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Control of the Hyperbolic Ill-posed Cauchy Problem by Controllability

Bylli André B. Guel[1,2], Sadou Tao[2], Somdouda Sawadogo[3]

[1] Laboratoire LANIBIO, Université Joseph KI-ZERBO, Burkina Faso

[2] Laboratoire MAINEGE, Université Ouaga 3S, Burkina Faso

[3] Département de Sciences Exactes, École Normale Supérieure, Burkina Faso

Correspondence: Bylli André B. Guel, Laboratoire LANIBIO, Université Joseph KI-ZERBO, Burkina Faso

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Abstract

The main purpose of this paper is the control of the hyperbolic ill-posed Cauchy problem. To do this, we adapt to the present case the controllability method previously introduced in the stationary case (Guel and Nakoulima 2023). So we interpret the problem as an inverse problem, and therefore a controllability problem. This point of view induces a regularization method that makes it possible, on the one hand, to characterize the existence of a regular solution to the problem. On the other hand, this method makes it possible to obtain a singular optimality system for the optimal control, without using any additional assumption, such as that of non-vacuity of the interior of the sets of admissible controls, an assumption that many analyses have had to use.

Keywords: singular distributed system, optimal control, singular optimality system, the ill-posed cauchy problem, controllability method, inverse problem

1. Introduction

In this paper, we are interested in the control of an ill-posed system relating to the Cauchy problem for an hyperbolic operator. It is a model example of a singular distributed system that occurs in several physical applications. It is the case in gravimetry, for what concerns the stationary case. But also in questions of the transport of electrical energy (Hadamard 1923), passing through the control of enzymatic reactions (Kernevez 1980 and the bibliography of this work) and the form of plasmas, for the evolution cases.

To introduce the problem, let $\Omega \subset \mathbb{R}^n$ be a bounded and regular domain of class C^2, with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are disjointed, regular and with superficial positive measures. For $T \in \mathbb{R}_+ \setminus \{0\}$, we denote by

$$Q = \Omega \times]0, T[\qquad \text{and} \qquad \Sigma = \Gamma \times]0, T[,$$

so that

$$\Sigma = \Sigma_0 \cup \Sigma_1, \qquad \text{with} \qquad \Sigma_0 = \Gamma_0 \times]0, T[\quad \text{and} \quad \Sigma_1 = \Gamma_1 \times]0, T[.$$

We consider in Q, the boundary value problem

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} - \Delta z = 0 \quad \text{in } Q, \\ z|_{t=0} = 0 = \left.\dfrac{\partial z}{\partial t}\right|_{t=0} \quad \text{in } \Omega, \\ z = v_0, \quad \dfrac{\partial z}{\partial \nu} = v_1 \quad \text{on } \Sigma_0, \end{cases} \tag{1}$$

where $v = (v_0, v_1)$ is given in $\left(L^2(\Sigma_0)\right)^2$.

The problem (1) is the ill-posed Cauchy problem for the wave operator. It is well known (Lions 1983 and Hadamard 1923) that this problem is ill-posed in Hadamard's sense, that is to say, for a given vector $v = (v_0, v_1)$, the problem does not always admits a solution, and it may lead to instability of the solution when it exists.

We therefore consider a priori pairs (v, z) such as

$$v = (v_0, v_1) \in \left(L^2(\Sigma_0)\right)^2 \qquad \text{and} \qquad z \in L^2(Q)\,, \tag{2}$$

and satisfying (1). It is said that such pairs constitute the set of control-state pairs.

