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Using Mathematical Models and Extensions to Solve a Robot's Equidistant Returning Path

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Abstract

The main purpose of this project is to study the Rusty the Robot problem where a robot travels equidistantly from the starting point back to the starting point. We mainly use mathematical models such as trigonometric functions, recursive sequences, and mathematical induction to explore, deduce and demonstrate findings, while using mathematical computer software for calculation and verification. In the beginning, Geogebra was used to explore the original question and allow us to obtain some properties and results. Then, the topic was extended and the relationship between the number of steps and angles was explored, as well as the recursive relationship between the landing points during the travel process. Next, we altered the starting positions of Rusty to see how this affects our findings. Finally, the two intersecting straight lines are extended into three straight lines intersecting at the same point. Based on the angles between them, we discuss the number of returning paths from the starting point to the starting point in an equidistant traveling method. Based on the mathematical model constructed, this study constructs a return path diagram for the robot to return to the starting point in an equidistant manner according to different starting point positions and obtains the recursive relationship of the relative positions on this return path. During the research process of this study, some interesting mathematical theories were obtained. It is expected that these results can be applied to related fields of AI robot movement patterns in the future.

Keywords: characteristic equation, complex plane, De Moivre's theorem, recursion formula, returning angle, returning path, trigonometric methods

1. Introduction

1.1 Original Problem

Two lines M and N meet at point O, making an angle $\theta \leq 90°$ there. (More precisely, θ is the measure of that angle in degrees). Rusty the Robot walks in the plane with steps that alternate between those lines M and N, starting from the intersection point O. Rusty is old and has stiff legs so all his steps have the same length (but they may be in any direction). Since Rusty is not allowed to return to a spot he just occupied, he might become stuck with no legal step to make. We can see Rusty's path because he leaves a trail of red rust behind him as he walks ("Ross Program", 2016).

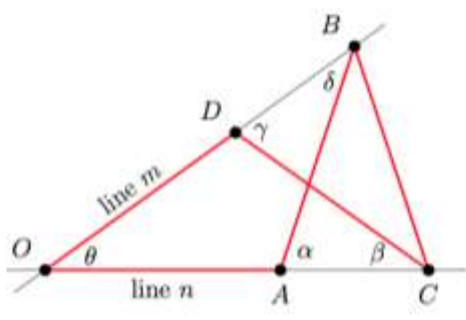

Figure 1. Part (a) of original problem

(a) For the angle θ pictured above, Rusty steps from O to A to B to C to D to O, returning to O in 5 steps. What is θ in this case?

(b) For which angle θ will Rusty return to O after 7 steps? What's the general rule? If Rusty returns to O after k steps, find (with proof!) the angle θ.

(c) Explore the following: for the 36 °situation in the picture above, suppose Rusty starts at some point P close to O on line M, such as in the figure below. What will his path be? Will Rusty return to his starting point P after some number of steps?

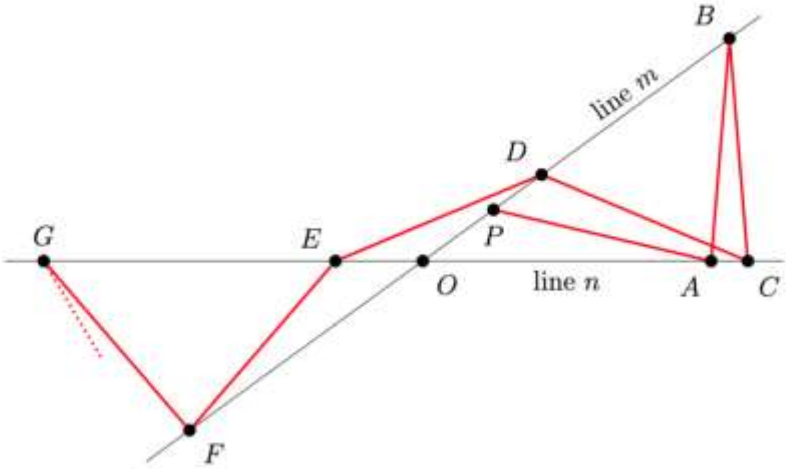

Figure 2. Part (c) of original problem

1.2 Research Motives

The "Rusty the Robot" problem is problem 4 of the 2016 Ross Summer Program Application. In the problem, Rusty the Robot walks along two lines that meet at an angle θ. All of Rusty's steps are the same length, and he leaves a trail of red rust behind him, marking his path. His goal is to start at the intersection between the two lines and walk in such a way so that he returns to his original spot. However, he is programmed so that he may not return to a spot on the two lines that he has been to already. For example, if θ = 90 degrees, Rusty can only make one legal step. Parts (a) and (b) of the question ask what θ is when Rusty returns to his original spot in 5 steps, 7 steps, and k steps. Part (c) of the question presents a scenario where Rusty doesn't start at the intersection of the two lines; instead, he starts at a point on one of the lines. Although this problem has been solved by many, this problem stood out to us because of its creativity and practicality. The researchers thought this concept could be applied to many things, such as how light reflections can be manipulated to shine at a specific location, or even how robotic systems navigate and map their surroundings (Raheem, Raafat, & Mahdi, 2022).

1.3 Procedure of the Study

First, we solve parts (a), (b), and (c) of the original problem using simple models and basic geometry. We also find mathematical models to explore deep solutions to the problem. For instance, the ratio of θ in relation to the number of steps.

Second, we extend the problem by finding the ratios of the distances between O and different points on the same line to the distance between O and the first step. We also find the recursive relationships for distance sequences from and to O. Meanwhile, we not only find the recursive relationships for scenarios not from and to O, but we also find the paths that Rusty takes during the recursive relationship scenarios.

Third, we consider three-line scenarios for certain angles and find the possible returning paths from and to O. We do the same for scenarios from and to a point not O.

2. Research Methods

2.1 Solution to the Original Problem

2.1.1 Solution to Part (a) of the Original Problem

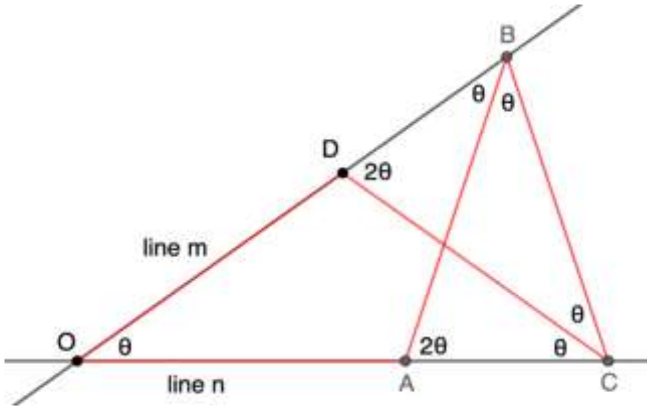

Figure 3. Solution to original problem's part (a)

Since $\overline{OD} = \overline{CD}$, $\angle DOC = \angle DCO = \theta$ in figure 3. So, based on the exterior angle theorem, $\angle CDB = 2\theta$. From the angle sum property of △BCD, $\theta + 2\theta + 2\theta = 180°$. Therefore, $\theta = 36°$.

We also found that there was no relationship between θ and the length of one step. This implies that the number of steps Rusty takes to return to his original position is not dependent on the length of one step Rusty takes.

2.1.2 Solution to Part (B) of the Original Problem and Finding the Relation Between θ and the Number of Steps

Since $\overline{OF} = \overline{FE}$, $\angle FOE = \angle FEO = \theta$. Similarly, since $\overline{EF} = \overline{ED}$, $\angle EFD = \angle EDF = 2\theta$ in figure 4. So, based on the exterior angle theorem, $\angle CDB = 3\theta$. From the angle sum property of △BCD, $\theta + 3\theta + 3\theta = 180°$. Therefore, $\theta = 180°/7$. Similar to the solution to 2.1.1, the implication of 2.1.2 was that the number of steps Rusty takes to return to his original position depends on θ, but not on the length of one of Rusty's steps.

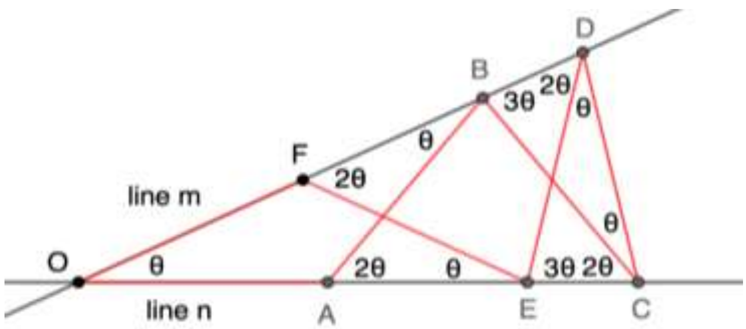

Figure 4. Solution to original problem's part (b)

We also noticed that the number of steps was always odd, which could be attributed to symmetry.

Through previous results, we can use GeoGebra to construct a diagram showing Rusty returning to his original point in 9 steps. Since $\overline{OA} = \overline{AB}$, $\angle AOB = \angle ABO = \theta$. Similarly, since $\overline{GH} = \overline{GF}$, $\angle GHF = \angle GFH = 2\theta$ in figure 5. Based on the same logic, $\angle EFD = \angle EDF = 2\theta$. So, from the angle sum property of △EFD, $\theta + 4\theta + 4\theta = 180°$. Therefore, $\theta = 20°$.

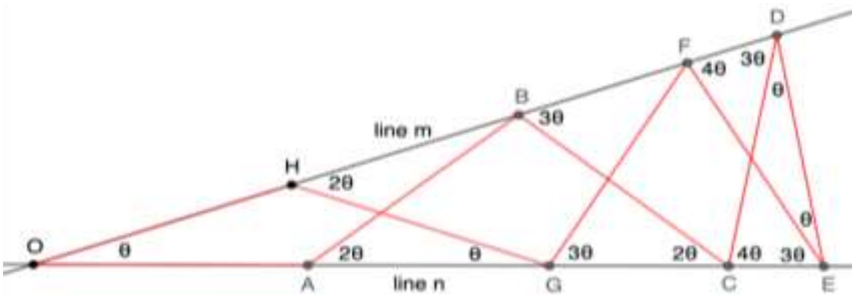

Figure 5. Rusty's path of 9 steps

2.1.3 Define Terms

To make subsequent proofs and research easier, we defined some terms for our use.

Definition 1 - Original point and non-original point: The original point is defined as the intersection of two lines. All other points are non-original points. For example, in figure 5, O is an original point and a start point. In figure 6, P_1 is a non-original point and a start point.

Definition 2 – Step Points: The step points are defined as the locations the robot passes through on his journey. The start point, whether an original point or not, counts as a step point. For example, in figure 6, line N has 5 step points, which are A,C,E,I, and G. Line M also has 5 step points, which are P,D,B,F, and H.

Definition 3 – Returning path: The returning path is defined as the robot's path from the start point back to the start point. This path will pass many step points. For example, in figure 6, the returning path is P→ A→ B →C →D→ E→ F→ G→ H→ I→ P.

Definition 4 – Equidistant Step Length: The step length is defined as the equidistant length of each of the robot's steps between two consecutive-forward step points. For example, in figure 6, $\overline{PA} = \overline{AB} = \overline{BC} = \cdots = \overline{IP}$.

Definition 5 – Step point sequence: The step point sequence is defined as the step points on the same line. The order of the sequence depends on the order of the returning path. For example, step point sequence for { P_n *} is defined as the step points on line N. In figure 5, the step point sequence is* P_1, P_2, P_3, *and* P_4 *for line N. This sequence correlates to step points A, C, E, and G. In the same way, {* Q_n *} is defined as the step points on line M. In figure 5, the step point sequence is* Q_1, Q_2, Q_3, *and* Q_4 *for line M. This sequence correlates to step points B, D, F, and H.*

Definition 6 – Step Distance Sequence: The step distance sequence is defined as the distance between a point in a step point sequence and the original point. For example, step distance sequence { a_n *} is defined as the distance from O to* P_n, *which is* $a_n = \overline{OP_n}$. *In the same way, {* b_n *} is defined as the distance from O to* Q_n, *which is* $b_n = \overline{OQ_n}$.

2.1.4 Initial Findings

Because m is always odd, both line M and line N will have $(m-1)/2$ points, respectively. We can conclude the results as follows:

2.1.4.1 Property of Reflection between P_n and Q_n

Lemma 1

If the robot walks *m* steps from the original point to itself, both line M and line N will have $(m-1)/2$ points. Then, the point P_k is reflected to the point $Q_{(m+1)/2-k}$ about the bisector line of M and N. That is, if P_i and Q_j are symmetric about the bisector line of M and N, the summation of index of P_i and Q_j is $(m+1)/2$.

2.1.4.2 The farthest point from O between P_1 to P_n and Q_1 to Q_n

Lemma 2

If the robot walks m steps from the original point to itself, both line M and line N will have $(m-1)/2$ points.

Then:

(1) When $m = 4k+1, k \in N$, P_{k+1} and Q_k are the farthest points from O

(2) When $m = 4k+3, k \in N$, P_{k+1} and Q_{k+1} are the farthest points from O.

2.1.4.3 The relationship between θ and m

Theorem 1

When there are two lines that pass through the original point and make the angle θ, and the robot takes m steps from the original point to itself, the relationship between θ and m is $\theta = 180° / m$.

Proof

By using mathematical induction on m:

1. When $m = 3$, there are 3 step distances. Lines M and N have a point that makes an equilateral triangle with point O. Hence, $\theta + \theta + \theta = 180°$, so $\theta = 60°$. Therefore, $m = 3$ is true.

When $m = 5$, there are 5 step distances. Lines M and N have two points as shown in figure 6.

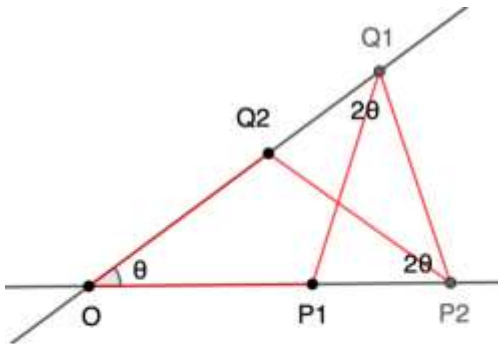

Figure 6. $m = 3$

Using the external angle theorem of a triangle, the triangle sum theorem, and the properties of an isosceles triangle, $\theta + 2\theta + 2\theta = 180°$, so $\theta = 36°$. Therefore, $m = 5$ is true.

