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Exponential Stability of Non-uniform Timoshenko Beam With Indefinite Damping Under Boundary Control

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Abstract

In this paper we study the stabilization of a non-uniform Timoshenko beam with indefinite damping terms under boundary control and prove how the damping terms can affect the decay rate asymptotically. Using the theory of perturbed problems, we obtain the stability and establish the spectrum determined growth condition for the problem. Moreover, when the two damping terms are indefinite, we provide a condition to obtain the exponential stability.

Keywords: beam equation, semigroup theory, asymptotic analysis, Riesz basis, exponential stability

1. Introduction

We study the following variable coefficients Timoshenko beam with two indefinite damping terms under boundary feedback controls given by:

$$
(S):\begin{cases}
m(x)\frac{\partial^2\omega}{\partial t^2}-\frac{\partial}{\partial x}\left(K(x)\left(\frac{\partial\omega}{\partial x}-\varphi\right)\right)+m(x)b_1(x)\frac{\partial\omega}{\partial t} & =0,\ \ 0\le x\le 1,\ \ t>0,\\
I_m(x)\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial}{\partial x}\left(EI(x)\left(\frac{\partial\omega}{\partial x}\right)\right)+K(x)\left(\varphi-\frac{\partial\omega}{\partial x}\right) & \\
\quad +I_m(x)b_2(x)\frac{\partial\varphi}{\partial t} & =0,\ \ 0\le x\le 1,\ \ t>0,\\
\omega(t,0)=\varphi(t,0) & =0,\ \ t>0\\
EI(1)\frac{\partial\varphi}{\partial x}(t,1)+\alpha\varphi(t,1)+\beta\frac{\partial\varphi}{\partial t}(t,1) & =0,\ \ t>0\\
K(1)\left(\varphi(t,1)-\frac{\partial\omega}{\partial x}(t,1)\right) & =0.\ \ t>0
\end{cases}
\tag{1}
$$

Here α and β are positive feedback constants, $\omega(t,x)$ is a lateral displacement and $\varphi(t,x)$ is the bending angle of the beam at position x and time t. The length of the beam is chosen to be unity. The coefficients $m(x)>0$ and $K(x)>0$ are respectively the mass density and the shear stiffness of a cross section. Furthermore, $I_m(x)>0$ is the rotatory inertia, $EI(x)>0$ is the flexural rigidity, $b_1(x)$ and $b_2(x)$ are continuously differentiable damping terms that are allowed to change signs in $[0,1]$.

Due to indefiniteness of functions $b_1(x)$ and $b_2(x)$ on interval $[0,1]$, we impose the following assumptions

$$
\int_0^1 b_1(\xi)\left(\frac{m(\xi)}{K(\xi)}\right)^{\frac{1}{2}}d\xi>0 \tag{2}
$$

and

$$
\int_0^1 b_2(\xi)\left(\frac{I_m(\xi)}{EI(\xi)}\right)^{\frac{1}{2}}d\xi>0, \tag{3}
$$

where $\left(\dfrac{K(x)}{m(x)}\right)^{\frac{1}{2}}$ and $\left(\dfrac{EI(x)}{I_m(x)}\right)^{\frac{1}{2}}$ are wave speeds. These assumptions, as we shall see, are necessary to ensure that the high frequencies of the system (S) are in left half complex plane. For convenience, we always assume that:

$$
m(x),\ K(x),\ I_m(x),\ EI(x)\in C^2[0,1]. \tag{4}
$$

The same problem was investigated in (Wang, 2004) with $\alpha = 0$ and $\beta = 0$. In (Feng et al, 2001) the authors study a constant coefficients Timoshenko beam system without damping with boundary controls and obtain Riesz basis property. The boundary stabilization and boundary control of Timoshenko beam system have been investigated by many researchers with different boundary dampers and controllers. We may cite the work of (Akian, 2022), (Feng et al, 2001), (Liping et al, 2019), (Nasser, 2011), (Nasser, 2013), (Wang, 2004) and (Xu & Feng, 2002).

There are two commonly used approaches to study these perturbed systems.

The first is due to (Huang, 1985), which says that if the resolvent $R(\lambda, A + B)$ is uniformly bounded along the imaginary axis, then the operator $A + B$ generates an exponentially stable semigroup on the energy space.

The second one is due to (Renardy, 1993) and (Xu & Feng, 2002) which says that the semigroup generated by the operator $A + B$ with bounded linear operator B will satisfy the spectrum determined growth condition if the operator A (not necessarily skew-adjoint) satisfies the following property:

There exists $N > 0$ such that the spectrum $\sigma(A) = \{\lambda_n\}$ of A is separated and simple when $|\lambda_n| \geq N$ and there is a sequence of generalized eigenfunctions of A that forms a Riesz basis in the state Hilbert space.

There is a third approach, namely the Riesz basis approach, which shows that the generalized eigenfunctions of system form a Riesz basis, and then deduces the spectrum determined growth condition and various stability results from the eigenvalue distribution of the system, see (Wang & Yung, 2006), (Wang et al, 2004) and (Wang, 2004).

In this paper, we use the second approach because we need the spectrum determined growth condition of system to prove the exponential stability. Eventually, asymptotic expressions of the eigenvalues of system (S) are obtained and the spectrum determined growth condition of system is deduced by second approach and stability is established.

The rest of the paper is organized as follows. In section 2, we convert system (S) into an evolution equation in an appropriate Hilbert space framework, and then prove that the system generates a C_0-semigroup.
In order to solve the eigenvalue problem, we shall use a space-scaling transformation to derive an equivalent eigenvalue boundary problem and this leads to much simpler asymptotic expansions. In section 3, we shall apply technique in (Wang, 2004) to the fundamental solutions of the eigenvalue boundary problem, and then use results to expand the characteristic determinant in deducing the asymptotic behavior of the eigenvalues. Furthermore, in the last section, we also obtain conditions for the exponential stability of the system for indefinite damping terms. We shall operate under the following conditions: $0 < \beta < r_2(1)EI(1)$ and $\beta > r_2(1)EI(1)$.

