = 2b_{n-1} \cos(\cos 180^\circ/7)$$

Therefore, the path of Rusty is $P(a_1) \rightarrow Q_1(b_1) \rightarrow P_1(a_2) \rightarrow Q_2(b_2) \rightarrow P_2(a_3) \rightarrow Q_3(b_3) \rightarrow P_3(a_4) \rightarrow Q_4(b_4) \rightarrow \cdots \rightarrow P_7(a_8) \rightarrow Q_7(b_7) \rightarrow P_7(a_9) \rightarrow Q_8(b_8) \rightarrow P_8(a_10) = P_1(a_1)$, which means Rusty returned to the starting point.

**Proof:**

Given that $\theta = \cos 180^\circ/7$, where $m$ is odd, through cosine law, we can get:

- $a_{n+1} + a_n = 2b_{n-1} \cos \theta$
- $b_{n+1} + b_n = 2a_n \cos \theta$

Claim: $a_{n+1} = a_1$, where $m$ is odd. We use the two recursive equations:

- $a_{n+1} + a_n = 2b_{n-1} \cos \theta$
- $b_{n+1} + b_n = 2a_n \cos \theta$
Adding both of these equations, we get \( a_{n+4} + 2a_n + a_{n+1} = 2 \cos \theta \cdot (b_{n-1} + b_n) = 4 \cos^2 \theta \cdot a_n \).

So, \( a_{n+4} + (2 - 4 \cos^2 \theta) \cdot a_n + a_{n+1} = 0 \). Using its characteristic equation, we get

\[ a_n = c_1 (\cos 2 \theta + i \sin 2 \theta)^n + c_2 (\cos 2 \theta - i \sin 2 \theta)^n, \]

where \( c_1 \) and \( c_2 \) are obtained by \( a_1 \) and \( a_2 \). So, we get

\[ c_1 = \frac{a_1 (\sin 4 \theta - i \cos 4 \theta) - a_2 (\sin 2 \theta - i \cos 2 \theta)}{\sin 2 \theta}, \quad c_2 = \frac{a_1 (\sin 4 \theta + i \cos 4 \theta) + a_2 (\sin 2 \theta + i \cos 2 \theta)}{\sin 2 \theta}. \]

Using De Moivre’s theorem,

\[ a_{m+1} = c_1 (\cos 2 \theta + i \sin 2 \theta)^{m+1} + c_2 (\cos 2 \theta - i \sin 2 \theta)^{m+1} \]

\[ = c_1 [\cos(2m + 2) \theta + i \sin(2m + 2) \theta] + c_2 [\cos(2m + 2) \theta - i \sin(2m + 2) \theta] \]

\[ = c_1 \{\cos(360^\circ + 2 \theta) + i \sin(360^\circ + 2 \theta)\} + c_2 \{\cos(360^\circ + 2 \theta) - i \sin(360^\circ + 2 \theta)\} \]

\[ = c_1 (\cos 2 \theta + i \sin 2 \theta) + c_2 (\cos 2 \theta - i \sin 2 \theta) \]

\[ = a_1 \text{ Q.E.D} \]

This means the robot needs \( 2m \) steps to travel from \( P_1 \) (not \( O \)) to \( P_1 \).

This result can be described in Theorem 4.

**Theorem 4**

Suppose that two lines \( M \) and \( N \) intersect at a point \( O \) at an unknown angle, such that points \( P_1 \) to \( P_n \) are on line \( N \) and points \( Q_1 \) to \( Q_n \) are on line \( M \). The distances from \( P_1 \) to \( O \) is \( a_1 \), the distance from \( P_2 \) to \( O \) is \( a_2 \), and so on. Similarly, the distances from \( Q_1 \) to \( O \) is \( b_1 \), the distances from \( Q_2 \) to \( O \) is \( b_2 \), and so on. Suppose that a random point on line \( N \) not \( O \), \( P_1 \), is chosen. If and only if the unknown angle is \( 180^\circ / m \) and \( m \) is odd, Rusty takes \( 2m \) steps from \( P_1 \) to return to \( P_1 \), and vice versa.

Also, when \( \theta = 180^\circ / m \) and \( m \) is an odd number, according to the cosine law of triangles, we can determine the recursive relationship between \( a_n \) and \( b_n \):

\[ a_{n+1} + a_n = 2b_{n-1} \cos(180^\circ / m), \quad b_{n+1} + b_n = 2a_{n} \cos(180^\circ / m). \]

Therefore, the path of Rusty is

\[ P(a_0) \rightarrow Q(b_0) \rightarrow P(a_1) \rightarrow Q(b_1) \rightarrow P(a_2) \rightarrow Q(b_2) \rightarrow \cdots \rightarrow P_m(a_m) \rightarrow Q_m(b_m) \rightarrow P_{m+1}(a_{m+1}) = P_1(a_1), \]

which means Rusty returned to the starting point.