2. Assume that the formula, $\theta = 180° / m$, is true for $m = 4k - 1$ and $m = 4k + 1$ ($k \in N$).

(1) Suppose $m = 4k - 1$ and $k \in N$, lines M and N have $2k - 1$ points, which are P_1 to P_{2k-1}, and Q_1 to Q_{2k-1}, as shown in figure 7.

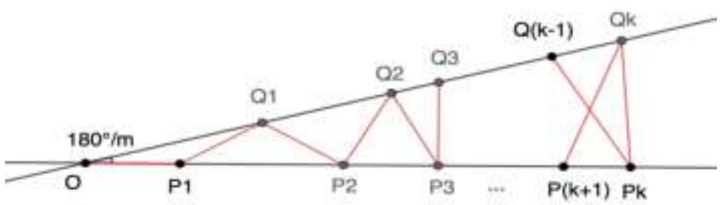

Figure 7. $m = 4k - 1$

Also, P_k and Q_k are the points that are farthest from O. Hence, $\angle OQ_{k-1}P_k = (2k-1)\theta = \angle OQ_kP_k$. So, $\theta + (2k-1)\theta + (2k-1)\theta = 180°$. That is, $\theta = 180°/(4k-1)$ is true and is identical to $\theta = 180°/m$ for $2k+1$.

If $m = 4k + 3$ and $k \in N$, lines M and N have $2k+1$ points, which are P_1 to P_{2k+1} and Q_1 to Q_{2k+1}. P_{k+1} and Q_{k+1} are the points farthest from O. By assumption, $\angle OP_{k+1}Q_k = 2k\theta$. Using the external angle theorem of a triangle, the triangle sum theorem, and the properties of an isosceles triangle, $\angle Q_{k+1}Q_kP_{k+1} = (2k+1)\theta = \angle Q_kQ_{k+1}P_{k+1}$, so $\theta + (2k+1)\theta + (2k+1)\theta = 180°$. That is, $\theta = 180°/(4k+3)$ is true for $m = 4k+3$.

(2) Suppose $m = 4k + 1$ and $k \in N$, lines M and N have $2k$ points, which are P_1 to P_{2k}, and Q_1 to Q_{2k}, as shown in figure 8.

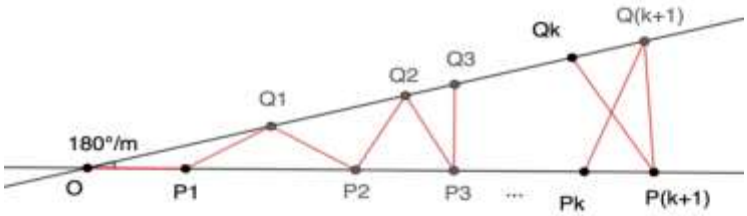

Figure 8. $m = 4k + 1$

Also, P_{k+1} and Q_k are the points that are farthest from O. Hence, $\angle OP_kQ_k = 2k\theta = \angle OP_{k+1}Q_k$. So, $\theta + 2k\theta + 2k\theta = 180°$. That is, $\theta = 180°/(4k+1)$ is true and is identical to $\theta = 180°/m$ for $m = 4k+1$.

If $m = 4k+5$ and $k \in N$, lines M and N have $2k+2$ points, which are P_1 to P_{2k+2} and Q_1 to Q_{2k+2}. P_{k+2} and Q_{k+1} are the points farthest from O. By assumption, $\angle OP_{k+1}Q_{k+1} = (2k+2)\theta$. Using the external angle theorem of a triangle, the triangle sum theorem, and the properties of an isosceles triangle, $\angle OP_{k+1}Q_{k+1} = (2k+2)\theta = \angle OP_{k+2}Q_{k+1}$, so $\theta + (2k+2)\theta + (2k+2)\theta = 180°$. That is, $\theta = 180°/(4k+5)$ is true for $m = 4k+5$.

Thus, using mathematical induction, the formula $\theta = 180°/m$ is true for odd numbers $m \ge 3$.

2.1.4.4 The order of a_1 to a_n and b_1 to b_n

Lemma 3

If the robot walks m steps from the original point to itself, both line M and line N will have $(m-1)/2$ points, which are P_1 to $P_{(m-1)/2}$ and Q_1 to $Q_{(m-1)/2}$. Their step distance sequences are { a_n } and { b_n }. Then,

(1) When $m = 4k+1, k \in N$, the order of a_1 to $a_{(m-1)/2}$ is $a_1 < a_{(m-1)/2} < a_2 < a_{(m-3)/2} < a_3 < a_{(m-5)/2} < \cdots < a_{(m-5)/4} < a_{(m+7)/4} < a_{(m-1)/4} < a_{(m+3)/4}$ and the order of b_1 to $b_{(m-1)/2}$ is $b_{(m-1)/2} < b_1 < b_{(m-3)/2} < b_2 < b_{(m-5)/2} < b_3 < \cdots < b_{(m+7)/4} < b_{(m-5)/4} < b_{(m+3)/4} < b_{(m-1)/4}$.

(2) When $m = 4k+3, k \in N$, the order of a_1 to $a_{(m-1)/2}$ is $a_1 < a_{(m-1)/2} < a_2 < a_{(m-3)/2} < a_3 < a_{(m-5)/2} < \cdots < a_{(m+5)/4} < a_{(m-3)/4} < a_{(m+1)/4}$ and the order of b_1 to $b_{(m-1)/2}$ is $b_{(m-1)/2} < b_1 < b_{(m-3)/2} < b_2 < b_{(m-5)/2} < b_3 < \cdots < b_{(m-3)/4} < b_{(m+5)/4} < b_{(m+1)/4}$.

2.1.4.5 The size of the exterior angle of $\angle OP_kQ_k$

Lemma 4

If the robot walks *m* steps from the original point to itself, both line M and line N will have $(m-1)/2$ points, which are P_1 to $P_{(m-1)/2}$ and Q_1 to $Q_{(m-1)/2}$.

(1) When $m = 4k+1, k \in N$, the size of the exterior angle of $\angle OP_kQ_k$ is $2k\theta$.

(2) When $m = 4k+3, k \in N$, the size of the exterior angle of $\angle OP_kQ_k$ is $2k\theta$.

2.1.5 Solution to Part (c) of the Original Problem

According to the results of 2.1.1 and 2.1.2, the number of steps and the length of steps are independent. We explored part (c) through Geogebra software. We first constructed two lines at a 36 ° angle, then constructed two points P and A randomly on the two lines as shown in figure 9. We constructed many circles using the two adjacent points: one being the center of the circle and one being a point on the circle. We did these operations until a circle passed through point P, the original point, as shown in the figure below. For example, initially, we made a circle with A being the center and AP being the radius, then found the intersection (point B) between the circle and line M. We then used point B as the center of another circle and AB as the radius and found the intersection (point C) between the circle and line N. Finally, we connected all the points together to construct Rusty's path returning to point P. We found that Rusty took 10 steps. However, we wondered if there was such a case in which Rusty would be unable to return to his original location, like if line PA was perpendicular to line M, or if line AB was perpendicular to line N.

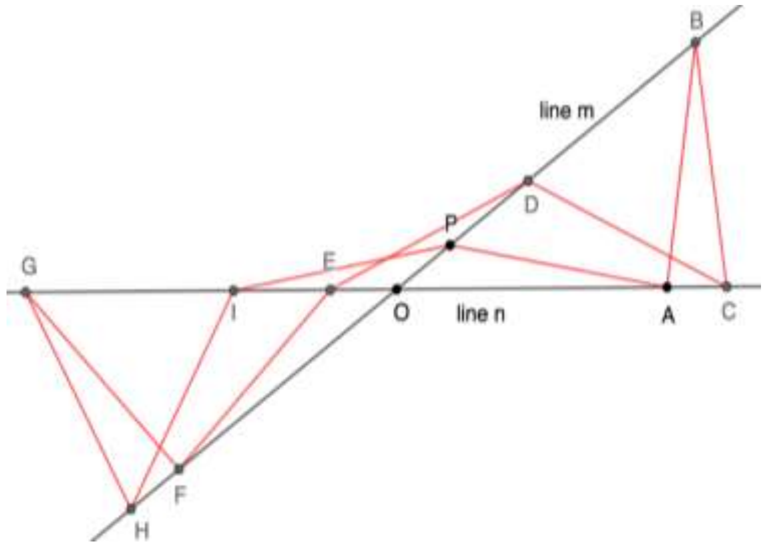

Figure 9. Solution to original problem’s part (c)

2.2 Extension Problems for Two Lines

2.2.0 Preliminary Theorem

Lemma 5

If m is an odd number and $\theta = 180°/m$, then $1 + \cos 2\theta + \cos 4\theta + \cos 6\theta + ... + \cos(2m-2)\theta = 0$.

Lemma 6

If m is an odd number and $\theta = 180°/m$, then

(1) When m≡1 (mod 4), then $1 + 2\cos 2\theta + 2\cos 4\theta + 2\cos 6\theta + ... + 2\cos(m-1)\theta/2 =$
$2\cos\theta + 2\cos 3\theta + 2\cos 5\theta + ... + 2\cos(m-3)\theta/2$.

(2) When m≡3 (mod 4), then $1 + 2\cos 2\theta + 2\cos 4\theta + 2\cos 6\theta + ... + 2\cos(m-3)\theta/2 =$
$2\cos\theta + 2\cos 3\theta + 2\cos 5\theta + ... + 2\cos(m-1)\theta/2$.

Lemma 7

If $k \in N$, then

(1) $1 + 2\cos 2\theta + 2\cos 4\theta + 2\cos 6\theta + ... + 2\cos 2k\theta = \dfrac{\sin(2k+1)\theta}{\sin\theta}$.

(2) $2\cos\theta + 2\cos 3\theta + 2\cos 5\theta + ... + 2\cos(2k-1)\theta = \dfrac{\sin 2k\theta}{\sin\theta}$.

2.2.1 Finding the Ratio of Step Length and Step Point Sequence on the Same Line

Result of m=7 case

Rusty’s path returning from O to O is O→A→B→C→D→E→F→O, as shown in figure 10. Using basic trigonometric principles, we found that the ratios of $\overline{OB}(=\overline{OE})/\overline{OA}$, $\overline{OC}(=\overline{OD})/\overline{OA}$, and $\overline{OF}/\overline{OA}$ are $2\cos\theta$, $1+2\cos 2\theta$, and 1, respectively.

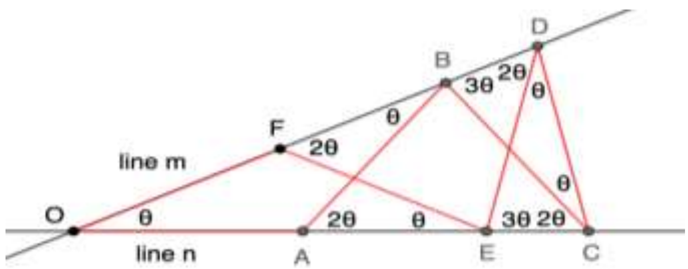

Figure 10. Ratios of *n*=7 case

By the definition of step distance sequence, these ratios are described as a_3/a_1, a_2/a_1, and a_1/a_1. These ratios are also described as b_1/a_1, b_2/a_1, and b_3/a_1.

Result of m=9 case

Rusty's path from O to O is O→A→B→C→D→E→F→G→H→O, as shown in figure 11. Using basic trigonometric principles, we found that the ratios of $\overline{OB}(=\overline{OG})/\overline{OA}$, $\overline{OC}(=\overline{OF})/\overline{OA}$, $\overline{OD}(=\overline{OE})/\overline{OA}$, and $\overline{OH}/\overline{OA}$ are $2\cos\theta$, $1+2\cos 2\theta$, $2\cos\theta+2\cos 3\theta$, and 1, respectively.

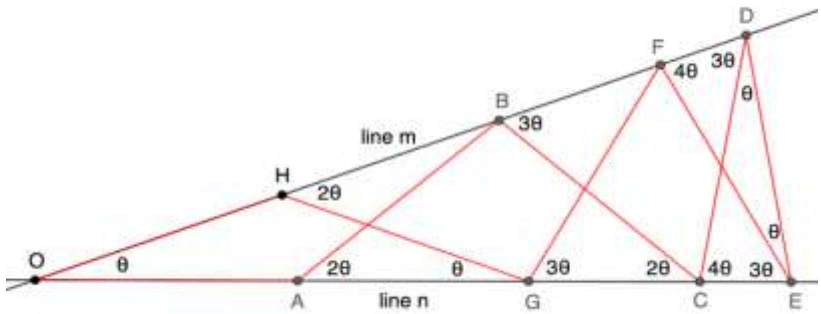

Figure 11. Ratios of *n*=9 case

By the definition of step distance sequence, these ratios are described as a_4/a_1, a_3/a_1, a_2/a_1, and a_1/a_1. These ratios are also described as b_1/a_1, b_2/a_1, b_3/a_1, and b_4/a_1.

Result of m=15 case

Rusty's path from O to O is O→A→B→C→D→E→F→G→H→I→J→K→L→M→O, as shown in figure 12. Using basic trigonometric principles, we found that the ratios of $\overline{OB}(=\overline{OM})/\overline{OA}$, $\overline{OC}(=\overline{OL})/\overline{OA}$, $\overline{OD}(=\overline{OK})/\overline{OA}$, $\overline{OE}(=\overline{OJ})/\overline{OA}$, $\overline{OF}(=\overline{OI})/\overline{OA}$, $\overline{OG}(=\overline{OH})/\overline{OA}$ and $\overline{ON}/\overline{OA}$ are $2\cos\theta$, $1+2\cos 2\theta$, $2\cos\theta+2\cos 3\theta$, $1+2\cos 2\theta+2\cos 4\theta$, $2\cos\theta+2\cos 3\theta+2\cos 5\theta$, $1+2\cos 2\theta+2\cos 4\theta+2\cos 6\theta$, and 1, respectively.

