Journal of Mathematics Research; Vol. 16, No. 2; April 2024
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Averaging Principle for BSDEs and one Barrier Reflected BSDEs With Non-Liphschitz Coefficients

Ibrahima Sané[1], Clément Manga[1] & Sadibou Aidara[2]

[1] Department of Mathematics Laboratory of Mathematics and Applications UFR Sciences and Technologies Assane SECK University of Ziguinchor, SENEGAL

[2] Department of Mathematics Laboratory of Mathematics and Applications UFR Sciences and Technologies Gaston Berger University of Saint-Louis, Saint-Louis, Senegal

Correspondence: Ibrahima Sané, Department of Mathematics Laboratory of Mathematics and Applications UFR Sciences and Technologies Assane SECK University of Ziguinchor, SENEGAL

Received: February 20, 2024 Accepted: March 29, 2024 Online Published: April 30, 2024

doi:10.5539/jmr.v16n2p62 URL: https://doi.org/10.5539/jmr.v16n2p62

Abstract

In this paper, the averaging principle for BSDEs and one barrier RBSDEs, with non Lipschitz coefficients, is investigated. An averaged BSDEs for the original BSDEs is proposed, as well as the one barrier RBSDEs, and their solutions are quantitatively compared. Under some appropriate assumptions, the solutions to original systems can be approximated by the solutions to averaged stochastic systems in the sense of mean square.

Keywords: reflected backward stochastic differential equation, isometry property, Holder's inequality, Gronwall lemma, non-lipschitz coefficients and Itô's representation formula

1. Introduction

The backward stochastic differential equations (BSDEs in short) were first studied by Pardoux, E. & Peng, S. G. (1990) and have the following type:

$$Y_t = \xi + \int_t^T g\left(r, Y_r, Z_r\right) dr - \int_t^T Z_r dW_r, \quad t \in [0, T], \tag{1}$$

where $\{W_t : 0 \leq t \leq T\}$ is a d-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$, the terminal value ξ is square integrable and g is mapping from $\Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$ to $\mathbb{R}$. They proved that equation (1) has a unique adapted and square integrable solution when g is globally Lipschitz. This pioneer work was extensively used in many fields like stochastic interpretation of solutions of PDEs and financial mathematics. Since then, several authors investigated BSDEs (see, among others, (Bahlali, K. 2001), (Lepeltier, J. P. & San Martin, J. 1997), (Lepeltier, J. P., Matoussi, A. & Xu, M. 2004), (Matoussi, A. 1997)). In all the above works, one notices that the coefficients of SDEs are usually assumed to satisfy the Lipschitz condition. However, many practical models of SDEs do not satisfy the Lipschitz condition. In view of the pressing need, the importance, and the impact on many diverse applications, it is necessary and also significant to consider some weaker conditions than the Lipschitz one. Fortunately, (Mao, X. 1995) and (Wang, Y. & Wang, X. 2003) have given much weaker conditions which are regarded as the so-called non Lipschitz conditions.

On the other hand, (EI Karoui, N., Kapoudjian, C., Pardoux, E., Peng, S. & Quenez, M. C. 1997) introduced the notion of a backward stochastic differential equation reflected to one continuous lower barrier (RBSDEs in short). That is a solution for such an equation associated with a coefficient g a terminal value ξ and a continuous barrier S_t, is a triple (K_t, Y_t, Z_t) of adapted process valued on $\mathbb{R}^{1+d+1}$, which satisfies a square integrability condition:

$$\begin{cases} Y_t = \xi + \int_t^T g\left(r, Y_r, Z_r\right) dr + K_T - K_t - \int_t^T Z_r dW_r, \quad t \in [0, T]; \\ Y_t \geq S_t, \quad t \in [0, T]; \\ K_0 = 0 \quad \text{and} \quad \int_0^T \left(Y_t - S_t\right) dK_t = 0. \end{cases} \tag{2}$$

They established that this equation has a unique smooth square integrable solution when g is Lipschitz. After that, many scholars have studied the solution of equation (2) under different conditions, such as (Matoussi, A. 1997) considered the case of continuous and at most linear growth in (y, z). (Hamadene, S. 2002) studied the case of a right-continuous with left limits barrier and (Lepeltier, J. P. & Xu, M. 2005) investigated the case of discontinuous barrier. For the monotonicity, general increasing growth conditions were investigated by (Lepeltier, J. P., Matoussi, A. & Xu, M. 2004).

On the contrary, averaging principle, which is usually used to approximate dynamical systems under random fluctuations, has long and rich history in multi-scale problems (see, e.g., (Khasminskii, R. Z. 1968), (N'Gorn, L. & N'Zi, M. 2001)). Recently, the averaging principle for BSDEs and one-barrier RBSDEs, with Lipschitz coefficients, were first studied by (Jing, Y. & Li, Z. 2021). However, motivated by the above works, the averaging principle for equation (1), even for equation (2), has not introduced at all. The main motivation of our work is to seek an answer to the following interesting question: compared with the general stochastic differential equations, do the backward stochastic differential equations have the averaging principle of solutions?

In this paper, we will consider this issue under non Lipschitz conditions. But, due to the characteristics of the equations of BSDEs and RBSEDs with a barrier, we should first consider the relationship among the random variables z and y and the function K_t, which is also one of the most challenging tasks in this paper.

This paper is organized as follows: In Section 2, we present some preliminaries and assumptions for the later use. In Section 3, we investigate the averaging principle for the BSDEs under some proper conditions. Then, the averaging principle for the RBSDEs with a barrier will be given in Section 4.