2. State Space Setup and Eigenvalue Problem

We start our investigation by formulating the problem on some state Hilbert space. Let

$$\mathcal{H} = V_E^1[0,1] \times L_\rho^2[0,1] \times V_E^1[0,1] \times L_{I_m}^2[0,1], \tag{5}$$

where

$$V_E^k[0,1] = \left\{ f \in H^k[0,1] : f(0) = 0 \right\}, \quad k = 1, 2, \tag{6}$$

and let $H^k[0,1]$ $(k = 1, 2)$ be the usual Sobolev space of order k. The inner product in $\mathcal{H}$ is defined by

$$\begin{aligned} \langle Y_1, Y_2 \rangle_{\mathcal{H}} &= \int_0^1 K(x)\left(\varphi_1 - \omega_1'\right)\overline{\left(\varphi_2 - \omega_2'\right)}dx + \int_0^1 m(x) z_1 \bar{z}_2 dx \\ &+ \int_0^1 EI\varphi_1'\overline{\varphi}_2' dx + \int_0^1 I_m \psi_1 \overline{\psi}_2 dx, + \alpha\varphi_1(1)\overline{\varphi_2(1)} \end{aligned} \tag{7}$$

where $Y_k = (\omega_k, z_k, \varphi_k, \psi_k)^\tau$ with $k = 1, 2$, in which the superscript τ denotes transpose of a vector or a matrix, and the notation ' denotes the derivative with respect to x.

In view of system (S), we define two linear operators $\mathcal{A}$ and $\mathcal{B}$ in Hilbert space $\mathcal{H}$ respectively by

$$D(\mathcal{A}) = \left\{ (\omega, z, \varphi, \psi)^\tau \in \mathcal{H} \;\middle|\; \begin{array}{c} \omega, \varphi \in V_E^2[0,1], z, \psi \in V_E^1[0,1] \\ EI(1)\varphi'(1) = -\alpha\varphi(1) - \beta\omega(1), \\ \varphi(1) = \omega'(1) \end{array} \right\}, \tag{8}$$

$$\mathcal{A}\begin{bmatrix} \omega \\ z \\ \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} z \\ \frac{1}{m}\left(K\left(\omega' - \varphi\right)\right)' \\ \psi \\ \frac{1}{I_m}\left(EI\varphi'\right)' - \frac{K}{I_m}\left(\varphi - \omega'\right) \end{bmatrix}, \quad \forall \begin{bmatrix} \omega \\ z \\ \varphi \\ \psi \end{bmatrix} \in D(\mathcal{A}), \tag{9}$$

and

$$\mathcal{B}\begin{bmatrix} \omega \\ z \\ \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ -b_1 z \\ 0 \\ -b_2\psi \end{bmatrix}, \ \forall \begin{bmatrix} \omega \\ z \\ \varphi \\ \psi \end{bmatrix} \in D(\mathcal{B}) = \mathcal{H}. \tag{10}$$

If we let $Y_t = (\omega_t, z_t, \varphi_t, \psi_t)^\tau$, then the system (S) can be rewritten into an abstract evolution equation in $\mathcal{H}$ as following:

$$\begin{cases} \frac{d}{dt}Y(t) = (\mathcal{A}+\mathcal{B})\,Y(t), & t > 0, \\ D(\mathcal{A}+\mathcal{B}) = D(\mathcal{A}), \\ Y(0) = (\omega(0), \omega_t(0), \varphi(0), \varphi_t(0))^\tau . \end{cases} \tag{11}$$

We have the following results.

Lemma 1. *The operator $\mathcal{A}$ given by* (8) *and* (9) *is skew-adjoint operator. See (Wang, 2004, Chapter 9)*

Theorem 1 *Let $\mathcal{A}$ and $\mathcal{B}$ be given by* (9) *and* (10), *then $\mathcal{A}+\mathcal{B}$ have compact resolvent and* $0 \in \rho(\mathcal{A}+\mathcal{B})$.

Proof. We first prove that $0 \in \rho(\mathcal{A}+\mathcal{B})$, which is equivalent to show that $(\mathcal{A}+\mathcal{B})^{-1}$ exists.
for any $G = (g_1, g_2, g_3, g_4)^\tau \in \mathcal{H}$, we must found a unique unique $F = (f_1, f_2, f_3, f_4)^\tau \in D(\mathcal{A}+\mathcal{B})$ so that

$$\begin{aligned}
(\mathcal{A}+\mathcal{B})\,F &= G, && (12)\\
f_2(x) &= g_1(x), && (13)\\
f_4(x) &= g_3(x), && (14)\\
(K(x)(f_1'(x) - f_3(x)))' &= m(x)(b_1(x)f_2(x) + g_2(x)), && (15)\\
\left(EI(x)f_3'(x)\right)' - K(x)(f_3(x) - f_1'(x)) &= I_m(x)(b_2(x)f_4(x) + g_4(x)). && (16)
\end{aligned}$$

Using (13) and (14) one obtains f_1 and f_3. From (15), (16) and the boundary conditions $f_1'(1) - f_3(1) = 0$, we get

$$-(f_1'(x) - f_3(x)) = \frac{1}{K(x)}\int_x^1 m(s)(b_1(s)g_1(s) + g_2(s))\,ds. \tag{17}$$