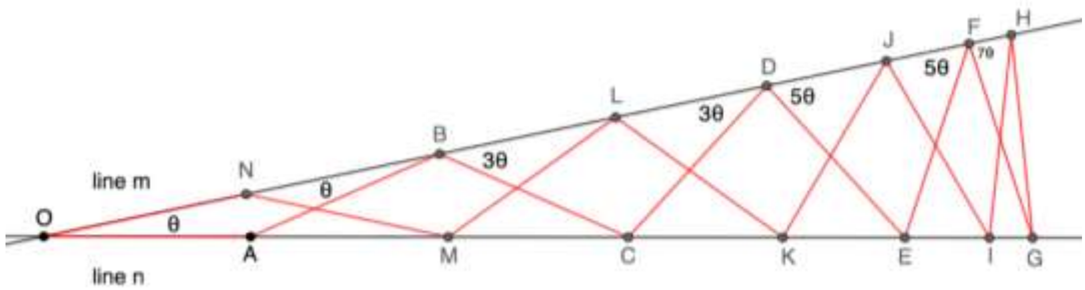

Figure 12. Ratios of *n*=15 case

Based off the deduction and conjecture of these three examples, the contents of the conjecture are described and proved as follows in Theorem 2.

Theorem 2

If the robot walks *m* steps from the original point to itself, both line M and line N will have $(m-1)/2$ points, which are P_1 to $P_{(m-1)/2}$ and Q_1 to $Q_{(m-1)/2}$., then

(1) the ratios of $a_{(m-1)/2}$ to a_1 are $a_i/a_1 = \frac{\sin(2i-1)\theta}{\sin\theta}$ for $i=1$ to $(m-1)/2$.

(2) the ratios of $b_{(m-1)/2}$ to b_1 are $b_i/a_1 = \frac{\sin 2i\theta}{\sin\theta}$ for $i=1$ to $(m-1)/2$.

Proof:

(1) When $m=4k+1$ and $k \in N$. That is, line M and line N both have $2k$ points, which are P_1 to P_{2k} and Q_1 to Q_{2k}, where P_{k+1} and Q_k are the farthest points from O. By lemma 4, the exterior angle of $\angle OP_iQ_i$ is $2i\theta$ for $i=1$ to $2k$, and $\angle OQ_iP_i=(2i-1)\theta$ for $i=1$ to $2k$.

In every ΔOP_iQ_i for $i=1$ to $2k$, we use law of sin to show that $\overline{OP_i}/\sin\angle OQ_iP_i=\overline{P_iQ_i}/\sin\angle P_iOQ_i$, which means $a_i/\sin(2i-1)\theta=a_1/\sin\theta$.

That is, $a_i/a_1=\sin(2i-1)\theta/\sin\theta$ for $i=1$ to $2k$.

On the other hand, by lemma 1 and lemma 7,

$b_i/a_1=a_{2k+1-i}/a_1=\sin[2(2k+1-i)-1)\theta/\sin\theta=\sin(180°-2i\theta)/\sin\theta=\sin(2i\theta)/\sin\theta$ for $i=1$ to $2k$.

That is, the ratios of $a_{(m-1)/2}$ to a_1 are $a_i/a_1=\dfrac{\sin(2i-1)\theta}{\sin\theta}$ for $i=1$ to $(m-1)/2$.

And the ratios of $b_{(m-1)/2}$ to b_1 are $b_i/a_1=\dfrac{\sin 2i\theta}{\sin\theta}$ for $i=1$ to $(m-1)/2$. QED

(2) When $m=4k+3$ and $k \in N$. That is, line M and line N both have $2k+1$ points, which are P_1 to P_{2k+1} and Q_1 to Q_{2k+1}, where P_{k+1} and Q_{k+1} are the farthest points from O. By lemma 4, the exterior angle of $\angle OP_iQ_i$ is $2i\theta$ for $i=1$ to $2k+1$, and $\angle OQ_iP_i=(2i-1)\theta$ for $i=1$ to $2k+1$.

In every ΔOP_iQ_i for $i=1$ to $2k+1$, we use law of sin to show that $\overline{OP_i}/\sin\angle OQ_iP_i=\overline{P_iQ_i}/\sin\angle P_iOQ_i$, which means $a_i/\sin(2i-1)\theta=a_1/\sin\theta$.

That is, $a_i/a_1=\sin(2i-1)\theta/\sin\theta$ for $i=1$ to $2k+1$.

On the other hand, by lemma 1 and lemma 6,

$b_i/a_1=a_{2k+2-i}/a_1=\sin[2(2k+2-i)-1)\theta/\sin\theta=\sin(180°-2i\theta)/\sin\theta=\sin(2i\theta)/\sin\theta$ for $i=1$ to $2k+1$.

That is, the ratios of $a_{(m-1)/2}$ to a_1 are $a_i/a_1=\dfrac{\sin(2i-1)\theta}{\sin\theta}$ for $i=1$ to $(m-1)/2$.

And the ratios of $b_{(m-1)/2}$ to b_1 are $b_i/a_1=\dfrac{\sin 2i\theta}{\sin\theta}$ for $i=1$ to $(m-1)/2$. QED

2.2.2 Finding the Recursive Relationship of the Points From and to O on Different Lines

According to the results of 2.2.1, we discovered that these points' locations had a relationship, which we explored further in 2.2.2. We assumed that the starting location was point O, the first point on line N was P_1, and the first point on line M was Q_1. The second point on line N and M were P_2 and Q_2, respectively, and the points before returning to point O were P_n and Q_n, respectively. The series from P_1 to P_n and the series from Q_1 to Q_n were also known as a_1 to a_n, and b_1 to b_n, respectively. For example, the step point sequence can also be shown with P_n and Q_n in figure 19 below.

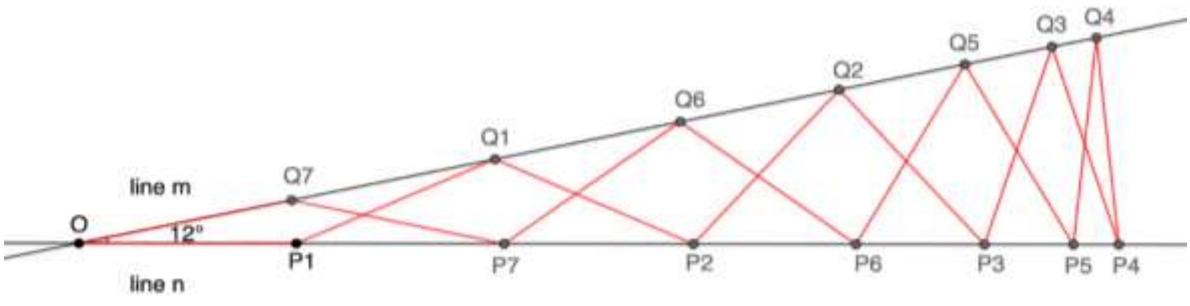

Figure 19. Relationship of 15 steps

We discovered that from P_1 to P_7 on line N, the step distance sequence first increases, then decreases. At the same time, Q_1 to Q_7 on line M follows the same rule. Hence, we sought to find out the characteristics regarding the step distance sequence. We used two examples, $\theta=36°$ and $\theta=180°/7$, to find out the recursive relationship between $<a_n>$ and $<b_n>$.

2.2.2.1 Case of $\theta=36°$.

Using figure 20 below, and the law of cosine, in ΔOP_1Q_1, $\overline{P_1Q_1}^2=a_1^2+b_1^2-2a_1b_1\cos 36°$. In ΔOP_2Q_1, $\overline{P_2Q_1}^2=a_2^2+b_1^2-2a_2b_1\cos 36°$. Finally, in ΔOP_2Q_2, $\overline{P_2Q_2}^2=a_2^2+b_2^2-2a_2b_2\cos 36°$. From the first and second equations, we can get $a_1+a_2=2b_1\cos 36°$. With the same logic, from the second and third equations, we get $b_1+b_2=2a_2\cos 36°$. We can conclude that $a_2+a_3=2b_2\cos 36°$, and $b_2+b_3=2a_3\cos 36°$.

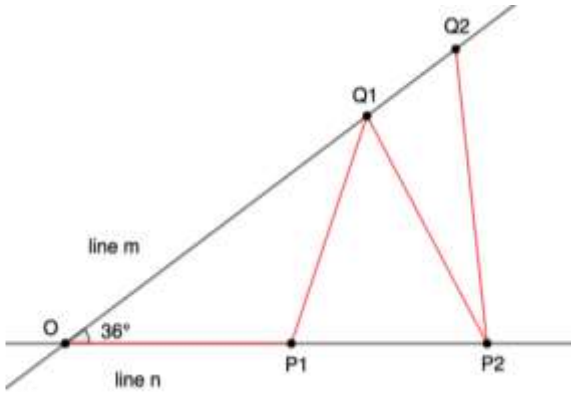

Figure 20. Relationship of $\theta=36°$ case

Therefore, the recursive formulas for the $<a_n>$ sequence and $<b_n>$ sequence are $a_{n-1}+a_n=2b_{n-1}\cos 36°$, and $b_{n-1}+b_n=2a_n\cos 36°$, respectively ("Recursive Formula", 2021). Coincidentally, $2\cos 36°$ is equal to the golden ratio, ϕ, and $\phi^2=\phi+1$, $\phi a_1=b_1$. Thus, we can conclude:

$a_2=\phi b_1-a_1$

$b_2=\phi a_2-b_1=\phi(\phi b_1-a_1)-b_1=(\phi^2-1)b_1-\phi a_1=\phi(b_1-a_1)$

$a_3=\phi b_2-a_2=\phi^2(b_1-a_1)-\phi b_1+a_1=(\phi^2-\phi)b_1-(1-\phi^2)a_1=b_1-\phi a_1=0$

This means that the robot's path is $O(0)\to P_1(a_1)\to Q_1(b_1)\to P_2(a_2)\to Q_2(b_2)\to O(a_3=0)$, as shown in figure 21.

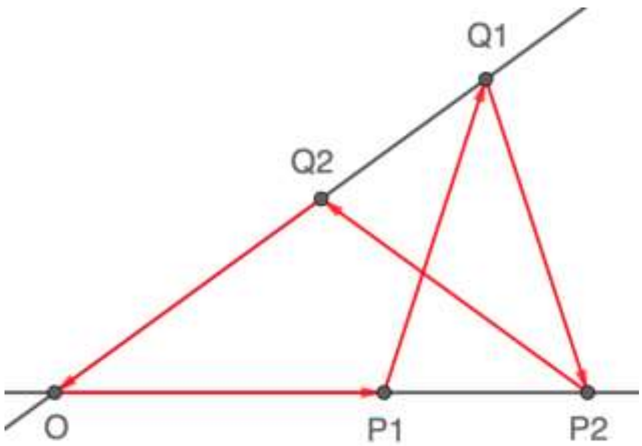

Figure 21. Returning path for $\theta=36°$ case

The meaning of $a_3=0$ means that the robot has returned to the original point because a_3 is the step distance from the origin to P_3. We also discovered that P_1 and P_2 on line N were symmetric to Q_2 and Q_1 on line M, respectively, with respect to the angle bisector of theta.

2.2.2.2 Case of $\theta=180°/7$.

Using figure 22 below, and the law of cosine, in ΔOP_1Q_1, $\overline{P_1Q_1}^2=a_1^2+b_1^2-2a_1b_1\cos(180°/7)$. In ΔOP_2Q_1, $\overline{P_2Q_1}^2=a_2^2+b_1^2-2a_2b_1\cos(180°/7)$. Finally, in ΔOP_2Q_2, $\overline{P_2Q_2}^2=a_2^2+b_2^2-2a_2b2\cos(180°/7)$. From the first and second equations, we can get $a_1+a_2=2b_1\cos(180°/7)$. With the same logic, from the second and third equations, we get $b_1+b_2=2a_2\cos(180°/7)$. We can conclude that $a_2+a_3=2b_2\cos(180°/7)$, and $b_2+b_3=2a_3\cos(180°/7)$.

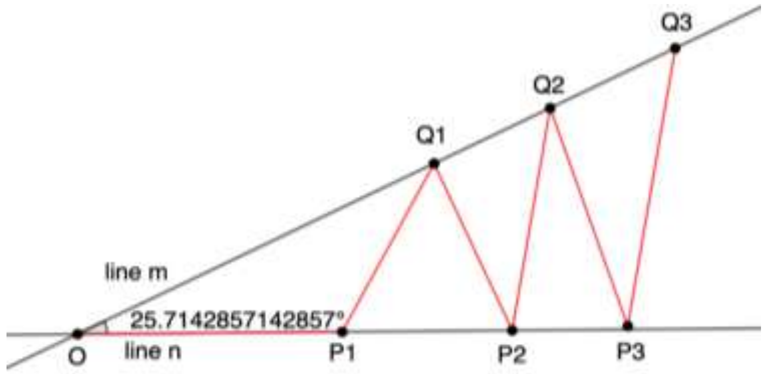

Figure 22. Relationship of $\theta = 180°/7$ case

Suppose that $2\cos(180°/7) = \lambda$. Then $a_{n-1} + a_n = \lambda b_{n-1}$, and $b_{n-1} + b_n = \lambda a_n$. While λ satisfies $\lambda^3 - \lambda^2 - 2\lambda + 1 = 0$ and $\lambda a_1 = b_1$, we can conclude:

$a_2 = \lambda b_1 - a_1$

$b_2 = \lambda a_2 - b_1 = \lambda(\lambda b_1 - a_1) - b_1 = (\lambda^2 - 1)b_1 - \lambda a_1$

$a_3 = \lambda b_2 - a_2 = (\lambda^3 - 2\lambda)b_1 - (\lambda^2 - 1)a_1 = (\lambda^2 - 1)b_1 - (\lambda^2 - 1)a_1$

$b_3 = \lambda a_3 - b_2 = (-\lambda^3 + \lambda^2 + 3\lambda - 1)\ b_1 + (-\lambda^3 + 2\lambda)a_1 = \lambda\ b_1 + (-\lambda^2 + 1)a_1$

$a_4 = \lambda b_3 - a_3 = \lambda(\lambda)\ b_1 + \lambda(-\lambda^2 + 1)a_1 - (\lambda^2 - 1)b_1 + (\lambda^2 - 1)a_1 = b_1 - \lambda a_1 = 0$

This means that the robot's path is $O(0) \to P_1(a_1) \to Q_1(b_1) \to P_2(a_2) \to Q_2(b_2) \to P_3(a_3) \to Q_3(b_3) \to O(a_4 = 0)$, as shown in figure 23.

