Journal of Mathematics Research; Vol. 16, No. 1; February 2024
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

A Proof of a Gantmacher's Result on Nonnegative Square Matrices by Means of M-matrices

Giorgio Giorgi

Correspondence: Department of Economics and Management, University of Pavia, Via S. Felice, 5 - 27100 PAVIA, Italy

Received: October 26, 2023 Accepted: January 9, 2024 Online Published: January 26, 2024

doi:10.5539/jmr.v16n1p46 URL: https://doi.org/10.5539/jmr.v16n1p46

Abstract

We give an elementary proof, by means of a basic characterization of nonsingular M-matrices, of a result of F. R. Gantmacher, concerning nonnegative decomposable square matrices.

Keywords: Gantmacher normal form, Perron-Frobenius theorem, Leontief modes, Sraffa modes

AMS Subject Classification: 15B48, 91B66

1. Introduction

One of the main results of the classical Perron-Frobenius theorem for an indecomposable nonnegative square matrix A, of order n, is that it is possible to obtain a positive eigenvector x^*, associated with the *Frobenius eigenvalue* or *dominant eigenvalue* of A, denoted by $\lambda^*(A)$, that is

$$\lambda^*(A) \geqq |\lambda|,$$

being $\lambda^*(A) > 0$ and being λ any other eigenvalue of A. A natural question arises: when A is not indecomposable (i. e. A is decomposable) it is possible to obtain a positive eigenvector x^* associated with $\lambda^*(A)$? The answer is in general negative, as it is well known; it is however affirmative under additional assumptions which are the relevant part of a nice result of Gantmacher (1959, 1966). The aim of the present paper is to give a simple and direct proof of the said result of Gantmacher, by means of the so-called *Theory of M-matrices.* The paper is organized as follows: Section 2 is concerned with basic notations, definitions and background material on Gantmacher's theorem and on M-matrices. Section 3 contains the main result, i. e. the proof of Gantmacher's theorem via M-matrices theory. The final Section 4 contains some remarks and conclusions.

2. Notations, Definitions and Background Material

We denote by [0] the *zero matrix* of order (m, n) and also the *zero vector* of $\mathbb{R}^n$. If A is a real matrix of order (m, n), we say that

- A is *nonnegative,* if $a_{ij} \geqq 0$, $\forall i = 1, ..., m$; $\forall j = 1, ..., n$, and we write $A \geqq [0]$.
- A is *semipositive,* if $A \geqq [0]$, $A \neq [0]$, and we write $A \geq [0]$.
- A is *positive,* if $a_{ij} > 0$, $\forall i = 1, ..., m$; $\forall j = 1, ..., n$, and we write $A > [0]$.

The same conventions and notations will be used to compare vectors $x \in \mathbb{R}^n$ with the zero vector $[0] \in \mathbb{R}^n$:

- If $x \geqq [0]$, then x is a *nonnegative vector.*
- If $x \geq [0]$, then x is a *semipositive vector.*
- If $x > [0]$, then x is a *positive vector.*

We recall that a square matrix A, of order n, is said to be *decomposable* (or *reducible*) if there exists a permutation matrix P such that

$$PAP^\top = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with A_{11} square submatrix of A (and hence also A_{22} is square) and at least one of the submatrices A_{12}, A_{21} a zero matrix. If A is not decomposable, it is said to be *indecomposable* (or *irreducible*).

