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The Upwind Finite Volume Element Method for Two-Dimensional Time Fractional Coupled Burgers' Equation

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Abstract

The finite volume element method for approximating a two-dimensional time fractional coupled Burgers' equation is presented. The linear finite volume element method is used for spatial discretization and the upwind technique is used for the nonlinear convective term to get the semi-discrete scheme. Further, the time-fractional derivative term is approximated by using L1 formula and the nonlinear convection term is treated by linearized upwind technique to get the fully discrete scheme. We prove that the semi-discrete scheme is convergent with one-order accuracy in space and the fully discrete scheme is convergent with one-order accuracy both in time and space in L^2-norm. Numerical experiments are presented finally to validate the theoretical analysis.

Keywords: two-dimensional time fractional coupled Burgers' equation, finite volume element method, upwind technique, convergence

1. Introduction

In this paper, we consider the following two-dimensional time fractional coupled Burgers' equation:

$$ {}_0^C\mathcal{D}_t^\alpha \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} = \mathbf{f}(\mathbf{x},t), (\mathbf{x},t)\in\Omega\times(0,T], \tag{1.1} $$

$$ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \mathbf{x}\in\Omega, \tag{1.2} $$

$$ \mathbf{u}(\mathbf{x},t)|_{\partial\Omega} = \eta(\mathbf{x},t), (\mathbf{x},t)\in\Omega\times(0,T], \tag{1.3} $$

where $\Omega\subset R^2$ is a bounded domain with boundary $\partial\Omega$, $\nu > 0$ is the viscosity. Symbols Δ, ∇ denote the Laplacian and gradient operators, respectively. $\mathbf{u} = \mathbf{u}(\mathbf{x},t) = (u(\mathbf{x},t), v(\mathbf{x},t))$ is the unknown vector function which represents the velocity of the fluid. The Caputo fractional derivative ${}_0^C\mathcal{D}_t^\alpha$ is defined as

$$ {}_0^C\mathcal{D}_t^\alpha \mathbf{u}(\mathbf{x},t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial\mathbf{u}(\mathbf{x},s)}{\partial s}\frac{1}{(t-s)^\alpha}ds, \quad 0<\alpha<1. $$

Fractional differential operators, different from integer differential operators, are non-local and very suitable for describing materials with memory and heredity in the real world. Fractional differential equations are widely used in the fields of anomalous diffusion, viscoelastic mechanics, fluid mechanics, boundary layer effect of pipeline, signal recognition processing and system identification (Baleanu et al., 2012; DElia, 2020; Li, Changpin and Chen, An, 2018) etc.

Burgers' equation is the simplest nonlinear time dependent partial differential equation which was first raised by Bateman in (Bateman, Harry, 1915) when he mentioned it as worthy of study and gave a special solution. It exists in plenty areas of applied mathematics, such as acoustic waves, heat conduction and modelling of dynamics (Caldwell, J et al. 1981; Cole, Julian D, 1951). Fractional Burgers' equation can be used to describe the physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe. The fractional derivative results from the cumulative effect of the wall friction through the boundary layer (Sugimoto, Nobumasa, 1991; Inc, Mustafa, 2008). Therefore, fractional Burgers' equations have been taken more and more seriously attention and numerical methods have been developed to provide numerical solutions for them. For example, Wang (Wang, Qi, 2006) discussed the adomian decomposition method for time and space fractional Burgers' equations. Esen and Tasbozan (Esen et al., 2015) proposed quadratic B-spline Galerkin method for one-dimensional time fractional Burgers' equation. Li et al. (Li, Dongfang et al. 2016) employed a linear implicit finite difference scheme for solving the generalized time fractional Burgers' equation. Cao et al. (Cao, Wen et al., 2017) presented discontinuous Galerkin method to solve a two-dimensional time fractional Burgers' equation with high and low Reynolds numbers on quasi-uniform triangular mesh. Wang et al. (Wang, Haifeng et al., 2021) proposed the weak Galerkin finite element method for a class of one-dimensional time fractional generalized Burgers' equation. Hussein and Ahmed Jabbar (Hussein, Ahmed Jabbar, 2020) proposed a weak Galerkin finite element method for solving two-dimensional time fractional coupled Burgers' equations on uniform triangular mesh. Qiao and Tang (Qiao, Leijie et al., 2022) introduce an accurate, robust, and efficient finite difference scheme with graded meshes for the time-fractional Burgers' equation. Zhang and Feng (Zhang, Yadong et al., 2023) present a local projection stabilization virtual element method for the time-fractional Burgers' equation with high Reynolds numbers. Kashif and Dwivedi (Mohd, Kashif etal., 2022) use non-standard finite difference scheme and Fibonacci collocation method for fractional order Burgers equation. AA AL-saedi and J Rashidinia (AA, AL-saedi etal., 2023) present a numerical scheme based on the Galerkin finite element method and cubic B-spline base function with quadratic weight function to approximate the numerical solution of the time-fractional Burgers equation.

The finite volume element method (FVEM) (Li, Ronghua et al., 2000; Cai, Zhiqiang et al., 1991; Cai, Zhiqiang, 1990; Ewing, Richard et al. 2000) is an important numerical method for solving partial differential equations. In some literature, FVEM is also called the control volume method, the covolume method, and the first-order generalized difference method. FVEM has been widely applied due to its various important features. For example, the grid is flexible and it inherits some physical conservation laws of the original problems locally which are very preferable in practical applications. And the computational effort is greater than in finite difference methods and less than finite element methods, while the accuracy is higher than finite difference methods and nearly the same as finite element methods. Liang (Liang, D, 1990; Liang, D, 1991) combined the upwind technique and the FVEM to solve the linear convection-dominated problems. Yang (Yang, Qing, 2013) used the upwind finite volume element method for two-dimensional integer-order Burgers' equation. The nonlinear convective term was dealt with upwind technique in order to avoid numerical oscillation. For the study of problem (1.1)-(1.3), there are still gaps in finite volume element method. In this paper, we will consider upwind finite volume element method for the approximation of the two-dimensional time coupled fractional Burgers' equation (1.1). The upwind approximation is applied to handle the nonlinear convection term in view of its advantage to solve the convection-dominated problems. The semi-discrete and fully discrete schemes are defined. The linear finite volume element format is used for spatial discretization, and the upwind technique is used for the nonlinear convective term to get the semi-discrete scheme. Then L1 formula is used to approximate the time-fractional derivative term, and the nonlinear convection term is treated by linearized upwind technique to get the fully discrete scheme. We prove that the semi-discrete scheme is convergent with one-order accuracy in space in L^2-norm and the fully discrete scheme is convergent with one-order accuracy both in time and space in L^2-norm. Some numerical experiments show that our method is effective for time fractional Burgers' equation.

The outline of this paper is as follows. In Section 2, we build the upwind finite volume element semi-discrete scheme and fully discrete scheme for the coupled fractional Burgers' equation, respectively. In section 3, we derive the L^2 norm error estimates for the semi-discrete scheme and the fully discrete scheme, respectively. In section 4, numerical experiments are presented to show the efficiency of proposed method and confirm our theoretical analysis.

2. The Approximation Schemes

In this section, we introduce some important function spaces and give the semi-discrete scheme and the fully discrete scheme for problem (1.1)-(1.3).

Set $\mathbf{X} = (H_0^1(\Omega))^2$, $\mathbf{Y} = (L^2(\Omega))^2$. Let $T_h = \{K\}$ be a quasi-uniform triangulation of domain Ω so that $\bar{\Omega} = \bigcup\limits_{K \in T_h} \{\bar{K}\}$ and N_h is the set of all nodal points of T_h, where $h = \max h_K$, h_K is the diameter of element K. Associated with the triangulation T_h, we introduce the velocity approximation space:

$$\mathbf{X}_h = \left\{ \mathbf{v}_h \in \mathbf{X} : \mathbf{v}_h|_K \in (P_1(K))^2 ,\ K \in T_h \right\},$$

where $P_1(K)$ is the set of linear polynomials on element K.

In order to define the finite volume element method, we need a dual partition associated with the primal partition T_h. We construct the barycenter dual partition T_h^* by connecting the barycenter to the midpoints of edges of each $K \in T_h$ by straight lines. Thus, for each nodal point P in T_h, there exists a polygonal surrounding it, which is called the dual element or the control volume at point P and denoted by K_P^*. So we have $T_h^* = \{K_P^*, P \in N_h\}$. The test function space is defined as

$$\mathbf{V}_h = \left\{ \mathbf{v}_h \in \left(L^2(\Omega)\right)^2 : \mathbf{v}_h|_{K_p^*} \in P_0\left(K_P^*\right),\ \forall P \in N_h,\ \mathbf{v}_h|_{K_p^*} = 0,\ \forall P \in \partial\Omega \right\},$$

where $P_0(K_P^*)$ is the constant set on element K_P^*. Clearly, the dimensions of $\mathbf{X}_h$ and $\mathbf{V}_h$ are the same.