Substituting this into (13) and using (14), (16), (17) and the boundary condition $f_3'(1) = -\alpha f_3(1) - \beta f_4(1)$, we get

$$\begin{aligned}
-\alpha f_3(1) - \beta f_4(1) - EI(x)f_3'(x) &= \int_x^1 I_m(r)(b_2(r)g_3(r) + g_4(r))\,dr \\
&+ \int_x^1\int_r^1 m(s)(b_1(s)g_1(s) + g_2(s))\,dsdr, \\
-\alpha f_3(1) - EI(x)f_3'(x) &= \int_x^1 I_m(r)(b_2(r)g_3(r) + g_4(r))\,dr \\
&+ \int_x^1\int_r^1 m(s)(b_1(s)g_1(s) + g_2(s))\,dsdr + \beta g_3(1), \\
\frac{-\alpha f_3(1)}{EI(x)} - f_3'(x) &= \frac{1}{EI(x)}\int_x^1 I_m(r)(b_2(r)g_3(r) + g_4(r))\,dr \\
&+ \frac{1}{EI(x)}\int_x^1\int_r^1 m(s)(b_1(s)g_1(s) + g_2(s))\,dsdr \\
&+ \frac{\beta}{EI(x)}g_3(1) \\
-\alpha f_3(1)\int_0^x \frac{1}{EI(\theta)}d\theta - \int_0^1 f_3'(\theta)d\theta &= \int_0^x \frac{1}{EI(\theta)}\int_\theta^1 I_m(r)(b_2(r)g_3(r) + g_4(r))\,drd\theta \\
&+ \int_0^x \frac{1}{EI(\theta)}\int_\theta^1\int_r^1 m(s)(b_1(s)g_1(s) + g_2(s))\,dsdrd\theta \\
&+ \int_0^x \frac{\beta}{EI(\theta)}g_3(1)d\theta.
\end{aligned}$$

Using the boundary conditions $f_3(0) = 0$ we obtain:

$$\begin{aligned}
-\alpha f_3(1)\int_0^x \frac{1}{EI(\theta)}d\theta - f_3(x) &= \int_0^x \frac{1}{EI(\theta)}\int_\theta^1 I_m(r)\,(b_2(r)g_3(r) + g_4(r))\,drd\theta \\
&+ \int_0^x \frac{1}{EI(\theta)}\int_\theta^1\int_r^1 m(s)\,(b_1(s)g_1(s) + g_2(s))\,dsdrd\theta \\
&+ \int_0^x \frac{\beta}{EI(\theta)}g_3(1)d\theta.
\end{aligned}$$

For $x = 1$, we get

$$-\alpha f_3(1)\int_0^1 \frac{1}{EI(\theta)}d\theta - f_3(1) = \chi$$

where

$$\begin{aligned}
\chi &= \int_0^1 \frac{1}{EI(\theta)}\int_\theta^1 I_m(r)\,(b_2(r)g_3(r) + g_4(r))\,drd\theta \\
&+ \int_0^1 \frac{1}{EI(\theta)}\int_\theta^1\int_r^1 m(s)\,(b_1(s)g_1(s) + g_2(s))\,dsdrd\theta \\
&+ \int_0^1 \frac{\beta}{EI(\theta)}g_3(1)d\theta
\end{aligned}$$

hence

$$f_3(1) = -\frac{\chi}{1 + \alpha\int_0^1 \frac{d\theta}{EI(\theta)}}.$$

Thus we obtain:

$$\begin{aligned}
\forall\, 0 < x < 1 \;\; f_3(x) &= -\int_0^x \frac{1}{EI(\theta)}\int_\theta^1 I_m(r)\,(b_2(r)g_3(r) + g_4(r))\,drd\theta \\
&- \int_0^x \frac{1}{EI(\theta)}\int_\theta^1\int_r^1 m(s)\,(b_1(s)g_1(s) + g_2(s))\,dsdrd\theta \\
&- \int_0^x \frac{\beta}{EI(\theta)}g_3(1)d\theta - \alpha f_3(1)\int_0^x \frac{1}{EI(\theta)}d\theta.
\end{aligned}$$

Moreover f_1 can also be obtained by substituting the above expression of $f_3(x)$ into (17) and using the boundary condition $f_1(0) = 0$ to yield

$$f_1(x) = -\int_0^x \frac{1}{K(t)}\int_t^1 m(s)\,(b_1(s)g_1(s) + g_2(s))\,dsdt + \int_0^x f_3(t)dt.$$

Thus $(\mathcal{A}+\mathcal{B})^{-1}$ exists and since $F = (f_1, f_2, f_3, f_4)^\tau \in D(\mathcal{A}+\mathcal{B})$ so $(\mathcal{A}+\mathcal{B})^{-1}$ is compact on $\mathcal{H}$ according to the Sobolev's Embedding Theorem.

Theorem 2 *Let $\mathcal{A}$ and $\mathcal{B}$ be given by* (9) *and* (10)*, then $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}$. Furthermore, since $\mathcal{B}$ is bounded, $\mathcal{A}+\mathcal{B}$ also generates a C_0-group on $\mathcal{H}$.*

Proof. See (Wang, 2004, Chapter 9, p. 167) for the detail of the demonstration.

Since the spectrum determined growth condition will later be shown to be true for $\mathcal{A}+\mathcal{B}$, the stability of (S) and (11) hinge on the behavior of the eigenvalues of $\mathcal{A}+\mathcal{B}$, and we are now in a position to study the eigenvalues problem.

Let $\lambda \in \sigma(\mathcal{A}+\mathcal{B})$ and $Y_\lambda = (\omega, z, \varphi, \psi)^\tau$, be an eigenfunction of $\mathcal{A}+\mathcal{B}$ corresponding to λ.