If $A \geq [0]$, square of order n, is indecomposable, it is well known that there is a real positive eigenvalue, $\lambda^*(A)$, called *Frobenius eigenvalue* or *dominant eigenvalue* of A, such that

$$\lambda^*(A) \geqq |\lambda|,$$

being λ any other eigenvalue of A, and that the problem

$$\begin{cases} Ax = \lambda x \\ \lambda > 0, \ x > [0] \end{cases} \tag{1}$$

has a solution if and only if $\lambda = \lambda^*(A)$. This is one of the statements of the *Perron-Frobenius Theorem,* in its "strong version", i. e. referred to semipositive indecomposable square matrices. More precisely, if $x^* > [0]$ is an eigenvector of A associated with $\lambda = \lambda^*(A)$, then all solutions of problem (1) are described by αx^*, with $\alpha > 0$ and arbitrary. The literature on the Perron-Frobenius theorem and its economic applications is quite relevant. We quote only Debreu and Herstein (1953), Gantmacher (1959, 1966), Lancaster and Tismenetsky (1985), Kemp and Kimura (1978), Murata (1977), Nikaido (1968, 1970), Pasinetti (1977), Seneta (1973), Takayama (1985), Woods (1978).

Furthermore, it can be proved (see Gantmacher (1959)) that $A \geq [0]$ is indecomposable if and only if systems (1) and (2) have a solution, where system (2) is:

$$\begin{cases} p^\top A = \lambda p^\top \\ \lambda > 0, \ p > [0]\,. \end{cases} \tag{2}$$

If $A \geq [0]$ is decomposable, it is no longer possible to obtain positive vectors x in system (1), but only *semipositive vectors* x. Furthermore, $\lambda^*(A)$ is not, in this case, the unique eigenvalue associated with a semipositive vector. Consider, e. g., the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix},$$

which has two semipositive eigenvectors $x^* \geq [0]$, respectively associated with $\lambda_1 = \lambda^*(A) = 3$ and $\lambda_2 = 2$.

It is possible to obtain an eigenvector $x^* > [0]$ also when $A \geq [0]$ is decomposable? Beyond trivial cases, such as, for example, a diagonal matrix αI, with $\alpha > 0$, we have already asserted that the answer is in general negative. However, under additional assumptions, the answer is positive and this is the content of a nice result due to Gantmacher (1959, 1966). This result has received some attention mainly within the analysis of some multi-sectoral economic models of the Leontief and Sraffa type. See Section 4 of the present paper.

A powerful generalization of the concept of a decomposable (semipositive) square matrix is the *Gantmacher normal form* (Gantmacher (1959, 1966)). Let be given a square matrix A, of order $n \geqq 2$ and, without loss of generality, let us suppose $A \geq [0]$ (this is not required in the original definition). Then, there exists a permutation matrix P such that

$$PAP^\top = \begin{bmatrix} A_{11} & [0] & \cdots & [0] & [0] & [0] & \cdots & [0] & [0] \\ [0] & A_{22} & \cdots & [0] & [0] & [0] & \cdots & [0] & [0] \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ [0] & [0] & \cdots & A_{gg} & [0] & [0] & \cdots & [0] & [0] \\ A_{h1} & A_{h2} & \cdots & A_{hg} & A_{hh} & [0] & \cdots & [0] & [0] \\ A_{i1} & A_{i2} & \cdots & A_{ig} & A_{ih} & A_{ii} & \cdots & [0] & [0] \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rg} & A_{rh} & A_{ri} & \cdots & A_{rr} & [0] \\ A_{s1} & A_{s2} & \cdots & A_{sg} & A_{sh} & A_{si} & \cdots & A_{sr} & A_{ss} \end{bmatrix} \tag{3}$$

where:

$*$ Each diagonal block $A_{11}, A_{22}, ..., A_{ss}$, is square, of order $k \geqq 1$ and indecomposable (or it is zero, if of order $k = 1$). The blocks $A_{11}, ..., A_{gg}, ..., A_{ss}$ are called *principal blocks,* while the blocks $A_{11}, ..., A_{gg}$, are called *isolated blocks.*

$*$ If $s = g$, we obtain a *block diagonal form:*

$$PAP^\top = \begin{bmatrix} A_{11} & [0] & \cdots & [0] \\ [0] & A_{22} & \cdots & [0] \\ \vdots & \vdots & \dots & \vdots \\ [0] & [0] & \cdots & A_{ss} \end{bmatrix}.$$

$*$ We have that A is *indecomposable,* if $s = g = 1$.