Remark 1 (see. Hadamard 1923)**.** *It is important to note that, when it exists, the solution to the ill-posed Cauchy problem is unique.*

$\mathcal{U}^0_{ad}$ and $\mathcal{U}^1_{ad}$ being two non-empty closed convex subsets of $L^2(\Sigma_0)$, a control-state pair (v, z) will be said admissible if

$$v = (v_0, v_1) \in \mathcal{U}_{ad} = \mathcal{U}^0_{ad} \times \mathcal{U}^1_{ad}, \tag{3}$$

with (v, z) satisfying (1). We use the notation $(v, z) \in \mathcal{A}$ to say that $\mathcal{A}$ is the set of admissible control-state pairs and assume (an example is given below)

$$\mathcal{A} \neq \emptyset. \tag{4}$$

Now we introduce the cost function

$$J(v, z) = \frac{1}{2}\|z - z_d\|^2_{L^2(Q)} + \frac{N_0}{2}\|v_0\|^2_{L^2(\Sigma_0)} + \frac{N_1}{2}\|v_1\|^2_{L^2(\Sigma_0)}, \tag{5}$$

where N_0, $N_1 \in \mathbb{R}_+ \setminus \{0\}$ and $z_d \in L^2(Q)$ are given. We are then interested in the control problem

$$\inf\{J(v, z)\,;\ (v, z) \in \mathcal{A}\}\,. \tag{6}$$

The non-vacuity assumption (4), the structure of the set of admissible control-state pairs (it is not difficult to show that $\mathcal{A}$ is a closed convex subset of $L^2(\Sigma_0) \times L^2(\Sigma_0) \times L^2(Q)$) and that of J easily show that the problem (6) admits a unique solution, the optimal control-state pair (u, y).

The cost function J being differentiable, the first order Euler-Lagrange conditions make it possible to establish that the optimal control-state pair (u, y) satisfies the optimality condition: $\forall\,(v, z) \in \mathcal{A}$,

$$(y - z_d\,,\, z - y)_{L^2(Q)} + N_0(u_0\,,\, v_0 - u_0)_{L^2(\Sigma_0)} + N_1(u_1\,,\, v_1 - u_1)_{L^2(\Sigma_0)} \geq 0. \tag{7}$$

Remains to characterize the optimal control-state pair (u, y) through a singular optimality system.

According to the literature on this problem, the origins of the Cauchy system control problem can be traced back to Lions 1983. Indeed in this book, presenting solutions to the main difficulties encountered in the enterprise of controlling singular distributed systems, J. L. Lions deals with the Cauchy system for an elliptic operator, considering a desired state of the trace on the boundary Γ_1. In order to obtain a decoupled singular distributed system, J. L. Lions uses the penalization method in the particular cases

1. $\mathcal{U}_{ad}^0 = L^2(\Gamma_0)\,, \quad \mathcal{U}_{ad}^1 \subsetneq L^2(\Gamma_0)$;
2. $\mathcal{U}_{ad}^0 \subsetneq L^2(\Gamma_0)\,, \quad \mathcal{U}_{ad}^1 = L^2(\Gamma_0)$.

If the strong convergence of the process is then obtained in the first case, the second requires recourse to the additional Slater type assumption that

$$\text{the interior of } \mathcal{U}_{ad}^0 \text{ is non-empty in } L^2(\Gamma_0)\,. \tag{8}$$

However, J. L. Lions conjectures that one should be able to solve the problem only with the usual assumptions of non-vacuity, convexity and closure of the control sets $\mathcal{U}_{ad}^0$ and $\mathcal{U}_{ad}^1$, without resorting to the Slater type assumption (8).

Many authors have studied the control of the ill-posed Cauchy problem. One of the first to take an interest in it was O. Nakoulima who, in Nakoulima 1994, showed that one could indeed do without the Slater type assumption. In this article quoted above, considering the distributed observation problem (as it is the case in the present paper) in the elliptic case, O. Nakoulima effectively manages well, via a regularization-penalization method, to do without that assumption, the sets of controls

$$\mathcal{U}_{ad}^0 = \mathcal{U}_{ad}^1 = \left(L^2(\Gamma_0)\right)^+ \tag{9}$$

then considered being of empty interior. The approach adopted considers the control problem as a "singular" limit of a sequence of well-posed control problems. These results did not, however, exhaust the problem, since they only concerned a particular case of constraints on controls.

A little later, G. Mophou and O. Nakoulima propose a new approach in Mophou and Nakoulima 2009. The authors use a regularization method (without penalization), called elliptic-elliptic regularization, and managed to obtain strong convergence of the process, but resort for that to the Slater type assumption.