Introduce the interpolation operators R_h: $(C(\bar{\Omega}))^2 \to \mathbf{X}_h$ and γ: $\mathbf{X}_h \to \mathbf{V}_h$, respectively. Assuming that $\mathbf{u} \in (H^2(\Omega))^2$, we can easily get the following interpolation estimates:

$$\|\mathbf{u} - R_h\mathbf{u}\|_s \leqslant h^{2-s}\|\mathbf{u}\|_2, \quad s = 0, 1. \tag{2.1}$$

Lemma 1. (Li, Ronghua et al., 2000) (i) The mapping γ is self-adjoint with respect to the L^2-inner product in $\mathbf{X}_h$:

$$(\mathbf{u}_h, \gamma\mathbf{v}_h) = (\gamma\mathbf{u}_h, \mathbf{v}_h), \quad \forall\mathbf{u}_h, \mathbf{v}_h \in \mathbf{X}_h. \tag{2.2}$$

In particular, if $\mathbf{u}_h(\cdot, t) \in \mathbf{X}_h$, $\mathbf{v}_h \in \mathbf{X}_h$, $t \in [0, T]$, it holds

$$\left({}_0^c\mathcal{D}_t^\alpha\mathbf{u}_h, \gamma\mathbf{v}_h\right) = \left(\gamma{}_0^c\mathcal{D}_t^\alpha\mathbf{u}_h, \mathbf{v}_h\right). \tag{2.3}$$

(ii) The norm $|||\mathbf{u}_h||| = (\mathbf{u}_h, \gamma\mathbf{u}_h)^{1/2}$ is equivalent to the usual L^2-norm in $\mathbf{X}_h$, i.e.

$$C_1\|\mathbf{u}_h\| \le |||\mathbf{u}_h||| \le C_2\|\mathbf{u}_h\|, \quad \forall\mathbf{u}_h \in \mathbf{X}_h, \tag{2.4}$$

where the positive constant C_1 and C_2 are independent of h.

We start by testing the Eq.(1.1) by $\mathbf{v} \in \mathbf{X}_h$,

$$\left({}_0^C\mathcal{D}_t^\alpha\mathbf{u}, \gamma\mathbf{v}\right) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \gamma\mathbf{v}) - \nu(\Delta\mathbf{u}, \gamma\mathbf{v}) = (\mathbf{f}, \gamma\mathbf{v}). \tag{2.5}$$

Using integration by parts and the fact that $\gamma\mathbf{v}$ is a piecewise constant function on T_h^*, we get

$$-\nu(\Delta\mathbf{u}, \gamma\mathbf{v}) = -\nu < \nabla\mathbf{u}\cdot\mathbf{n}, \gamma\mathbf{v} >,$$
$$((\mathbf{u}\cdot\nabla)\mathbf{u}, \gamma\mathbf{v}) =< \mathbf{u}\cdot\mathbf{n}, \mathbf{u}\cdot\gamma\mathbf{v} > -(\mathbf{u}, (\nabla\cdot\mathbf{u})\gamma\mathbf{v}),$$

where

$$< \nabla\mathbf{u}\cdot n, \gamma\mathbf{v} >= \sum_{P_i\in N_h}\mathbf{v}(P_i)\cdot\int_{\partial K_{P_i}^*}\nabla\mathbf{u}\cdot\mathbf{n}ds,$$

$$< \mathbf{u}\cdot\mathbf{n}, \mathbf{u}\cdot\gamma\mathbf{v} >= \sum_{P_i\in N_h}\mathbf{v}(P_i)\cdot\int_{\partial K_{P_i}^*}(\mathbf{u}\cdot\mathbf{n})\mathbf{u}ds,$$

$$(\mathbf{u}, (\nabla\cdot\mathbf{u})\gamma\mathbf{v}) = \sum_{P_i\in N_h}\mathbf{v}(P_i)\cdot\int_{K_{P_i}^*}(\nabla\cdot\mathbf{u})\mathbf{u}ds.$$

Define

$$A(\mathbf{u}, \mathbf{v}) = -\nu < \nabla\mathbf{u}\cdot\mathbf{n}, \gamma\mathbf{v} >,$$
$$b_1(\mathbf{u}, \mathbf{w}, \mathbf{v}) = -(\mathbf{u}, (\nabla\cdot\mathbf{w})\gamma\mathbf{v}),$$
$$b_2(\mathbf{u}, \mathbf{w}, \mathbf{v}) =< \mathbf{u}\cdot\mathbf{n}, \mathbf{w}\cdot\gamma\mathbf{v} > .$$

So (2.5) can be rewritten as

$$\left({}_0^C\mathcal{D}_t^\alpha\mathbf{u}, \gamma\mathbf{v}\right) + A(\mathbf{u}, \mathbf{v}) + b_1(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b_2(\mathbf{u}, \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \gamma\mathbf{v}), \quad \forall\mathbf{v} \in \mathbf{X}_h. \tag{2.6}$$

Lemma 2. (Li, Ronghua et al., 2000) For the bilinear form $A(\cdot, \cdot)$, we have the following conclusions.

(i) For $\mathbf{u}_h, \mathbf{v}_h \in \mathbf{X}_h$, one has

$$A(\mathbf{u}_h, \mathbf{v}_h) = A(\mathbf{v}_h, \mathbf{u}_h). \tag{2.7}$$

(ii) There exists a positive constant C such that

$$|A(\mathbf{u} - R_h\mathbf{u}, \mathbf{v}_h)| \le Ch\|\mathbf{u}\|_2\|\mathbf{v}_h\|_1, \quad \forall\mathbf{u} \in (H^2(\Omega))^2, \mathbf{v}_h \in \mathbf{X}_h. \tag{2.8}$$

(iii) There exists a positive constant μ such that

$$|A(\mathbf{u}_h, \mathbf{u}_h)| \ge \mu\|\mathbf{u}_h\|_1^2, \quad \forall\mathbf{u}_h \in \mathbf{X}_h. \tag{2.9}$$

Lemma 3. *Let $\boldsymbol{u}_h(\cdot, t) \in \boldsymbol{X}_h$, then*

$$\left({}_0^c\mathcal{D}_t^\alpha\boldsymbol{u}_h, \gamma\boldsymbol{u}_h\right) = \frac{1}{2}{}_0^C\mathcal{D}_t^\alpha |||\boldsymbol{u}_h|||^2. \tag{2.10}$$

Proof. It is known by the definition of the fractional derivative that

$$\begin{aligned}
\left({}_0^c\mathcal{D}_t^\alpha \mathbf{u}_h, \gamma\mathbf{u}_h\right) &= \int_\Omega {}_0^C\mathcal{D}_t^\alpha \mathbf{u}_h \cdot \gamma\mathbf{u}_h d\mathbf{x} = \int_\Omega \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\mathbf{u}_{h,\tau}}{(t-\tau)^\alpha} d\tau \cdot \gamma\mathbf{u}_h d\mathbf{x} \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \int_\Omega \mathbf{u}_{h,\tau} \cdot \gamma\mathbf{u}_h d\mathbf{x} d\tau \\
&= \frac{1}{2\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \frac{d}{d\tau} \int_\Omega \mathbf{u}_h \cdot \gamma\mathbf{u}_h d\mathbf{x} d\tau \\
&= \frac{1}{2\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \frac{d}{d\tau} \left(\mathbf{u}_h, \gamma\mathbf{u}_h\right) d\tau \\
&= \frac{1}{2\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \frac{d}{d\tau} ||| \mathbf{u}_h |||^2 d\tau \\
&= \frac{1}{2} {}_0^C\mathcal{D}_t^\alpha ||| \mathbf{u}_h |||^2 .
\end{aligned}$$

□

We will approximate $b_2(\mathbf{u}, \mathbf{v}, \mathbf{w})$ by using the upwind technique. Let $\Lambda_i = \left\{j : P_j \text{ is adjoint with } P_i\right\}$. Assuming that $j \in \Lambda_i$, let $\Gamma_{ij} = \partial K_{P_i}^* \cap \partial K_{P_j}^*$ and γ_{ij} is the length of Γ_{ij}. Denote by $\mathbf{n}_{ij}$ the unit outward normal vector of Γ_{ij} when Γ_{ij} is regarded as the boundary of $K_{P_i}^*$. Define

$$\beta_{ij}(\mathbf{u}) = \int_{\Gamma_{ij}} \mathbf{u} \cdot \mathbf{n}_{ij} ds.$$

Let

$$\beta_{ij}^+(\mathbf{u}) = \max\left\{\beta_{ij}(\mathbf{u}), 0\right\}, \quad \beta_{ij}^-(\mathbf{u}) = \max\left\{-\beta_{ij}(\mathbf{u}), 0\right\},$$

$$\int_{\partial K_{P_i}^*} (\mathbf{u}\cdot\mathbf{n})\mathbf{v} ds \approx \sum_{j\in\Lambda_i} \left\{\beta_{ij}^+(\mathbf{u})\mathbf{v}(P_i) - \beta_{ij}^-(\mathbf{u})\mathbf{v}\left(P_j\right)\right\}.$$

The upwind approximation form of the nonlinear term $b_2(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is defined by the form

$$b_{2h}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{P_i\in N_h} \sum_{j\in\Lambda_i} \left\{\beta_{ij}^+(\mathbf{u})\mathbf{v}(P_i) - \beta_{ij}^-(\mathbf{u})\mathbf{v}\left(P_j\right)\right\} \cdot \mathbf{w}(P_i).$$