**2.3 Extension Problems for three lines**

**2.3.1 Finding the recursive relationship of the points from and to \( O \) on three lines**

First, we examine case 1, where \( \theta_1 = \theta_2 = 36^\circ \). In figure 26, the returning path is \( O \rightarrow A \rightarrow B \rightarrow C \rightarrow O \), which is 4 steps. In figure 27, the returning path is \( O \rightarrow A \rightarrow B \rightarrow C \rightarrow D \rightarrow O \), which is 5 steps. In figure 28, the returning path is \( O \rightarrow A \rightarrow B \rightarrow C \rightarrow D \rightarrow C' \rightarrow B \rightarrow A' \rightarrow O \), which is 8 steps. We noticed that point \( C \) in the first diagram is symmetrical to \( C \) in the returning path of two lines when \( \theta = 36^\circ \) in figure 29.
Second, we examine case 2, where $\theta_1 = \theta_2 = 180^\circ / 7$. In figure 30, the returning path is $O \rightarrow A \rightarrow B \rightarrow A' \rightarrow O$, which is 4 steps. In figure 31, the returning path is $O \rightarrow A \rightarrow B \rightarrow C' \rightarrow D \rightarrow E \rightarrow F \rightarrow O$, which is 7 steps. In figure 32, the returning path is $O \rightarrow A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow B \rightarrow A' \rightarrow O$, which is 8 steps. In figure 33, the returning path is $O \rightarrow A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow E' \rightarrow D \rightarrow C \rightarrow B \rightarrow A' \rightarrow O$, which is 12 steps. We noticed that points $A'$, $C'$, and $E'$ are symmetrical to $A$, $C$, and $E$, respectively, in the returning path when $\theta = 180^\circ / 7$ in figure 34.
Third, we examine case 3, where $\theta_1 = \theta_2 = 20^\circ$. In figure 35, the returning path is $O \to A \to B \to A' \to O$, which is 4 steps. In figure 36, the returning path is $O \to A \to B \to C' \to D \to C \to B \to A' \to O$, which is 8 steps. In figure 37, the returning path is $O \to A \to B \to C' \to D \to E \to F \to G' \to H \to O$, which is 9 steps. In figure 38, the returning path is $O \to A \to B \to C' \to D \to E \to F \to E' \to D \to C \to B \to A' \to O$, which is 12 steps. In figure 39, the returning path is $O \to A \to B \to C' \to D \to E \to F \to G' \to H \to G \to F \to E' \to D \to C \to B \to A' \to O$, which is 16 steps. We noticed that points $A'$, $G'$, $C'$, and $E'$ are symmetrical to $A$, $G$, $C$, and $E$, respectively, in the returning path when $\theta = 20^\circ$ in figure 40.

Based on the findings of the first three cases, we gathered the information into a table, seen in table 1.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>Steps of returning path</th>
</tr>
</thead>
<tbody>
<tr>
<td>$180^\circ/3$</td>
<td>3,4</td>
</tr>
<tr>
<td>$180^\circ/5$</td>
<td>4,5,8</td>
</tr>
<tr>
<td>$180^\circ/7$</td>
<td>4,7,8,12</td>
</tr>
<tr>
<td>$180^\circ/9$</td>
<td>4,8,9,12,16</td>
</tr>
<tr>
<td>$180^\circ/11$</td>
<td>4,8,11,12,16,20</td>
</tr>
</tbody>
</table>
Note. The possible number of steps it takes to return to O for certain angles between three intersecting lines.

Based on conjecture, the results are as follows:

Knowing that three lines, L, M, and N, all intersect at one point, O, \( \theta_1 = \theta_2 = 180^\circ / m \), and \( m \) is an odd number greater than or equal to 3, the returning path starts and ends at point O, and follows \( L \rightarrow M \rightarrow N \rightarrow M \rightarrow L \rightarrow M \rightarrow N \rightarrow M \rightarrow L \), and so on. Then, the number of steps is the set of \( \{ 4k \mid 1 \leq k \leq (m+1)/2, k \in N, m \text{ is odd} \} \cup \{ m \mid m = 2k + 1, k \in N \} \).

Proof:

We will split our proof into two sections. The first part is proven through mathematical induction. When \( m = 3 \) and \( \theta = 180^\circ / 3 = 60^\circ \), clearly the returning path from O to O is \( O \rightarrow A \rightarrow B \rightarrow O \) and has 3 steps, with A being on line L and B being on line M. By reflection, a quadrilateral \( OABA' \) is produced from the figure of the returning path, where \( A' \) is the reflection of A about line M. Hence, the returning path is \( O \rightarrow A \rightarrow B \rightarrow A' \rightarrow O \) and has 4 steps. So, this case \( m = 3 \) stands. Next, when \( m = 5 \) and \( \theta = 180^\circ / 5 = 36^\circ \), the returning path from O to O is \( O \rightarrow A \rightarrow B \rightarrow C \rightarrow D \rightarrow O \) and has 5 steps, with A and C being on line L and B and D being on line M. By reflection, two quadrilaterals \( OABA' \) and \( CBC'D \) are produced from the figure of the returning path, where A' and C' are reflections of A and C about line M. Hence, the returning paths are \( O \rightarrow A \rightarrow B \rightarrow A' \rightarrow O \) and \( O \rightarrow A \rightarrow B \rightarrow C' \rightarrow D \rightarrow C \rightarrow B \rightarrow A' \rightarrow O \) and have 4 and 8 steps. So, this case \( m = 5 \) is true.