Still in the elliptic case, one of the latest results to our knowledge concerns the work of A. Berhail and A. Omrane (cf. Berhail and Omrane 2015). Thanks to which, using the low and no-regret control notion, the authors managed to characterize the optimal solution through a strong and decoupled singular optimality system, but this in the particular unconstrained case $\mathcal{U}_{ad}^0 = \mathcal{U}_{ad}^1 = L^2(\Gamma_0)$.

In the evolution cases, the bibliography refers to Barry and Ndiaye 2013 and Barry and Ndiaye 2014, in which the authors, M. Barry, G. B. Ndiaye and O. Nakoulima take up the idea of penalization method proposed in the elliptic case by J. L. Lions. The authors then obtained results similar to those obtained in the stationary case.

Nevertheless, in general, the problem remains open. Indeed, as shown by the literature review above, almost all of the work carried out concerns only specific cases of controls (v_0, v_1), such as the following:

- $\mathcal{U}_{ad}^0 = \mathcal{U}_{ad}^1 = L^2(\Sigma_0)$, the "unconstrained" case;
- $\mathcal{U}_{ad}^0 = \mathcal{U}_{ad}^1 = \left(L^2(\Sigma_1)\right)^+$,
- or with the additional Slater type assumption that

$$\text{the interiors of } \mathcal{U}_{ad}^0 \text{ and/or } \mathcal{U}_{ad}^1 \text{ are non-empty in } L^2(\Gamma_0)\,. \tag{10}$$

In this paper, we adapt to the hyperbolic case, an original method recently proposed in Guel and Nakoulima 2023. The point of view adopted consists in interpreting the initial problem (1) as an inverse problem, and therefore a controllability problem. This approach induces a regularization method that makes it possible, on the one hand, to characterize the existence of a regular solution to the problem. On the other hand, this method makes it possible to obtain a strong and decoupled singular optimality system for the optimal control, without using any additional assumption, such as that of non-vacuity of the interior of the sets of admissible controls, an assumption that many analyses have had to use.

The rest of the paper is organized as follows. Section 2 is devoted to interpreting the initial problem as an inverse problem. In Section 3, we return to the control problem, starting by regularizing it via the controllability results previously obtained. After establishing the convergence of the process in Section 3.2, then the approached optimality system in Section 3.3, we end in Section 3.4 with the singular optimality system for the initial problem.

Example 1 (Non-vacuity of the set of admissible control-state pairs)**.** *Suppose that*

$$\begin{cases} \mathcal{U}_{ad}^{0} = L^2(\Sigma_0)\,, \\ \mathcal{U}_{ad}^{1}\left(\subset L^2(\Sigma_0)\right) \textit{ is convex closed and containing at least} \\ \textit{one function } v_1 \in H^1\Big(0,T;H^{1/2}(\Gamma_0)\Big) \cap H^{3/2}\Big(0,T;L^2(\Gamma_0)\Big)\,. \end{cases}$$

Then, the set of admissible control-state pairs is non-empty. Indeed, given

$$v_1 \in \mathcal{U}_{ad}^{1} \cap H^1\Big(0,T;H^{1/2}(\Gamma_0)\Big) \cap H^{3/2}\Big(0,T;L^2(\Gamma_0)\Big),$$

we build a solution ξ of

$$\begin{cases} \dfrac{\partial^2 \xi}{\partial t^2} - \Delta \xi = 0 \quad \textit{in } Q, \\ \xi|_{t=0} = 0 = \left.\dfrac{\partial \xi}{\partial t}\right|_{t=0} \quad \textit{in } \Omega, \\ \xi = 0 \quad \textit{on } \Sigma_0, \quad \dfrac{\partial \xi}{\partial \nu} = v_1 \quad \textit{on } \Sigma_1. \end{cases}$$

What defines, and that in a unique way

$$\xi \in H^{2,2}(Q) = H^0\Big(0,T;H^2(\Omega)\Big) \cap H^2\Big(0,T;H^0(\Omega)\Big) \qquad \textit{with} \qquad \xi|_{\Sigma} \in H^{3/2,3/2}(\Sigma) \subset L^2(\Sigma)\,.$$

Thus, the control-state pair $\left(\xi|_{\Sigma_0}, v_1, \xi\right)$ is admissible.