2. Preliminaries and Assumptions

2.1 Preliminaries

Let Ω be a non-empty set, $\mathcal{F}$ a s-algebra of sets of Ω and $\mathcal{P}$ a probability measure defined on $\mathcal{F}$. The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ defines a probability space, which is assumed to be complete.

For a fixed real number $0 < T = +\infty$, we define the following sets:

- $\mathcal{L}^2(\Omega, \mathcal{F}_T, \mathbb{P}) := \left\{\xi : \mathcal{F}_T - \text{measurable random variable}, \mathbb{E}\left[|\xi|^2\right] < \infty\right\}.$

- $\mathcal{M}^2([0,T];\mathbb{R}) := \{\phi : \phi$ is jointly measurable processes such that
$$\|\phi\|^2_{M^2} = \mathbb{E}\left[\int_0^T |\phi_t|^2\, dt\right] < \infty, \text{ and } \phi_t \text{ is } \mathcal{F}_t - \text{measurable, for any } t \in [0,T]\right\}$$

- $\mathcal{S}^2([0,T];\mathbb{R}) := \{\phi : \phi$ is continuous processes with values in $\mathbb{R}$ such that
$$\|\phi\|^2_{S^2} = \mathbb{E}[\sup_{0\leq t\leq T} |\phi_t|^2] < \infty, \text{ and } \phi_t \text{ is } \mathcal{F}_t - \text{measurable, for any } t \in [0,T]\right\}.$$

- $\mathcal{S}_\infty([0,T];\mathbb{R}) := \{\phi : \phi$ is continuous processes with values in $\mathbb{R}$ such that
$$\|\phi\|^2_{S_\infty} = \mathbb{E}[\sup_{\omega\in\Omega}\sup_{0\leq t\leq T} |\phi_t(\omega)|] < \infty, \text{ and } \phi_t \text{ is } \mathcal{F}_t - \text{measurable, for any } t \in [0,T]\right\}.$$

2.2 Assumptions

In order to study the qualitative properties of the solution to equations (1) and (2), we impose some assumptions on the coefficient functions, which will enable us to solve it.

- **(A1)** $\xi \in \mathcal{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$;
- **(A2)** $g(.,0,0) \in \mathcal{M}^2([0,T];\mathbb{R})$;
- **(A3)** The obstacle $S \in \mathcal{S}^2([0,T];\mathbb{R})$;
- **(A4)** $K_t \in \mathcal{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ is continuous and $S_T < \xi$.
- **(A5)** $\mathbb{Q}$ be the set of all non decreasing, continuous and concave function:
$\rho(.) : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\rho(0) = 0,\ \rho(s) > 0$ for $s > 0$ and
$$\int_0^{+\infty} \frac{du}{\rho(u)} = +\infty.$$

For any $\rho \in \mathbb{Q}$, we can find a pair of positive constants a and b such that $\rho(v) \leq a + bv$ for all $v \geq 0$.

- **(A6)** Let $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that for any $(y,z) \in \mathbb{R} \times \mathbb{R}^d$, g is $\mathcal{F}_t$−measurable.
Then, for all $(t, y_i, z_i) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, $i = 1, 2$,

$$|g(t,y_1,z_1) - g(t,y_2,z_2)|^2 \leq \rho\left(|y_1 - y_2|^2\right) + \rho\left(|z_1 - z_2|^2\right).$$

3. Averaging Principle for BSDEs

In this section, we are going to investigate the averaging principle for the BSDEs under non Lipschitz coefficients. Let us consider the standard form of equation (1):

$$Y_t^\varepsilon = \xi + \varepsilon \int_t^T g\left(r, Y_r^\varepsilon, Z_r^\varepsilon\right) dr - \sqrt{\varepsilon} \int_t^T Z_r^\varepsilon dW_r, \quad t \in [0,T]. \tag{3}$$

According to the second part, equation (3) also has an adapted unique and square integrable solution. We will examine whether the solution Y_t^ε can be approximated by $\overline{Y}_t$ solution of the simplified equation:

$$\overline{Y}_t = \xi + \varepsilon \int_t^T \overline{g}\left(\overline{Y}_r, \overline{Z}_r\right) dr - \sqrt{\varepsilon} \int_t^T \overline{Z}_r dW_r, \quad t \in [0,T]; \tag{4}$$

$\left(\overline{Y}_t, \overline{Z}_t\right)$ having the same properties as $(Y_t^\varepsilon, Z_t^\varepsilon)$, $\overline{g} : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ is a measurable function satisfying (**A6**) and the additional inequalities.

- **(A7)** For any $t \in [0,T_1] \subset [0,T]$, we have

$$\frac{1}{T_1 - t} \int_t^{T_1} |g(s,Y_s,Z_s) - \bar{g}(Y_s,Z_s)|^2\, ds \leq \phi(T_1 - t)\left(1 + |Y|^2 + |Z|^2\right)$$

where $\phi(T_1)$ is a bounded function.

In what follows, we establish the result which will be useful in the sequel.