Then $(\mathcal{A}+\mathcal{B})\,Y_\lambda = \lambda Y_\lambda$ implies

$$z = \lambda\omega, \quad \psi = \lambda\varphi$$

and that ω and φ must satisfy the following characteristic equations

$$\begin{cases} m(x)\,\lambda^2\omega - (K(x)\,\omega')' + (K(x)\varphi)' + m(x)b_1(x)\lambda\omega = 0, & 0 \le x \le 1 \\ I_m(x)\lambda^2\varphi - (EI(x)\,\varphi')' - K(x)\,(\omega' - \varphi) + I_m(x)\,b_2(x)\,\lambda\varphi = 0, & 0 \le x \le 1 \end{cases} \tag{18}$$

with boundary conditions

$$\begin{cases} \omega(0) = 0, \quad \varphi(0) = 0, \\ \varphi'(1) = 0, \quad \varphi(1) - \omega'(1) = 0. \end{cases} \tag{19}$$

For convenience, we let

$$\begin{cases} a_1(x) = \frac{K'(x)}{K(x)}, \quad r_1(x) = \sqrt{\frac{m(x)}{K(x)}}, \\ a_2(x) = \frac{EI'(x)}{EI(x)}, \quad a_3(x) = \frac{K(x)}{EI(x)}, \quad r_2(x) = \sqrt{\frac{I_m(x)}{EI(x)}}. \end{cases} \tag{20}$$

which has been stated in (Wang, 2004). Here $\frac{1}{r_1(x)}$ and $\frac{1}{r_2(x)}$ are usually called wave of the system (S) and not equal in general. Now (18) becomes

$$\begin{cases} \omega'' - \varphi' + a_1(\omega' - \varphi) - b_1 r_1^2 \lambda \omega - r_1^2 \lambda^2 \omega = 0, \ \ 0 < x < 1, \\ \varphi'' - a_2 \varphi' + a_3(\omega' - \varphi) - b_2 r_2^2 \lambda \varphi - r_2^2 \lambda^2 \varphi = 0, 0 < x < 1. \end{cases} \tag{21}$$

And if we let

$$\omega_1(x) = \omega(x), \quad \omega_2(x) = \omega'(x), \quad \varphi_1(x) = \varphi(x), \quad \varphi_2(x) = \varphi'(x), \tag{22}$$

$$\Phi(x) = (\omega_1(x), \omega_2(x), \varphi_1(x), \varphi_2(x))^{\tau}, \tag{23}$$

then (22) becomes

$$T^D(x, \lambda)\Phi(x) = 0, \tag{24}$$

where

$$\begin{aligned} T^D(x, \lambda)\Phi(x) &= \Phi'(x) + A(x, \lambda)\Phi(x) \\ &= \Phi'(x) + A_0(x)\Phi(x) - \lambda A_1(x)\Phi(x) - \lambda^2 A_2(x)\Phi(x), \end{aligned} \tag{25}$$

$$A(x, \lambda) = A_0(x)\Phi(x) - \lambda A_1(x)\Phi(x) - \lambda^2 A_2(x)\Phi(x) \tag{26}$$

and $A_0(x)$, $A_1(x)$ and $A_2(x)$ are three matrix functions see (Wang, 2004)

$$\begin{aligned} A_0(x) &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & a_1(x) & -a_1(x) & -1 \\ 0 & 0 & 0 & -1 \\ 0 & a_3(x) & -a_3(x) & a_2(x) \end{bmatrix}, \\ A_1(x) &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ b_1(x) r_1^2(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_2(x) r_2^2(x) & 0 \end{bmatrix}, \\ A_2(x) &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ r_1^2(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r_2^2(x) & 0 \end{bmatrix}. \end{aligned}$$

Under the same procedure, the boundary conditions (19) can be written as follows:

$$T^R\Phi(x) = W^0\Phi(0) + W^1\Phi(1) = 0, \tag{27}$$

with

$$W^0(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad W^1(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha + \lambda\beta}{EI(1)} & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}. \tag{28}$$

We get the following result:

Theorem 3 *The characteristics equations* (18) *together with boundary conditions* (19) *are equivalent to the first order linear system* (25) *with boundary conditions* (27)*.*

Proof. See (Wang, 2004, Chapter 2, pp. 24-25)

3. Asymptotic Behavior of Eigenfrequencies of System (S)

We want to find an asymptotic expression for eigenvalues of $\mathcal{A} + \mathcal{B}$. It is accomplished by expanding the characteristic determinant with asymptotic expressions of the fundamental matrix solutions of (25). A crucial step is an invertible matrix transformation which is very powerful in the sense that it can be applied to a lot other coupled problems as well, see (Wang & Yung, 2006), (Wang, 2004) and (Xu & Feng, 2002)

Theorem 4 *For $\lambda \in \mathbb{C}$ with $|\lambda|$ large enough. the first order linear system* (25) *has fundamental matrix of solutions given by:*

$$\widehat{\Phi}(.,\lambda) = P(.,\lambda)\widehat{\Psi}(.,\lambda), \tag{29}$$

where $P(.,\lambda)$ is invertible matrix defined by

$$P(x,\lambda) = \begin{bmatrix} r_1(x)\lambda & r_1(x)\lambda & 0 & 0 \\ r_1^2(x)\lambda^2 & -r_1^2(x)\lambda^2 & 0 & 0 \\ 0 & 0 & r_2(x)\lambda & r_2(x)\lambda \\ 0 & 0 & r_2^2(x)\lambda^2 & -r_2^2(x)\lambda^2 \end{bmatrix}, \tag{30}$$

and $\widehat{\Psi}(.,\lambda)$ can be given by:

$$\widehat{\Psi}(x,\lambda) = \left(\widehat{\Psi}_0(x) + O\left(\lambda^{-1}\right)\right)E(x,\lambda), \tag{31}$$

with

$$E(x,\lambda) = \begin{bmatrix} \exp(\lambda R_1(x)) & 0 & 0 & 0 \\ 0 & \exp(-\lambda R_1(x)) & 0 & 0 \\ 0 & 0 & \exp(\lambda R_2(x)) & 0 \\ 0 & 0 & 0 & \exp(-\lambda R_2(x)) \end{bmatrix} \tag{32}$$