2. Controllability for the Hyperbolic Ill-posed Cauchy Problem

We adapt here to the hyperbolic case the idea of controllability, previously introduced in Guel and Nakoulima 2023. Which consists in interpreting the ill-posed Cauchy problem (1) as a system of inverse problems.

We establish, as in the stationary case, that when it exists, the solution of the hyperbolic ill-posed Cauchy problem coincides with the common solution of the system of inverse problems mentioned above and we also manage to characterize the existence of a regular solution to the hyperbolic Cauchy system.

Starting by the initial problem (1), we consider the systems

$$\begin{cases} \dfrac{\partial^2 y_1}{\partial t^2} - \Delta y_1 = 0 \quad \text{in } Q, \\ y_1|_{t=0} = 0 = \dfrac{\partial y_1}{\partial t}\Big|_{t=0} \quad \text{in } \Omega, \\ y_1 = v_0 \quad \text{on } \Sigma_0, \end{cases} \tag{11}$$

and

$$\begin{cases} \dfrac{\partial^2 y_2}{\partial t^2} - \Delta y_2 = 0 \quad \text{in } Q, \\ y_2|_{t=0} = 0 = \dfrac{\partial y_2}{\partial t}\Big|_{t=0} \quad \text{in } \Omega, \\ \dfrac{\partial y_2}{\partial \nu} = v_1 \quad \text{on } \Sigma_0, \end{cases} \tag{12}$$

for which we set ourselves the objective of observing

$$\frac{\partial y_1}{\partial \nu} = v_1 \qquad \text{and} \qquad y_2 = v_0 \quad \text{on } \Sigma_0. \tag{13}$$

Having, for each of the problems (11) and (12), a datum and an "observation" on the border Σ_0 and no information on the border Σ_1, we look at the whole (11)(12)(13) as a system of inverse problems, setting ourselves the problem of knowing how to complete (11) and (12) on the border Σ_1, with "dummy controls" w_1 and w_2, respectively, which can guarantee to make the observations (13).

More precisely, we consider the following problem, said inverse problem (also said exact controllability problem): given $(v_0, v_1) \in \left(L^2(\Sigma_0)\right)^2$, find $(w_1, w_2) \in \left(L^2(\Sigma_1)\right)^2$ such that, if y_1 and y_2 are respective solutions of

$$\begin{cases} \dfrac{\partial^2 y_1}{\partial t^2} - \Delta y_1 = 0 \quad \text{in } Q, \\ y_1|_{t=0} = 0 = \dfrac{\partial y_1}{\partial t}\Big|_{t=0} \quad \text{in } \Omega, \\ y_1 = v_0 \quad \text{on } \Sigma_0, \quad \dfrac{\partial y_1}{\partial \nu} = w_1 \quad \text{on } \Sigma_1, \end{cases} \tag{14}$$

and

$$\begin{cases} \dfrac{\partial^2 y_2}{\partial t^2} - \Delta y_2 = 0 \quad \text{in } Q, \\ y_2|_{t=0} = 0 = \dfrac{\partial y_2}{\partial t}\Big|_{t=0} \quad \text{in } \Omega, \\ \dfrac{\partial y_2}{\partial \nu} = v_1 \quad \text{on } \Sigma_0, \quad y_2 = w_2 \quad \text{on } \Sigma_1, \end{cases} \tag{15}$$

then y_1 and y_2 further satisfy the conditions (13).