Using the Heaviside function

$$H(r) = \begin{cases} 1, & r \geq 0, \\ 0, & r < 0, \end{cases}$$

we can write $b_{2h}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ as

$$b_{2h}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{P_i\in N_h} \sum_{j\in\Lambda_i} \beta_{ij}(\mathbf{u}) \left\{H\left(\beta_{ij}(\mathbf{u})\right)\mathbf{v}(P_i) + \left(1 - H\left(\beta_{ij}(\mathbf{u})\right)\right)\mathbf{v}\left(P_j\right)\right\} \cdot \mathbf{w}(P_i). \tag{2.11}$$

Lemma 4. (Yang, Qing, 2016) For $\mathbf{u} \in \left(W^{0,\infty}(\Omega)\right)^2$, $\mathbf{v} \in \left(H_0^1(\Omega)\right)^2$, $\mathbf{u}_h \in \mathbf{X}_h$ and $\mathbf{w}_h \in \mathbf{X}_h$, one has

$$\begin{aligned}
&|b_2(\mathbf{u}, \mathbf{v}, \mathbf{w}_h) - b_{2h}(\mathbf{u}_h, R_h\mathbf{v}, \mathbf{w}_h)| \\
&\quad \leq |\mathbf{w}_h|_1 \left\{h\|\mathbf{u}\|_\infty |\mathbf{v}|_1 + \|\mathbf{v}\|_\infty \left(\|\mathbf{u} - \mathbf{u}_h\| + h|\mathbf{u} - \mathbf{u}_h|_1\right)\right\}.
\end{aligned} \tag{2.12}$$

Then, the semi-discrete finite volume element scheme of problem (1.1)-(1.3) is as follows: seek $u_h(\cdot, t) \in \mathbf{X}_h$ for $t \in [0, T]$ such that

$$\left({}_0^C\mathcal{D}_t^\alpha \mathbf{u}_h, \gamma\mathbf{v}_h\right) + A(\mathbf{u}_h, \mathbf{v}_h) + b_1(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + b_{2h}(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \gamma\mathbf{v}_h), \quad \forall\mathbf{v}_h \in \mathbf{X}_h, \tag{2.13}$$

$$\mathbf{u}_h(\mathbf{x}, 0) = R_h\mathbf{u}_0(\mathbf{x}). \tag{2.14}$$

Let $\mathcal{T}_\tau = \{t_n | t_n = n\tau, 0 \leq n \leq N\}$ be a uniform partition of $[0, T]$ with the time step $\tau = T/N$. Based on a piecewise linear interpolation, the L1-approximation to the Caputo fractional derivative is given by

$$\begin{aligned}
{}_0^C\mathcal{D}_t^\alpha u|_{t=t_n} &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^n \frac{u\left(x, t_j\right) - u\left(x, t_{j-1}\right)}{\tau} \int_{t_{j-1}}^{t_j} \frac{1}{(t_n - s)^\alpha} ds + Q^n \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^n a_{n-j}\left(u\left(x, t_j\right) - u\left(x, t_{j-1}\right)\right) + Q^n,
\end{aligned}$$

where

$$a_i = (i+1)^{1-\alpha} - i^{1-\alpha}.$$

If $u \in C^2([0,T]; L^2(\Omega))$, the truncation error Q^n satisfies (Sayevand, 2016)

$$\|Q^n\| = O(\tau^{2-\alpha}).$$

For a sequence $\{\omega^n\}_{n=0}^N$, we define

$$D_\tau^\alpha \omega^n := \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^n a_{n-j}\delta_t\omega^j = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^n b_{n-j}\omega^j, \quad n = 1, \cdots, N, \tag{2.15}$$

where $\delta_t\omega^n = \omega^n - \omega^{n-1}$ and

$$b_0 = a_0, \quad b_n = -a_{n-1}, \quad b_{n-j} = a_{n-j} - a_{n-j-1}, \quad j = 1, \cdots, n-1.$$

With the above notations, the fully discrete finite volume element scheme for (1.1)−(1.3) seeks $\mathbf{u}_h^n \in \mathbf{X}_h$ such that

$$\left(D_\tau^\alpha \mathbf{u}_h^n, \gamma\mathbf{v}_h\right) + A\left(\mathbf{u}_h^n, \mathbf{v}_h\right) + b_1\left(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v}_h\right) + b_{2h}\left(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v}_h\right) = (\mathbf{f}^n, \gamma\mathbf{v}_h), \quad n \geq 1, \forall \mathbf{v}_h \in \mathbf{X}_h, \tag{2.16}$$

$$\mathbf{u}_h^0 = R_h\mathbf{u}_0. \tag{2.17}$$

3. Error Analysis

In this section, we present the error estimates for the semi-discrete scheme (2.13)-(2.14) and the fully discrete scheme (2.16)-(2.17) in L^2-norm.

3.1 Error Estimates for the Semi-Discrete Scheme

Lemma 5. (Zhu, Tao, 2018) Let $0 < T < \infty, \beta \in (0,1)$, $a(t)$, $l(t)$ and $u(t)$ are continuous, nonnegative functions on $[0,T)$ with

$$u(t) \leq a(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} l(s)u(s)ds. \tag{3.1}$$

Then

$$u(t) \leq \left(A(t) + \int_0^t L(s)A(s)\exp\left(\int_s^t L(\tau)d\tau\right)ds\right)^\alpha, \quad t \in [0,T). \tag{3.2}$$

If $a(t)$ is nondecreasing on $[0,T)$, then the above inequality is reduced to

$$u(t) \leq \left(A(t)\exp\left(\int_0^t L(s)ds\right)\right)^\alpha, \tag{3.3}$$

where $A(t) = 2^{\frac{1}{\alpha}-1}a^{\frac{1}{\alpha}}(t), L(t) = \frac{2^{\frac{1}{\alpha}-1}}{\Gamma^{\frac{1}{\alpha}}(\beta)}\left(\Gamma\left(\frac{\beta-\alpha}{1-\alpha}\right)\Gamma\left(\frac{1-\beta}{1-\alpha}\right)\right)^{\frac{1-\alpha}{\alpha}} t^{\frac{\beta-\alpha}{\alpha}} l^{\frac{1}{\alpha}}(t)$ and $0 < \alpha < \beta < 1$. If $a(t) \equiv 0$ on $[0,T)$, then

$$u(t) \equiv 0. \tag{3.4}$$

Theorem 6. *Let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the solutions of (1.1)-(1.3) and (2.13)-(2.14), respectively. Also assume that $\boldsymbol{u}$ satisfies the necessary regularities. Then, for $t \in [0,T]$ and sufficiently small $h > 0$,*

$$\|\boldsymbol{u}(t) - \boldsymbol{u}_h(t)\| \leq Ch, \tag{3.5}$$

where C is a constant independent of h and dependent on principally $\|\boldsymbol{u}_0\|_2$, $\|\boldsymbol{u}\|_{L^\infty((H^2(\Omega))^2)}$, $\max\limits_{0\leq t\leq T} {}_0^C\mathcal{D}_t^\alpha\|\boldsymbol{u}\|_2$ and $\|\boldsymbol{u}\|_{L^\infty((W^{1,\infty}(\Omega))^2)}$.

Proof. Subtract (2.13) from (2.6) to obtain that

$$\begin{aligned}\left({}_0^C\mathcal{D}_t^\alpha\mathbf{u} - {}_0^C\mathcal{D}_t^\alpha\mathbf{u}_h, \gamma\mathbf{v}_h\right) + A\left(\mathbf{u}-\mathbf{u}_h, \mathbf{v}_h\right) + b_1\left(\mathbf{u}, \mathbf{u}, \mathbf{v}_h\right) - b_1\left(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h\right)& \\ + b_2\left(\mathbf{u}, \mathbf{u}, \mathbf{v}_h\right) - b_{2h}\left(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h\right) = 0.&\end{aligned} \tag{3.6}$$

Letting $\mathbf{u} - \mathbf{u}_h = (\mathbf{u} - R_h\mathbf{u}) + (R_h\mathbf{u} - \mathbf{u}_h) = \mathbf{E} + \mathbf{e}$, we rewrite (3.6) as

$$\begin{aligned}\left({}_0^C\mathcal{D}_t^\alpha\mathbf{e}, \gamma\mathbf{v}_h\right) + A\left(\mathbf{e}, \mathbf{v}_h\right) + b_1\left(\mathbf{u}, \mathbf{u}, \mathbf{v}_h\right) - b_1\left(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h\right) + b_2\left(\mathbf{u}, \mathbf{u}, \mathbf{v}_h\right)& \\ - b_{2h}\left(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h\right) = -\left({}_0^C\mathcal{D}_t^\alpha\mathbf{E}, \gamma\mathbf{v}_h\right) - A\left(\mathbf{E}, \mathbf{v}_h\right).&\end{aligned} \tag{3.7}$$

We choose $\mathbf{v}_h = \mathbf{e}$ in (3.7) to get

$$\begin{aligned}\left({}_0^C\mathcal{D}_t^\alpha \mathbf{e}, \gamma\mathbf{e}\right) + A\left(\mathbf{e}, \mathbf{e}\right) = &-\left({}_0^C\mathcal{D}_t^\alpha \mathbf{E}, \gamma\mathbf{e}\right) - A\left(\mathbf{E}, \mathbf{e}\right) \\ &- \left[b_1\left(\mathbf{u}, \mathbf{u}, \mathbf{e}\right) - b_1\left(\mathbf{u}_h, \mathbf{u}_h, \mathbf{e}\right)\right] - \left[b_2\left(\mathbf{u}, \mathbf{u}, \mathbf{e}\right) - b_{2h}\left(\mathbf{u}_h, \mathbf{u}_h, \mathbf{e}\right)\right].\end{aligned} \tag{3.8}$$