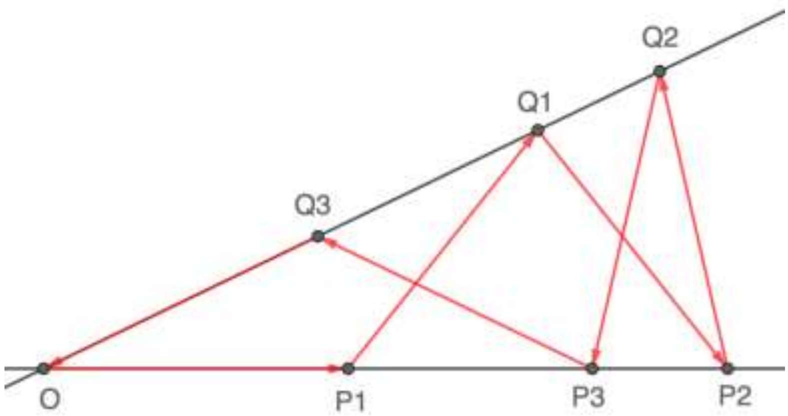

Figure 23. Returning path for $\theta = 180°/7$ case

The meaning of $a_4 = 0$ means that the robot has returned to the original point because a_4 is the step distance from the origin to P_4. We also discovered that P_1, P_2, and P_3 on line N were symmetric to Q_3, Q_2 and Q_1 on line M, respectively, with respect to the angle bisector of θ.

Finally, we consider the general recursive formula, where $\theta = 180°/m$, where m is an odd number. The distance from O to P_n is a_n, and the distance from O to Q_n is b_n. According to the cosine law of triangles, we can determine the recursive relationship between a_n and b_n: $a_{n-1} + a_n = 2b_{n-1}\cos(180°/m)$, $b_{n-1} + b_n = 2a_n\cos(180°/m)$

where $2\cos(180°/m)$ satisfies the equation of $(m-1)/2$ power and a cosine form. For example, $2\cos 36°$ satisfies $(2\cos 36°)^2 = 2\cos 36° + 1$. Also, $2\cos(180°/7)$ satisfies $[2\cos(180°/7)]^3 - [2\cos(180°/7)]^2 - 2[2\cos(180°/7)] + 1 = 0$.

Therefore, the path of Rusty is $O(0) \to P_1(a_1) \to Q_1(b_1) \to P_2(a_2) \to Q_2(b_2) \to P_3(a_3) \to Q_3(b_3) \to \ldots \to P_{(m-1)/2}(a_{(m-1)/2}) \to Q_{(m-1)/2}(b_{(m-1)/2}) \to P_{(m+1)/2}(a_{(m+1)/2} = 0) = O$, which means Rusty returned to the original point.

Proof:

From the cosine law of ΔOP_1Q_1, ΔOP_2Q_1, and ΔOP_2Q_2, we can get $a_1 + a_2 = 2b_1\cos\theta$ and $b_1 + b_2 = 2a_2\cos\theta$. From this, we can get $a_{n-1} + a_n = 2b_{n-1}\cos\theta$ and $b_{n-1} + b_n = 2a_n\cos\theta$. Next, we can get $a_{n+1} = (4\cos^2\theta - 2)a_n - a_{n-1}$. Using its characteristic equation, we can get $x^2 = (4\cos^2\theta - 2)x - 1$. The two roots are $\cos 2\theta + i\sin 2\theta$ and $\cos 2\theta - i\sin 2\theta$. So, a_n can be expressed with a combination of $(\cos 2\theta + i\sin 2\theta)^n$ and $(\cos 2\theta - i\sin 2\theta)^n$. $a_n = c_1(\cos 2\theta + i\sin 2\theta)^n + c_2(\cos 2\theta - i\sin 2\theta)^n$. Here, the values of c_1 and c_2 will be determined by a_1 and a_2.

We also have to find the relationship between a_1 and a_2.

$a_1 \cdot \cos 2\theta = a_2 - a_1^2$, which implies that $a_2 = (3 - 4\sin^2\theta)a_1$.

Finally, we have to prove $a_{(m+1)/2} = 0$, where *m* is odd.

$a_{(m+1)/2} = c_1(\cos 2\theta + i\sin 2\theta)^{(m+1)/2} + c_2(\cos 2\theta - i\sin 2\theta)^{(m+1)/2}$.

By De Moivre's Theorem ("De Moivre's Theorem", 2023), this can be written as

$c_1[\cos(m+1)\theta + i\sin(m+1)\theta] + c_2[\cos(m+1)\theta - i\sin(m+1)\theta]$. Here $c_1 = \dfrac{a_2 - a_1(\cos 2\theta - i\sin 2\theta)}{2\sin 2\theta[\cos(90° + 2\theta) + i\sin(90° + 2\theta)]}$,

$c_2 = \dfrac{a_2 - a_1(\cos 2\theta + i\sin 2\theta)}{-2\sin 2\theta[\cos(90° - 2\theta) + i\sin(90° - 2\theta)]}$. $m\theta = 180°$, and $a_{(m+1)/2} = c_1(-\cos\theta - i\sin\theta) + c_2(-\cos\theta + i\sin\theta)$

$$= \frac{a_2 \cdot 2\sin\theta - a_1 \cdot 2\sin 3\theta}{2\sin 2\theta} = \frac{2\sin\theta[a_2 - a_1(3 - 4\sin^2\theta)]}{2\sin 2\theta}$$

$$= \frac{2\sin\theta[a_2 - a_2]}{2\sin 2\theta} = 0$$ .Q.E.D

This result can be described in Theorem 3.

Theorem 3

Suppose that two lines M and N intersect at a point O at an unknown angle, such that points P_1 to P_n are on line N and points Q_1 to Q_n are on line M. The distances from P_1 to O is a_1, the distance from P_2 to O is a_2, and so on. Similarly, the distances from Q_1 to O is b_1, the distances from Q_2 to O is b_2, and so on. If and only if the unknown angle is $180°/m$, Rusty takes m steps from O to O to return, and vice versa.

Also, when $\theta = 180°/m$ and m is an odd number, according to the cosine law of triangles, we can determine the recursive relationship between a_n and b_n:

$$a_{n-1} + a_n = 2b_{n-1}\cos(180°/m),\quad b_{n-1} + b_n = 2a_n\cos(180°/m).$$

Therefore, the path of Rusty is $O \to P_1(a_1) \to Q_1(b_1) \to P_2(a_2) \to Q_2(b_2) \to P_3(a_3) \to Q_3(b_3) \to \ldots \to P_{(m-1)/2}(a_{(m-1)/2}) \to Q_{(m-1)/2}(b_{(m-1)/2}) \to P_{(m+1)/2}(a_{(m+1)/2} = 0) = O$, which means Rusty returned to the starting point.

The meaning of $a_{(m+1)/2} = 0$ means that the robot has returned to the original point because $a_{(m+1)/2}$ is the step distance from the origin to $P_{(m+1)/2}$. We also discovered that P_1 to $P_{(m+1)/2}$ on line N were symmetric to $Q_{(m+1)/2}$ to Q_1 on line M, respectively, with respect to the angle bisector of theta.

2.2.3 Finding the Recursive Relationship of the Points not From and to O on Different Lines

First, we consider case 1, where $\theta = 36°$ in figure 24. According to cosine law of ΔOP_1Q_1, ΔOP_2Q_1, and ΔOP_2Q_2, we can conclude:

$$l^2 = a_1^2 + b_1^2 - 2a_1b_1\cos 36°$$

$$l^2 = a_2^2 + b_1^2 - 2a_2b_1\cos 36°$$

$$l^2 = a_2^2 + b_2^2 - 2a_2b_2\cos 36°$$

Therefore, $a_1 + a_2 = 2b_1\cos 36°$ and $b_1 + b_2 = 2a_2\cos 36°$

We can conclude that $a_2 + a_3 = 2b_2\cos 36°$, and $b_2 + b_3 = 2a_3\cos 36°$.

Therefore, the recursive formulas for the $<a_n>$ sequence and $<b_n>$ sequence are $a_{n-1} + a_n = 2b_{n-1}\cos 36°$, and $b_{n-1} + b_n = 2a_n\cos 36°$, respectively. We can conclude that $a_n + a_{n+1} = 2b_n\cos 36°$ is also true. Adding the two recursive formulas for the $<a_n>$ sequence together, we get $a_{n-1} + 2a_n + a_{n+1} = 2\cos 36°(b_{n-1} + b_n) = 4\cos^2 36° \cdot a_n$. Simplified, this is $a_{n+1} = 2(2\cos^2 36° - 1) \cdot a_n - a_{n-1}$.

Suppose that $2(2\cos^2 36°-1)=\alpha+\beta$, and that $\alpha\beta=1$. Then, $a_{n+1}=2(2\cos^2 36°-1)\cdot a_n-a_{n-1}$ becomes

$a_{n+1}=(\alpha+\beta)a_n-\alpha\beta a_{n-1}$. We can conclude that $a_n-\alpha a_n=\beta^{n-2}(a_2-\alpha a_1)$, and

$a_n-\beta a_n=\alpha^{n-2}(a_2-\beta a_1)$. Simplifying even more, we get the formula of an, which is

$a_n=\alpha^n\dfrac{(a_2-\beta a_1)}{\alpha(\alpha-\beta)}+\beta^n\dfrac{(a_2-\alpha a_1)}{\beta(\beta-\alpha)}$, for all natural numbers *n*, and $\alpha+\beta=2(2\cos^2 36°-1)=\dfrac{\sqrt{5}-1}{2}$, and $\alpha\beta=1$.

According to $a_{n+1}=(\alpha+\beta)a_n-\alpha\beta a_{n-1}$, let $\alpha+\beta=\frac{\sqrt{5}-1}{2}=k$. Then, $k^2+k=1$, and $a_{n+1}=ka_n-a_{n-1}$.

For *n*=2, *n*=3, *n*=4, and *n*=5, we get:

$a_3=ka_2-a_1$, $a_4=ka_3-a_2=-ka_2-ka_1$, $a_5=ka_4-a_3=-a_2+ka_1$

$a_6=ka_5-a_4=k(-a_2+ka_1)-(-ka_2-ka_1)=a_1$

This means that the robot's path is $P_1(a_1)\to Q_1(b_1)\to P_2(a_2)\to Q_2(b_2)\to P_3(a_3)\to Q_3(b_3)\to P_4(a_4)\to Q_4(b_4)\to P_5(a_5)\to Q_5(b_5)\to P_6(a_6=a_1)=P_1(a_1)$, meaning when the robot starts at a point P_1 not O, he returns to P_1 in ten steps.

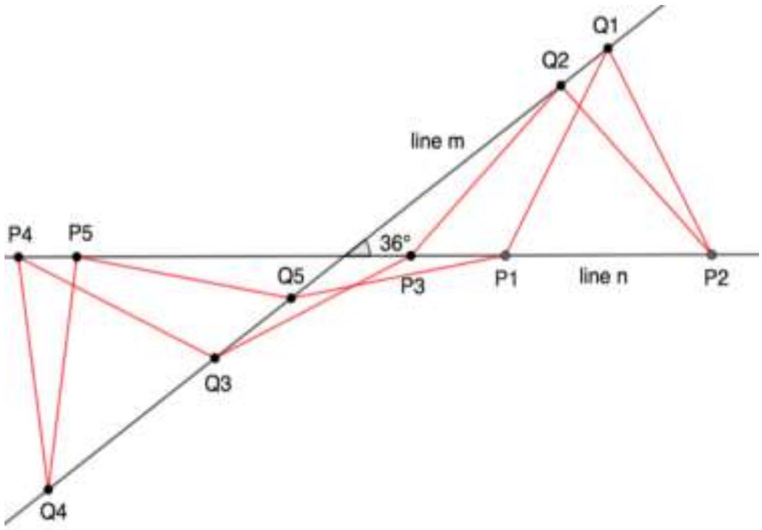

Figure 24. Returning path for $\theta=36°$ case

Second, we consider case 2, where $\theta=180°/7$ in figure 25. According to the cosine law of ΔOP_1Q_1, ΔOP_2Q_1, and ΔOP_2Q_2, we can conclude:

$l^2=a_1^2+b_1^2-2a_1b_1(\cos 180°/7)$

$l^2=a_2^2+b_1^2-2a_2b_1\cos(\cos 180°/7)$

$l^2=a_2^2+b_2^2-2a_2b_2\cos(\cos 180°/7)$

Therefore, $a_1+a_2=2b_1\cos(\cos 180°/7)$ and $b_1+b_2=2a_2\cos(\cos 180°/7)$

We can conclude that $a_2+a_3=2b_2\cos(\cos 180°/7)$, and $b_2+b_3=2a_3\cos(\cos 180°/7)$.

Therefore, the recursive formulas for the $<a_n>$ sequence and $<b_n>$ sequence are $a_{n-1}+a_n=2b_{n-1}\cos(\cos 180°/7)$, and $b_{n-1}+b_n=2a_n\cos(\cos 180°/7)$, respectively. We can conclude that $a_n+a_{n+1}=2b_n\cos(\cos 180°/7)$ is also true. Adding the two recursive formulas for the $<a_n>$ sequence together, we get

$a_{n-1}+2a_n+a_{n+1}=2(b_{n-1}+b_n)\cos(\cos 180°/7)=4\cos^2(\cos 180°/7)\cdot a_n$. Simplified, this is

$a_{n+1}=2[2\cos^2(\cos 180°/7)-1]\cdot a_n-a_{n-1}$.

Suppose that $2[2\cos^2(\cos 180°/7)-1]=\alpha+\beta$, and that $\alpha\beta=1$. Then, $a_{n+1}=2[2\cos^2(\cos 180°/7)-1]\cdot a_n-a_{n-1}$ becomes $a_{n+1}=(\alpha+\beta)a_n-\alpha\beta a_{n-1}$. We can conclude that $a_n-\alpha a_n=\beta^{n-2}(a_2-\alpha a_1)$, and $a_n-\beta a_n=\alpha^{n-2}(a_2-\beta a_1)$.