Suppose the case \( m = 2n + 1 \), where \( n \in N \) and \( \theta = 180^\circ / (2n+1) \), stands. There are \((m+1)/2\) points each on both lines L and M. Then, \((m+1)/2\) quadrilaterals are produced, which correspond to \((m+1)/2\) returning paths. So, the number of steps is the set of \( \{ 4k \mid 1 \leq k \leq (m+1)/2, k \in N, m \text{ is odd} \} \). Next, when \( m = 2n + 3 \), where \( n \in N \) and \( \theta = 180^\circ / (2n+3) \), stands, the returning path from O to O is \( O \rightarrow P_1(a_1) \rightarrow Q_1(b_1) \rightarrow P_1(a_2) \rightarrow Q_1(b_2) \rightarrow P_1(a_3) \rightarrow Q_1(b_3) \rightarrow \ldots \rightarrow P_{(m-1)/2}(a_{(m-1)/2}) \rightarrow Q_{(m-1)/2}(b_{(m-1)/2}) \rightarrow O \), and has \( 2n + 3 \) steps, where \( P_1 \) to \( P_{(m-1)/2} \) are on line L, and \( Q_1 \) to \( Q_{(m-1)/2} \) are on line M. By reflection, \((m+1)/2\) quadrilaterals, such as \( OOP_1P_1' \), \( Q_1P_1Q_1Q_1' \), and so on, are produced from the figure of the returning path. Hence, the returning paths are \( O \rightarrow P_1(a_1) \rightarrow Q_1(b_1) \rightarrow P_1(a_2) \rightarrow Q_1(b_2) \rightarrow O \) and \( O \rightarrow P_1(a_1) \rightarrow Q_1(b_1) \rightarrow P_1(a_2) \rightarrow Q_1(b_2) \rightarrow P_1(a_3) \rightarrow Q_1(b_3) \rightarrow \ldots \rightarrow P_{(m-1)/2}(a_{(m-1)/2}) \rightarrow Q_{(m-1)/2}(b_{(m-1)/2}) \rightarrow O \), which has m steps. The second part is proven by the reflection of the original figure’s returning path. According to the previous findings, \( P_1 \) to \( P_{(m-1)/2} \) are on line L, and \( Q_1 \) to \( Q_{(m-1)/2} \) are on line M. \( P_1, P_2, \ldots, P_{(m+1)/2} \) are reflected about line M to create \( P_1', P_2', \ldots, P_{(m+1)/2}' \). Then, there are two situations regarding the returning paths. The first is when \( (m-1)/2 \) is even. Its path is \( O \rightarrow P_1(a_1) \rightarrow Q_1(b_1) \rightarrow P_1(a_2) \rightarrow Q_1(b_2) \rightarrow P_1(a_3) \rightarrow Q_1(b_3) \rightarrow \ldots \rightarrow P_{(m-1)/2}(a_{(m-1)/2}) \rightarrow Q_{(m-1)/2}(b_{(m-1)/2}) \rightarrow O \), which has m steps. The second case is when \( (m-1)/2 \) is odd. Its path is \( O \rightarrow P_1(a_1) \rightarrow Q_1(b_1) \rightarrow P_2(a_2) \rightarrow Q_1(b_2) \rightarrow P_2(a_3) \rightarrow Q_1(b_3) \rightarrow \ldots \rightarrow P_{(m-1)/2}(a_{(m-1)/2}) \rightarrow Q_{(m-1)/2}(b_{(m-1)/2}) \rightarrow O \), which also has m steps. Q.E.D.

This result can be described in Theorem 5.

Theorem 5

Suppose that 3 lines, L, M, and N intersect at a point O, and that their intersecting angles are the same. According to the intersecting angles, the table shows the possible number of steps it takes to return to O for certain angles between three intersecting lines.

Suppose that the angle \( \theta_1 \) bound by lines L and M equals the angle \( \theta_2 \) bound by lines M and N. If \( \theta_1 = \theta_2 = 180^\circ / m \), and \( m \) is an odd number greater than or equal to 3, the returning path starts and ends at point O, and follows \( L \rightarrow M \rightarrow N \rightarrow M \rightarrow L \rightarrow M \rightarrow N \rightarrow M \rightarrow L \), and so on. Then, the number of steps is the set of \( \{ 4k \mid 1 \leq k \leq (m+1)/2, k \in N, m \text{ is odd} \} \cup \{ m \mid m = 2k + 1, k \in N \} \).