Using Lemmas 2 and 3 for the left two terms in (3.8), we have

$$\begin{aligned}\frac{1}{2}{}_0^C\mathcal{D}_t^\alpha\, |||\,\mathbf{e}|||^2 + \mu\|\mathbf{e}\|_1^2 \le &-\left({}_0^C\mathcal{D}_t^\alpha \mathbf{E}, \gamma\mathbf{e}\right) - A\left(\mathbf{E}, \mathbf{e}\right) \\ &- \left[b_1\left(\mathbf{u}, \mathbf{u}, \mathbf{e}\right) - b_1\left(\mathbf{u}_h, \mathbf{u}_h, \mathbf{e}\right)\right] - \left[b_2\left(\mathbf{u}, \mathbf{u}, \mathbf{e}\right) - b_{2h}\left(\mathbf{u}_h, \mathbf{u}_h, \mathbf{e}\right)\right].\end{aligned} \tag{3.9}$$

For the first term on the right-hand side of (3.9), using Young's inequality, we get

$$\left|\left({}_0^C\mathcal{D}_t^\alpha \mathbf{E}, \gamma\mathbf{e}\right)\right| \le C(\|{}_0^C\mathcal{D}_t^\alpha \mathbf{E}\|^2 + \|\mathbf{e}\|^2), \tag{3.10}$$

For the second term, applying Lemma 2, we obtain

$$|A\left(\mathbf{E}, \mathbf{e}\right)| \le Ch\|\mathbf{u}\|_2\|\mathbf{e}\|_1 \le Ch^2\|\mathbf{u}\|_2^2 + \varepsilon\|\mathbf{e}\|_1^2. \tag{3.11}$$

For $\left\|{}_0^C\mathcal{D}_t^\alpha \mathbf{E}\right\|$ in (3.10), we have

$$\begin{aligned}\left\|{}_0^C\mathcal{D}_t^\alpha \mathbf{E}\right\| &= \left\|\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{(\mathbf{u}-R_h\mathbf{u})_\tau}{(t-\tau)^\alpha}d\tau\right\| \le \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\|(\mathbf{u}-R_h\mathbf{u})_\tau\|}{(t-\tau)^\alpha}d\tau \\ &\le \frac{h^2}{\Gamma(1-\alpha)}\int_0^t \frac{\|\mathbf{u}_\tau\|_2}{(t-\tau)^\alpha}d\tau \le Ch^2{}_0^C\mathcal{D}_t^\alpha\, \|\mathbf{u}\|_2\,.\end{aligned} \tag{3.12}$$

Now we bound the last two terms on the right-hand side of (3.9). We need the following induction hypothesis:

$$\left(\log\frac{1}{h}\right)^{1/2}\|\mathbf{e}(s)\| \to 0,\ h \to 0,\ 0 \le s \le t,\ 0 < t \le T. \tag{3.13}$$

Also we have the following norm inequalities

$$\|\mathbf{w}\|_\infty \le C\left(\log\frac{1}{h}\right)^{1/2}\|\mathbf{w}\|_1\,, \quad \forall \mathbf{w} \in \mathbf{X}_h, \tag{3.14}$$

$$\|\mathbf{w}\|_1 \le Ch^{-1}\|\mathbf{w}\|\,, \quad \forall \mathbf{w} \in \mathbf{X}_h. \tag{3.15}$$

Using hypothesis (3.13), (3.14) and Young's inequality, we have

$$\begin{aligned}&|b_1\left(\mathbf{u}, \mathbf{u}, \mathbf{e}\right) - b_1\left(\mathbf{u}_h, \mathbf{u}_h, \mathbf{e}\right)| \\ &= |(\mathbf{u}, (\nabla\cdot\mathbf{u})\gamma\mathbf{e}) - (\mathbf{u}_h, (\nabla\cdot\mathbf{u}_h)\,\gamma\mathbf{e})| \\ &= |(\mathbf{u}, (\nabla\cdot\mathbf{u})\gamma\mathbf{e}) - (\mathbf{u}_h, (\nabla\cdot\mathbf{u})\gamma\mathbf{e}) + (\mathbf{u}_h, (\nabla\cdot\mathbf{u})\gamma\mathbf{e}) - (\mathbf{u}_h, (\nabla\cdot\mathbf{u}_h)\,\gamma\mathbf{e})| \\ &\le C\left(\|\nabla\cdot\mathbf{u}\|_\infty\|\mathbf{u}-\mathbf{u}_h\| + \|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|\left(\|\mathbf{e}\|_\infty + \|R_h\mathbf{u}\|_\infty\right)\right)\|\mathbf{e}\| \\ &\le C\left(\|\mathbf{E}\|_1 + \|\mathbf{e}\|_1 + (\|\mathbf{E}\|_1 + \|\mathbf{e}\|_1)\left((\log\frac{1}{h})^{1/2}\|\mathbf{e}\|_1 + \|\mathbf{u}\|_\infty\right)\right)\|\mathbf{e}\| \\ &\le C\left(\|\mathbf{e}\|_1 + \|\mathbf{E}\|_1 + (\log\frac{1}{h})^{1/2}\|\mathbf{e}\|_1\left(\|\mathbf{e}\|_1 + \|\mathbf{E}\|_1\right)\right)\|\mathbf{e}\| \\ &\le C\left(\|\mathbf{E}\|_1^2 + \|\mathbf{e}\|^2\right) + \varepsilon\|\mathbf{e}\|_1^2 + C(\log\frac{1}{h})^{1/2}\|\mathbf{e}\|\|\mathbf{e}\|_1^2 \\ &\le Ch^2\|\mathbf{u}\|_2^2 + 2\varepsilon\|\mathbf{e}\|_1^2 + C\|\mathbf{e}\|^2.\end{aligned} \tag{3.16}$$

Next, we write

$$\begin{aligned}&|b_2\left(\mathbf{u}, \mathbf{u}, \mathbf{e}\right) - b_{2h}\left(\mathbf{u}_h, \mathbf{u}_h, \mathbf{e}\right)| \\ &\le |b_2\left(\mathbf{u}, \mathbf{u}, \mathbf{e}\right) - b_{2h}\left(\mathbf{u}_h, R_h\mathbf{u}, \mathbf{e}\right)| + |b_{2h}\left(\mathbf{u}_h, \mathbf{e}, \mathbf{e}\right)| \\ &= D_1 + D_2,\end{aligned} \tag{3.17}$$

where $D_1 = |b_2(\mathbf{u},\mathbf{u},\mathbf{e}) - b_{2h}(\mathbf{u}_h, R_h\mathbf{u},\mathbf{e})|$ and $D_2 = |b_{2h}(\mathbf{u}_h,\mathbf{e},\mathbf{e})|$. By choosing $\mathbf{v}=\mathbf{u}$, $\mathbf{v}_h=\mathbf{u}_h$ and $\mathbf{w}_h=\mathbf{e}$ in Lemma 4, using (3.15) and Young's inequality, we can obtain

$$\begin{aligned} D_1 &\leq |\mathbf{e}|_1 \{h\|\mathbf{u}\|_\infty|\mathbf{u}|_1 + \|\mathbf{u}\|_\infty(\|\mathbf{u}-\mathbf{u}_h\| + h|\mathbf{u}-\mathbf{u}_h|_1)\} \\ &\leq C\left(h^2|\mathbf{u}|_1^2 + \|\mathbf{E}\|_1^2 + \|\mathbf{e}\|^2\right) + \varepsilon\|\mathbf{e}\|_1^2. \end{aligned} \tag{3.18}$$

By the definition of $b_{2h}(\cdot,\cdot,\cdot)$ and then by (3.13)-(3.15), we have

$$\begin{aligned} D_2 &= |\sum_{P_i\in N_h}\mathbf{e}(P_i)\cdot\sum_{j\in\Lambda_i}\int_{\Gamma_{ij}}\left(\mathbf{u}_h\cdot\mathbf{n}_{ij}\right)ds\times\left[H\left(\beta_{ij}\right)\mathbf{e}(P_i) + \left(1-H\left(\beta_{ij}\right)\right)\mathbf{e}\left(P_j\right)\right]| \\ &\leq \frac{1}{2}\sum_{K\in T_h}\sum_{i,j\in\Lambda_K}\left|\mathbf{e}(P_i)-\mathbf{e}\left(P_j\right)\right|\times\left|H\left(\beta_{ij}\right)\mathbf{e}(P_i) + \left(1-H\left(\beta_{ij}\right)\right)\mathbf{e}\left(P_j\right)\right| \\ &\times\int_{\Gamma_{ij}\cap K}\left|\mathbf{e}\cdot\mathbf{n}_{ij}\right|ds \\ &+\frac{1}{2}\sum_{K\in T_h}\sum_{i,j\in\Lambda_K}\left|\mathbf{e}(P_i)-\mathbf{e}\left(P_j\right)\right| \\ &\times\left|H\left(\beta_{ij}\right)\mathbf{e}(P_i) + \left(1-H\left(\beta_{ij}\right)\right)\mathbf{e}\left(P_j\right)\right|\times\int_{\Gamma_{ij}\cap K}\left|R_h\mathbf{u}\cdot\mathbf{n}_{ij}\right|ds \\ &\leq C\left(\|\mathbf{e}\|_1(\|\mathbf{e}\| + h|\mathbf{e}|_1)\|\mathbf{e}\|_\infty + \|R_h\mathbf{u}\|_\infty\|\mathbf{e}\|_1\|\mathbf{e}\|\right) \\ &\leq C\left(\|\mathbf{e}\|^2 + \|\mathbf{e}\|(\log\frac{1}{h})^{1/2}\|\mathbf{e}\|_1^2\right) + \varepsilon\|\mathbf{e}\|_1^2 \\ &\leq C\|\mathbf{e}\|^2 + 2\varepsilon\|\mathbf{e}\|_1^2. \end{aligned} \tag{3.19}$$