Lemma 1 *Let $u \in [0,T]$ and the assumptions* (**A1**) *to* (**A7**) *holds, then there exist two constant L_1 and C such that*

$$\mathbb{E}\int_u^T \left|Z_s^\varepsilon - \overline{Z}_s\right|^2 ds \leq L_1 \mathbb{E}\int_u^T \left|Y_s^\varepsilon - \overline{Y}_s\right|^2 ds + C\,(T-u). \tag{5}$$

Proof. Applying Itô's formula to $\left|Y_t^\varepsilon - \overline{Y}_t\right|^2$ and taking the mathematical expectation, we obtain

$$\begin{aligned}
&\mathbb{E}\left|Y_t^\varepsilon - \overline{Y}_t\right|^2 + \varepsilon\mathbb{E}\int_u^T \left|Z_s^\varepsilon - \overline{Z}_s\right|^2 ds \\
&\qquad = 2\varepsilon\mathbb{E}\int_u^T \left(Y_s^\varepsilon - \overline{Y}_s\right)\left(g(s,Y_s^\varepsilon,Z_s^\varepsilon) - \overline{g}(Y_s,Z_s)\right) ds \\
&\qquad \leq 2\varepsilon\mathbb{E}\int_u^T \left(Y_s^\varepsilon - \overline{Y}_s\right)\left(g(s,Y_s^\varepsilon,Z_s^\varepsilon) - g(s,\overline{Y}_s,\overline{Z}_s)\right) ds \\
&\qquad + 2\varepsilon\mathbb{E}\int_u^T \left(Y_s^\varepsilon - \overline{Y}_s\right)\left(g(s,\overline{Y}_s,\overline{Z}_s) - \overline{g}(\overline{Y}_s,\overline{Z}_s)\right) ds \\
&\qquad = E_1 + E_2.
\end{aligned} \tag{6}$$

For E_1, by using the condition (**A6**) and Holder's inequality, for any $\alpha > 0$, $2ab \leq \alpha a^2 + b^2/\alpha$, we deduce that

$$\begin{aligned}
E_1 \leq& \alpha\varepsilon\mathbb{E}\int_u^T \left|Y_s^\varepsilon - \overline{Y}_s\right|^2 ds + \frac{\varepsilon}{\alpha}\mathbb{E}\int_u^T \left|g(s,Y_s^\varepsilon,Z_s^\varepsilon) - g(s,\overline{Y}_s,\overline{Z}_s)\right|^2 ds \\
\leq& \alpha\varepsilon\mathbb{E}\int_u^T \left|Y_s^\varepsilon - \overline{Y}_s\right|^2 ds + \frac{\varepsilon}{\alpha}\mathbb{E}\int_u^T \rho\left(\left|Y_s^\varepsilon - \overline{Y}_s\right|^2\right) ds + \frac{\varepsilon}{\alpha}\mathbb{E}\int_u^T \rho\left(\left|Z_s^\varepsilon - \overline{Z}_s\right|^2\right) ds \\
\leq& \frac{b\varepsilon}{\alpha}\mathbb{E}\int_u^T \left|Z_s^\varepsilon - \overline{Z}_s\right|^2 ds + \left(\alpha\varepsilon + \frac{b\varepsilon}{\alpha}\right)\mathbb{E}\int_u^T \left|Y_s^\varepsilon - \overline{Y}_s\right|^2 ds + \frac{2a\varepsilon}{\alpha}(T-u).
\end{aligned} \tag{7}$$

For E_2, by using Young's inequality and assumption (**A7**), we have

$$
\begin{aligned}
E_2 &= \varepsilon \mathbb{E} \int_u^T 2(Y_s^\varepsilon - \overline{Y}_s)[g(s, \overline{Y}_s, \overline{Z}_s) - \overline{g}(\overline{Y}_s, \overline{Z}_s)]ds \\
&\leq \varepsilon \mathbb{E} \int_u^T \left|Y_s^\varepsilon - \overline{Y}_s\right|^2 ds + \varepsilon \mathbb{E} \int_u^T \left|g(s, \overline{Y}_s, \overline{Z}_s) - \overline{g}(\overline{Y}_s, \overline{Z}_s)\right|^2 ds \\
&\leq \varepsilon \mathbb{E} \int_u^T \left|Y_s^\varepsilon - \overline{Y}_s\right|^2 ds + C_1 \varepsilon (T-u)
\end{aligned} \tag{8}
$$

where $C_1 = \sup_{u \leq s \leq T} \phi(s-u) \left(1 + \sup_{u \leq s \leq T} \mathbb{E}(|\overline{Y}_s|^2) + \sup_{u \leq s \leq T} \mathbb{E}(|\overline{Z}_s|^2)\right)$.

Putting pieces together, we deduce from (6) that

$$
\begin{aligned}
\mathbb{E}\left|Y_t^\varepsilon - \overline{Y}_t\right|^2 + \varepsilon \mathbb{E} \int_u^T \left|Z_s^\varepsilon - \overline{Z}_s\right|^2 ds \leq \frac{b\varepsilon}{\alpha} \mathbb{E} \int_u^T \left|Z_s^\varepsilon - \overline{Z}_s\right|^2 ds \\
+ \varepsilon \left(\alpha + \frac{b}{\alpha} + 1\right) \mathbb{E} \int_u^T \left|Y_s^\varepsilon - \overline{Y}_s\right|^2 ds \\
+ \varepsilon \left(\frac{2a}{\alpha} + C_1\right)(T-u).
\end{aligned} \tag{9}
$$

Now, we can choose $\alpha = 2b$, then

$$
\begin{aligned}
\varepsilon \mathbb{E} \int_u^T \left|Z_s^\varepsilon - \overline{Z}_s\right|^2 ds &\leq 2\left(2b + \frac{3}{2}\right) \varepsilon \mathbb{E} \int_u^T \left|Y_s^\varepsilon - \overline{Y}_s\right|^2 ds \\
&\quad + 2(\frac{a}{b} + C_1)\varepsilon (T-u) - 2\mathbb{E}\left|Y_t^\varepsilon - \overline{Y}_t\right|^2 \\
&\leq (4b+3)\, \varepsilon \mathbb{E} \int_u^T \left|Y_s^\varepsilon - \overline{Y}_s\right|^2 ds + 2(\frac{a}{b} + C_1)\varepsilon (T-u)
\end{aligned} \tag{10}
$$

Thus,

$$
\mathbb{E} \int_u^T \left|Z_s^\varepsilon - \overline{Z}_s\right|^2 ds \leq L_1 \mathbb{E} \int_u^T \left|Y_s^\varepsilon - \overline{Y}_s\right|^2 ds + C\,(T-u), \tag{11}
$$

where $L_1 = (4b+3)$ and $C = 2(\frac{a}{b} + C_1)$. This completes the proof.