$$R_1(x) = \int_0^x r_1(\xi)\,d\xi, \quad R_2(x) = \int_0^x r_2(\xi)\,d\xi, \tag{33}$$

$$\widehat{\Psi}_0(x) = diag\left(C_1(x), C_2(x), C_3(x), C_4(x)\right), \tag{34}$$

$$\begin{cases} c_1(x) = \int_0^x (a_1(\xi) - b_1(\xi)r_1(\xi) + 3a_4(\xi))\,d\xi, \\ c_2(x) = \int_0^x (a_1(\xi) + b_1(\xi)r_1(\xi) + 3a_4(\xi))\,d\xi, \\ c_3(x) = \int_0^x (a_2(\xi) - b_2(\xi)r_2(\xi) + 3a_5(\xi))\,d\xi, \\ c_4(x) = \int_0^x (a_2(\xi) + b_2(\xi)r_2(\xi) + 3a_5(\xi))\,d\xi, \end{cases} \quad \begin{cases} C_1(x) = \exp\left(-\frac{1}{2}c_1(x)\right), \\ C_2(x) = \exp\left(-\frac{1}{2}c_2(x)\right), \\ C_3(x) = \exp\left(-\frac{1}{2}c_3(x)\right), \\ C_4(x) = \exp\left(-\frac{1}{2}c_4(x)\right). \end{cases} \tag{35}$$

Proof. for the details of the proof, See (Wang, 2004, Chapter 2, pp. 24-32).

Theorem 5 *Let $\Delta(\lambda)$ the characteristic determinant of the first order linear system* (25) *with boundary conditions expression of* (27)*, then an asymptotic of $\Delta(\lambda)$ is given by*

$$\begin{aligned} \Delta(\lambda) = {} & \mu\lambda^6\left(C_2(1)[1]_1 e^{-\lambda R_1(1)} + C_1(1)[1]_1 e^{\lambda R_1(1)}\right) \\ & \times\left(C_3(1)[1]_1\left(\frac{\beta}{EI(1)r_2(1)} + 1\right)e^{\lambda R_2(1)} + C_4(1)[1]_1\left(\frac{-\beta}{EI(1)r_2(1)} + 1\right)e^{-\lambda R_2(1)}\right), \end{aligned} \tag{36}$$

where

$$\mu = -r_1(0)r_2(0)r_1^2(1)r_2^2(1). \tag{37}$$

Thus, (25) *and* (27) *are strongly regular, and so asymptotically, the eigenvalues of determinant characteristic $\Delta(\lambda)$* (36) *are simple and separated.*

Proof. The eigenvalues of the first order linear system (25) and (27) are given by the zeros of the characteristic determinant

$$\Delta(\lambda) = \det\left(T^R\Phi(.,\lambda)\right), \quad \lambda \in \mathbb{C}, \tag{38}$$

where T^R is given by (27) and $\Phi(.,\lambda)$ is any fundamental matrix solutions of the equation $T^D(x,\lambda)\Phi(x) = 0$. So, by substituting (29)-(35) into (38) together with the boundary conditions (27), we can obtain the asymptotic eigenfrequencies. Note that:

$$T^R\Phi(x,\lambda) = W^0P(0,\lambda)\Psi(0,\lambda) + W^1P(1,\lambda)\Psi(1,\lambda). \tag{39}$$

A computation using (28) (30) give us

$$W^0 P(0,\lambda) = \begin{bmatrix} r_1(0)\lambda & r_1(0)\lambda & 0 & 0 \\ 0 & 0 & r_2(0)\lambda & r_2(0)\lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$W^1 P(1,\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{r_2(1)\lambda(\alpha+\lambda\beta)}{EI(1)} + r_2^2(1)\lambda^2 & \frac{r_2(1)\lambda(\alpha+\lambda\beta)}{EI(1)} - r_2^2(1)\lambda^2 \\ -r_1^2(1)\lambda^2 & r_1^2(1)\lambda^2 & r_2(1)\lambda & r_2(1)\lambda \end{bmatrix}.$$

Denote

$$[a]_1 = a + O\left(\lambda^{-1}\right).$$

Since, $\widehat{\Psi}_0(0) = I$ and $E(0,\lambda) = I$, so a direct computation gives us

$$\begin{aligned} W^1 P(1,\lambda)\Psi(1,\lambda) \;=\; & \left[\begin{matrix} 0 & 0 \\ 0 & 0 \\ \lambda^2[0]_1 e^{\lambda R_1(1)} & \lambda^2[0]_1 e^{-\lambda R_1(1)} \\ -r_1^2(1)C_1(1)[1]_1\lambda^2 e^{\lambda R_1(1)} & r_1^2(1)C_2(1)[1]_1\lambda^2 e^{-\lambda R_1(1)} \end{matrix}\right. \\ & \left.\begin{matrix} 0 & 0 \\ 0 & 0 \\ \left(\frac{\beta r_2(1)}{EI(1)} + r_2^2(1)\right)C_3(1)\lambda^2[1]_1 e^{\lambda R_2(1)} & \left(\frac{\beta r_2(1)}{EI(1)} - r_2^2(1)\right)C_4(1)\lambda^2[1]_1 e^{-\lambda R_2(1)} \\ r_2(1)C_3(1)\lambda[1]_1 e^{\lambda R_2(1)} & r_2(1)C_4(1)\lambda[1]_1 e^{-\lambda R_2(1)} \end{matrix}\right], \end{aligned}$$

$$W^0 P(0,\lambda)\Psi(0,\lambda) = \begin{bmatrix} r_1(0)\lambda[0]_1 & r_1(0)\lambda[0]_1 & \lambda[0]_1 & \lambda[0]_1 \\ \lambda[0]_1 & \lambda[0]_1 & r_2(0)\lambda[0]_1 & r_2(0)\lambda[0]_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and hence,