Now, substituting (3.10)-(3.19) into (3.9), using Lemma 3, we find

$$\frac{1}{2}{}_0^C\mathcal{D}_t^\alpha |||\,\mathbf{e}\,|||^2 + \mu\|\mathbf{e}\|_1^2 \leq Ch^2\left(h^2({}_0^C\mathcal{D}_t^\alpha\|\mathbf{u}\|_2)^2 + \|\mathbf{u}\|_2^2\right) + C\|\mathbf{e}\|^2 + 6\varepsilon\|\mathbf{e}\|_1^2. \tag{3.20}$$

Taking $\varepsilon = \frac{\mu}{6}$, we have

$${}_0^C\mathcal{D}_t^\alpha |||\,\mathbf{e}|||^2 \leq Ch^2\left(h^2({}_0^C\mathcal{D}_t^\alpha\|\mathbf{u}\|_2)^2 + \|\mathbf{u}\|_2^2\right) + C\|\mathbf{e}\|^2. \tag{3.21}$$

Therefore, making a fractional integral of order α on both sides of (4.21), noting that $\mathbf{e}(0)=0$, so

$$\begin{aligned} |||\mathbf{e}(t)|||^2 &\leq \frac{Ch^4}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}({}_0^C\mathcal{D}_t^\alpha\|\mathbf{u}(\tau)\|_2)^2d\tau + \frac{Ch^2}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\|\mathbf{u}(\tau)\|_2^2d\tau \\ &+\frac{C}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\|\mathbf{e}(\tau)\|^2d\tau \\ &\leq Ch^2 + \frac{C}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\|\mathbf{e}(\tau)\|^2d\tau. \end{aligned} \tag{3.22}$$

Noting Lemma 1, we have

$$\|\mathbf{e}(t)\|^2 \leq Ch^2 + \frac{C}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\|\mathbf{e}(\tau)\|^2d\tau. \tag{3.23}$$

Using Lemma 5 in (3.23), we get

$$\|\mathbf{e}(t)\| \leq Ch. \tag{3.24}$$

By Lemma 2, we know

$$\|\mathbf{E}(t)\| \leq Ch^2\|\mathbf{u}(t)\|_2 \leq Ch^2(\|\mathbf{u}_0\|_2 + \int_0^t\|\mathbf{u}(\tau)\|_2d\tau). \tag{3.25}$$

Combining (3.24) with (3.25) and using triangle inequality, we obtain

$$\|\mathbf{u}(t)-\mathbf{u}_h(t)\| \leq Ch, \tag{3.26}$$

where C depends on $\|\mathbf{u}_0\|_2$, $\|\mathbf{u}\|_{L^\infty((H^2(\Omega))^2)}$, $\max\limits_{0\leq t\leq T}{}_0^C\mathcal{D}_t^\alpha\|\mathbf{u}\|_2$ and $\|\mathbf{u}\|_{L^\infty((W^{1,\infty}(\Omega))^2)}$.

We now prove the hypothesis (3.13). Noting that $\|\mathbf{e}(0)\| = 0$, so it holds when $s = 0$ clearly. Suppose it holds when $0 \le s < t$, we know from (3.24) that

$$(\log \frac{1}{h})^{1/2}\|\mathbf{e}(t)\| \le C(\log \frac{1}{h})^{1/2}h \to 0, h \to 0, \tag{3.27}$$

i.e. (3.13) is true for $s = t$. Thus, the hypothesis (3.13) holds for any $t \in [0, T]$. □

3.2 Error Estimates for the Fully Discrete Scheme

Lemma 7. (Li, Dongfang, 2018) Suppose that the nonnegative sequence $\{\omega^n\}_{n=0}^N$ and $\{g^n\}_{n=0}^N$ satisfy

$$D_\tau^\alpha \omega^n \le \lambda_1\omega^n + \lambda_2\omega^{n-1} + g^n, \quad n \ge 1, \tag{3.28}$$

where λ_1 and λ_2 are both positive constants independent of the time step τ. Then, there exists a positive constant τ^* such that, when $\tau \le \tau^*$,

$$\omega^n \le 2\left(\omega^0 + \frac{t_n^\alpha}{\Gamma(1+\alpha)} \max_{0\le j\le n} g^j\right) E_\alpha\left(2\lambda t_n^\alpha\right), \quad 1 \le n \le N, \tag{3.29}$$

here, $E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1+k\alpha)}$ is the Mittag-Leffler function and $\lambda = \lambda_1 + \frac{\lambda_2}{2-2^{1-\alpha}}$.

Theorem 8. *Let $\boldsymbol{u}$ and $\{\boldsymbol{u}_h^n\}_{n=0}^N$ be the solutions of (1.1)-(1.3) and (2.16)-(2.17), respectively. Also assume that $\boldsymbol{u}$ satisfies the necessary regularities and the discretization parameters obey the relation $\tau = O(h)$. Then*

$$\max_{0\le n\le N} \|\boldsymbol{u}^n - \boldsymbol{u}_h^n\| \le C(h+\tau), \tag{3.30}$$

where C depends on $\|\boldsymbol{u}_0\|_2$, $\|\boldsymbol{u}\|_{L^\infty((H^2(\Omega))^2)}$, $\|\boldsymbol{u}\|_{L^\infty((H^1(\Omega))^2)}$ and $\|\boldsymbol{u}\|_{L^\infty((W^{1,\infty}(\Omega))^2)}$.

Proof. Subtract (2.16) from (2.6) to obtain that

$$\begin{aligned}\left({}_0^C\mathcal{D}_t^\alpha \mathbf{u}^n - D_\tau^\alpha \mathbf{u}_h^n, \gamma\mathbf{v}_h\right) + A\left(\mathbf{u}^n - \mathbf{u}_h^n, \mathbf{v}_h\right) + b_1\left(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}_h\right) - b_1\left(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v}_h\right)&\\ + b_2\left(\mathbf{u}^n, \mathbf{u}^n, \mathbf{v}_h\right) - b_{2h}\left(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v}_h\right) = 0.&\end{aligned} \tag{3.31}$$

Taking $\mathbf{v}_h = \mathbf{e}^n$, we have

$$\begin{aligned}\left(D_\tau^\alpha \mathbf{e}^n, \gamma\mathbf{e}^n\right) + A\left(\mathbf{e}^n, \mathbf{e}^n\right) = &-\left({}_0^C\mathcal{D}_t^\alpha \mathbf{u}^n - D_\tau^\alpha \mathbf{u}^n, \gamma\mathbf{e}^n\right) - \left(D_\tau^\alpha \mathbf{E}^n, \gamma\mathbf{e}^n\right) - A\left(\mathbf{E}^n, \mathbf{e}^n\right)\\ &- [b_1\left(\mathbf{u}^n, \mathbf{u}^n, \mathbf{e}^n\right) - b_1\left(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{e}^n\right)]\\ &- [b_2\left(\mathbf{u}^n, \mathbf{u}^n, \mathbf{e}^n\right) - b_{2h}\left(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{e}^n\right)].\end{aligned} \tag{3.32}$$

For the left two terms of (4.32), from Lemma 3, we have

$$\begin{aligned}\left(D_\tau^\alpha \mathbf{e}^n, \gamma\mathbf{e}^n\right) &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(a_0\mathbf{e}^n - \sum_{j=1}^{n-1}\left(a_{n-j-1} - a_{n-j}\right)\mathbf{e}^j - a_{n-1}e^0, \gamma\mathbf{e}^n\right)\\ &\ge \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(a_0 |||\, \mathbf{e}^n |||^2 - \sum_{j=1}^{n-1}\left(a_{n-j-1} - a_{n-j}\right)\frac{|||\mathbf{e}^j|||^2 + |||\,\mathbf{e}^n|||^2}{2} - a_{n-1}\frac{|||\mathbf{e}^0|||^2 + |||\,\mathbf{e}^n|||^2}{2}\right)\\ &= \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\left(a_0 |||\, \mathbf{e}^n |||^2 - \sum_{j=1}^{n-1}\left(a_{n-j-1} - a_{n-j}\right) |||\, \mathbf{e}^j |||^2 - a_{n-1} |||\, \mathbf{e}^0|||^2\right)\\ &= \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\sum_{j=0}^{n} b_{n-j} |||\, \mathbf{e}^j|||^2\\ &= \frac{1}{2}D_\tau^\alpha |||\, \mathbf{e}^n|||^2,\end{aligned} \tag{3.33}$$

and

$$|A\left(\mathbf{e}^n, \mathbf{e}^n\right)| \ge \mu\,\|\mathbf{e}^n\|_1^2\,. \tag{3.34}$$