Now, we claim the main theorem showing the relationship between the solution Y_t^ε of equation (3) and $\overline{Y}_t$ solution of the averaged equation (4). It shows that the solution of the averaged (4) converges to that of the original (3) in mean square sense.

theorem 1 *Assume that assumptions* **(A1)-(A3)** *and* **(A7)** *are satisfied. For* $\delta_1 > 0$ *small enough, there exists* $\varepsilon_1 \in]0, \varepsilon_0]$ *and* $\beta \in]0, 1]$ *such that for all* $\varepsilon \in]0, \varepsilon_1]$ *we have*

$$
\sup_{T\varepsilon^{1-\beta} \leq t \leq T} \mathbb{E}\left|Y_t^\varepsilon - \overline{Y}_t\right|^2 \leq \delta_1.
$$

Proof. Using the elementary inequality and the isometry property, we derive that

$$
\begin{aligned}
\mathbb{E}\left|Y_t^\varepsilon - \overline{Y}_t\right|^2 &= \mathbb{E}\left|\varepsilon \int_u^T \left[g(s, Y_s^\varepsilon, Z_s^\varepsilon) - \overline{g}(\overline{Y}_s, \overline{Z}_s)\right] ds - \sqrt{\varepsilon} \int_u^T \left(Z_s^\varepsilon - \overline{Z}_s\right) dW_s\right|^2 \\
&\leq 2\varepsilon^2 \mathbb{E}\left|\int_u^T \left[g(s, Y_s^\varepsilon, Z_s^\varepsilon) - \overline{g}(\overline{Y}_s, \overline{Z}_s)\right] ds\right|^2 + 2\varepsilon \mathbb{E}\left|\int_u^T \left(Z_s^\varepsilon - \overline{Z}_s\right) dW_s\right|^2 \\
&\leq 4\varepsilon^2 \mathbb{E}\left|\int_u^T \left[g(s, Y_s^\varepsilon, Z_s^\varepsilon) - g(s, \overline{Y}_s, \overline{Z}_s)\right] ds\right|^2 \\
&\quad + 4\varepsilon^2 \mathbb{E}\left|\int_u^T \left[g(s, \overline{Y}_s, \overline{Z}_s) - \overline{g}(\overline{Y}_s, \overline{Z}_s)\right] ds\right|^2 + 2\varepsilon \mathbb{E} \int_u^T \left|Z_s^\varepsilon - \overline{Z}_s\right|^2 ds \\
&= D_1 + D_2 + D_3
\end{aligned} \tag{12}
$$

Applying Holder's inequality and the assumption (**A6**), we obtain

$$\begin{aligned} D_1 \leq & 4(T-u)\varepsilon^2 \mathbb{E} \int_u^T \left| g(s, Y_s^\varepsilon, Z_s^\varepsilon) - \overline{g}(\overline{Y}_s, \overline{Z}_s) \right|^2 ds \\ & \leq 4(T-u)\varepsilon^2 \left(\mathbb{E} \int_u^T \rho\left(\left| Y_s^\varepsilon - \overline{Y}_s \right|^2 \right) ds + \mathbb{E} \int_u^T \rho\left(\left| Z_s^\varepsilon - \overline{Z}_s \right|^2 \right) ds \right) \\ & \leq 8a(T-u)\varepsilon^2 + 4(T-u)b\varepsilon^2 \mathbb{E} \int_u^T \left[\left| Y_s^\varepsilon - \overline{Y}_s \right|^2 + \left| Z_s^\varepsilon - \overline{Z}_s \right|^2 \right] ds \end{aligned} \tag{13}$$

Then, together with Holder's inequality and the assumption (**A7**), we get

$$\begin{aligned} D_2 \leq & 4(T-u)\varepsilon^2 \mathbb{E} \int_u^T \left| g(s, \overline{Y}_s, \overline{Z}_s) - \overline{g}(\overline{Y}_s, \overline{Z}_s) \right|^2 ds \\ \leq & 4(T-u)^2 \varepsilon^2 \mathbb{E} \int_u^T \frac{1}{T-u} \left| g(s, \overline{Y}_s, \overline{Z}_s) - \overline{g}(\overline{Y}_s, \overline{Z}_s) \right|^2 ds \\ \leq & 4(T-u)^2 \varepsilon^2 \sup_{u \leq s \leq T} [\phi(s-u)] \left(1 + \sup_{u \leq s \leq T} \mathbb{E} \left| \overline{Y}_s \right|^2 + \sup_{u \leq s \leq T} \mathbb{E} \left| \overline{Z}_s \right|^2 \right) \\ = & C_2 (T-u)^2 \varepsilon^2, \end{aligned} \tag{14}$$

where $C_2 = 4 \sup_{u \leq s \leq T} [\phi(s-u)] \left(1 + \sup_{u \leq s \leq T} \mathbb{E} \left| \overline{Y}_s \right|^2 + \sup_{u \leq s \leq T} \mathbb{E} \left| \overline{Z}_s \right|^2 \right)$.
By the Lemma 1 we have

$$D_3 = 2\varepsilon \mathbb{E} \int_u^T \left| Z_s^\varepsilon - \overline{Z}_s \right|^2 ds \leq 2L_1 \varepsilon \mathbb{E} \int_u^T |Y_s^\varepsilon - \overline{Y}_s|^2 ds + 2C' \varepsilon (T-u). \tag{15}$$