Remark 2. *The symmetric character of the roles played by y_1 and y_2 in the formulation of the controllability problem is obvious. Consequently, one could very well be satisfied with only one of these states in the definition of the problem,* *thus considering one or the other of problems* (14) *and* (15) *with the corresponding observation objective in* (13)*. This is evidenced by the first part of the proof of Theorem 1.*

As far as the present analysis is concerned, it is precisely this symmetric nature of the roles of y_1 *and* y_2 *that motivates their simultaneous use (which facilitates, perhaps for a short time, the continuation of the analysis), but also the wish to remain faithful to the framework of Cauchy's problem.*

Remark 3 (Well-defined nature of the controllability problem, see for instance Lions and Magenes 1968)**.** *For* $z \in L^2(Q)$ *with*

$$\frac{\partial^2 z}{\partial t^2} - \Delta z = 0,$$

we know that

$$z(t)|_\Sigma \in H^{-1/2}(\Gamma)\,, \qquad \left.\frac{\partial z}{\partial \nu}(t)\right|_\Sigma \in H^{-3/2}(\Gamma)\,, \quad a.e.\ t \in (0,T),$$

$$z(0), z(T) \in H^{-1}(\Omega) \qquad and \qquad \frac{\partial z}{\partial t}(0),\ \frac{\partial z}{\partial t}(T) \in H^{-2}(\Omega)\,.$$

Thus, seeking, within the framework of controllability problems, functions of $L^2(\Sigma_1)$ *making it possible to reach, or if not, approaching, the targets fixed still in* $L^2(\Sigma_0)$*, it is necessary that the accessible states* y_1 *and* y_2 *are in* $H^{2,2}(Q)$*.*

Hence the necessity, within the framework of the problem of optimal control, to consider that it is, beyond the non-vacuity assumption of $\mathcal{A} \neq \emptyset$*, the set*

$$\left\{(v,z) \in \mathcal{A} :\ z \in H^{2,2}(Q)\right\}$$

which is non-empty.

Remark 4. *If the system* (11)(12)(13) *admits a solution, then this last one verifies*

$$y_1 = z = y_2,$$

where $(v = (v_0, v_1)\,, z)$ *constitutes a control-state pair for the Cauchy problem.*

Remark 5. *Problems* (14) *and* (15)*, mixed Dirichlet-Neumann problems for the wave operator are then two well-posed problems in the sense of Hadamard.*

With these notations, conditions (13) become

$$\left.\frac{\partial y_1}{\partial \nu}(v_0, w_1)\right|_{\Sigma_0} = v_1 \qquad \text{and} \qquad y_2(v_1, w_2)|_{\Sigma_0} = v_0. \tag{16}$$

Finally, and to fix the vocabulary, we will say that the problem (14)(15)(16) constitutes a problem of exact controllability and that, system (14)(15) is exactly controllable in (v_1, v_0) if there exist $w_1, w_2 \in L^2(\Sigma_1)$ satisfying (16).

Remark 6. *By linearity of mappings*

$$(v_0, w_1) \longmapsto y_1(v_0, w_1) = y_1(v_0, 0) + y_1(0, w_1)$$

and

$$(v_1, w_2) \longmapsto y_2(v_1, w_2) = y_2(v_1, 0) + y_2(0, w_2)\,,$$

the exact controllability problem (14)(15)(16) *is equivalent to the following:*

$$\begin{cases} \text{Find } w_1, w_2 \in L^2(\Sigma_1) \ \text{ such that the solutions} \\ \\ y_1(0, w_1) \quad \text{and} \quad y_2(0, w_2) \ \text{ verify} \\ \\ \dfrac{\partial y_1}{\partial \nu}(0, w_1) = 0\,, \quad y_2(0, w_2) = 0 \quad \text{on } \Sigma_0, \end{cases} \tag{17}$$

which translates the controllability of the system $(y_1(0, w_1), y_2(0, w_2))$ *in* $(0, 0)$.

Then, we approach the problem (17) by density, establishing for this purpose the following proposition.

Proposition 1. *Let us denote by*

$$E_1 = \left\{ \frac{\partial y_1}{\partial \nu}(0, w_1)\bigg|_{\Sigma_0} ;\ w_1 \in L^2(\Sigma_1) \right\} \qquad \text{and} \qquad E_2 = \left\{ y_2(0, w_2)|_{\Sigma_0} ;\ w_2 \in L^2(\Sigma_1) \right\} \tag{18}$$

the sets of zero and one orders traces, on Σ_0*, of the reachable states* y_1 *and* y_2*, respectively.*

Then, we have that

$$\text{sets } E_1 \text{ and } E_2 \text{ are dense in } L^2(\Sigma_0), \tag{19}$$

and then we speak of the approached controllability of the system $(y_1(0, w_1), y_2(0, w_2))$.