The five terms on the right-hand side of (3.32) are represented by T_1, T_2, ..., T_5, respectively. So, (3.32) can be rewritten as

$$\frac{1}{2}D_\tau^\alpha \,|||\, \mathbf{e}^n \,|||^2 + \mu\, \|\mathbf{e}^n\|_1^2 \le T_1 + T_2 + \cdots + T_5. \tag{3.35}$$

Now, let's estimate the terms $T_1, T_2, ..., T_5$ one by one. Using the property of L1-approximation we have

$$|T_1| \le C \left\| {}_0^C\mathcal{D}_t^\alpha \mathbf{u}^n - D_\tau^\alpha \mathbf{u}^n \right\| \|\mathbf{e}^n\| \le C \left\|\mathbf{Q}_n\right\| \|\mathbf{e}^n\| \le C(\left\|\mathbf{Q}_n\right\|^2 + \|\mathbf{e}^n\|^2). \tag{3.36}$$

For T_2, we first have

$$\begin{aligned} |D_\tau^\alpha \mathbf{E}^n| &= \left| \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^n a_{n-j} \delta_t \mathbf{E}^j \right| \\ &= \left| \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^n a_{n-j} \int_{t_{j-1}}^{t_j} (R_h - I)\mathbf{u}_t dt \right| \\ &\le \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^n a_{n-j} \tau \max_{t_0 \le t \le t_n} |(R_h - I)\mathbf{u}_t| \\ &\le \frac{(n\tau)^{1-\alpha}}{\Gamma(2-\alpha)} \max_{t_0 \le t \le t_n} |(R_h - I)\mathbf{u}_t|. \end{aligned} \tag{3.37}$$

Then using Young's inequality we get

$$\begin{aligned} |T_2| &\le \|D_\tau^\alpha \mathbf{E}^n\| \|\mathbf{e}^n\| \le C \frac{(n\tau)^{1-\alpha}}{\Gamma(2-\alpha)} \max_{t_0 \le t \le t_n} \|(R_h - I)\mathbf{u}_t\| \, \|\mathbf{e}^n\| \le C T^{1-\alpha} h^2 \max_{t_0 \le t \le t_n} \|\mathbf{u}_t\|_2 \, \|\mathbf{e}^n\| \\ &\le C(h^4 \max_{t_0 \le t \le t_n} \|\mathbf{u}_t\|_2^2 + \|\mathbf{e}^n\|^2). \end{aligned} \tag{3.38}$$

For the next term, by applying Lemma 2, we have

$$|T_3| \le |A\left(\mathbf{u}^n - R_h \mathbf{u}^n, \mathbf{e}^n\right)| \le Ch \, \|\mathbf{u}^n\|_2 \, \|\mathbf{e}^n\|_1 \le Ch^2 \, \|\mathbf{u}^n\|_2^2 + \varepsilon \, \|\mathbf{e}^n\|_1^2 \,. \tag{3.39}$$

Next, we make the following induction hypothesis:

$$\|\mathbf{e}^s\| \,(\log \frac{1}{h})^{1/2} \to 0, \quad h \to 0, \; 0 \le s \le n-1, \; n \le N. \tag{3.40}$$

For T_4, using the similar argument as (3.15) and noting (3.40), we deduce that

$$\begin{aligned} |T_4| &\le C \left\{ \|\nabla \cdot \mathbf{u}^n\|_\infty \|\mathbf{u}_n - \mathbf{u}_h^{n-1}\| \|\mathbf{e}^n\| + \|\nabla \cdot (\mathbf{u}^n - \mathbf{u}_h^n)\| (\|\mathbf{e}^n\|_\infty \|\mathbf{e}^{n-1}\| + \|R_h \mathbf{u}^{n-1}\|_\infty \|\mathbf{e}^n\|) \right\} \\ &\le C \left(\left(\tau \int_{t_{n-1}}^{t^n} \left\| \frac{\partial \mathbf{u}}{\partial t} \right\|^2 dt \right)^{1/2} + \|\mathbf{E}^{n-1}\| + \|\mathbf{e}^{n-1}\| \right) \|\nabla \cdot \mathbf{u}^n\|_\infty \|\mathbf{e}^n\| \\ &+ C\left(\|\mathbf{E}^n\|_1 + \|\mathbf{e}^n\|_1\right) \left((\log \frac{1}{h})^{1/2} \|\mathbf{e}^{n-1}\| \|\mathbf{e}^n\|_1 + \|\mathbf{u}^{n-1}\|_\infty \|\mathbf{e}^n\| \right) \\ &\le C \left(\tau^2 \max_{t_0 \le t \le t_n} \|\mathbf{u}_t\|^2 + \|\mathbf{E}^{n-1}\|^2 + \|\mathbf{E}^n\|_1^2 + \|\mathbf{e}^{n-1}\|^2 + \|\mathbf{e}^n\|^2 \right) + 2\varepsilon \|\mathbf{e}^n\|_1^2. \end{aligned} \tag{3.41}$$

Next, we write

$$|T_5| \le \left| b_2\left(\mathbf{u}^n, \mathbf{u}^n, \mathbf{e}^n\right) - b_{2h}\left(\mathbf{u}_h^{n-1}, R_h \mathbf{u}^n, \mathbf{e}\right) \right| + \left| b_{2h}\left(\mathbf{u}_h^n, \mathbf{e}^n, \mathbf{e}^n\right) \right| = E_1 + E_2, \tag{3.42}$$

where $E_1 = \left| b_2\left(\mathbf{u}^n, \mathbf{u}^n, \mathbf{e}^n\right) - b_{2h}\left(\mathbf{u}_h^{n-1}, R_h \mathbf{u}^n, \mathbf{e}^n\right) \right|$ and $E_2 = \left| b_{2h}\left(\mathbf{u}_h^n, \mathbf{e}^n, \mathbf{e}^n\right) \right|$. E_1 and E_2 can be handled as D_1 and D_2 in Theorem 6. Thus, we have

$$\begin{aligned} E_1 &\le C\,|\mathbf{e}^n|_1 \, \|\mathbf{u}^n\|_\infty \left\{ h|\mathbf{u}^n|_1 + \left(\left\| \mathbf{u}^n - \mathbf{u}_h^{n-1} \right\| + h \left| \mathbf{u}^n - \mathbf{u}_h^{n-1} \right|_1 \right) \right\} \\ &\le C\,|\mathbf{e}^n|_1 \, \|\mathbf{u}^n\|_\infty \{ h|\mathbf{u}^n|_1 + \|\mathbf{E}^{n-1}\| + \|\mathbf{e}^{n-1}\| + \left(\tau \int_{t_{n-1}}^{t^n} \left\| \frac{\partial \mathbf{u}}{\partial t} \right\|^2 dt \right)^{1/2} \\ &+ h \left(\|\mathbf{E}^{n-1}\|_1 + \|\mathbf{e}^{n-1}\|_1 + \left(\tau \int_{t_{n-1}}^{t^n} \left\| \frac{\partial \mathbf{u}}{\partial t} \right\|_1^2 dt \right)^{1/2} \right) \} \\ &\le C \left(\|\mathbf{E}^{n-1}\|^2 + \|\mathbf{e}^{n-1}\|^2 + \tau^2 \max_{t_0 \le t \le t_n} \|\mathbf{u}_t\|_1^2 \right) + \varepsilon \|\mathbf{e}^n\|_1^2, \end{aligned} \tag{3.43}$$

and

$$\begin{aligned}
E_2 \leq & \frac{1}{2}\sum_{K\in T_h}\sum_{i,j\in\Lambda_K}\left|\mathbf{e}^n(P_i)-\mathbf{e}^n\left(P_j\right)\right|\times\left|H\left(\beta_{ij}(\mathbf{u}_h^{n-1})\right)\mathbf{e}^n(P_i)+\left(1-H\left(\beta_{ij}(\mathbf{u}_h^{n-1})\right)\right)\mathbf{e}^n\left(P_j\right)\right| \\
& \times\int_{r_{ij}\cap K}\left|\mathbf{e}^{n-1}\cdot\mathbf{n}_{ij}\right|ds+\frac{1}{2}\sum_{K\in T_k}\sum_{i,j\in\Lambda_K}\left|\mathbf{e}^n(P_i)-\mathbf{e}^n\left(P_j\right)\right| \\
& \times\left|H\left(\beta_{ij}(\mathbf{u}_h^{n-1})\right)\mathbf{e}^n(P_i)+\left(1-H\left(\beta_{ij}(\mathbf{u}_h^{n-1})\right)\right)\mathbf{e}^n\left(P_j\right)\right|\int_{\Gamma_{ij}\cap K}\left|R_h\mathbf{u}^{n-1}\cdot\mathbf{n}_{ij}\right|ds \\
\leq & C\left(\|\mathbf{e}^n\|_1\left(\|\mathbf{e}^{n-1}\|+h|\mathbf{e}^{n-1}|_1\right)\|\mathbf{e}^n\|_\infty+\left\|R_h\mathbf{u}^{n-1}\right\|_\infty\|\mathbf{e}^n\|_1\|\mathbf{e}^n\|\right) \\
\leq & C\left(\|\mathbf{e}^n\|^2+\|\mathbf{e}^{n-1}\|\left(\log\frac{1}{h}\right)^{1/2}\|\mathbf{e}^n\|_1^2\right)+\varepsilon\|\mathbf{e}^n\|_1^2 \\
\leq & C\|\mathbf{e}^n\|^2+2\varepsilon\|\mathbf{e}^n\|_1^2.
\end{aligned} \tag{3.44}$$