Simplifying even more, we get the formula of a_n, which is $a_n=\alpha^n\frac{(a_2-\beta a_1)}{\alpha(\alpha-\beta)}+\beta^n\frac{(a_2-\alpha a_1)}{\beta(\beta-\alpha)}$, for all natural numbers *n*, and $\alpha+\beta=2[2\cos^2(\cos 180°/7)-1]$, and $\alpha\beta=1$. According to $a_{n+1}=(\alpha+\beta)a_n-\alpha\beta a_{n-1}$, let

$2[2\cos^2(\cos 180°/7)-1]=k$. Then, $k^3+k^2-2k=1$, and $a_{n+1}=ka_n-a_{n-1}$.

For *n*=2, *n*=3, *n*=4, *n*=5, *n*=6, and *n*=7, we get:

$a_3=ka_2-a_1$, $a_4=ka_3-a_2=(k^2-1)a_2-ka_1$, $a_5=ka_4-a_3=-(k^2-1)a_2-(k^2-1)a_1$

$a_6=ka_5-a_4=-ka_2+(k^2-1)a_1$, $a_7=ka_6-a_5=-a_2+ka_1$

$a_8=ka_7-a_6=-ka_2+k^2a_1+ka_2-(k^2-1)a_1=a_1$

This means that the robot's path is $P_1(a_1)\to Q_1(b_1)\to P_2(a_2)\to Q_2(b_2)\to P_3(a_3)\to Q_3(b_3)\to P_4(a_4)\to Q_4(b_4)\to P_5(a_5)\to Q_5(b_5)$ $\to P_6(a_6)\to Q_6(b_6)\to P_7(a_7)\to Q_7(b_7)\to P_8(a_8=a_1)=P_1(a_1)$, meaning when the robot starts at a point P_1 not O, he returns to P_1 in 14 steps.

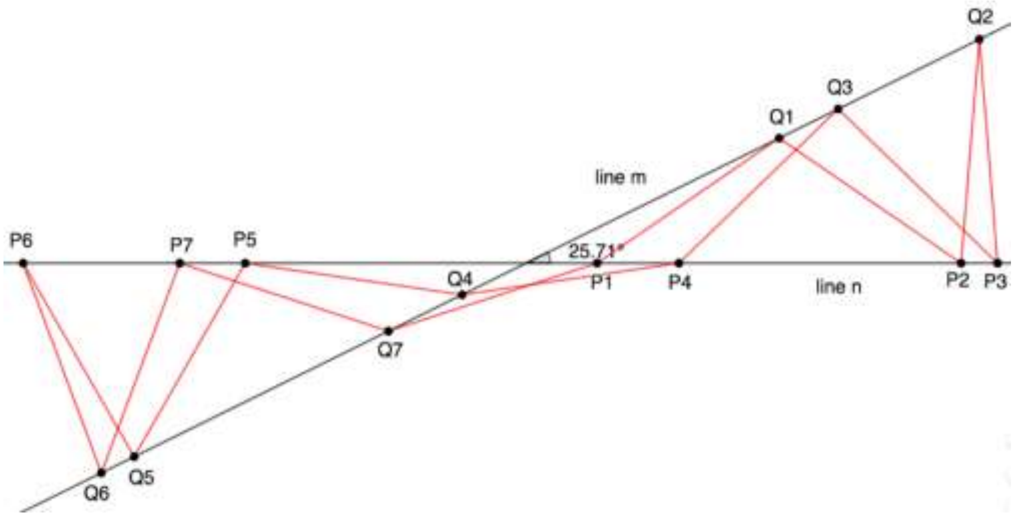

Figure 25. Returning path for $\theta=180°/7$ case

Finally, we consider the general recursive formula, where $\theta=180°/7$, where *m* is an odd number. The distance from O to P_n is a_n, and the distance from *O* to Q_n is b_n. According to the cosine law of triangles, we can determine the recursive relationship between a_n and b_n:

$$a_{n-1}+a_n=2b_{n-1}\cos(\cos 180°/m),\ b_{n-1}+b_n=2a_n\cos(\cos 180°/m)$$

Therefore, the path of Rusty is $P_1(a_1)\to Q_1(b_1)\to P_2(a_2)\to Q_2(b_2)\to P_3(a_3)\to Q_3(b_3)\to\ldots\to P_m(a_m)\to Q_m(b_m)\to P_{m+1}(a_{m+1}=a_1)$ $=P_1(a_1)$, which means Rusty returned to the starting point.

Proof:

Given that $\theta=\cos 180°/m$, where *m* is odd, through cosine law, we can get:

$a_{n-1}+a_n=2b_{n-1}\cos\theta$

$b_{n-1}+b_n=2a_n\cos\theta$

Claim: $a_{m+1}=a_1$, where *m* is odd. We use the two recursive equations:

$a_{n-1}+a_n=2b_{n-1}\cos\theta$

$b_{n-1}+b_n=2a_n\cos\theta$

Adding both of these equations, we get $a_{n-1}+2a_n+a_{n+1}=2\cos\theta\cdot(b_{n-1}+b_n)=4\cos^2\theta\cdot a_n$.

So, $a_{n-1}+(2-4\cos^2\theta)\cdot a_n+a_{n+1}=0$. Using its characteristic equation, we get

$1+(2-4\cos^2\theta)x+x^2=0$. Solving for the two roots, we get $\cos 2\theta \pm i\sin 2\theta$. Then, we get

$a_n=c_1(\cos 2\theta+i\sin 2\theta)^n+c_2(\cos 2\theta-i\sin 2\theta)^n$, where c_1 and c_2 are obtained by a_1 and a_2. So, we get

$c_1=\dfrac{a_1(\sin 4\theta-i\cos 4\theta)-a_2(\sin 2\theta-i\cos 2\theta)}{\sin 2\theta}$, $c_2=\dfrac{a_1(\sin 4\theta+i\cos 4\theta)+a_2(\sin 2\theta+i\cos 2\theta)}{\sin 2\theta}$. Using De Moivre's theorem,

$$\begin{aligned}
a_{m+1}&=c_1(\cos 2\theta+i\sin 2\theta)^{m+1}+c_2(\cos 2\theta-i\sin 2\theta)^{m+1}\\
&=c_1[\cos(2m+2)\theta+i\sin(2m+2)\theta]+c_2[\cos(2m+2)\theta-i\sin(2m+2)\theta]\\
&=c_1[\cos(360^\circ+2\theta)+i\sin(360^\circ+2\theta)]+c_2[\cos(360^\circ+2\theta)-i\sin(360^\circ+2\theta)]\\
&=c_1(\cos 2\theta+i\sin 2\theta)+c_2(\cos 2\theta-i\sin 2\theta)\\
&=a_1 \quad \text{Q.E.D}
\end{aligned}$$

This means the robot needs 2*m* steps to travel from P_1 (not O) to P_1.

This result can be described in Theorem 4.

Theorem 4

Suppose that two lines M and N intersect at a point O at an unknown angle, such that points P_1 to P_n are on line N and points Q_1 to Q_n are on line M. The distances from P_1 to O is a_1, the distance from P_2 to O is a_2, and so on. Similarly, the distances from Q_1 to O is b_1, the distances from Q_2 to O is b_2, and so on. Suppose that a random point on line N not O, P_1, is chosen. If and only if the unknown angle is $180^\circ/m$ and m is odd, Rusty takes 2m steps from P_1 to return to P_1, and vice versa.

Also, when $\theta=180^\circ/m$ and m is an odd number, according to the cosine law of triangles, we can determine the recursive relationship between a_n and b_n:

$$a_{n-1}+a_n=2b_{n-1}\cos(180^\circ/m),\ b_{n-1}+b_n=2a_n\cos(180^\circ/m).$$

Therefore, the path of Rusty is $P_1(a_1)\to Q_1(b_1)\to P_2(a_2)\to Q_2(b_2)\to P_3(a_3)\to Q_3(b_3)\to\ldots\to P_m(a_m)\to Q_m(b_m)\to P_{m+1}(a_{m+1}=a_1)=P_1(a_1)$, which means Rusty returned to the starting point.

2.3 Extension Problems for three lines

2.3.1 Finding the recursive relationship of the points from and to O on three lines

First, we examine case 1, where $\theta_1=\theta_2=36^\circ$. In figure 26, the returning path is $O\to A\to B\to C\to O$, which is 4 steps. In figure 27, the returning path is $O\to A\to B\to C\to D\to O$, which is 5 steps. In figure 28, the returning path is $O\to A\to B\to C\to D\to C'\to B\to A'\to O$, which is 8 steps. We noticed that point C in the first diagram is symmetrical to C in the returning path of two lines when $\theta=36^\circ$ in figure 29.

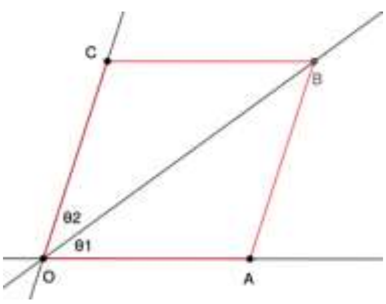

Figure 26. Three lines, 4 steps

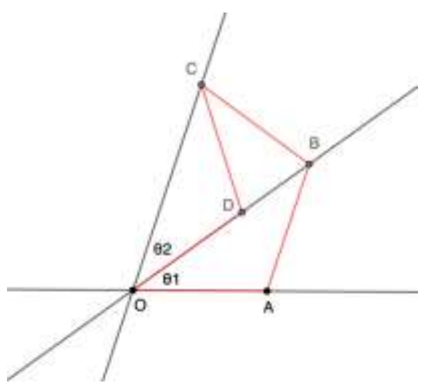

Figure 27. Three lines, 5 steps

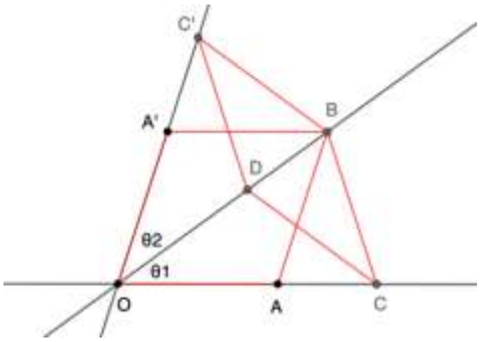

Figure 28. Three lines, 8 steps

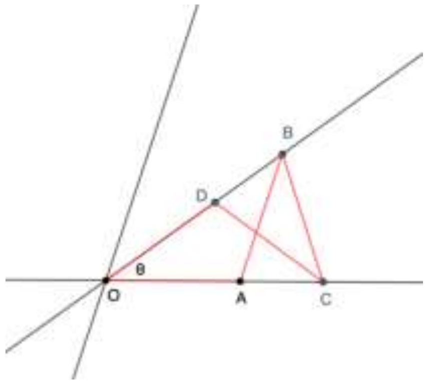

Figure 29. Two lines, $\theta = 36°$

Second, we examine case 2, where $\theta_1 = \theta_2 = 180°/7$. In figure 30, the returning path is $O \to A \to B \to A' \to O$, which is 4 steps. In figure 31, the returning path is $O \to A \to B \to C' \to D \to E \to F \to O$, which is 7 steps. In figure 32, the returning path is $O \to A \to B \to C \to D \to C' \to B \to A' \to O$, which is 8 steps. In figure 33, the returning path is $O \to A \to B \to C' \to D \to E \to F \to E' \to D \to C \to B \to A' \to O$, which is 12 steps. We noticed that points A', C', and E' are symmetrical to A, C, and E, respectively, in the returning path when $\theta = 180°/7$ in figure 34.

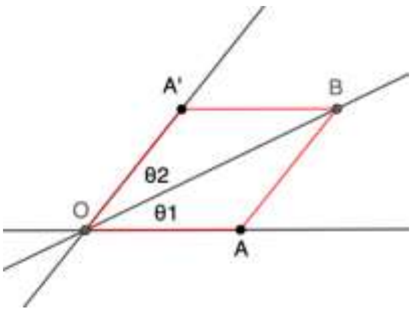

Figure 30. Three lines, 4 steps

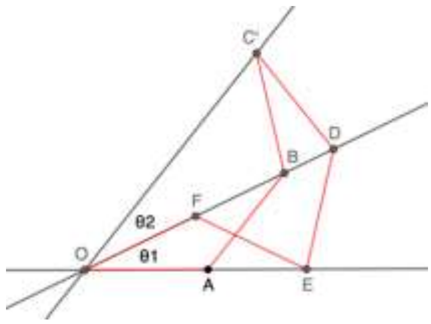

Figure 31. Three lines, 7 steps

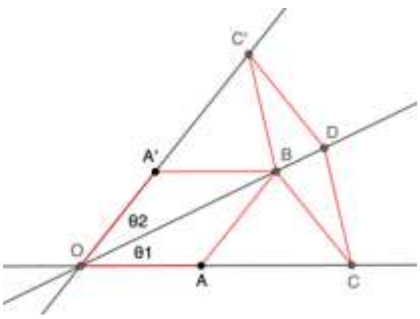

Figure 32. Three lines, 8 steps

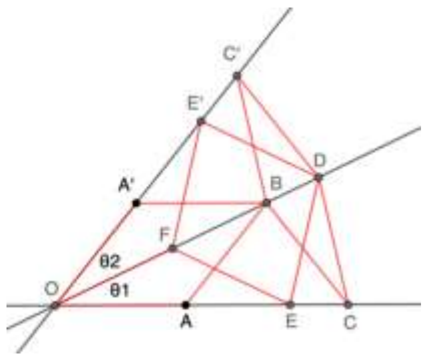

Figure 33. Three lines, 12 steps

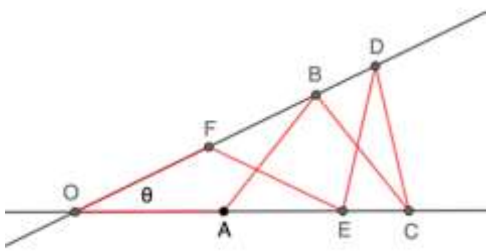

Figure 34. Two lines, $\theta = 180°/7$

Third, we examine case 3, where $\theta_1 = \theta_2 = 20°$. In figure 35, the returning path is $O \to A \to B \to A' \to O$, which is 4 steps. In figure 36, the returning path is $O \to A \to B \to C' \to D \to C \to B \to A' \to O$, which is 8 steps. In figure 37, the returning path is $O \to A \to B \to C' \to D \to E \to F \to G' \to H \to O$, which is 9 steps. In figure 38, the returning path is $O \to A \to B \to C' \to D \to E \to F \to E' \to D \to C \to B \to A' \to O$, which is 12 steps. In figure 39, the returning path is $O \to A \to B \to C' \to D \to E \to F \to G' \to H \to G \to F \to E' \to D \to C \to B \to A' \to O$, which is 16 steps. We noticed that points A', G', C', and E' are symmetrical to A, G, C, and E, respectively, in the returning path when $\theta = 20°$ in figure 40.