$$\begin{aligned} T^R\Phi(.,\lambda) \;=\; & \left[\begin{matrix} r_1(0)\lambda[0]_1 & r_1(0)\lambda[0]_1 \\ \lambda[0]_1 & \lambda[0]_1 \\ \lambda^2[0]_1 e^{\lambda R_1(1)} & \lambda^2[0]_1 e^{-\lambda R_1(1)} \\ -r_1^2(1)C_1(1)[1]_1\lambda^2 e^{\lambda R_1(1)} & r_1^2(1)C_2(1)[1]_1\lambda^2 e^{-\lambda R_1(1)} \end{matrix}\right. \\ & \left.\begin{matrix} \lambda[0]_1 & \lambda[0]_1 \\ r_2(0)\lambda[0]_1 & r_2(0)\lambda[0]_1 \\ \left(\frac{\beta r_2(1)}{EI(1)} + r_2^2(1)\right)C_3(1)\lambda^2[1]_1 e^{\lambda R_2(1)} & \left(\frac{\beta r_2(1)}{EI(1)} - r_2^2(1)\right)C_4(1)\lambda^2[1]_1 e^{-\lambda R_2(1)} \\ r_2(1)C_3(1)\lambda[1]_1 e^{\lambda R_2(1)} & r_2(1)C_4(1)\lambda[1]_1 e^{-\lambda R_2(1)} \end{matrix}\right]. \end{aligned}$$

thus, we get

$$\begin{aligned} \Delta(\lambda) \;=\; & \det\left(T^R\Phi(.,\lambda)\right) \\ =\; & \lambda^6 \det\left[\begin{matrix} r_1(0)[0]_1 & r_1(0)[0]_1 \\ [0]_1 & [0]_1 \\ [0]_1 e^{\lambda R_1(1)} & [0]_1 e^{-\lambda R_1(1)} \\ -r_1^2(1)C_1(1)[1]_1 e^{\lambda R_1(1)} & r_1^2(1)C_2(1)[1]_1 e^{-\lambda R_1(1)} \end{matrix}\right. \\ & \left.\begin{matrix} [0]_1 & [0]_1 \\ r_2(0)[0]_1 & r_2(0)[0]_1 \\ \left(\frac{\beta r_2(1)}{EI(1)} + r_2^2(1)\right)C_3(1)[1]_1 e^{\lambda R_2(1)} & \left(\frac{\beta r_2(1)}{EI(1)} - r_2^2(1)\right)C_4(1)[1]_1 e^{-\lambda R_2(1)} \\ r_2(1)C_3(1)[1]_1 e^{\lambda R_2(1)} & r_2(1)C_4(1)[1]_1 e^{-\lambda R_2(1)} \end{matrix}\right]. \end{aligned}$$

and so,

$$\begin{aligned}\Delta(\lambda) &= \lambda^6 \det\begin{bmatrix} r_1(0)[0]_1 & r_1(0)[0]_1 \\ -r_1^2(1)C_1(1)[1]_1 e^{\lambda R_1(1)} & r_1^2(1)C_2(1)[1]_1 e^{-\lambda R_1(1)} \end{bmatrix} \\ &\quad \times \det\begin{bmatrix} r_2(0)[0]_1 & r_2(0)[0]_1 \\ \left(\frac{\beta r_2(1)}{EI(1)} + r_2^2(1)\right)C_3(1)[1]_1 e^{\lambda R_2(1)} & \left(\frac{\beta r_2(1)}{EI(1)} - r_2^2(1)\right)C_4(1)[1]_1 e^{-\lambda R_2(1)} \end{bmatrix} \\ &= -\lambda^6 r_1(0) r_2(0) r_1^2(1) r_2^2(1)\left(C_2(1)[1]_1 e^{-\lambda R_1(1)} + C_1(1)[1]_1 e^{\lambda R_1(1)}\right) \\ &\quad \times\left(C_3(1)[1]_1\left(\frac{\beta}{EI(1)r_2(1)} + 1\right)e^{\lambda R_2(1)} + C_4(1)[1]_1\left(\frac{-\beta}{EI(1)r_2(1)} + 1\right)e^{-\lambda R_2(1)}\right).\end{aligned}$$

Thus, (25) and (27) are strongly regular, and so asymptotically, the eigenvalues of determinant characteristic $\Delta(\lambda)$ (36) are simple and separated.

We have the following result on the spectrum of $\mathcal{A} + \mathcal{B}$.

Theorem 6 *Let $\mathcal{A} + \mathcal{B}$ be given by* (8)-(10)*, then each $\lambda \in \sigma(\mathcal{A} + \mathcal{B})$ is algebraically simple when $|\lambda|$ is large enough, and has asymptotic expression given by*

$$\lambda_{1k} = \frac{1}{2R_1(1)}\left(-\int_0^1 b_1(\xi)\left(\frac{m(\xi)}{K(\xi)}\right)^{\frac{1}{2}} d\xi + (1+2k)\pi i\right) + O\left(k^{-1}\right),\ |k| \geq N_1,\ k \in \mathbb{Z}, \tag{40}$$

where, for $0 < \beta < r_2(1)EI(1)$

$$\lambda_{2k} = \frac{\gamma_1}{2R_2(1)}\left(-\int_0^1 b_2(\xi)\left(\frac{I_m(\xi)}{EI(\xi)}\right)^{\frac{1}{2}} d\xi + (1+2k)\pi i\right) + O\left(k^{-1}\right),\ |k| \geq N_2,\ k \in \mathbb{Z},, \tag{41}$$

and for $\beta > r_2(1)EI(1)$

$$\lambda_{2k} = \frac{\gamma_2}{2R_2(1)}\left(-\int_0^1 b_2(\xi)\left(\frac{I_m(\xi)}{EI(\xi)}\right)^{\frac{1}{2}} d\xi + (2k)\pi i\right) + O\left(k^{-1}\right),\ |k| \geq N_3,\ k \in \mathbb{Z},, \tag{42}$$

where N_1, N_2 and N_3 are large enough positive integers, $R_1(1)$ and $R_2(1)$ of (33)*, given by*

$$R_1(1) = \int_0^1 r_1(\xi)\, d\xi = \int_0^1 \left(\frac{m(\xi)}{K(\xi)}\right)^{\frac{1}{2}} d\xi$$

or

$$R_2(1) = \int_0^1 r_2(\xi)\, d\xi = \int_0^1 \left(\frac{I_m(\xi)}{EI(\xi)}\right)^{\frac{1}{2}} d\xi.$$