2.3.2 Finding the Recursive Relationship of the Points Not From and to O on Three Lines

First, we examine case 1, where \( \theta_1 = \theta_2 = 36^\circ \). In figure 41, the returning path is \( P \rightarrow Q \rightarrow P' \rightarrow Q \rightarrow P \), which is 4 steps. In figure 42, the returning path is \( P_1 \rightarrow Q_1 \rightarrow R_1 \rightarrow Q_1 \rightarrow P_1 \rightarrow Q_1 \rightarrow R_1 \rightarrow Q_1 \rightarrow P_1 \), which is 8 steps. In figure 43, the returning path is \( P_1 \rightarrow Q_1 \rightarrow R_1 \rightarrow Q_1 \rightarrow P_1 \rightarrow Q_1 \rightarrow R_1 \rightarrow Q_1 \rightarrow P_1 \), which is 10 steps. In figure 44, the returning path is \( P_1 \rightarrow Q_1 \rightarrow R_1 \rightarrow Q_1 \rightarrow P_1 \rightarrow Q_1 \rightarrow R_1 \rightarrow Q_1 \rightarrow P_1 \), which is 12 steps. In figure 45, the returning path is \( P_1 \rightarrow Q_1 \rightarrow R_1 \rightarrow Q_1 \rightarrow P_1 \rightarrow Q_1 \rightarrow R_1 \rightarrow Q_1 \rightarrow P_1 \), which is 16 steps. We noticed that certain points on line R were symmetrical to certain points on line P in the returning path of two lines when \( \theta = 36^\circ \) in figure 46.
Second, we examine case 2, where $\theta_1 = \theta_2 = 20^\circ$. In figure 47, the returning path is $P_1 \rightarrow Q \rightarrow R \rightarrow Q \rightarrow P_1$, which is 4 steps. In figure 48, the returning path is $P_1 \rightarrow Q_2 \rightarrow R \rightarrow Q_1 \rightarrow P_1$, which is 8 steps. In figure 49, the returning path is $P_1 \rightarrow Q \rightarrow R \rightarrow Q_1 \rightarrow P_2 \rightarrow Q \rightarrow R \rightarrow Q_2 \rightarrow P_1$, which is 12 steps. In figure 50, the returning path is $P_1 \rightarrow Q \rightarrow R \rightarrow Q_1 \rightarrow P_2 \rightarrow Q \rightarrow R \rightarrow Q_2 \rightarrow P_1$, which is 16 steps. In figure 51, the returning path is $P_1 \rightarrow Q \rightarrow R \rightarrow Q_1 \rightarrow P_2 \rightarrow Q \rightarrow R \rightarrow Q_2 \rightarrow P_3 \rightarrow Q \rightarrow R \rightarrow Q_3 \rightarrow P_1$, which is 18 steps. In figure 52, the returning path is $P_1 \rightarrow Q \rightarrow R \rightarrow Q_1 \rightarrow P_2 \rightarrow Q \rightarrow R \rightarrow Q_2 \rightarrow P_3 \rightarrow Q \rightarrow R \rightarrow Q_3 \rightarrow P_4 \rightarrow Q \rightarrow R \rightarrow Q_4 \rightarrow P_1$, which is 20 steps. In figure 53, the returning path is $P_1 \rightarrow Q \rightarrow R \rightarrow Q_1 \rightarrow P_2 \rightarrow Q \rightarrow R \rightarrow Q_2 \rightarrow P_3 \rightarrow Q \rightarrow R \rightarrow Q_3 \rightarrow P_4 \rightarrow Q \rightarrow R \rightarrow Q_4 \rightarrow P_5 \rightarrow Q \rightarrow R \rightarrow Q_5 \rightarrow P_1$, which is 24 steps. In figure 54, the returning path is $P_1 \rightarrow Q \rightarrow R \rightarrow Q_1 \rightarrow P_2 \rightarrow Q \rightarrow R \rightarrow Q_2 \rightarrow P_3 \rightarrow Q \rightarrow R \rightarrow Q_3 \rightarrow P_4 \rightarrow Q \rightarrow R \rightarrow Q_4 \rightarrow P_5 \rightarrow Q \rightarrow R \rightarrow Q_5 \rightarrow P_6 \rightarrow Q \rightarrow R \rightarrow Q_6 \rightarrow P_7 \rightarrow Q \rightarrow R \rightarrow Q_7 \rightarrow P_8 \rightarrow Q \rightarrow R \rightarrow Q_8 \rightarrow P_9 \rightarrow Q \rightarrow R \rightarrow Q_9 \rightarrow P_1$, which is 28 steps. In figure 55, the returning path is $P_1 \rightarrow Q \rightarrow R \rightarrow Q_1 \rightarrow P_2 \rightarrow Q \rightarrow R \rightarrow Q_2 \rightarrow P_3 \rightarrow Q \rightarrow R \rightarrow Q_3 \rightarrow P_4 \rightarrow Q \rightarrow R \rightarrow Q_4 \rightarrow P_5 \rightarrow Q \rightarrow R \rightarrow Q_5 \rightarrow P_6 \rightarrow Q \rightarrow R \rightarrow Q_6 \rightarrow P_7 \rightarrow Q \rightarrow R \rightarrow Q_7 \rightarrow P_8 \rightarrow Q \rightarrow R \rightarrow Q_8 \rightarrow P_9 \rightarrow Q \rightarrow R \rightarrow Q_9 \rightarrow P_1$, which is 32 steps. In figure 56, the returning path is $P_1 \rightarrow Q \rightarrow R \rightarrow Q_1 \rightarrow P_2 \rightarrow Q \rightarrow R \rightarrow Q_2 \rightarrow P_3 \rightarrow Q \rightarrow R \rightarrow Q_3 \rightarrow P_4 \rightarrow Q \rightarrow R \rightarrow Q_4 \rightarrow P_5 \rightarrow Q \rightarrow R \rightarrow Q_5 \rightarrow P_6 \rightarrow Q \rightarrow R \rightarrow Q_6 \rightarrow P_7 \rightarrow Q \rightarrow R \rightarrow Q_7 \rightarrow P_8 \rightarrow Q \rightarrow R \rightarrow Q_8 \rightarrow P_9 \rightarrow Q \rightarrow R \rightarrow Q_9 \rightarrow P_1$. 