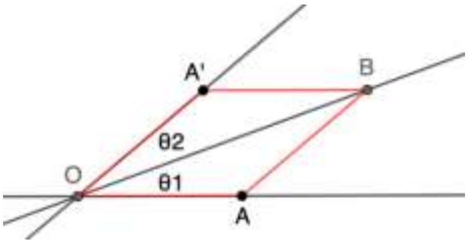

Figure 35. Three lines, 4 steps

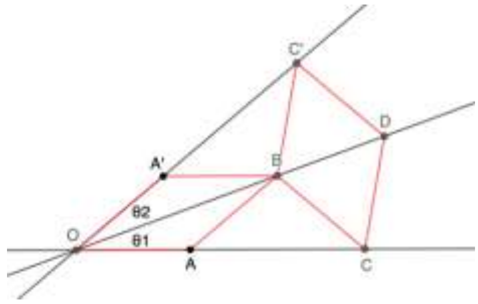

Figure 36. Three lines, 8 steps

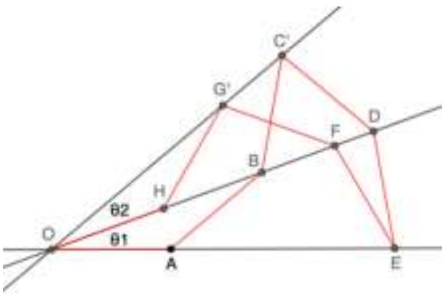

Figure 37. Three lines, 9 steps

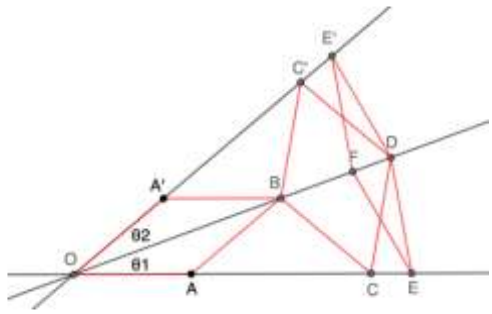

Figure 38. Three lines, 12 steps

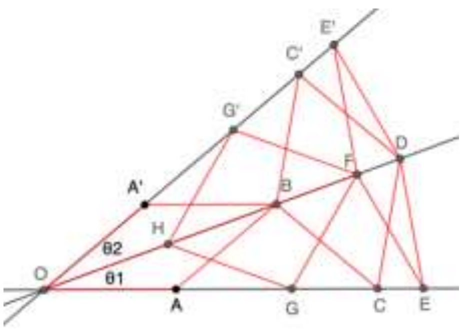

Figure 39. Three lines, 16 steps

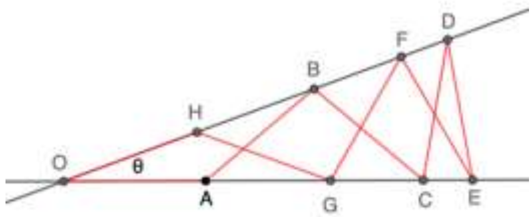

Figure 40. Two lines, $\theta = 20°$

Based on the findings of the first three cases, we gathered the information into a table, seen in table 1.

Table 1. Steps of Returning Path for Three Lines (from and to O)

θ	Steps of returning path
$180°/3$	3,4
$180°/5$	4,5,8
$180°/7$	4,7,8,12
$180°/9$	4,8,9,12,16
$180°/11$	4,8,11,12,16,20

$180°/13$	4,8,12,13,16,20,24
$180°/15$	4,8,12,15,16,20,24,28

Note. The possible number of steps it takes to return to O for certain angles between three intersecting lines.

Based on conjecture, the results are as follows:

Knowing that three lines, L, M, and N, all intersect at one point, O, $\theta_1=\theta_2=180°/m$, and *m* is an odd number greater than or equal to 3, the returning path starts and ends at point O, and follows L→M→N→M→L→M→N→M→L, and so on. Then, the number of steps is the set of $\{4k\,|\,1\le k\le (m+1)/2, k\in N, m\ is\ odd\}\cup\{m\,|\,m=2k+1, k\in N\}$.

Proof:

We will split our proof into two sections. The first part is proven through mathematical induction. When $m=3$ and $\theta=180°/3=60°$, clearly the returning path from *O* to *O* is $O\to A\to B\to O$ and has 3 steps, with A being on line L and B being on line M. By reflection, a quadrilateral $OABA'$ is produced from the figure of the returning path, where A' is the reflection of A about line M. Hence, the returning path is $O\to A\to B\to A'\to O$ and has 4 steps. So, this case $m=3$ stands. Next, when $m=5$ and $\theta=180°/5=36°$, the returning path from O to O is $O\to A\to B\to C\to D\to O$ and has 5 steps, with A and C being on line L and B and D being on line M. By reflection, two quadrilaterals $OABA'$ and $CBC'D$ are produced from the figure of the returning path, where A' and C' are reflections of A and C about line M. Hence, the returning paths are $O\to A\to B\to A'\to O$ and $O\to A\to B\to C'\to D\to C\to B\to A'\to O$ and have 4 and 8 steps. So, this case $m=5$ is true.

Suppose the case $m=2n+1$, where $n\in N$ and $\theta=180°/(2n+1)$, stands. There are $(m+1)/2$ points each on both lines L and M. Then, $(m+1)/2$ quadrilaterals are produced, which correspond to $(m+1)/2$ returning paths. So, the number of steps is the set of $\{\{4k\,|\,1\le k\le (m-1)/2, k\in N, m\ is\ odd\}$. Next, when $m=2n+3$, where $n\in N$ and $\theta=180°/(2n+3)$, stands, the returning path from O to O is $O\to P_1(a_1)\to Q_1(b_1)\to P_2(a_2)\to Q_2(b_2)\to P_3(a_3)\to Q_3(b_3)\to\ldots\to P_{n+1}(a_{n+1})\to Q_{n+1}(b_{n+1})\to O$, and has $2n+3$ steps, where P_1 to P_{n+1} are on line L, and Q_1 to Q_{n+1} are on line M. By reflection, $(m+1)/2$ quadrilaterals, such as $OP_1Q_1P_1'$, $P_2Q_2P_2'Q_1$, and so on, are produced from the figure of the returning path. Hence, the returning paths are $O\to P_1(a_1)\to Q_1(b_1)\to P_1'(a_1)\to O$ and $O\to P_1(a_1)\to Q_1(b_1)\to P_2'(a_2)\to Q_2(b_2)\to P_2(a_2)\to Q_1(b_1)\to P_1'(a_1)\to O$, which have 4 and 8 steps. Then, every time m is increased by 2, an additional quadrilateral is produced and the number of steps increases by 4. So, the number of steps is the set of $\{\{4k\,|\,1\le k\le (m+1)/2, k\in N, m\ is\ odd\}$, and this case $m=2n+3$ also stands.

The second part is proven by the reflection of the original figure's returning path. According to the previous findings, P_1 to $P_{(m-1)/2}$ are on line L, and Q_1 to $Q_{(m-1)/2}$ are on line M. P_2, P_4, …, $P_{(m-1)/2}$ are reflected about line M to create P_2', P_4', …, $P_{(m-1)/2}$. Then, there are two situations regarding the returning paths. The first is when $(m-1)/2$ is even. Its path is $O\to P_1(a_1)\to Q_1(b_1)\to P_2'(a_2)\to Q_2(b_2)\to P_3(a_3)\to Q_3(b_3)\to P_4'(a_4)\to\ldots\to P_{(m-1)/2}'(a_{(m-1)/2})\to Q_{(m-1)/2}(b_{(m-1)/2})\to O$, which has m steps. The second is when $(m-1)/2$ is odd. Its path is $O\to P_1(a_1)\to Q_1(b_1)\to P_2'(a_2)\to Q_2(b_2)\to P_3(a_3)\to Q_3(b_3)\to P_4'(a_4)\to\ldots\to P_{(m-1)/2}'(a_{(m-1)/2})\to Q_{(m-3)/2}(b_{(m-3)/2})\to P_{(m-1)/2}(a_{(m-1)/2})\to Q_{(m-1)/2}(b_{(m-1)/2})\to O$, which also has *m* steps. Q.E.D.

This result can be described in Theorem 5.

Theorem 5

Suppose that 3 lines, L, M, and N intersect at a point O, and that their intersecting angles are the same. According to the intersecting angles, the table shows the possible number of steps it takes to return to O for certain angles between three intersecting lines.

Suppose that the angle θ_1 bound by lines L and M equals the angle θ_2 bound by lines M and N. If $\theta_1=\theta_2=180°/m$, and m is an odd number greater than or equal to 3, the returning path starts and ends at point O, and follows L→M→N→M→L→M→N→M→L, and so on. Then, the number of steps is the set of $\{4k\,|\,1\le k\le (m+1)/2, k\in N, m\ is\ odd\}\cup\{m\,|\,m=2k+1, k\in N\}$.

2.3.2 Finding the Recursive Relationship of the Points not From and to O on Three Lines

First, we examine case 1, where $\theta_1=\theta_2=36°$. In figure 41, the returning path is $P\to Q_1\to P'\to Q_2\to P$, which is 4 steps. In figure 42, the returning path is $P_1\to Q_1\to R_1\to Q_2\to P_2\to Q_1\to R_2\to Q_3\to P_1$, which is 8 steps. In figure 43, the returning path is $P_1\to Q_1\to R_1\to Q_2\to P_2\to Q_3\to R_3\to Q_4\to P_3\to Q_5\to P_1$, which is 10 steps. In figure 44, the returning path is $P_1\to Q_1\to R_1\to Q_2\to P_2\to Q_1\to R_2\to Q_3\to P_3\to Q_4\to R_3\to Q_3\to P_1$, which is 12 steps. In figure 45, the returning path is $P_1\to Q_1\to R_1\to Q_2\to P_2\to Q_1\to R_2\to Q_3\to P_3\to Q_4\to P_4\to Q_5\to R_3\to Q_4\to P_3\to Q_3\to P_1$, which is 16 steps. We noticed that certain points on line R were symmetrical to certain points on line P in the returning path of two lines when $\theta=36°$ in figure 46.