Substituting the previously mentioned estimates (3.36)-(3.44) into (3.35), we get

$$\begin{aligned}
\frac{1}{2}D_\tau^\alpha \, ||| \, \mathbf{e}^n \, |||^2 + \mu\|\mathbf{e}^n\|_1^2 \leq & C(\|\mathbf{e}^{n-1}\|^2+\|\mathbf{e}^n\|^2+\tau^2\max_{t_0\leq t\leq t_n}\|\mathbf{u}_t\|_1^2+h^2\|\mathbf{u}^n\|_2^2+h^4\max_{t_0\leq t\leq t_n}\|\mathbf{u}_t\|_2^2 \\
& +\|\mathbf{E}^{n-1}\|^2+\|\mathbf{E}^n\|_1^2+\|\mathbf{Q}^n\|^2)+6\varepsilon\|\mathbf{e}^n\|_1^2.
\end{aligned} \tag{3.45}$$

Taking $\varepsilon=\mu/6$, we have

$$D_\tau^\alpha \, ||| \, \mathbf{e}^n \, |||^2 \leq C(\|\mathbf{e}^n\|^2+\|\mathbf{e}^{n-1}\|^2)+O(\tau^2+h^2). \tag{3.46}$$

By Lemmas 1 and 7, there exists a positive constant τ^* such that, when $\tau<\tau^*$,

$$\|\mathbf{e}^n\| \leq C(\tau+h). \tag{3.47}$$

Now we prove the introduction hypothesis (3.40). Noting that $\mathbf{u}_h^0=R_h\mathbf{u}_0$, so it holds true when $s=0$ obviously. Suppose it holds when $0\leq s\leq n-1$, from (3.47) and the assumption $\tau=O(h)$, we have

$$\|\mathbf{e}^n\|\,(\log\frac{1}{h})^{1/2}\to 0, \quad h\to 0. \tag{3.48}$$

So it is true when $0\leq s\leq n$. Thus, we know that the hypothesis (3.40) holds for any $1\leq n\leq N$. Using triangular inequality and the interpolation theory we get

$$\|\mathbf{u}^n-\mathbf{u}_h^n\| \leq C(\tau+h). \tag{3.49}$$

□

4. Numerical Examples

In this section, we will show the numerical performances of proposed finite volume element method on the convergence and efficiency by several examples.

Example 1. In this example, the exact solutions of coupled Burgers' equation are given by

$$u(x,y,t)=t^{\alpha+1}sin(\pi x)sin(\pi y), \qquad v(x,y,t)=t^{\alpha+1}sin(\pi x)cos(\pi y).$$

Then we have the source term functions

$$\begin{aligned}
f_1(x,y,t)=&\Gamma(1+\alpha)t\sin(\pi x)\sin(\pi y)+\pi t^{2\alpha+2}\sin(\pi x)\cos(\pi x)(\sin(\pi y))^2 \\
&+\pi t^{2\alpha+2}(\sin(\pi x))^2(\sin(\pi y))^2+2\nu\pi^2t^{\alpha+1}\sin(\pi x)\sin(\pi y),
\end{aligned}$$

and

$$\begin{aligned}
f_2(x,y,t)=&\Gamma(1+\alpha)t\sin(\pi x)\cos(\pi y)+\pi t^{2\alpha+2}\sin(\pi x)\cos(\pi x)\sin(\pi y))\cos(\pi y) \\
&-\pi t^{2\alpha+2}(\sin(\pi x))^2\sin(\pi y)\cos(\pi y)+2\nu\pi^2t^{\alpha+1}\sin(\pi x)\cos(\pi y)
\end{aligned}$$

by computing accordingly. We take the viscosity $\nu=1$, the spatial interval $\Omega=(0,1)\times(0,1)$ and the time interval $[0,T]=[0,1]$. In all runs, we use the uniform mesh $h=\tau$ and choose the time $t=1$. Table 1 shows the numerical results of $\|u-u_h\|$ and $\|v-v_h\|$ for $\alpha=0.3$ and $\alpha=0.7$. As seen in the table, the errors decrease by a factor of about two as

h decreases by the factor of two. This indicates that all L^2-norm error estimates are of first-order convergence, which is consistent with our theoretical analysis. Surface plots of the exact solution u and the numerical solution u_h for $\alpha = 0.3$ are given in Figure 1 and Figure 2. Figure 3 shows the error between the exact and numerical solutions. As can be seen, there is a good agreement between the exact solution and numerical solution.

Table 1. Numerical results for $\alpha = 0.3$ and $\alpha = 0.7$

h	$\alpha = 0.3$ $\|\|u - u_h\|\|$	order	$\alpha = 0.3$ $\|\|v - v_h\|\|$	order	$\alpha = 0.7$ $\|\|u - u_h\|\|$	order	$\alpha = 0.7$ $\|\|v - v_h\|\|$	order
$\frac{1}{16}$	7.1904e − 03	-	2.7828e − 03	-	6.0983e − 03	-	2.9354e − 03	-
$\frac{1}{32}$	3.1573e − 03	1.18	1.1943e − 03	1.22	2.6444e − 03	1.21	1.2921e − 03	1.18
$\frac{1}{64}$	1.4709e − 03	1.10	5.5057e − 04	1.11	1.2289e − 03	1.11	6.0639e − 04	1.09
$\frac{1}{128}$	7.0889e − 04	1.05	2.6416e − 04	1.06	5.9325e − 04	1.05	2.9432e − 04	1.04

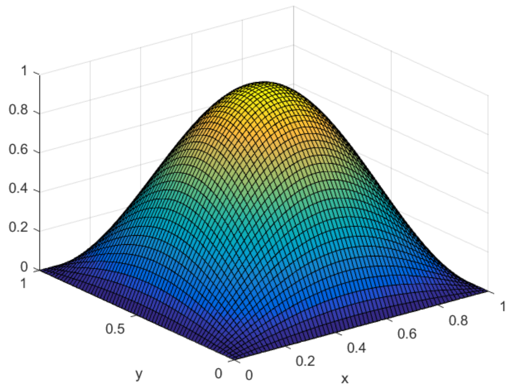

Figure 1. The exact solution u for $t = 1$, $\nu = 1$ and $\alpha = 0.3$

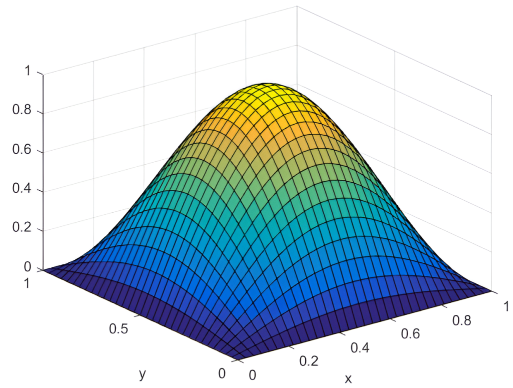

Figure 2. The numerical solution u_h for $t = 1$, $\nu = 1$, $h = 1/32$ and $\alpha = 0.3$

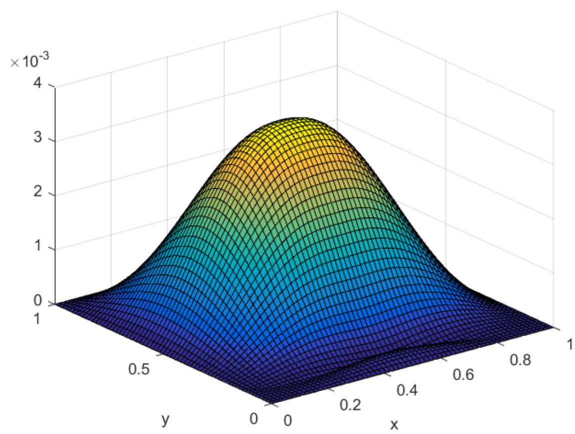

Figure 3. The error $u - u_h$ for $t = 1$, $\nu = 1$, $h = 1/64$ and $\alpha = 0.3$

Example 2. We consider the following time fractional Burgers' equation:

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \nu\Delta u = 0, \quad \mathbf{x} = (x, y) \in \Omega, t \in (0, T] \\ {}_0^C\mathcal{D}_t^\alpha v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \nu\Delta v = 0, \quad (\mathbf{x}, t) \in \Omega \times (0, T], \\ u(\mathbf{x}, 0) = sin(\pi x)sin(\pi y), \quad v(\mathbf{x}, 0) = xy(x - 1)(y - 1), \\ u = 0, \quad v = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times (0, T]. \end{cases}$$

where $\Omega = (0,1) \times (0,1)$, $T = 1$ and $\nu = 0.01$. In this example, the exact solution is not given beforehand, so we take the numerical solution with $h = \frac{1}{150}$ as the reference solution to calculate the error and convergence rate. In Table 2, we present the numerical results for $\alpha = 0.2$ and $\alpha = 0.8$. It can be seen in the table that the format is convergent with one-order accuracy. Surface plots of the reference solution u and numerical solution u_h for $\alpha = 0.8$ are given in Figure 4 and Figure 5. As can be seen from the table and figure, it is consistent with the results of our theoretical analysis.