$*$ If $s > g$ (i. e. if A is in a *block triangular form*), then it holds

$$\begin{aligned}
\left[A_{h1}, A_{h2}, ..., A_{hg}\right] &\neq [0]\,, \text{ i. e.} \geq [0] \text{ under our assumptions;}\\
\left[A_{i1}, A_{i2}, ..., A_{ig}, A_{ih}\right] &\neq [0]\,, \text{ i. e.} \geq [0] \text{ under our assumptions;}\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\\
\left[A_{s1}, A_{s2}, ..., A_{sg}, A_{sh}, ..., A_{sr}\right] &\neq [0]\,, \text{ i. e. } \geq [0] \text{ under our assumptions;}
\end{aligned}$$

The form (3), called *Gantmacher normal form,* is unique, provided the following operations are not considered relevant:

(a) To operate permutations within the blocks $A_{11}, A_{22}, ..., A_{gg}$.

(b) To operate permutations within the blocks $A_{hh}, A_{ii}, ..., A_{ss}$ (this operation is not always allowed).

(c) To operate permutations of the lines of any one of the blocks sub (a) and (b).

In some questions related to multi-sectoral economic models, it is more convenient to make reference to a form which is, in a sense, a transpose of the form (3), i. e. the following one:

$$PAP^{\top} = \begin{bmatrix} A_{11} & [0] & \cdots & [0] & A_{1h} & \cdots & A_{1r} & A_{1s} \\ [0] & A_{22} & \cdots & [0] & A_{2h} & \cdots & A_{2r} & A_{2s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ldots & \vdots & \vdots \\ [0] & [0] & \cdots & A_{gg} & A_{gh} & \cdots & A_{gr} & A_{gs} \\ [0] & [0] & \cdots & [0] & A_{hh} & \cdots & A_{hr} & A_{hs} \\ \vdots & \vdots & \ldots & \ldots & \ldots & \ldots & \vdots & \vdots \\ [0] & [0] & \cdots & \cdots & \cdots & \cdots & A_{rr} & A_{rs} \\ [0] & [0] & \ldots & \cdots & \cdots & \cdots & [0] & A_{ss} \end{bmatrix}. \tag{4}$$

We may note that the first scheme (3) is useful when a_{ij} describes a relation from j towards i, whereas the second scheme (4) is useful when the said relation is from i towards j, for example when, in a Leontief input-output model or in a Sraffa model with no joint production, a_{ij} measures the quantity of the i-th product which is the unitary input of the j-th industry.

We point out that every square matrix (not necessarily nonnegative) admits both the normal form (3) and the normal form (4), but in these forms the respective pairs $(g; s)$ are not necessarily equal, nor are necessarily equal the respective principal blocks. Consider, e. g., the following example.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

With $P = I$, we obtain the normal form (4), whereas with

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

we obtain the normal form (3):

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

Hence in the first case we have $(g; s) = (1; 3)$ and in the second case we have $(g; s) = (2; 3)$.

The result of Gantmacher, quoted in the Introduction, is given by the following theorem (Gantmacher (1959), page 92, Theorem 6).

Theorem 1. There is a positive eigenvector corresponding to the dominant eigenvalue $\lambda^*(A)$ of the matrix $A \geqq [0]$, square of order n, if and only if

(i) each of the isolated blocks $A_{11}, A_{22}, ..., A_{gg}$ in the normal form (3) of the matrix A has $\lambda^*(A)$ as its eigenvalue;

(ii) when $g < s$, none of the blocks $A_{hh}, ..., A_{ss}$ possesses this property.

We shall give a rather simple and direct proof of Theorem 1 by means of some basic properties of (nonsingular) M-matrices. See, e. g., Bapat and Raghavan (1997), Berman and Plemmons (1994), Fiedler and Ptàk (1962), Giorgi (2022), Magnani and Meriggi (1981), Plemmons (1977), Poole and Boullion (1974), Varga (1962).