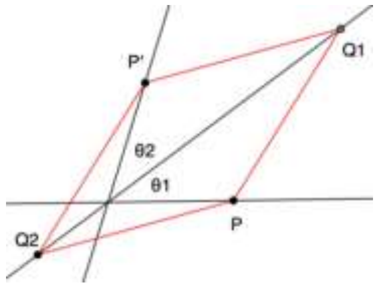

Figure 41. Three lines, 4 steps

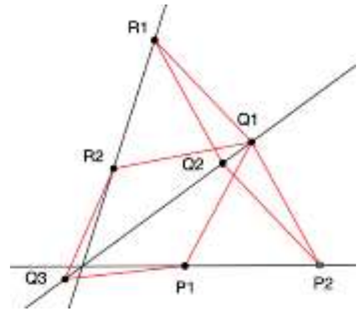

Figure 42. Three lines, 8 steps

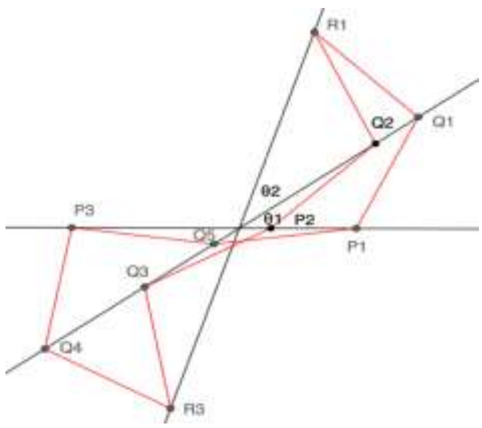

Figure 43. Three lines, 10 steps

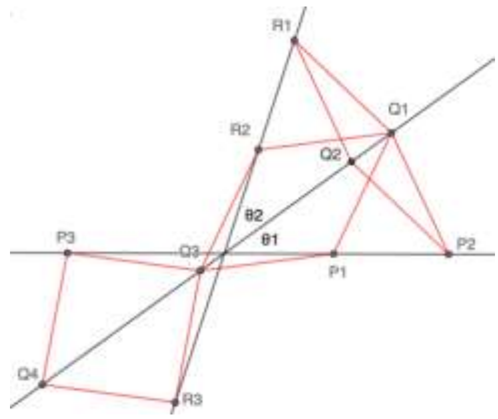

Figure 44. Three lines, 12 steps

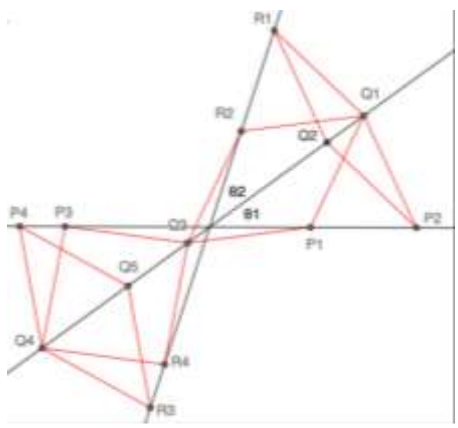

Figure 45. Three lines, 16 steps

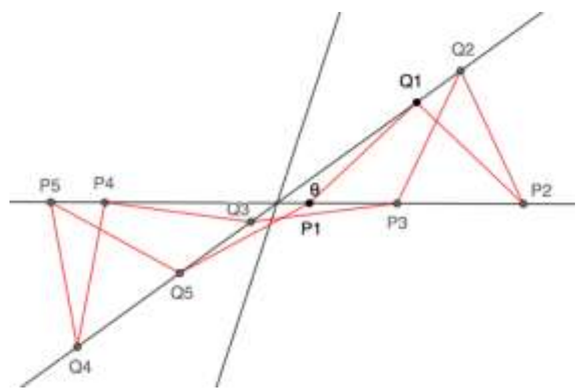

Figure 46. Two lines, $\theta = 36°$

Second, we examine case 2, where $\theta_1 = \theta_2 = 20°$. In figure 47, the returning path is $P_1 \to Q_1 \to R_1 \to Q_9 \to P_1$, which is 4 steps. In figure 48, the returning path is $P_1 \to Q_1 \to R_1 \to Q_9 \to P_9 \to Q_8 \to R_9 \to Q_9 \to P_1$, which is 8 steps. In figure 49, the returning path is $P_1 \to Q_1 \to R_2 \to Q_2 \to P_2 \to Q_1 \to R_1 \to Q_9 \to P_9 \to Q_8 \to R_9 \to Q_9 \to P_1$, which is 12 steps. In figure 50, the returning path is $P_1 \to Q_1 \to R_2 \to Q_2 \to P_3 \to Q_3 \to R_3 \to Q_2 \to P_2 \to Q_1 \to R_1 \to Q_9 \to P_9 \to Q_8 \to R_9 \to Q_9 \to P_1$, which is 16 steps. In figure 51, the returning path is $P_1 \to Q_1 \to R_2 \to Q_2 \to P_3 \to Q_3 \to R_4 \to Q_4 \to P_5 \to Q_5 \to R_6 \to Q_6 \to P_7 \to Q_7 \to R_8 \to Q_8 \to P_9 \to Q_9 \to P_1$, which is 18 steps. In figure 52, the returning path is $P_1 \to Q_1 \to R_2 \to Q_2 \to P_3 \to Q_3 \to R_4 \to Q_4 \to P_4 \to Q_3 \to R_3 \to Q_2 \to P_2 \to Q_1 \to R_1 \to Q_9 \to P_9 \to Q_8 \to P_9 \to Q_9 \to P_1$, which is 20 steps. In figure 53, the returning path is $P_1 \to Q_1 \to R_2 \to Q_2 \to P_3 \to Q_3 \to R_4 \to Q_4 \to P_4 \to Q_3 \to R_3 \to Q_2 \to P_2 \to Q_1 \to R_1 \to Q_9 \to P_9 \to Q_8 \to R_8 \to Q_7 \to P_8 \to Q_8 \to R_9 \to Q_9 \to P_1$, which is 24 steps. In figure 54, the returning path i $P_1 \to Q_1 \to R_2 \to Q_2 \to P_3 \to Q_3 \to R_4 \to Q_4 \to P_4 \to Q_3 \to R_3 \to Q_2 \to P_2 \to Q_1 \to R_1 \to Q_9 \to P_9 \to Q_8 \to R_8 \to Q_7 \to P_7 \to Q_6 \to R_7 \to Q_7 \to P_8 \to Q_8 \to R_9 \to Q_9 \to P_1$, which is 28 steps. In figure 55, the returning path is $P_1 \to Q_1 \to R_2 \to Q_2 \to P_3 \to Q_3 \to R_4 \to Q_4 \to P_5 \to Q_5 \to R_5 \to Q_4 \to P_4 \to Q_3 \to R_3 \to Q_2 \to P_2 \to Q_1 \to R_1 \to Q_9 \to P_9 \to Q_8 \to R_8 \to Q_7 \to P_7 \to Q_6 \to R_7 \to Q_7 \to P_8 \to Q_8 \to R_9 \to Q_9 \to P_1$, which is 32 steps. In figure 56, the returning path is $P_1 \to Q_1 \to R_2 \to Q_2 \to P_3 \to Q_3 \to R_4 \to Q_4 \to P_5 \to Q_5 \to R_6 \to Q_6 \to P_6 \to Q_5 \to R_5 \to Q_4 \to P_4 \to Q_3 \to R_3 \to Q_2 \to P_2 \to Q_1 \to R_1 \to Q_9 \to P_9 \to Q_8 \to R_8 \to Q_7 \to P_7 \to Q_6 \to R_7 \to Q_7 \to P_8 \to Q_8$

$\rightarrow R_9 \rightarrow Q_9 \rightarrow P_1$, which is 36 steps. We noticed that certain points on line R were symmetrical to certain points on line P in the returning path of two lines when $\theta = 20°$ in figure 57.

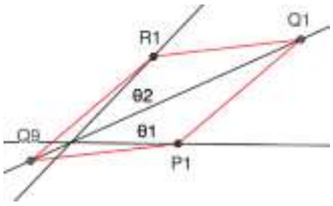

Figure 47. Three lines, 4 steps

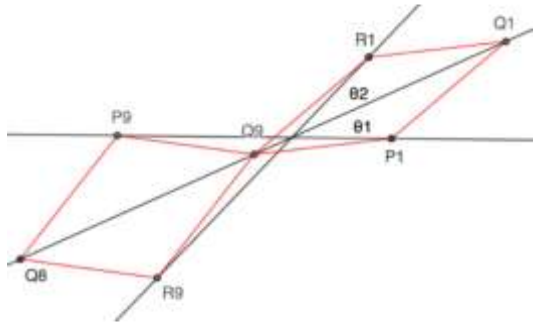

Figure 48. Three lines, 8 steps

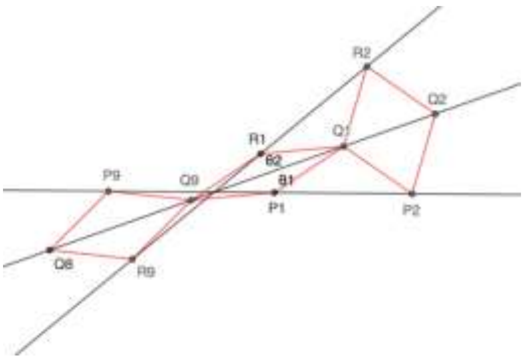

Figure 49. Three lines, 12 steps

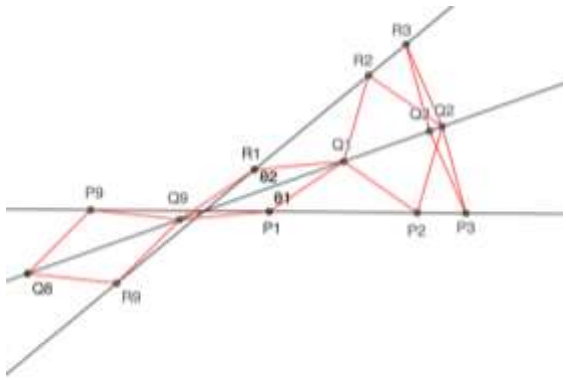

Figure 50. Three lines, 16 steps

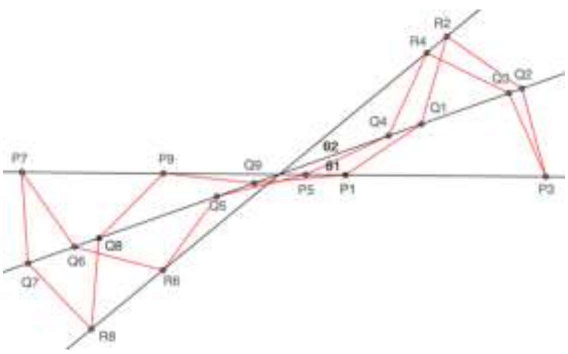

Figure 51. Three lines, 18 steps

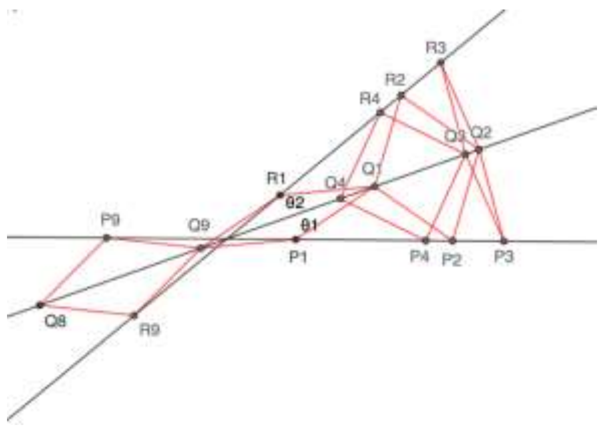

Figure 52. Three lines, 20 steps

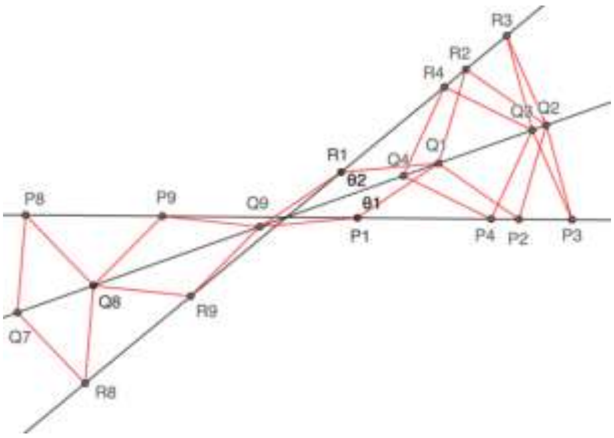

Figure 53. Three lines, 24 steps

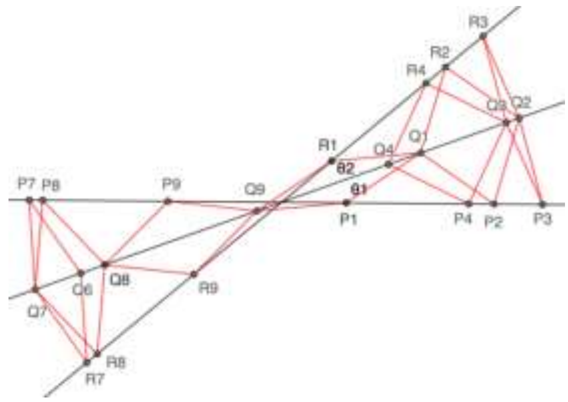

Figure 54. Three lines, 28 steps

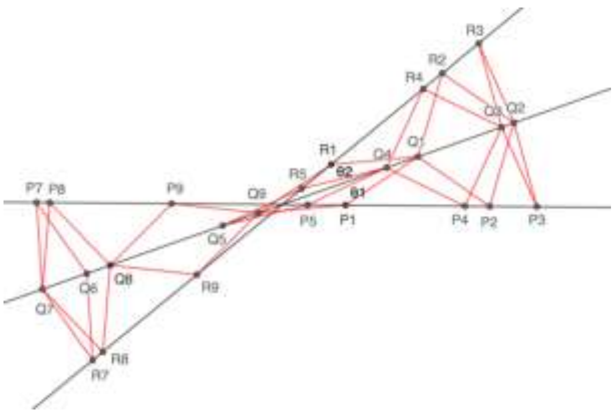

Figure 55. Three lines, 32 steps

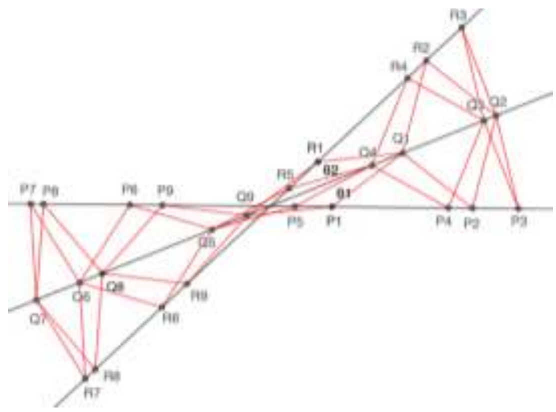

Figure 56. Three lines, 36 steps

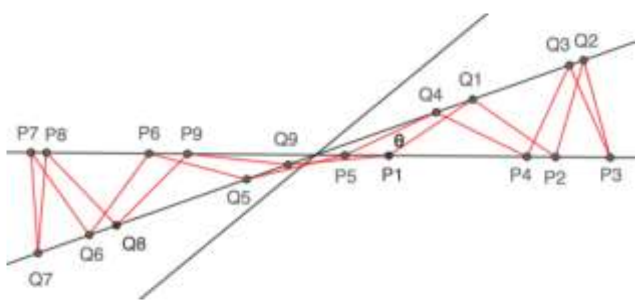

Figure 57. Two lines, $\theta = 20°$

Based on the findings of the previous two cases, we gathered the information into a table, seen in table 2.

Table 2. Steps of Returning Path for Three lines (not from and to O)

θ	Steps of returning path
$180°/3$	4,6,8,12
$180°/5$	4,8,10,12,16, 20
$180°/7$	4,8,12,14,16,20,24, 28
$180°/9$	4,8,12,16,18,20,24,28,32,36
$180°/11$	4,8,12,16,20,22,24,28,32,36,40,44
$180°/13$	4,8,12,16,20,24,26,28,32,36,40,44,48,52
$180°/15$	4,8,12,16,20,24,28,30,32,36,40,44,48,52,56,60

Note. The possible number of steps it takes to return to a point not O for certain angles between three intersecting lines.

Based on conjecture, the results are as follows:

Knowing that three lines, L, M, and N, all intersect at one point, O, suppose that the angle θ_1 bound by lines L and M equals the angle θ_2 bound by lines M and N. If $\theta_1 = \theta_2 = 180°/m$, and m is an odd number greater than or equal to 3, the returning path starts and ends at point O, and follows L→M→N→M→L→M→N→M→L, and so on. Then, the number of steps is the set of $\{4k \mid 1 \le k \le m, k \in N, m\ is\ odd\} \cup \{2m \mid m = 2k+1, k \in N\}$.

Proof:

We will split our proof into two sections.

(1) The first part is proven through mathematical induction. When $m = 3$ and $\theta = 180°/3 = 60°$, clearly the returning path from P_1 to P_1 is $P_1 \to Q_1 \to P_2 \to Q_2 \to P_3 \to Q_3 \to P_1$ in figure 58 and has 6 steps, with P_1, P_2, P_3 being on line L and Q_1, Q_2, Q_3 being on line M.