Proof. It is clear that E_1 and E_2 constitute vector subspaces of $L^2(\Sigma_0)$. Hence, by the Hahn-Banach Theorem, E_1 and E_2 are dense in $L^2(\Sigma_0)$ if and only if their orthogonal $E_1^\perp$ and $E_2^\perp$ are reduced to $\{0\}$.

Let $k_1 \in E_1^\perp$; so we have

$$\forall\, w_1 \in L^2(\Sigma_1), \qquad \left(k_1, \frac{\partial y_1}{\partial \nu}(0, w_1)\right)_{L^2(\Sigma_0)} = 0.$$

But, by definition of $y_1(0, w_1)$, we have

$$\begin{cases} \dfrac{\partial^2 y_1}{\partial t^2}(0, w_1) - \Delta y_1(0, w_1) = 0 \quad \text{in } Q, \\[2ex] y_1(0, w_1)|_{t=0} = 0, \quad \dfrac{\partial y_1}{\partial t}(0, w_1)\bigg|_{t=0} = 0 \quad \text{in } \Omega, \\[2ex] y_1(0, w_1) = 0 \quad \text{on } \Sigma_0, \quad \dfrac{\partial y_1}{\partial \nu}(0, w_1) = w_1 \quad \text{on } \Sigma_1. \end{cases}$$

So that, taking $\varphi \in C^\infty\left(\overline{Q}\right)$, such that

$$\begin{cases} \dfrac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi = 0 \quad \text{in } Q, \\[2ex] \varphi|_{t=T} = 0, \quad \dfrac{\partial \varphi}{\partial t}\bigg|_{t=T} = 0 \quad \text{in } \Omega, \\[2ex] \varphi = k_1 \quad \text{on } \Sigma_0, \quad \dfrac{\partial \varphi}{\partial \nu} = 0 \quad \text{on } \Sigma_1, \end{cases} \tag{20}$$

it comes that

$$\left(\frac{\partial^2 y_1}{\partial t^2} - \Delta y_1, \varphi\right)_{L^2(Q)} = \left(\frac{\partial^2 y_1}{\partial t^2}, \varphi\right)_{L^2(Q)} - (\Delta y_1, \varphi)_{L^2(Q)} = 0$$

which is equivalent to

$$\begin{aligned} \left(\frac{\partial y_1}{\partial t}(T), \varphi(T)\right)_{L^2(\Omega)} - \left(\frac{\partial y_1}{\partial t}(0), \varphi(0)\right)_{L^2(\Omega)} - \left(y_1(T), \frac{\partial \varphi}{\partial t}(T)\right)_{L^2(\Omega)} \\ + \left(y_1(0), \frac{\partial \varphi}{\partial t}(0)\right)_{L^2(\Omega)} + \left(y_1, \frac{\partial^2 \varphi}{\partial t^2}\right)_{L^2(Q)} - (y_1, \Delta\varphi)_{L^2(Q)} \\ - \left(\frac{\partial y_1}{\partial \nu}, \varphi\right)_{L^2(\Sigma)} + \left(y_1, \frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma)} = 0, \end{aligned}$$

that is to say

$$\left(y_1\,,\,\frac{\partial^2\varphi}{\partial t^2}\right)_{L^2(Q)}-(y_1\,,\,\Delta\varphi)_{L^2(Q)}-\left(\frac{\partial y_1}{\partial\nu}\,,\,\varphi\right)_{L^2(\Sigma_0)}-(w_1\,,\,\varphi)_{L^2(\Sigma_1)}+\left(y_1\,,\,\frac{\partial\varphi}{\partial\nu}\right)_{L^2(\Sigma_1)}=0,$$

either again

$$-\left(\frac{\partial y_1}{\partial\nu}\,,\,k_1\right)_{L^2(\Sigma_0)}-(w_1\,,\,\varphi)_{L^2(\Sigma_1)}=0. \tag{21}$$