Furthermore, the assumptions (2) *and* (3) *respectively imply that*

$$Re\lambda_{1k} \to -\frac{1}{2R_1(1)}\int_0^1 b_1(\xi)\left(\frac{m(\xi)}{K(\xi)}\right)^{\frac{1}{2}} d\xi < 0,\ when\ k \to \infty, \tag{43}$$

and for $0 < \beta < r_2(1)EI(1)$ *and* $\beta > r_2(1)EI(1)$*:*

$$Re\lambda_{2k} \to -\frac{\gamma_i}{2R_2(1)}\int_0^1 b_2(\xi)\left(\frac{I_m(\xi)}{EI(\xi)}\right)^{\frac{1}{2}} d\xi < 0,\ when\ k \to \infty\ for\ \ i = 1;\ 2. \tag{44}$$

Proof. If the characteristic determinant $\Delta(\lambda) = 0$, we get

$$\left(C_2(1)[1]_1 e^{-\lambda R_1(1)} + C_1(1)[1]_1 e^{\lambda R_1(1)}\right)\times$$
$$\left(C_3(1)[1]_1\left(\frac{\beta}{EI(1)r_2(1)} + 1\right)e^{\lambda R_2(1)} + C_4(1)[1]_1\left(\frac{-\beta}{EI(1)r_2(1)} + 1\right)e^{-\lambda R_2(1)}\right) = 0$$

which is equivalent to

$$C_2(1)[1]_1 e^{-\lambda R_1(1)} + C_1(1)[1]_1 e^{\lambda R_1(1)} = 0 \tag{45}$$

where

$$C_3(1)[1]_1\left(\frac{\beta}{EI(1)r_2(1)}+1\right)e^{\lambda R_2(1)}+C_4(1)[1]_1\left(\frac{-\beta}{EI(1)r_2(1)}+1\right)e^{-\lambda R_2(1)}=0. \tag{46}$$

Equation (45), can be written as

$$C_2(1)e^{-\lambda R_1(1)}+C_1(1)e^{\lambda R_1(1)}+O\left(\lambda^{-1}\right)=0. \tag{47}$$

By Rouch?'s Theorem, the roots of (47) can be estimated by those of

$$C_2(1)e^{-\lambda R_1(1)}+C_1(1)e^{\lambda R_1(1)}=0,$$

which, using (35) , yields

$$e^{2\lambda R_1(1)}=-\frac{C_2(1)}{C_1(1)}=-\exp\left(-\frac{1}{2}\left(c_2(x)-c_1(x)\right)\right)$$

which roots are given by

$$\lambda_{1k}=\frac{1}{2R_1(1)}\left(-\frac{1}{2}\left(c_2(x)-c_1(x)\right)+\pi i+2k\pi i\right),\ \ k\in\mathbb{Z}, \tag{48}$$

with $c_1(x), c_2(x)$ defined by (35) . Thus, the roots of (47) satisfy

$$\lambda_{1k}=\frac{1}{2R_1(1)}\left(-\frac{1}{2}\left(c_2(x)-c_1(x)\right)+(1+2k)\pi i\right)+O\left(k^{-1}\right),\ \ |k|\geq N_1,\ \ k\in\mathbb{Z}, \tag{49}$$

where N_1 is a large enough positive integer. We can repeat the same arguments for equation (46) and conclude for eigenvalues λ_{2k}

First case: $\mathbf{0<\beta<r_2(1)EI(1)}$

$$\lambda_{2k}=-\frac{\gamma_1}{2R_2(1)}\left(-\frac{1}{2}\left(c_4(x)-c_3(x)\right)+(1+2k)\pi i\right)+O\left(k^{-1}\right),\ \ |k|\geq N_2,\ \ k\in\mathbb{Z}, \tag{50}$$

for

$$\gamma_1=\frac{r_2(1)EI(1)-\beta}{r_2(1)EI(1)+\beta}>0$$

Second case: $\mathbf{\beta>r_2(1)EI(1)}$

$$\lambda_{2k}=\frac{\gamma_2}{2R_2(1)}\left(-\frac{1}{2}\left(c_4(x)-c_3(x)\right)+(2k)\pi i\right)+O\left(k^{-1}\right),\ \ |k|\geq N_3,\ \ k\in\mathbb{Z}, \tag{51}$$

for

$$\gamma_2=\frac{-r_2(1)EI(1)+\beta}{r_2(1)EI(1)+\beta}>0$$

with N_2 and N_3 are large enough positive integers $c_3(x), c_4(x)$ defined by (35).