Table 2. Numerical results for $\alpha = 0.2$ and $\alpha = 0.8$

	$\alpha = 0.2$		$\alpha = 0.2$		$\alpha = 0.8$		$\alpha = 0.8$	
h	$\lVert u - u_h \rVert$	order	$\lVert v - v_h \rVert$	order	$\lVert u - u_h \rVert$	order	$\lVert v - v_h \rVert$	order
$\frac{1}{25}$	9.1067e − 03	-	5.9229e − 04	-	8.7381e − 04	-	4.9997e − 05	-
$\frac{1}{30}$	7.5809e − 03	1.01	4.9387e − 04	0.97	7.0824e − 04	1.15	4.0657e − 05	1.13
$\frac{1}{50}$	4.1389e − 03	1.18	2.7062e − 04	1.18	3.6608e − 04	1.29	2.1167e − 05	1.28
$\frac{1}{75}$	2.1722e − 03	1.59	1.4230e − 04	1.58	1.8710e − 04	1.66	1.0862e − 05	1.65

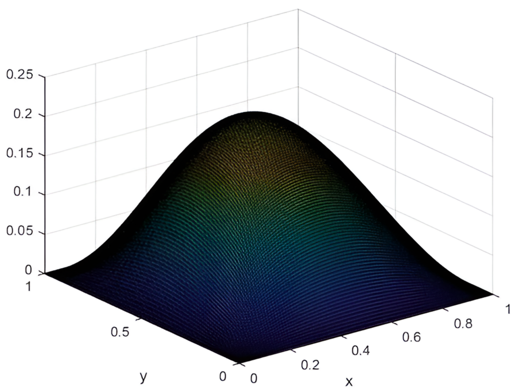

Figure 4. The reference solution u for $t = 1$, $\nu = 0.01$, $h = 1/150$ and $\alpha = 0.8$

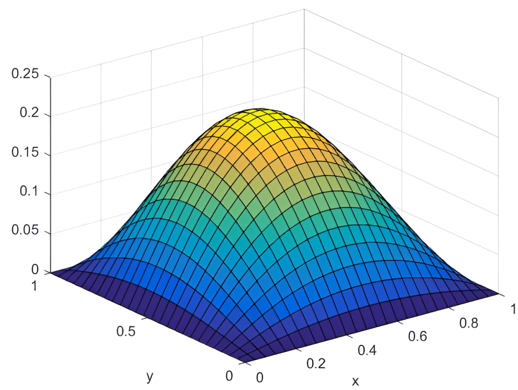

Figure 5. The numerical solution u_h for $t = 1$, $\nu = 0.01$, $h = 1/25$ and $\alpha = 0.8$

Example 3. We consider the following time fractional Burgers' equation:

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \nu\Delta u = 0, & \mathbf{x} = (x,y) \in \Omega, t \in (0,T] \\ {}_0^C\mathcal{D}_t^\alpha v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \nu\Delta v = 0, & (\mathbf{x}, t) \in \Omega \times (0,T], \\ u(\mathbf{x},0) = 0, \quad v(\mathbf{x},0) = 0, & \end{cases}$$

where $\Omega = (0,1) \times (0,1)$, $T = 1$ and $\nu = 0.001$, and the boundary value functions $t \in (0,T]$

$$u(x,y,t) = \begin{cases} 0, y = 1, 0 < x < 0.3 \\ 1, y = 1, 0.3 \le x < 1 \\ 0, x = 0 \text{ or } x = 1 \text{ or } y = 0 \end{cases}, \quad v(x,y,t) = \begin{cases} 0, x = 0, 0 < y < 0.3 \\ 1, x = 0, 0.3 \le y < 1 \\ 0, x = 1 \text{ or } y = 0 \text{ or } y = 1 \end{cases}$$

In order to show that our method keeps stable when ν is smaller, we give the the numerical solution by FVEM with and without upwinding technique for $\nu = 0.001$ in Figure 6 and Figure 7. We can see that our method can perform well but the approximation without upwinding technique produces nonphysical oscillations. These are seen much more clearly in Figure 8 and Figure 9, which shows that the oscillations will increase further when ν is smaller and smaller. In contrast, our method still keeps stable.

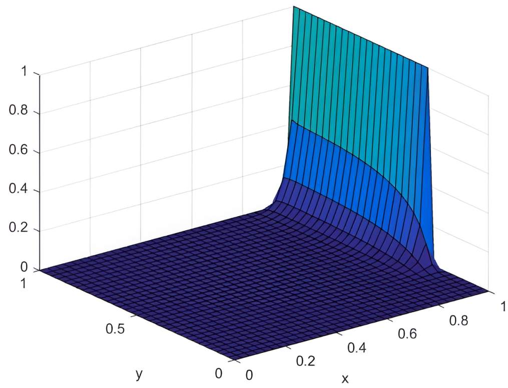

Figure 6. The numerical solution u_h with upwinding for $t = 1$, $\nu = 0.001$, $h = 1/32$ and $\alpha = 0.3$

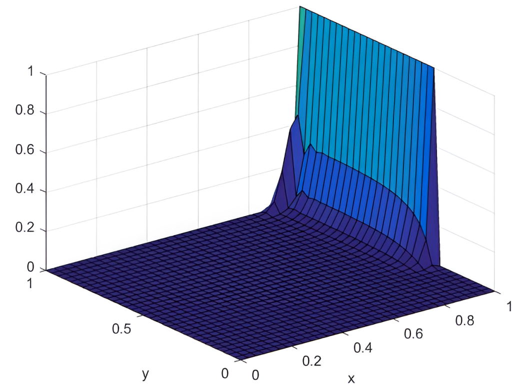

Figure 7. The numerical solution $\tilde{u}_h$ without upwinding for $t = 1$, $\nu = 0.001$, $h = 1/32$ and $\alpha = 0.3$

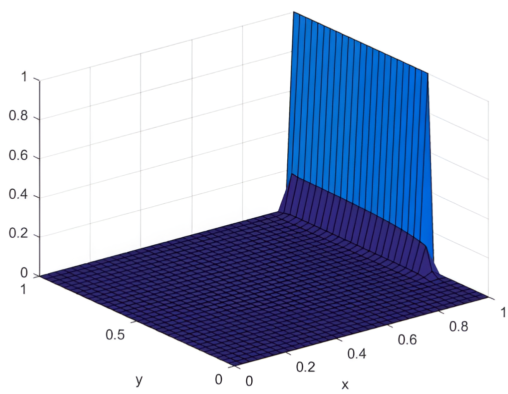

Figure 8. The numerical solution u_h with upwinding for $t = 1$, $\nu = 0.0002$, $h = 1/32$ and $\alpha = 0.3$

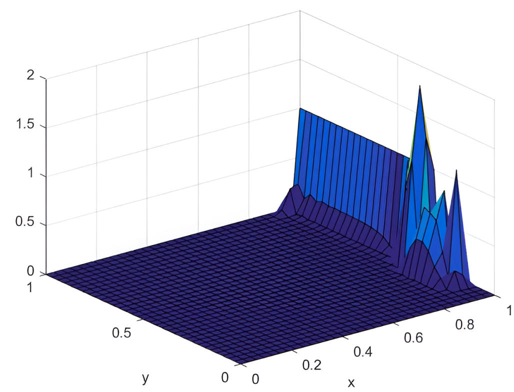

Figure 9. The numerical solution $\tilde{u}_h$ without upwinding for $t = 1$, $\nu = 0.0002$, $h = 1/32$ and $\alpha = 0.3$

5. Conclusions

In this article, a upwind finite volume element scheme has been derived for the two-dimensional time fractional coupled Burgers' equation. We discrete the equation using L1 formula in the time direction. We then discrete the resulting equations in space domain using upwind finite volume element method. The convergence analyses of semi-discrete scheme and fully discrete scheme are proved. Then, the numerical experiments are given to verify the effectiveness of the proposed scheme. And our method keeps stable when ν is smaller by contrast with finite volume element method without upwinding technique. Due to the influence of the linear scheme and the upwinding technique, the convergence order of the scheme is $O(\tau + h)$. In future works, we will use this method to solve more nonlinear fractional partial differential equations, such as nonlinear convection-diffusion equation and Navier-Stokes equation. And for the Caputo fractional derivative with $\alpha \in (0, 1)$, we will try to use other approximation methods (such as L1-2, L2-1_σ formulas) and high-precision upwinding technique to improve the convergence order.

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Authors contributions

Mr. Lv was responsible for creating the mathematical models and drafting the manuscript. Dr. Yang was responsible for reviewing the research questions and revising the manuscript. Both authors approved the final manuscript.

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Data sharing statement

No additional data are available.