89
→R →Q →P, which is 36 steps. We noticed that certain points on line R were symmetrical to certain points on line P in the returning path of two lines when θ = 20° in figure 57.
Based on the findings of the previous two cases, we gathered the information into a table, seen in table 2.
Table 2. Steps of Returning Path for Three lines (not from and to O)

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>Steps of returning path</th>
</tr>
</thead>
<tbody>
<tr>
<td>180°/3</td>
<td>4, 6, 8, 12</td>
</tr>
<tr>
<td>180°/5</td>
<td>4, 8, 10, 12, 16, 20</td>
</tr>
<tr>
<td>180°/7</td>
<td>4, 8, 12, 14, 16, 20, 24, 28</td>
</tr>
<tr>
<td>180°/9</td>
<td>4, 8, 12, 16, 18, 20, 24, 28, 32, 36</td>
</tr>
<tr>
<td>180°/11</td>
<td>4, 8, 12, 16, 20, 22, 24, 28, 32, 36, 40, 44</td>
</tr>
<tr>
<td>180°/13</td>
<td>4, 8, 12, 16, 20, 24, 26, 28, 32, 36, 40, 44, 48, 52</td>
</tr>
<tr>
<td>180°/15</td>
<td>4, 8, 12, 16, 20, 24, 28, 30, 32, 36, 40, 44, 48, 52, 56, 60</td>
</tr>
</tbody>
</table>

Note. The possible number of steps it takes to return to a point not O for certain angles between three intersecting lines. Based on conjecture, the results are as follows:

Knowing that three lines, L, M, and N, all intersect at one point, O, suppose that the angle $\theta_1$ bound by lines L and M equals the angle $\theta_2$ bound by lines M and N. If $\theta_1 = \theta_2 = 180^\circ / m$, and m is an odd number greater than or equal to 3, the returning path starts and ends at point O, and follows L→M→N→L→M→N→M→L, and so on. Then, the number of steps is the set of: $\{4k | 1 \leq k \leq m, k \in N, m \text{ is odd}\} \cup \{2m | m = 2k + 1, k \in N\}$.

Proof:

We will split our proof into two sections.

(1) The first part is proven through mathematical induction. When $m = 3$ and $\theta = 180^\circ / 3 = 60^\circ$, clearly the returning path from $P_1$ to $P_i$ is $P_1 \rightarrow Q_1 \rightarrow P_1 \rightarrow Q_1 \rightarrow P_1 \rightarrow Q_1 \rightarrow P_1$, and has 6 steps, with $P_1$, $P_2$, $P_3$ being on line L and $Q_1$, $Q_2$, $Q_3$ being on line M.

By reflection, three quadrilaterals $P_1Q_1R_1Q_2$, $P_1Q_1R_2Q_3$, $P_1Q_1R_3Q_2$ are produced from the figure of the returning path, where $R_1$, $R_2$, $R_3$ is the reflection of $P_1$, $P_2$, $P_3$ about line M. Hence, the returning path is $P_1 \rightarrow Q_1 \rightarrow R_1 \rightarrow Q_1 \rightarrow P_1$, and there are 4, 8 and 12 steps, respectively. So, this case $m = 3$ is true.