Taking into account the relations (13), (14) and (15) in (12) we obtain

$$\begin{aligned} \mathbb{E}|Y_t^\varepsilon - \overline{Y}_t|^2 \quad & \leq (4b(T-u)\varepsilon + 2L_1)\varepsilon \mathbb{E} \int_u^T |Y_s^\varepsilon - \overline{Y}_s|^2 ds \\ & + (8a\varepsilon + 2C' + C_2 \varepsilon (T-u))\, \varepsilon (T-u) \\ & \leq (4b(T-u)\varepsilon + 2L_1)\varepsilon \int_u^T \mathbb{E}|Y_s^\varepsilon - \overline{Y}_s|^2 ds \\ & + (8a\varepsilon + 2C' + C_2 \varepsilon (T-u))\, \varepsilon (T-u) \end{aligned}$$

By taking the sup

$$\begin{aligned} \sup_{u \leq t \leq T} \mathbb{E}|Y_t^\varepsilon - \overline{Y}_t|^2 \quad & \leq C_3 \varepsilon \int_u^T \sup_{u \leq t \leq s} \mathbb{E}|Y_t^\varepsilon - \overline{Y}_t|^2 ds + C_4 \\ & C_3 = (4b(T-u)\varepsilon + 2L_1), \qquad C_4 = (8a\varepsilon + 2C' + C_2 \varepsilon (T-u))\, \varepsilon (T-u) \end{aligned}$$

Thanks to Gronwall's inequality, we obtain

$$\sup_{u \leq t \leq T} \mathbb{E} \left| Y_t^\varepsilon - \overline{Y}_t \right|^2 \leq \left[(8a\varepsilon + 2C' + C_2 \varepsilon (T-u))\, \varepsilon (T-u) \right] e^{(T-u)[(4b(T-u)\varepsilon + 2L_1)\varepsilon]}. \tag{16}$$

Obviously, the above estimate implies that there exist $\beta \in [0, 1]$, for every $t \in \left[T\varepsilon^{1-\beta}, T \right]$,

$$\sup_{T\varepsilon^{1-\beta} \leq t \leq T} \mathbb{E} \left| Y_t^\varepsilon - \overline{Y}_t \right|^2 \leq C_5 \varepsilon, \tag{17}$$

in which

$$C_5 = T(1 - \varepsilon^{1-\beta}) \left[8a\varepsilon + 2C' + C_2 \varepsilon T (1 - \varepsilon^{1-\beta}) \right] e^{T(1-\varepsilon^{1-\beta})\left[4bT\varepsilon(1-\varepsilon^{1-\beta}) + 2L_1 \right]\varepsilon}$$

is constant . Consequently, for any number $\delta_1 > 0$, we can choose $\varepsilon_1 \in [0, \varepsilon_0]$ making

$$\sup_{T\varepsilon^{1-\beta} \leq t \leq T} \mathbb{E} \left| Y_t^\varepsilon - \overline{Y}_t \right|^2 \leq \delta_1, \tag{18}$$

where $\varepsilon \in [0, \varepsilon_1]$. This completes the proof.

4. Averaging Principle for RBSDEs With a Barrier

In this section, we will continue to study the averaging principle for the RBSDEs under non Lipschitz condition. Firstly, let us consider the standard form of equation (2):

$$\begin{cases} Y_t^\varepsilon = \xi + \int_t^T g\left(r, Y_r^\varepsilon, Z_r^\varepsilon\right) dr + K_T - K_t - \sqrt{\varepsilon} \int_t^T Z_r^\varepsilon dW_r, \quad t \in [0, T]; \\ Y_t^\varepsilon \geq S_t, \quad t \in [0, T]; \\ K_0 = 0 \quad \text{and} \quad \int_0^T \left(Y_t^\varepsilon - S_t\right) dK_t = 0. \end{cases} \tag{19}$$

In fact, according to Section 3 , it is easy to find that equation (19) also has a unique solution. Then, we consider the simplified system:

$$\begin{cases} \overline{Y}_t = \xi + \int_t^T \overline{g}\left(\overline{Y}_r, \overline{Z}_r\right) dr + \overline{K}_T - \overline{K}_t - \sqrt{\varepsilon} \int_t^T \overline{Z}_r dW_r, \quad t \in [0, T]; \\ \overline{Y}_t \geq S_t, \quad t \in [0, T]; \\ \overline{K}_0 = 0 \quad \text{and} \quad \int_0^T \left(\overline{Y}_t - S_t\right) d\overline{K}_t = 0. \end{cases} \tag{20}$$

lemma 2 *Let $u \in [0, T]$ and the assumptions (**A1**) to (**A7**) holds, then there exist two constant L_2 and C such that*

$$\mathbb{E} \int_u^T \left|Z_s^\varepsilon - \overline{Z}_s\right|^2 ds \leq L_2 \mathbb{E} \int_u^T \left|Y_s^\varepsilon - \overline{Y}_s\right|^2 ds + C(T - u). \tag{21}$$