Let be given a (real) square matrix A, of order n, such that

$$a_{ij} \leqq 0, \ \forall i \neq j.$$

These matrices are also called *Z-matrices.* The following result is fundamental.

★ Let be given a Z-matrix C, of order n. Then C is an *M-matrix* (more precisely: a *nonsingular M-matrix*) if any one of the following equivalent condition is satisfied (some authors call these matrices *K-matrices*).

M 1) There exists a vector $x \geq [0]$ such that $Cx > [0]$.

M 2) There exists a vector $x > [0]$ such that $Cx > [0]$.

M 3) There exists a vector $y > [0]$ such that the system

$$\begin{cases} Cx = y \\ x \geq [0] \end{cases}$$

has a solution.

M 4) For any $y \geqq [0]$ the system

$$\begin{cases} Cx = y \\ x \geqq [0] \end{cases}$$

has a solution.

M 5) C verifies the so-called "Hawkins-Simon" conditions, i. e. all *leading principal minors* of C (or *North-West principal minors* of C) are positive:

$$c_{11} > 0; \ \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} > 0; \ \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} > 0; \ ...; \ |C| > 0.$$

M 6) All principal minors of C are positive, i. e. $-C$ is an "Hicksian matrix" (see, e. g., Kemp and Kimura (1978), Murata (1977), Takayama (1985)).

M 7) C^{-1} exists and it holds $C^{-1} \geq [0]$ (note that C^{-1} has all semipositive lines).

M 8) If C is written in the form

$$C = \rho I - A, \ \rho \in \mathbb{R}, \ A \geqq [0],$$

which is always possible, then $\rho > \lambda^*(A)$.

M 9) When C is written in the form of characterization M 8), then we have

$$C^{-1} = (\rho I - A)^{-1} = \sum_{k=0}^{+\infty} \frac{1}{\rho^{k+1}} (A)^k \geq [0],$$

where $(A)^k$ denotes the k-th power of A. The above series is known as the "C. Neumann series".

Nikaido (1968) calls M 1) the "weak solvability condition" and M 4) the "strong solvability condition". See also Takayama (1985). The above list of equivalent properties is not exhaustive: at present more than 60 equivalent properties characterizing M-matrices are known. See, e. g., Giorgi (2022). Furthermore, if C is indecomposable, the previous equivalent conditions M 1), M 2), M 3) and M 7) can be given in the following versions:

M 1)′ There exists $x \geq [0]$ such that $Cx \geq [0]$.

M 2)′ There exists $x > [0]$ such that $Cx \geq [0]$.

M 3)′ There exists $y \geq [0]$ such that the system

$$\begin{cases} Cx = y \\ x \geq [0] \end{cases}$$

has a solution.

$M\ 7)'$ C^{-1} exists and it holds $C^{-1} > [0]$.

3. Main Result

Now we prove Theorem 1 by means of properties of M-matrices recalled in the previous section. First we note that it holds ($|A|$ denotes the determinant of A)

$$\left|PAP^{\top} - \lambda I\right| = |A_{11} - \lambda I| \cdot |A_{22} - \lambda I| \cdot \ldots \cdot |A_{ss} - \lambda I|,$$

and that $|P| = \left|P^{\top}\right| = 1$, $P^{\top} = P^{-1}$. Hence we have

$$\begin{aligned} \left|PAP^{\top} - \lambda I\right| &= \left|PAP^{\top} - \lambda PP^{\top}\right| = \left|P(A - \lambda I)P^{\top}\right| = |P| \cdot |A - \lambda I| \cdot \left|P^{\top}\right| = \\ &= |A - \lambda I|. \end{aligned}$$

Therefore

$$|A - \lambda I| = |A_{11} - \lambda I| \cdot |A_{22} - \lambda I| \cdot \ldots \cdot |A_{ss} - \lambda I|,$$

and, with $\lambda = 0$,

$$|A| = |A_{11}| \cdot |A_{22}| \cdot \ldots \cdot |A_{ss}|.$$

Therefore, $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if λ is an eigenvalue of at least one of the principal blocks of Gantmacher's normal form of A. In other words, the union of the spectra of the said blocks gives the spectrum of A.