We can further simplify (49) and (50) by using (35) and (18) to conclude that

$$\begin{aligned}
c_2(1)-c_1(1) &= \int_0^1\left(a_1(\xi)+b_1(\xi)r_1(\xi)+3a_4(\xi)\right)d\xi \\
&\quad -\int_0^1\left(a_1(\xi)-b_1(\xi)r_1(\xi)+3a_4(\xi)\right)d\xi \\
&= 2\int_0^1 b_1(\xi)r_1(\xi)\,d\xi \\
&= 2\int_0^1 b_1(\xi)\left(\frac{m(\xi)}{K(\xi)}\right)^{\frac{1}{2}}d\xi
\end{aligned}$$

and

$$c_4(1)-c_3(1)=2\int_0^1 b_2(\xi)r_2(\xi)\,d\xi=2\int_0^1 b_2(\xi)\left(\frac{I_m(\xi)}{EI(\xi)}\right)^{\frac{1}{2}}d\xi.$$

Furthermore, the assumptions (2) and (3) respectively imply that

$$Re\lambda_{1k} \to -\frac{1}{2R_1(1)}\int_0^1 b_1(\xi)\left(\frac{m(\xi)}{K(\xi)}\right)^{\frac{1}{2}} d\xi < 0,\ when\ k \to \infty, \tag{52}$$

and for $0 < \beta < r_2(1)EI(1)$ and $\beta > r_2(1)EI(1)$:

$$Re\lambda_{2k} \to -\frac{\gamma_i}{2R_2(1)}\int_0^1 b_2(\xi)\left(\frac{I_m(\xi)}{EI(\xi)}\right)^{\frac{1}{2}} d\xi < 0,\ when\ k \to \infty\ for\ \ i = 1;\ 2. \tag{53}$$

4. Riesz Basis Property and the Exponential Stability of the System (S) Under the Conditions $0 < \beta < r_2(1)EI(1)$ and $\beta > r_2(1)EI(1)$

We now ready to investigate Riesz basis property and the exponential stability of the system (11) under the conditions $0 < \beta < r_2(1)EI(1)$ and $\beta > r_2(1)EI(1)$.

Theorem 7 *Let $\mathcal{A} + \mathcal{B}$ given by* (8)-(10)*. Then generalized eigenfunctions of the operator $\mathcal{A} + \mathcal{B}$ of system* (11) *are complete in $\mathcal{H}$.*

Proof. See (Wang, 2004, Chapter 9, p. 174) for the details of the proof.

Theorem 8 *The system* (11) *is Riesz-spectral system (in the sense that eigenfunctions form a Riesz basis in $\mathcal{H}$) and so it satisfies the spectrum determined growth condition.*

Proof. See (Wang, 2004, Chapter 9, p. 174) for the details of the proof.

We now ready to discuss the stability of the Timoshenko beam system (S). Under assumptions (2) and (3), the Theorems 9.3.3 and 9.4.2 imply that for each $\varepsilon > 0$ there are at most finitely many eigenvalues lying outside of two strips:

$$-\frac{1}{2R_1(1)}\int_0^1 b_1(\xi)\left(\frac{m(\xi)}{K(\xi)}\right)^{\frac{1}{2}} d\xi - \varepsilon \le Re\lambda \le -\frac{1}{2R_1(1)}\int_0^1 b_1(\xi)\left(\frac{m(\xi)}{K(\xi)}\right)^{\frac{1}{2}} d\xi + \varepsilon$$

and for $i = 1;\ 2$

$$-\frac{\gamma_i}{2R_2(1)}\int_0^1 b_2(\xi)\left(\frac{I_m(\xi)}{EI(\xi)}\right)^{\frac{1}{2}} d\xi - \xi \le Re\lambda \le -\frac{\gamma_i}{2R_2(1)}\int_0^1 b_2(\xi)\left(\frac{I_m(\xi)}{EI(\xi)}\right)^{\frac{1}{2}} d\xi + \xi.$$

We consider the important following result:

Theorem 9 *Suppose that assumptions* (8)-(20) *hold.*
1. If both $b_1(x)$ and $b_2(x)$ are non-negative and exist intervals I_1 and I_2 in $[0; 1]$ *such that:*

$$b_1(x)|_{I_1} > 0\ \ and\ \ b_2(x)|_{I_2} > 0$$

then the system (11) *is exponentially stable.*
2. If $b_1(x)$ and $b_2(x)$ are indefinite, then the system (11) *is exponentially stable when conditions:*

$$\max_{x\in[0,1]}\{b_1^-(x), b_2^-(x)\} < \left|s(\mathcal{A} + \mathcal{B}^+)\right|, \tag{54}$$

is satisfied.

Proof. We use the same reasoning in (Wang, 2004, Chapter 9, pp. 174-176) and (Xu & Feng, 2002) with

$$0 = Re\langle(\mathcal{A} + \mathcal{B})Y_\lambda, Y_\lambda\rangle = -|\lambda|^2\int_0^1 \left(m(x)b_1(x)|\omega|^2 + I_m(x)b_2(x)|\varphi|^2\right)dx.$$

and $Y_\lambda = (\omega, \lambda\omega, \varphi, \lambda\varphi)^\tau \in \mathcal{A} + \mathcal{B}$ and λ an eigenvalue of $\mathcal{A} + \mathcal{B}$.

5. Conclusion

The boundary feedback stabilization problem of a hybrid system has been studied extensively in the last decade. Many important results have been obtained. Among them, most of studies in the literatures are concerned with Euler-Bernoulli and Rayleigh beams; there are a few results for Timoshenko beams, we may cite the work of Akian (2022), (Feng et al, 2001), (Liping et al, 2019), (Nasser, 2011), (Nasser, 2013), (Wang, 2004, Chapter 9) and (Xu & Feng, 2002), which are

mainly focused on the stability of the closed-loop system. Though it is important to obtain the exponential stability of the system, it is also very interesting to study the rate of the exponential decay of the system. In our work, we obtain the exponential stability of the system under conditions.
In perspective, we can take an interest in condition $\beta = r_2(1)EI(1)$ by carrying out an asymptotic expansion to order 2 of the eigenvalues in order to study the exponential stability of the system.

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Authors contributions

Dr. Kouassi Ayo Ayébié Hermith was responsible for study design. Dr. Kouassi Ayo Ayébié Hermith drafted the manuscript, Professor Touré Kidjégbo and Dr Diop Fatou N'diaye revised it. All authors read and approved the final manuscript. The authors contributed equally to the study.

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