Proof. If equations (19) and (20) are satisfied, according to the property of K_T , $\overline{K}_T$ in (**A4**), then applying Itô's formula to $\left|Y_t^\varepsilon - \overline{Y}_t\right|^2$ and taking the mathematical expectation, we obtain

$$\begin{aligned} \mathbb{E}\left|Y_t^\varepsilon - \overline{Y}_t\right|^2 + \varepsilon \mathbb{E} \int_u^T \left|Z_s^\varepsilon - \overline{Z}_s\right|^2 ds = {} & 2\varepsilon \mathbb{E} \int_u^T \left(Y_s^\varepsilon - \overline{Y}_s\right)\left(g(s, Y_s^\varepsilon, Z_s^\varepsilon) - \overline{g}(\overline{Y}_s, \overline{Z}_s)\right) ds \\ & + 2\mathbb{E} \int_u^T \left(Y_s^\varepsilon - \overline{Y}_s\right) d\left(K_s - \overline{K}_s\right) \end{aligned} \tag{22}$$

Since $Y_t^\varepsilon \geq S_t$, $\int_0^T \left(Y_t^\varepsilon - S_t\right) dK_t = 0$, $\overline{Y}_t \geq S_t$, $\int_0^T \left(\overline{Y}_t - S_t\right) d\overline{K}_t = 0$.
Then

$$\begin{aligned} 2\mathbb{E} \int_u^T \left(Y_s^\varepsilon - \overline{Y}_s\right) d\left(K_s - \overline{K}_s\right) &= 2\mathbb{E} \int_u^T \left(Y_s^\varepsilon - S_s\right) d\left(K_s - \overline{K}_s\right) - 2\mathbb{E} \int_u^T \left(\overline{Y}_s - S_s\right) d\left(K_s - \overline{K}_s\right) \\ &= 2\mathbb{E} \int_u^T \left(Y_s^\varepsilon - S_s\right) dK_s - 2\mathbb{E} \int_u^T \left(Y_s^\varepsilon - S_s\right) d\overline{K}_s \\ &\quad - 2\mathbb{E} \int_u^T \left(\overline{Y}_s - S_s\right) dK_s - 2\mathbb{E} \int_u^T \left(\overline{Y}_s - S_s\right) d\overline{K}_s \\ &\leq 2\mathbb{E} \int_u^T \left(Y_s^\varepsilon - S_s\right) dK_s + 2\mathbb{E} \int_u^T \left(\overline{Y}_s - S_s\right) d\overline{K}_s \\ &\leq 2\mathbb{E} \int_0^T \left(Y_s^\varepsilon - S_s\right) dK_s + 2\mathbb{E} \int_0^T \left(\overline{Y}_s - S_s\right) d\overline{K}_s \\ &= 0. \end{aligned} \tag{23}$$

It follows that

$$\mathbb{E}\left|Y_t^\varepsilon - \overline{Y}_t\right|^2 + \varepsilon \mathbb{E} \int_u^T \left|Z_s^\varepsilon - \overline{Z}_s\right|^2 ds = 2\varepsilon \mathbb{E} \int_u^T \left(Y_s^\varepsilon - \overline{Y}_s\right)\left(g(s, Y_s^\varepsilon, Z_s^\varepsilon) - \overline{g}(Y_s, Z_s)\right) ds \tag{24}$$

Recalling equations (6) to (11), it is easy to obtain the result of Lemma 2.

lemma 3 *Let $u \in [0,T]$ and the assumptions* (**A1**) *to* (**A7**) *holds, then there exist two constant* Θ *and* Θ_0 *such that*

$$\mathbb{E}\left|K_t^{\varepsilon}-\overline{K}_t\right|^2 \leq \Theta\mathbb{E}\int_u^T \left|Y_s^{\varepsilon}-\overline{Y}_s\right|^2 ds+\Theta_0. \tag{25}$$

Proof. By equations (19) and (20), we obtain

$$\left|K_t^{\varepsilon}-\overline{K}_t\right|=\left|\varepsilon\int_u^T [g(s,Y_s^{\varepsilon},Z_s^{\varepsilon})-\overline{g}(Y_s,Z_s)]\,ds-\sqrt{\varepsilon}\int_u^T\left(Z_s^{\varepsilon}-\overline{Z}_s\right)dW_s+K_T-\overline{K}_T\right|, \tag{26}$$

applying Holder's inequality, isometry property to $\left|K_t^{\varepsilon}-\overline{K}_t\right|$ and taking the mathematical expectation, it follows that

$$\begin{aligned}\mathbb{E}\left|K_t^{\varepsilon}-\overline{K}_t\right|^2 &\leq 3\varepsilon^2\mathbb{E}\left|\varepsilon\int_u^T [g(s,Y_s^{\varepsilon},Z_s^{\varepsilon})-\overline{g}(Y_s,Z_s)]\,ds\right|^2+3\varepsilon\mathbb{E}\left|\int_u^T\left(Z_s^{\varepsilon}-\overline{Z}_s\right)dW_s\right|^2\\
&\leq 3(T-u)\varepsilon^2\mathbb{E}\int_u^T\left|g(s,Y_s^{\varepsilon},Z_s^{\varepsilon})-\overline{g}(Y_s,Z_s)\right|^2ds+3\varepsilon\mathbb{E}\int_u^T\left|Z_s^{\varepsilon}-\overline{Z}_s\right|^2ds\\
&\leq 6(T-u)\varepsilon^2\mathbb{E}\int_u^T\left|g(s,Y_s^{\varepsilon},Z_s^{\varepsilon})-g(s,\overline{Y}_s,\overline{Z}_s)\right|^2ds+3\varepsilon\mathbb{E}\int_u^T\left|Z_s^{\varepsilon}-\overline{Z}_s\right|^2ds\\
&+6(T-u)^2\varepsilon^2\mathbb{E}\int_u^T\frac{1}{T-u}\left|g(s,\overline{Y}_s,\overline{Z}_s)-\overline{g}(\overline{Y}_s,\overline{Z}_s)\right|^2ds,\end{aligned} \tag{27}$$

where we used the fact that $\mathbb{E}\left|K_T-\overline{K}_T\right|=0$.