Proof of Theorem 1.

If A is indecomposable, then problem (1), here rewritten and renumbered, i. e.

$$\begin{cases} Ax = \lambda x \\ \lambda > 0,\ x > [0] \end{cases} \tag{5}$$

is solved by means of the Perron-Frobenius theorem, in its strong version. Now, let us consider A decomposable and given in its Gantamacher normal form (3). In order to solve the above problem, we have to find a scalar $\lambda \in \mathbb{R}$ and vectors x^1, $x^2, ..., x^s$ solutions of

$$\begin{aligned} A_{11}x^1 &= \lambda x^1 \\ A_{22}x^2 &= \lambda x^2 \end{aligned}$$

$$\cdots\cdots\cdots$$

$$A_{gg}x^g = \lambda x^g$$

$$\left[A_{h1},\ A_{h2}, ..., A_{hg}\right]\begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^g \end{bmatrix} + A_{hh}x^h = \lambda x^h$$

$$[A_{i1},\ A_{i2}, ..., A_{ih}]\begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^h \end{bmatrix} + A_{ii}x^i = \lambda x^i$$

$$\cdots\cdots\cdots$$

$$[A_{s1},\ A_{s2}, ..., A_{sr}]\begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^r \end{bmatrix} + A_{ss}x^s = \lambda x^s,$$

with $\lambda > 0$, $x^1 > [0]$, $x^2 > [0]$, ..., $x^s > [0]$.

Consider the first g subsystems. If some block $A_{11}, ..., A_{gg}$ is a scalar equal to zero, obviously the problem has no solution. Therefore, we do not take into consideration this case. Hence we assume that the "isolated blocks" $A_{11}, ..., A_{gg}$, are semipositive and indecomposable. By applying to these blocks the Perron-Frobenius theorem in its strong version, we obtain that it must hold

$$\lambda = \lambda^*(A) = \lambda^*(A_{11}) = \lambda^*(A_{22}) = ... = \lambda^*(A_{gg}).$$

Every subsystem, associated with the first g blocks, will admit a corresponding positive Frobenius eigenvector, associated with $\lambda^*(A)$, say $\alpha_1 x^{*1}$, $\alpha_2 x^{*2}, ..., \alpha_g x^{*g}$, with $\alpha_1 > 0$, $\alpha_2 > 0, ..., \alpha_g > 0$ and arbitrary. If $g = s$, there is no other issue to prove. If $g < s$, we have, with reference to the other $(s - g)$ subsystems,

$$[\lambda^*(A_{11})I - A_{hh}]\, x^h = \left[A_{h1},\ A_{h2}, ..., A_{hg}\right] \begin{bmatrix} x^{*1} \\ x^{*2} \\ \vdots \\ x^{*g} \end{bmatrix}$$

$$[\lambda^*(A_{11})I - A_{ii}]\, x^i = \left[A_{i1},\ A_{i2}, ..., A_{ig},\ A_{ih}\right] \begin{bmatrix} x^{*1} \\ x^{*2} \\ \vdots \\ x^{*g} \\ x^h \end{bmatrix}$$

$$\cdots\cdots\cdots$$

$$[\lambda^*(A_{11})I - A_{ss}]\, x^s = \left[A_{s1},\ A_{s2}, ..., A_{sg},\ A_{sh}, ..., A_{sr}\right] \begin{bmatrix} x^{*1} \\ x^{*2} \\ \vdots \\ x^{*g} \\ x^h \\ \vdots \\ x^r \end{bmatrix}$$