Suppose the case $m = 2n+1$, where $n \in N$ and $\theta = 180^\circ / (2n + 1)$, stands. There are $(m+1)/2$ points each on both lines L and M. Then, $(m+1)/2$ quadrilaterals are produced, which correspond to $(m+1)/2$ returning paths. So, the number of steps is the set of: $\{4k | 1 \leq k \leq m, k \in N, m \text{ is odd}\}$. Next, when $m = 2n+3$, where $n \in N$ and $\theta = 180^\circ / (2n + 3)$, stands, the returning path from $P_1$ to $P_i$ is $O \rightarrow P_1(a) \rightarrow Q_1(b) \rightarrow P_1(a) \rightarrow Q_1(b) \rightarrow P_1(a) \rightarrow Q_1(b) \rightarrow \ldots \rightarrow P_{m+1}(a_{m+1}) \rightarrow Q_{m+1}(b_{m+1}) \rightarrow O$, and has $2n+3$ steps, where $P_i$ to $P_{m+1}$ are on line L and $Q_i$ to $Q_{m+1}$ are on line M. By reflection, $m + 1$ quadrilaterals, such as $P_1Q_1R_1Q_2$, $P_1Q_1R_2Q_3$, $P_1Q_1R_3Q_2$, and so on, are produced from the figure of the returning path. Hence, the returning paths are $P_1 \rightarrow Q_1 \rightarrow R_1 \rightarrow Q_1 \rightarrow P_1$ and $P_1 \rightarrow Q_1 \rightarrow R_1 \rightarrow Q_1 \rightarrow P_1 \rightarrow Q_1 \rightarrow R_1 \rightarrow Q_1 \rightarrow P_1$, which have 4 and 8 steps. Then, every time m is increased by 2, an additional quadrilateral is produced and the number of steps increases by 4. So, the number of steps is the set of $\{4k | 1 \leq k \leq m, k \in N, m \text{ is odd}\}$, and this case $m = 2n+3$ also stands.

(2) The second part $\{2m | m = 2k - 1, k \in N\}$ is also proven by mathematical induction.
When \( m = 3 \) and \( \theta = 180^\circ / 3 = 60^\circ \), clearly the returning path from \( P_1 \) to \( P_1 \) is \( P_1 \rightarrow Q_1 \rightarrow P_2 \rightarrow Q_2 \rightarrow P_3 \rightarrow Q_3 \rightarrow P_4 \), in figure 61 and has 6 steps, with \( P_1, P_2, P_3 \) being on line L and \( Q_1, Q_2, Q_3 \) being on line M. By reflection, three quadrilaterals \( P_1Q_1R_1Q_2, P_1Q_2R_2Q_3, P_1Q_3R_3Q_4 \) are produced from the figure of the returning path, where \( R_1, R_2, R_3 \) is the reflection of \( P_1, P_2, P_3 \) about line M. Hence, the returning path is \( P_1 \rightarrow Q_1 \rightarrow R_1 \rightarrow Q_2 \rightarrow P_2 \rightarrow Q_3 \rightarrow P_3 \rightarrow Q_4 \rightarrow P_4 \), which is 6 steps. So, this case \( m = 3 \) is true.

Suppose the case \( m = 2k + 1 \), where \( k \in N \) and \( \theta = 180^\circ / (2k + 1) \) is true, the returning path from \( P_1 \) to \( P_1 \) is by the reflection of the original figure’s returning path. That is, the number of steps is \( 2m \) for \( m = 2k + 1, k \in N \) and its path is \( P_1 \rightarrow Q_1 \rightarrow R_1 \rightarrow Q_2 \rightarrow P_2 \rightarrow Q_3 \rightarrow P_3 \rightarrow \ldots \rightarrow R_{m-1} \rightarrow Q_m \rightarrow P_m \rightarrow Q_{m+1} \rightarrow Q_{m+2} \rightarrow P_{m+1} \rightarrow Q_{m+2} \rightarrow P_{m+2} \rightarrow Q_{m+3} \rightarrow P_{m+3} \rightarrow \ldots \rightarrow R_1 \rightarrow Q_1 \rightarrow P_1 \), which is \( 2m \) steps.

Consider the case \( m+1 = 2k + 3 \), where \( k \in N \) and \( \theta = 180^\circ / (2k + 3) \). According to the previous findings, \( P_1 \) to \( P_{n+1} \) are on line L, and \( Q_1 \) to \( Q_{n+1} \) are on line M. \( P_1, P_2, \ldots, P_{n+1} \) are reflected about line M to create \( R_1, R_2, \ldots, R_{n+1} \).

So, its path is \( P_1 \rightarrow Q_1 \rightarrow R_1 \rightarrow Q_2 \rightarrow P_2 \rightarrow Q_3 \rightarrow P_3 \rightarrow \ldots \rightarrow R_{n+1} \rightarrow Q_{n+1} \rightarrow P_{n+1} \rightarrow Q_{n+2} \rightarrow P_{n+2} \rightarrow Q_{n+3} \rightarrow P_{n+3} \rightarrow \ldots \rightarrow Q_1 \rightarrow P_1 \), which is \( 2m+2 \) steps. Therefore, the number of steps is \( 2m+2 \) for \( m = 2k + 1, k \in N \). Q.E.D.

This result can be described in Theorem 6.

**Theorem 6**

Suppose that 3 lines, L, M, and N intersect at a point O, and that their intersecting angles are the same. According to the intersecting angles, the table shows the possible number of steps it takes to return to a point not O for certain angles between three intersecting lines.