Owing to the assumptions (**A6**) and (**A7**), we have

$$\begin{aligned}\mathbb{E}\left|K_t^{\varepsilon}-\overline{K}_t\right|^2 &\leq 6(T-u)\varepsilon^2\mathbb{E}\int_u^T\left(\rho\left(\left|Y_s^{\varepsilon}-\overline{Y}_s\right|^2\right)+\rho\left(\left|Z_s^{\varepsilon}-\overline{Z}_s\right|^2\right)\right)\\
&+6(T-u)^2\varepsilon^2\sup_{u\leq s\leq T}\left[\phi^2(s)\right]\left(1+\sup_{u\leq s\leq T}\mathbb{E}\left(\left|\overline{Y}_s\right|\right)+\sup_{u\leq s\leq T}\mathbb{E}\left(\left|\overline{Z}_s\right|\right)\right)\\
&+3\varepsilon\mathbb{E}\int_u^T\left|Z_s^{\varepsilon}-\overline{Z}_s\right|^2ds\\
&\leq 6(T-u)\varepsilon^2b\mathbb{E}\int_u^T\left|Y_s^{\varepsilon}-\overline{Y}_s\right|^2ds+6(T-u)\varepsilon^2b\mathbb{E}\int_u^T\left|Z_s^{\varepsilon}-\overline{Z}_s\right|^2ds\\
&+C_6(T-u)^2\varepsilon^2+3\varepsilon\mathbb{E}\int_u^T\left|Z_s^{\varepsilon}-\overline{Z}_s\right|^2ds\end{aligned} \tag{28}$$

where $C_6=12a+6\sup_{u\leq s\leq T}\left[\phi^2(s)\right]\left(1+\sup_{u\leq s\leq T}\mathbb{E}\left(\left|\overline{Y}_s\right|\right)+\sup_{u\leq s\leq T}\mathbb{E}\left(\left|\overline{Z}_s\right|\right)\right)$. Mean while, from Lemma 2, we get

$$\begin{aligned}\mathbb{E}\left|K_t^{\varepsilon}-\overline{K}_t\right|^2 &\leq \left(6(T-u)\varepsilon^2b+L_26(T-u)\varepsilon^2b+3L_2\varepsilon\right)\mathbb{E}\int_u^T\left|Y_s^{\varepsilon}-\overline{Y}_s\right|^2ds\\
&+6(T-u)^2C\varepsilon^2b+3\varepsilon C(T-u).\end{aligned} \tag{29}$$

It is enough to take $\Theta=6(T-u)\varepsilon^2b+L_26(T-u)\varepsilon^2b+3L_2\varepsilon$ and $\Theta_0=6(T-u)^2C\varepsilon^2b+3\varepsilon C(T-u)$. This completes the proof.

Now, we claim the main theorem showing the relationship between solution processes Y_t^{ε} to the original (19) and $\overline{Y}_t$ to the averaged (20). It shows that the solution of the averaged (20) converges to that of the original (19) in mean square sense.

Theorem 2 *Assume that assumptions* **(A1)-(A3)** *and* **(A7)** *are satisfied. For $\delta_1>0$ small enough, there exists $\varepsilon_1\in\,]0,\varepsilon_0]$ and $\beta\in\,]0,1]$ such that for all $\varepsilon\in\,]0,\varepsilon_1]$ we have*

$$\sup_{T\varepsilon^{1-\beta}\leq t\leq T}\mathbb{E}\left|Y_t^{\varepsilon}-\overline{Y}_t\right|^2\leq\delta_2.$$

Proof. T Together with the elementary equality and the isometry property, we deduce that

$$\begin{aligned}\sup_{u\le t\le T}\mathbb{E}\left|Y_t^{\varepsilon}-\overline{Y}_t\right|^2 &\le 4\varepsilon^2\mathbb{E}\left|\int_u^T\left[g(s,Y_s^{\varepsilon},Z_s^{\varepsilon})-\overline{g}(\overline{Y}_s,\overline{Z}_s)\right]ds\right|^2+4\sup_{u\le t\le T}\mathbb{E}\left|K_t^{\varepsilon}-\overline{K}_t\right|^2\\ &\quad+4\varepsilon\sup_{u\le t\le T}\mathbb{E}\left|\int_t^T(Z_s^{\varepsilon}-\overline{Z}_s)dW_s\right|^2\\ &=J_1+J_2+J_3.\end{aligned} \tag{30}$$

Recalling (13) and (14), we have

$$\begin{aligned}J_1 &\le C_2(T-u)^2\varepsilon^2+8a(T-u)\varepsilon^2+4(T-u)b\varepsilon^2\sup_{u\le t\le T}\mathbb{E}\int_u^T\left|Y_s^{\varepsilon}-\overline{Y}_s\right|^2ds\\ &\quad+4L_1(T-u)b\varepsilon^2\sup_{u\le t\le T}\mathbb{E}\int_u^T\left|Y_s^{\varepsilon}-\overline{Y}_s\right|^2ds+4C^{'}b(T-u)^2\varepsilon^2\\ &\le\left(4(T-u)b\varepsilon^2+4L_1(T-u)b\varepsilon^2\right)\sup_{u\le t\le T}\mathbb{E}\int_t^T\left|Y_s^{\varepsilon}-\overline{Y}_s\right|^2ds\\ &\quad+C_3(T-u)^2\varepsilon^2+8a(T-u)\varepsilon^2\end{aligned} \tag{31}$$

where $C_3=4C^{'}b+C_2$.