with $x^h > [0]\,, x^i > [0]\,, ..., x^r > [0]\,, x^s > [0]\,.$

We note that the matrices

$$[\lambda^*(A_{11})I - A_{hh}]\,, ..., [\lambda^*(A_{11})I - A_{ss}]$$

are indecomposable Z-matrices and that the right-hand side members of the above systems are all semipositive vectors. Hence these systems will have positive solutions if and only if their related matrices are M-matrices, that is (property M 8) of the characterizations of M-matrices) if and only if

$$\begin{array}{rcl} \lambda^*(A) & = & \lambda^*(A_{11}) = ... = \lambda^*(A_{gg}) > \lambda^*(A_{hh}), \\ \lambda^*(A) & > & \lambda^*(A_{ii}), ..., \lambda^*(A) > \lambda^*(A_{ss}). \end{array} \tag{6}$$

In conclusion, problem (5) admits a solution if and only if, if $g = s$, it holds

$$\lambda = \lambda^*(A) = \lambda^*(A_{11}) = \lambda^*(A_{22}) = ... = \lambda^*(A_{gg}),$$

and, if $g < s$, relation (6) holds. □

4. Some Remarks and Conclusions

The Gantmacher normal form has received some attention as a tool to obtain further results from the classical Perron-Frobenius theorem (see, e. g., Odiard (1971)). The same normal form and in particular Theorem 1 have been used mainly in the analysis of some multi-sectoral economic models, such as the *Leontief models* and the *Sraffa models* (with no joint production). See, e. g., Giorgi and Magnani (1978), Lippi (1979), Pasinetti (1977), Szyld (1985), Varri (1979), Zaghini (1967). See also the interesting letters exchanged between P. K. Newman and P. Sraffa, in appendix to the paper of K. Bharadway (1970). In particular, the conditions of Gantmacher can be useful to analyze the nature and properties of those commodities, called by Sraffa "non-basic commodities", and to analyze the construction and properties of what Sraffa calls "standard system". If A, square of order n and semipositive, is a matrix whose elements are the physical inputs per unit of physical outputs, then Sraffa (1960) calls "basic commodities" those commodities which enter, either directly or indirectly, in the production of all n commodities. The other commodities, which have not this property (i. e. they enter, either directly or indirectly, only in the production of a proper subset of the whole economic system), are called "non-basic commodities". It is possible to remark that a production system contains only basic commodities if and only if A is indecomposable. Otherwise, i. e. if A is decomposable, there are also non-basic commodities. More precisely, if in the Gantmacher normal form we have

• $g = s = 1$ (i. e. A is indecomposable), then every commodity is a basic commodity.

• $g = s > 1$, then no commodity is a basic commodity.

• $1 < g < s$, then no commodity is a basic commodity.

• $g = 1 < s$, then the basic commodities are all the ones referred to the block A_{11}.

The result of Gantmacher is relevant also in the study of the construction and properties of what Sraffa calls "standard system": given the technological matrix $A \geq [0]$, square of order n, we have to find the pairs (x, μ) solution of the system

$$\begin{cases} x = Ax + \mu Ax \\ x > [0]\,,\ \mu \geqq 0. \end{cases}$$

Obviously, this problem can be rewritten in the form

$$\begin{cases} Ax = \frac{1}{1+\mu} x \\ x > [0]\,,\ \mu \geqq 0, \end{cases}$$

i. e.

$$\begin{cases} Ax = \lambda x \\ x > [0]\,,\ 0 < \lambda \leqq 1. \end{cases}$$

If A is decomposable, i. e. there are also non-basic commodities, this problem has a solution if and only if the Gantmacher conditions are satisfied. The same is true if we consider a price system of a decomposable Sraffa model (with no joint production), where the value of the whole surplus produced by the system is used to reward the means of production, at a common rate of profit $r \geqq 0$, i. e. we have to solve the system

$$(1 + r)p^{\top} A = p^{\top},\ p > [0]\,,\ r \geqq 0.$$

In the present paper we have given a simple proof of the Gantmacher conditions on decomposable semipositive square matrices, by means of some elementary considerations and by means of a basic characterization of nonsingular M-matrices.