As

$$k_1\in E_1^{\perp}\iff\left(\frac{\partial y_1}{\partial\nu}\,,\,k_1\right)_{L^2(\Sigma_0)}=0,$$

then (21) becomes

$$\forall\, w_1\in L^2(\Sigma_1)\,,\qquad (w_1\,,\,\varphi)_{L^2(\Sigma_1)}=0, \tag{22}$$

taking $w_1=\varphi$ on Σ_1, we have

$$\|\varphi\|^2_{L^2(\Sigma_1)}=0\qquad \textit{i.e.}\qquad \varphi=0\quad\text{on }\Sigma_1.$$

So therefore, it comes from (20), that φ satisfies the ill-posed Cauchy problem

$$\begin{cases}\dfrac{\partial^2\varphi}{\partial t^2}-\Delta\varphi=0\quad\text{in } Q,\\[2ex] \varphi|_{t=T}=0,\quad \left.\dfrac{\partial\varphi}{\partial t}\right|_{t=T}=0\quad\text{in }\Omega,\\[2ex] \varphi=0,\quad \dfrac{\partial\varphi}{\partial\nu}=0\quad\text{on }\Sigma_1.\end{cases} \tag{23}$$

But then, due to the uniqueness, when it exists, of the solution of such problem, we obtain that $\varphi\equiv 0$, and consequently, that

$$\varphi|_{\Sigma_0}=0\qquad \textit{i.e.}\qquad k_1=0.$$

This last equality being valid for all $w_1\in L^2(\Sigma_1)$, we deduce that:

$$\forall\, k_1\in E_1^{\perp},\qquad k_1=0.$$

Which means $E_1^{\perp}=\{0\}$, so that E_1 is well dense in $L^2(\Sigma_0)$.

Analogeously, one conclude at the same result for E_2. □

Remark 7. *The approximate controllability problem* (19) *expresses the following idea: failing to find* $w_1, w_2\in L^2(\Sigma_1)$ *making it possible to reach the targets*

$$\left.\frac{\partial y_1}{\partial\nu}\right|_{\Sigma_0}=0\qquad \textit{and}\qquad y_2|_{\Sigma_0}=0$$

fixed by the exact controllability problem (17)*, one can obtain sequences*

$$(w_{1\varepsilon})_\varepsilon,(w_{2\varepsilon})_\varepsilon\subset L^2(\Sigma_1)\,,$$

by the through which the fixed targets can be approached to ε *close, and this, for all* $\varepsilon>0$.

The two corollaries which follow specify this result, first for the exact controllability problem (17) *then, by linearity of the problem (cf. Remark* (6)*), for the exact controllability problem* (14)(15)(16).

Corollary 1. *For all $\varepsilon > 0$, it exists $w_{1\varepsilon}$, $w_{2\varepsilon} \in L^2(\Sigma_1)$ such that*

$$y_{1\varepsilon} = y_1(0, w_{1\varepsilon}),\ y_{2\varepsilon} = y_2(0, w_{2\varepsilon}) \in H^{2,2}(Q)$$

are unique solutions of

$$\begin{cases} \dfrac{\partial^2 y_{1\varepsilon}}{\partial t^2} - \Delta y_{1\varepsilon} = 0, \quad \text{in } Q, \\ y_{1\varepsilon}\big|_{t=0} = 0, \quad \dfrac{\partial y_{1\varepsilon}}{\partial t}\Big|_{t=0} = 0 \quad \text{in } \Omega, \\ y_{1\varepsilon} = 0 \quad \text{on } \Sigma_0, \quad \dfrac{\partial y_{1\varepsilon}}{\partial \nu} = w_{1\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{24}$$

$$\begin{cases} \dfrac{\partial^2 y_{2\varepsilon}}{\partial t^2} - \Delta y_{2\varepsilon} = 0 \quad \text{in } Q, \\ y_{2\varepsilon}\big|_{t=0} = 0, \quad \dfrac{\partial y_{2\varepsilon}}{\partial t}\Big|_{t=0} = 0 \quad \text{in } \Omega, \\ \dfrac{\partial y_{2\varepsilon}}{\partial \nu} = 0 \quad \text{on } \Sigma_0, \quad y_{2\varepsilon} = w_{2\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{25}$$

$$\left\| \frac{\partial y_{1\varepsilon}}{\partial \nu} \right\|_{L^2(\Sigma_0)} < \varepsilon \qquad \text{and} \qquad \left\| y_{2\varepsilon} \right\|_{L^2(\Sigma_0)} < \varepsilon. \tag{26}$$