For J_2, combining the isometry property with Lemma produces

$$\begin{aligned}J_2 &\le 4\varepsilon\sup_{u\le t\le T}\mathbb{E}\int_u^T\left|Z_s^{\varepsilon}-\overline{Z}_s\right|^2ds\\ &\le 4L_2\varepsilon\sup_{u\le t\le T}\mathbb{E}\int_u^T\left|Y_s^{\varepsilon}-\overline{Y}_s\right|^2ds+4C^{'}(T-u)\varepsilon\end{aligned} \tag{32}$$

By virtue of Lemma , we obtain

$$J_3\le 4\Theta\sup_{u\le t\le T}\mathbb{E}\int_t^T\left|Y_s^{\varepsilon}-\overline{Y}_s\right|^2ds+4\Theta_0. \tag{33}$$

Now, plug equations (31) to (33) into (30) for any $u\le t\le T$, and we get

$$\begin{aligned}\sup_{u\le t\le T}\mathbb{E}\left|Y_t^{\varepsilon}-\overline{Y}_t\right|^2 &\le \Theta_1\sup_{u\le t\le T}\mathbb{E}\int_u^T\left|Y_s^{\varepsilon}-\overline{Y}_s\right|^2ds+C_3(T-u)^2\varepsilon^2\\ &\quad+8a(T-u)\varepsilon^2+4C^{'}(T-u)\varepsilon+4\Theta_0\end{aligned} \tag{34}$$

where $\Theta_1=\left(4\Theta+4L_2\varepsilon+4(T-u)b\varepsilon^2+4L_1(T-u)b\varepsilon^2\right)$.
Indeed, for evry number δ_2 and in terms of Gronwall's inequality, we obtain

$$\sup_{T\varepsilon^{1-\beta}\le t\le T}\mathbb{E}\left|Y_t^{\varepsilon}-\overline{Y}_t\right|^2\le\delta_2. \tag{35}$$

This completes the proof.

Acknowledgements

The autours thanks the referees and editor for their careful reading and helpful suggestions which lead to a much improved version of this paper.

References

Aidara, S. (2019). Anticipated backward doubly stochastic differential equations with non-Liphschitz coefficients. *Applied Mathematics and Nonlinear Sciences, 4*(1). https://doi.org/10.2478/AMNS.2019.1.00002

Aidara, S., & Sow, A. (2015). Anticipated BDSDEs driven by Lévy process with non-Lipschitz coefficients. *Random Operators and Stochastic Equations, 23*(3), 195-207. https://doi.org/10.1515/rose-2014-0040

Bahlali, K. (2001). Backward stochastic differential equations with locally Lipschitz coefficient. *Comptes Rendus de l'Academie des Sciences - Series I-Mathematics, 333*5, 481-486. https://doi.org/10.1515/rose.2002.10.4.335

EI Karoui, N., Kapoudjian, C., Pardoux, E., Peng, S., & Quenez, M. C. (1997). Reflected solutions of backward SDE and related obstacle problems for PDEs. *The Annals of Probability, 25*2, 702-737.

Hamadene, S. (2002). Reflected BSDEs with discontinuous barrier and application. *Stochastics: An International Journal of Probability and Stochastic Processes, 74*(3), 571-596.

Jing, Y., & Li, Z. (2021). *Averaging Principle for Backward Stochastic Differential Equations*, Discrete Dynamics in Nature and Society, Article ID 6615989, 10 pages. https://doi.org/10.1155/2021/6615989

Khasminskii, R. Z. (1968). *On the principle of averaging the it stochastic differential equations*. Kybernetika (Prague), vol. 4, 260-279.

Lepeltier, J. P., & San Martin, J. (1997). Backward stochastic differential equations with continuous coefficient. *Statistics and Probability Letters, 32*(4), 425-430.

Lepeltier, J. P., Matoussi, A., & Xu, M. (2004). Reflected *BSDEs under Momotonicity and General Increasing Growth Conditions*, Preprint Universite du Maine, France.

Lepeltier, J. P., & Xu, M. (2005). Penalization method for reflected backward stochastic differential equations with one r.c.l.l. barrier. *Statistics and Probability Letters, 75*(1), 58-66. https://doi.org/10.1016/j.spl.2005.05.016

Mao, X. (1995). Adapted solutions of backward stochastic differential equations with non-Lipschitz coefficients. *Stochastic Processes and Their Applications, 58*(2), 281-292. https://doi.org/10.1016/0304-4149(95)00024-2

Matoussi, A. (1997). Reflected solutions of backward stochastic differential equations with continuous coefficient. *Statistics and Probability Letters, 34*(4), 347-354. https://doi.org/10.1016/S0167-7152(96)00202-7

N'Gorn, L., & N'Zi, M. (2001). Averaging principle for multi valued stochastic differential equations. *Random Operators and Stochastic Equations, 9*, 399-407. https://doi.org/10.1515/rose.2001.9.4.399

Pardoux, E., & Peng, S. G. (1990). Adapted solution of a backward stochastic differential equation. *Systems and Control Letters, 14*(1), 55-61. https://doi.org/10.1016/0167-6911(90)90082-6

Wang, Y., & Wang, X. (2003). Adapted solutions of backward SDE with non-Lipschitz coefficient. *Chinese Journal of Applied Probability and Statisties, 19*, 245-251.

Copyrights