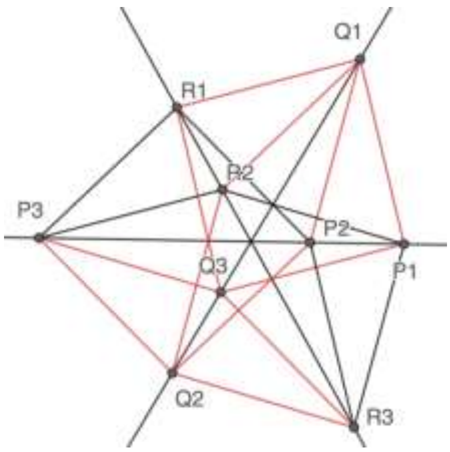

Figure 58. Three lines, $\theta = 60°$

By reflection, three quadrilaterals $P_1Q_1R_1Q_2$, $P_2Q_2R_2Q_1$, $P_3Q_3R_3Q_2$ are produced from the figure of the returning path, where R_1, R_2, R_3 is the reflection of P_1, P_2, P_3 about line M. Hence, the returning path is $P_1 \to Q_1 \to R_1 \to Q_2 \to P_1$, $P_1 \to Q_1 \to R_2 \to Q_2 \to P_3 \to Q_3 \to P_1$, $P_1 \to Q_1 \to R_1 \to Q_3 \to P_3 \to Q_2 \to R_3 \to Q_3 \to P_1$ and $P_1 \to Q_1 \to R_1 \to Q_3 \to P_3 \to Q_2 \to R_2 \to Q_1 \to P_2 \to Q_2 \to R_3 \to Q_3 \to P_1$ and there are 4, 8 and 12 steps, respectively. So, this case $m = 3$ is true.

Suppose the case $m = 2n+1$, where $n \in N$ and $\theta = 180°/(2n+1)$, stands. There are $(m+1)/2$ points each on both lines L and M. Then, $(m+1)/2$ quadrilaterals are produced, which correspond to $(m+1)/2$ returning paths. So, the number of steps is the set of { $\{4k \mid 1 \le k \le m, k \in N, m\ is\ odd\}$. Next, when $m = 2n+3$, where $n \in N$ and $\theta = 180°/(2n+3)$, stands, the returning path from P_1 to P_1 is $O \to P_1(a_1) \to Q_1(b_1) \to P_2(a_2) \to Q_2(b_2) \to P_3(a_3) \to Q_3(b_3) \to \ldots \to P_{n+1}(a_{n+1}) \to Q_{n+1}(b_{n+1}) \to O$, and has $2n+3$ steps, where P_1 to P_{n+1} are on line L, and Q_1 to Q_{n+1} are on line M. By reflection, $m+1$ quadrilaterals, such as $P_1Q_1R_1Q_2$, $P_2Q_2R_2Q_1$, $P_3Q_3R_3Q_2$, and so on, are produced from the figure of the returning path. Hence, the returning paths are $P_1 \to Q_1 \to R_1 \to Q_2 \to P_1$ and $P_1 \to Q_1 \to R_1 \to Q_3 \to P_3 \to Q_2 \to R_3 \to Q_3 \to P_1$, which have 4 and 8 steps. Then, every time m is increased by 2, an additional quadrilateral is produced and the number of steps increases by 4. So, the number of steps is the set of { $\{4k \mid 1 \le k \le m, k \in N, m\ is\ odd\}$, and this case $m = 2n+3$ also stands.

(2)The second part $\{2m \mid m = 2k-1, k \in N\}$ is also proven by mathematical induction.

When $m=3$ and $\theta=180°/3=60°$, clearly the returning path from P_1 to P_1 is $P_1 \to Q_1 \to P_2 \to Q_2 \to P_3 \to Q_3 \to P_1$ in figure 61 and has 6 steps, with P_1, P_2, P_3 being on line L and Q_1, Q_2, Q_3 being on line M. By reflection, three quadrilaterals $P_1Q_1R_1Q_2$, $P_2Q_2R_2Q_1$, $P_3Q_3R_3Q_2$ are produced from the figure of the returning path, where R_1, R_2, R_3 is the reflection of P_1, P_2, P_3 about line M. Hence, the returning path is $P_1 \to Q_1 \to R_2 \to Q_2 \to P_3 \to Q_3 \to P_1$, which is 6 steps. So, this case $m=3$ is true.

Suppose the case $m=2k+1$, where $k \in N$ and $\theta = 180°/(2k+1)$ is true, the returning path from P_1 to P_1 is by the reflection of the original figure's returning path. That is, the number of steps is $2m$ for $m=2k+1, k \in N$ and its path is $P_1 \to Q_1 \to R_2 \to Q_2 \to P_3 \to Q_3 \to R_4 \to Q_4 \to P_5 \to \ldots \to R_{m-3} \to Q_{m-3} \to P_{m-2} \to Q_{m-2} \to R_{m-1} \to Q_{m-1} \to P_m \to Q_m \to P_1$, which is $2m$ steps.

Consider the case $m+1=2k+3$, where $k \in N$ and $\theta = 180°/(2k+3)$. According to the previous findings, P_1 to P_{m+1} are on line L, and Q_1 to Q_{m+1} are on line M. P_1, P_2, …, P_{m+1} are reflected about line M to create R_1, R_2, …, R_{m+1}.

So, its path is $P_1 \to Q_1 \to R_2 \to Q_2 \to P_3 \to Q_3 \to R_4 \to Q_4 \to P_5 \to \ldots \to R_{m-2} \to Q_{m-2} \to P_{m-1} \to Q_{m-1} \to R_m \to Q_m \to P_{m+1} \to Q_{m+1} \to P_1$, which is $2m+2$ steps. Therefore, the number of steps is $2m+2$ for $m=2k+1, k \in N$. Q.E.D.

This result can be described in Theorem 6.

Theorem 6

Suppose that 3 lines, L, M, and N intersect at a point O, and that their intersecting angles are the same. According to the intersecting angles, the table shows the possible number of steps it takes to return to a point not O for certain angles between three intersecting lines.

Suppose that the angle θ_1 bound by lines L and M equals the angle θ_2 bound by lines M and N. If $\theta_1=\theta_2=180°/m$, and m is an odd number greater than or equal to 3, the returning path starts and ends at point O, and follows L→M→N→M→L→M→N→M→L, and so on. Then, the number of steps is the set of $\{4k \mid 1 \le k \le m, k \in N, m\ is\ odd\} \cup \{2m \mid m\ is\ odd\}$.

3. Main Results

Theorem 1

When there are two lines that pass through the original point and make the angle *θ*, and the robot takes *m* steps from the original point to itself, the relationship between *θ* and *m* is $\theta = 180°/m$.

Theorem 2

Suppose that two lines M and N intersect at a point O at an unknown angle, such that points P_1 to P_n are on line N and points Q_1 to Q_n are on line M. The distances from P_1 to O is a_1, the distance from P_2 to O is a_2, and so on. Similarly, the distances from Q_1 to O is b_1, the distances from Q_2 to O is b_2, and so on. Suppose Rusty is able to return from a point O to O. Then we have two cases, $m=2n+1, n \in N$:

(1) the ratios of a_i to a_1 are $a_i/a_1 = \dfrac{\sin(2i-1)\theta}{\sin\theta}$ for $i=1$ to $(m-1)/2$.

(2) the ratios of b_i to b_1 are $b_i/a_1 = \dfrac{\sin 2i\theta}{\sin\theta}$ for $i=1$ to $(m-1)/2$.

Theorem 3

Suppose that two lines M and N intersect at a point O at an unknown angle, such that points P_1 to P_n are on line N and points Q_1 to Q_n are on line M. The distances from P_1 to O is a_1, the distance from P_2 to O is a_2, and so on. Similarly, the distances from Q_1 to O is b_1, the distances from Q_2 to O is b_2, and so on. If and only if the unknown angle is $180°/m$, Rusty takes m steps from O to O to return, and vice versa.

Also, when $\theta=180°/m$ and $m=2n+1, n \in N$, according to the cosine law of triangles, we can determine the recursive relationship between a_n and b_n:

$$a_{n-1}+a_n = 2b_{n-1}\cos(180°/m), \quad b_{n-1}+b_n = 2a_n\cos(180°/m).$$

Therefore, the path of Rusty is $O \to P_1(a_1) \to Q_1(b_1) \to P_2(a_2) \to Q_2(b_2) \to P_3(a_3) \to Q_3(b_3) \to \ldots \to P_{(m-1)/2}(a_{(m-1)/2}) \to Q_{(m-1)/2}(b_{(m-1)/2}) \to P_{(m+1)/2}(a_{(m+1)/2}=0)=O$, which means Rusty returned to the starting point.

Theorem 4

Suppose that two lines M and N intersect at a point O at an unknown angle, such that points P_1 to P_n are on line N and points Q_1 to Q_n are on line M. The distances from P_1 to O is a_1, the distance from P_2 to O is a_2, and so on. Similarly, the distances from Q_1 to O is b_1, the distances from Q_2 to O is b_2, and so on. Suppose that a random point on line N not O, P_1, is chosen. If and only if the unknown angle is $180°/m$ and $m=2n+1, n \in N$, Rusty takes 2m steps from P_1 to return to P_1, and vice versa.

Also, when $\theta = 180°/m$ and $m=2n+1, n \in N$, according to the cosine law of triangles, we can determine the recursive relationship between a_n and b_n:

$$a_{n-1} + a_n = 2b_{n-1}\cos(180°/m), \quad b_{n-1} + b_n = 2a_n \cos(180°/m).$$

Therefore, the path of Rusty is $P_1(a_1) \to Q_1(b_1) \to P_2(a_2) \to Q_2(b_2) \to P_3(a_3) \to Q_3(b_3) \to \ldots \to P_m(a_m) \to Q_m(b_m) \to P_{m+1}(a_{m+1} = a_1) = P_1(a_1)$, which means Rusty returned to the starting point.

Theorem 5

Suppose that 3 lines, L, M, and N intersect at a point O, and that their intersecting angles are the same. According to the intersecting angles, the table shows the possible number of steps it takes to return to O for certain angles between three intersecting lines.

Suppose that the angle θ_1 bound by lines L and M equals the angle θ_2 bound by lines M and N. If $\theta_1 = \theta_2 = 180°/m$, and m is an odd number greater than or equal to 3, the returning path starts and ends at point O, and follows L→M→N→M→L→M→N→M→L, and so on. Then, the number of steps is the set of $\{4k \mid 1 \le k \le (m+1)/2, k \in N, m \text{ is odd}\} \cup \{m \mid m = 2k+1, k \in N\}$.

Theorem 6

Suppose that 3 lines, L, M, and N intersect at a point O, and that their intersecting angles are the same. According to the intersecting angles, the table shows the possible number of steps it takes to return to a point not O for certain angles between three intersecting lines.

Suppose that the angle θ_1 bound by lines L and M equals the angle θ_2 bound by lines M and N. If $\theta_1 = \theta_2 = 180°/m$, and m is an odd number greater than or equal to 3, the returning path starts and ends at point O, and follows L→M→N→M→L→M→N→M→L, and so on. Then, the number of steps is the set of $\{4k \mid 1 \le k \le m, k \in N, m \text{ is odd}\} \cup \{2m \mid \{m \mid m = 2k+1, k \in N\}\}$.

4. Discussion

4.1 Conclusion

Regarding Rusty starting at a point O, we found that any angle would work as long as it was odd, smaller than 90°, and a divisor of 180. This is because if two lines meet at an angle larger than 90°, one angle must be larger and the other must be smaller than 90°. Also, since we initially considered the odd values of m, the angle will be considered to be a divisor of 180. Meanwhile, we disregarded cases where m was even because Rusty would never return to the original point in these cases.

Regarding Rusty starting at a point not O, we discovered that any point P_1 we chose would still guarantee Rusty's return. We concluded this was because no matter how much we dilate a point, the returning path would still be the same shape. Also, since we initially considered the values of m, we found the number of steps, 2m, is a divisor of 360. If 2m is a divisor of 360, it doesn't guarantee that the angle of intersection will be an integer. Conversely, if the angle of intersection is an integer, in some cases, Rusty will start at a point not O and return to this point in some integer steps. For example, when $\theta = 63°$, Rusty can return to a point not O in 40 steps. There are many other cases like this. We took all our findings and summarized them into six theorems.

4.2 Limitations and Original Aspects

We included the fact that we could only consider odd numbers of steps. We found that even numbers of steps, like 6 steps, were impossible if Rusty's goal were to return to the original point. Another limitation we faced was how, during the investigation of 3 lines, the two angles between the three lines must have been equal. We tried doing two different angles, but decided not to pursue this idea because we couldn't find anything that suggested that Rusty would return to his original point. Finally, we considered a project where we could write some sort of code that automatically draws out diagrams of two or three lines once we put in the appropriate data. However, we also decided not to pursue this idea because of time and lack of coding knowledge.

One original aspect of our project was that we considered three lines. Another original aspect was that we discovered the relationship between number of steps and θ not only when Rusty starts at the intersection, but also when Rusty starts at a random point. Finally, using GeoGebra tools like constructing circles and reflections was also a unique research method.

5. Future Outlook

This project has only skimmed the surface of what could be discovered in terms of the Rusty the robot problem. Here is a list of topics we discussed, but decided to leave for a future project due to their complex and time-consuming natures:

(1) Four lines seems to be an appropriate topic to consider. Because we already discovered two line and three line scenarios, four lines would be an appropriate next step.

(2) Multiple lines that intersect into the shape of a circle was also an idea we discussed. For example, 5 lines where $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = 36°$ make a circular shape.

(3) We also considered cases of not three lines, but three 3D planes (Yang et al., 2016). Would there still be a correlation between the angle and the number of steps somehow?

(4) We thought about three lines that don't intersect at one point, unlike the three-line scenarios we discussed. If the third line was positioned in a special way, for example, parallel, then would a relationship or pattern appear?

(5) As discussed in the conclusion, there are certain thetas where Rusty can return in some number of steps that don't satisfy the property that $360°/\theta$ is an integer, such as $\theta = 75°$ (24 steps) or $\theta = 50°$ (36 steps). Why is this the case?

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Authors contributions

Mr. Louis Tsai was responsible for creating the mathematical models and drafting the manuscript. Dr. Cheng-Hua Tsai was responsible for reviewing the research questions and revising the manuscript. Both authors approved the final manuscript.

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Data sharing statement

No additional data are available.

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