Suppose that the angle \( \theta_1 \) bound by lines L and M equals the angle \( \theta_2 \) bound by lines M and N. If \( \theta_1 = \theta_2 = 180^\circ / m \), and m is an odd number greater than or equal to 3, the returning path starts and ends at point O, and follows \( L \rightarrow M \rightarrow N \rightarrow L \rightarrow M \rightarrow N \rightarrow M \rightarrow L \), and so on. Then, the number of steps is the set of \( \{4k \mid 1 \leq k \leq m, k \in N, m \text{ is odd} \} \cup \{2m \mid m \text{ is odd} \} \).

**3. Main Results**

**Theorem 1**

When there are two lines that pass through the original point and make the angle \( \theta \), and the robot takes \( m \) steps from the original point to itself, the relationship between \( \theta \) and \( m \) is \( \theta = 180^\circ / m \).

**Theorem 2**

Suppose that two lines M and N intersect at a point O at an unknown angle, such that points \( P_i \) to \( P_n \) are on line N and points \( Q_i \) to \( Q_n \) are on line M. The distances from \( P_i \) to O is \( a_i \), the distance from \( P_{i+1} \) to O is \( a_{i+1} \), and so on. Similarly, the distances from \( Q_i \) to O is \( b_i \), the distances from \( Q_{i+1} \) to O is \( b_{i+1} \), and so on. Suppose Rusty is able to return from a point O to O. Then we have two cases, \( m=2n+1, n \in N \):

1. the ratios of \( a_i \) to \( a_i \) are \( a_i = \frac{\sin(2i-1)\theta}{\sin \theta} \) for \( i = 1 \) to \( (m-1)/2 \).

2. the ratios of \( b_i \) to \( b_i \) are \( b_i = \frac{2\theta}{\sin \theta} \) for \( i = 1 \) to \( (m-1)/2 \).

**Theorem 3**

Suppose that two lines M and N intersect at a point O at an unknown angle, such that points \( P_i \) to \( P_n \) are on line N and points \( Q_i \) to \( Q_n \) are on line M. The distances from \( P_i \) to O is \( a_i \), the distance from \( P_{i+1} \) to O is \( a_{i+1} \), and so on. Similarly, the distances from \( Q_i \) to O is \( b_i \), the distances from \( Q_{i+1} \) to O is \( b_{i+1} \), and so on. If and only if the unknown angle is \( 180^\circ / m \), Rusty takes \( m+1 \) steps from O to O to return, and vice versa.

Also, when \( \theta = 180^\circ / m \) and \( m = 2n+1, n \in N \), according to the cosine law of triangles, we can determine the recursive relationship between \( a_n \) and \( b_n \):

\[ a_{n+1} = a_n - 2b_n \cos(180^\circ / m), \ b_{n+1} + b_n = 2a_n \cos(180^\circ / m) \]

Therefore, the path of Rusty is \( O \rightarrow P_1(a_1) \rightarrow Q_1(b_1) \rightarrow P_2(a_1) \rightarrow Q_2(b_1) \rightarrow P_3(a_1) \rightarrow Q_3(b_1) \rightarrow \ldots \rightarrow P_{m-1}(a_{m-1},b_{m-1}) \rightarrow Q_{m-1}(a_{m-1},b_{m-1}) \rightarrow Q_m(a_{m-1},b_{m-1}) = O \), which means Rusty returned to the starting point.

**Theorem 4**

Suppose that two lines M and N intersect at a point O at an unknown angle, such that points \( P_i \) to \( P_n \) are on line N and points \( Q_i \) to \( Q_n \) are on line M. The distances from \( P_i \) to O is \( a_i \), the distance from \( P_{i+1} \) to O is \( a_{i+1} \), and so on. Similarly,
the distances from \( Q_1 \) to \( b_1 \), the distances from \( Q_2 \) to \( b_2 \), and so on. Suppose that a random point on line \( N \) not \( O \), \( P_1 \), is chosen. If and only if the unknown angle is 180° / \( m \) and \( m = 2n + 1, n \in N \), Rusty takes 2m steps from \( P_1 \) to return to \( P_1 \), and vice versa.

Also, when \( \theta = 180° / m \) and \( m = 2n + 1, n \in N \), according to the cosine law of triangles, we can determine the recursive relationship between \( a_n \) and \( b_n \):

\[
a_{n+1} + a_n = 2b_n \cos(180° / m), \quad b_{n+1} + b_n = 2a_n \cos(180° / m).
\]

Therefore, the path of Rusty is \( P(a_1) \rightarrow Q_1(b_1) \rightarrow P(a_2) \rightarrow Q_1(b_2) \rightarrow P(a_3) \rightarrow Q_1(b_3) \rightarrow \ldots \rightarrow P_m(a_m) \rightarrow Q_n(b_n) \rightarrow P_{m+1}(a_{m+1} = a_1) = P(a_1) \), which means Rusty returned to the starting point.

**Theorem 5**

Suppose that 3 lines, \( L, M, \) and \( N \) intersect at a point \( O \), and that their intersecting angles are the same. According to the intersecting angles, the table shows the possible number of steps it takes to return to \( O \) for certain angles between three intersecting lines.