References

Bapat, R. B., & Raghavan, T. E. S. (1997). *Nonnegative Matrices and Applications.* Cambridge Univ. Press, Cambridge.

Berman, A., & Plemmons, R. (1994). *Nonnegative Matrices in the Mathematical Sciences.* SIAM, Philadelphia.

Bharadway, K. (1970). On the maximum number of switches between two production systems. *Schweizerische Zeitschrift für Volkswirtschaft and Statistik, 106*, 409-429.

Debreu, G., & Herstein, I. N. (1953). Nonnegative square matrices. *Econometrica, 21*, 597-607.

Fiedler, M., & Ptàk, V. (1962). On matrices with non-positive off-diagonal elements and positive principal minors. *Czech. Math. Journal, 12*(87), 382-400.

Gantmacher, F. R. (1959). *Applications of the Theory of Matrices.* Interscience, New York.

Gantmacher, F. R. (1966). *Théorie des Matrices.* Dunod, Paris.

Giorgi, G. (2022). Nonsingular M-matrices: a Tour in the Various Characterizations and in Some Related Classes. *DEM Working Paper Series, Department of Economics and Management.* University of Pavia (economiaweb.unipv.it).

Giorgi, G., & Magnani, U. (1978). Problemi aperti nella teoria dei modelli multisettoriali di produzione congiunta. *Rivista Internazionale di Scienze Sociali, LXXXVI*(4), 435-468.

Kemp, M. C., & Kimura, Y. (1978). *Introduction to Mathematical Economics.* Springer verlag, New York.

Lancaster, P. L., & Tismenetsky, M. (1985). *The Theory of Matrices*. Academic Press, Orlando, New York, etc.

Lippi, M. (1979). *I Prezzi di Produzione.* Il Mulino, Bologna.

Magnani, U., & Meriggi, M. R. (1981). Characterizations of K-matrices; in G. Castellani and P. Mazzoleni (Eds.). *Mathematical Programming and Its Economic Applications.* Franco Angeli Editore, Milan, 535-547.

Murata, Y. (1977). *Mathematics for Stability and Optimization of Economic Systems.* Academic Press, New York.

Nikaido, H. (1968). *Convex Structures and Economic Theory.*

Nikaido, H. (1972). *Introduction to Sets and Mappings in Modern Economics.* North-Holland, Amsterdam.

Odiard, C. (1971). Un corollaire du théorème de Perron-Frobenius. *Revue Française d'Informatique et de Recherche Opérationnelle, RIRO, 5*, R-2, 124-129.

Pasinetti, L. (1977). *Lectures on the Theory of Production.*

Plemmons, R. (1977). M-matrix characterizations I C Nonsingular M-matrices. *Linear Algebra and Its Appl., 18*, 175-188.

Poole, D. G., & Boullion, T. (1974). A survey on M-matrices. *SIAM Review, 16*(4), 419-427.

Seneta, E. (1973). *Non-Negative Matrices.* J. Wiley, New York.

Sraffa, P. (1960). *Production of Commodities by Means of Commodities: Prelude to a Critique of Economic Theory.* Cambridge Univ. Press, Cambridge.

Szyld, D. B. (1985). Conditions for the existence of a balanced growth solution for the Leontief dynamic input-output model. *Econometrica, 53*, 1411-1419.

Varga, R. S. (1962). *Matrix Iterative Analysis.* Prentice-Hall, Englewood Cliffs, N. J.

Varri, P. (1979). Basic and non-basic commodities in Mr. Sraffa's price system, *Metroeconomica, XXXI*, 55-72.

Copyrights