Suppose that the angle \( \theta_1 \) bound by lines \( L \) and \( M \) equals the angle \( \theta_2 \) bound by lines \( M \) and \( N \). If \( \theta_1 = \theta_2 = 180° / m \), and \( m \) is an odd number greater than or equal to \( 3 \), the returning path starts and ends at point \( O \), and follows \( L \rightarrow M \rightarrow N \rightarrow M \rightarrow L \rightarrow M \rightarrow N \rightarrow M \rightarrow L \), and so on. Then, the number of steps is the set of \( \{k | 1 \leq k \leq (m+1)/2, k \in N, m is odd\} \cup \{m | m = 2k + 1, k \in N\} \).

**Theorem 6**

Suppose that 3 lines, \( L, M, \) and \( N \) intersect at a point \( O \), and that their intersecting angles are the same. According to the intersecting angles, the table shows the possible number of steps it takes to return to a point not \( O \) for certain angles between three intersecting lines.

Suppose that the angle \( \theta_1 \) bound by lines \( L \) and \( M \) equals the angle \( \theta_2 \) bound by lines \( M \) and \( N \). If \( \theta_1 = \theta_2 = 180° / m \), and \( m \) is an odd number greater than or equal to \( 3 \), the returning path starts and ends at point \( O \), and follows \( L \rightarrow M \rightarrow N \rightarrow M \rightarrow L \rightarrow M \rightarrow N \rightarrow M \rightarrow L \), and so on. Then, the number of steps is the set of \( \{k | 1 \leq k \leq m, k \in N, m is odd\} \cup \{2m | m = 2k + 1, k \in N\} \).

**4. Discussion**

**4.1 Conclusion**

Regarding Rusty starting at a point \( O \), we found that any angle would work as long as it was odd, smaller than 90°, and a divisor of 180. This is because if two lines meet at an angle larger than 90°, one angle must be larger and the other must be smaller than 90°. Also, since we initially considered the odd values of \( m \), the angle will be considered to be a divisor of 180. Meanwhile, we disregarded cases where \( m \) was even because Rusty would never return to the original point in these cases.

Regarding Rusty starting at a point not \( O \), we discovered that any point \( P_1 \) we chose would still guarantee Rusty’s return. We concluded this was because no matter how much we dilate a point, the returning path would still be the same shape. Also, since we initially considered the values of \( m \), we found the number of steps, 2m, is a divisor of 360. If 2m is a divisor of 360, it doesn’t guarantee that the angle of intersection will be an integer. Conversely, if the angle of intersection is an integer, in some cases, Rusty will start at a point not \( O \) and return to this point in some integer steps. For example, when \( \theta = 63° \), Rusty can return to a point not \( O \) in 40 steps. There are many other cases like this. We took all our findings and summarized them into six theorems.

**4.2 Limitations and Original Aspects**

We included the fact that we could only consider odd numbers of steps. We found that even numbers of steps, like 6 steps, were impossible if Rusty’s goal were to return to the original point. Another limitation we faced was how, during the investigation of 3 lines, the two angles between the three lines must have been equal. We tried doing two different angles, but decided not to pursue this idea because we couldn’t find anything that suggested that Rusty would return to his original point. Finally, we considered a project where we could write some sort of code that automatically draws out diagrams of two or three lines once we put in the appropriate data. However, we also decided not to pursue this idea because of time and lack of coding knowledge.

One original aspect of our project was that we considered three lines. Another original aspect was that we discovered the relationship between number of steps and \( \theta \) not only when Rusty starts at the intersection, but also when Rusty starts at a random point. Finally, using GeoGebra tools like constructing circles and reflections was also a unique research method.
5. Future Outlook

This project has only skimmed the surface of what could be discovered in terms of the Rusty the robot problem. Here is a list of topics we discussed, but decided to leave for a future project due to their complex and time-consuming natures:

(1) Four lines seems to be an appropriate topic to consider. Because we already discovered two line and three line scenarios, four lines would be an appropriate next step.

(2) Multiple lines that intersect into the shape of a circle was also an idea we discussed. For example, 5 lines where \(\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = 36^\circ\) make a circular shape.

(3) We also considered cases of not three lines, but three 3D planes (Yang et al., 2016). Would there still be a correlation between the angle and the number of steps somehow?

(4) We thought about three lines that don’t intersect at one point, unlike the three-line scenarios we discussed. If the third line was positioned in a special way, for example, parallel, then would a relationship or pattern appear?

(5) As discussed in the conclusion, there are certain thetas where Rusty can return in some number of steps that don’t satisfy the property that \(360^\circ/\theta\) is an integer, such as \(\theta = 75^\circ\) (24 steps) or \(\theta = 50^\circ\) (36 steps). Why is this the case?

References


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Authors contributions

Mr. Louis Tsai was responsible for creating the mathematical models and drafting the manuscript. Dr. Cheng-Hua Tsai was responsible for reviewing the research questions and revising the manuscript. Both authors approved the final manuscript.

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