Corollary 2. *For all $v_0, v_1 \in L^2(\Sigma_0)$ and $\varepsilon > 0$, it exists $w_{1\varepsilon}$, $w_{2\varepsilon} \in L^2(\Sigma_1)$ such that*

$$y_1(v_0, w_{1\varepsilon}),\ y_2(v_1, w_{2\varepsilon}) \in H^{2,2}(Q)$$

are unique solutions of

$$\begin{cases} \dfrac{\partial^2 y_1}{\partial t^2}(v_0, w_{1\varepsilon}) - \Delta y_1(v_0, w_{1\varepsilon}) = 0 \quad \text{in } Q, \\ y_1(v_0, w_{1\varepsilon})|_{t=0} = 0, \quad \dfrac{\partial y_1}{\partial t}(v_0, w_{1\varepsilon})\Big|_{t=0} = 0 \quad \text{in } \Omega, \\ y_1(v_0, w_{1\varepsilon}) = v_0 \quad \text{on } \Sigma_0, \quad \dfrac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) = w_{1\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{27}$$

$$\begin{cases} \dfrac{\partial^2 y_2}{\partial t^2}(v_1, w_{2\varepsilon}) - \Delta y_2(v_1, w_{2\varepsilon}) = 0 \quad \text{in } Q, \\ y_2(v_1, w_{2\varepsilon})|_{t=0} = 0, \quad \dfrac{\partial y_2}{\partial t}(v_1, w_{2\varepsilon})\Big|_{t=0} = 0 \quad \text{in } \Omega, \\ \dfrac{\partial y_2}{\partial \nu}(v_1, w_{2\varepsilon}) = v_1 \quad \text{on } \Sigma_0, \quad y_2(v_1, w_{2\varepsilon}) = w_{2\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{28}$$

$$\left\| \frac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) - v_1 \right\|_{L^2(\Sigma_0)} < \varepsilon \qquad \text{and} \qquad \| y_2(v_1, w_{2\varepsilon}) - v_0 \|_{L^2(\Sigma_0)} < \varepsilon. \tag{29}$$

Moreover, we establish

Theorem 1. *Given* $v = (v_0, v_1) \in \left(L^2(\Sigma_0)\right)^2$*, the ill-posed Cauchy problem*

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} - \Delta z = 0 \quad \textit{in } Q, \\ z|_{t=0} = 0, \quad \left.\dfrac{\partial z}{\partial t}\right|_{t=0} = 0 \quad \textit{in } \Omega, \\ z = v_0, \quad \dfrac{\partial z}{\partial \nu} = v_1 \quad \textit{on } \Sigma_0, \end{cases} \tag{30}$$

admits a regular solution $z \in H^{2,2}(Q)$ *if and only if either of the sequences* $(w_{1\varepsilon})_\varepsilon$ *or* $(w_{2\varepsilon})_\varepsilon$ *is bounded in* $L^2(\Sigma_1)$*.*

Proof.

1. Let $\varepsilon > 0$. According to Corollary 2, it exists $w_{1\varepsilon}$, $w_{2\varepsilon} \in L^2(\Sigma_1)$, such that there exist

$$y_1(v_0, w_{1\varepsilon}),\ y_2(v_1, w_{2\varepsilon}) \in H^{2,2}(Q)$$

solutions of (27),(28) and (29). Then, we generate

$$(w_{1\varepsilon})_\varepsilon,\ (w_{2\varepsilon})_\varepsilon \subset L^2(\Sigma_1) \qquad \text{and} \qquad (y_1(v_0, w_{1\varepsilon}))_\varepsilon,\ (y_2(v_1, w_{2\varepsilon}))_\varepsilon \subset H^{2,2}(Q)\,.$$

Assuming that the sequence $(w_{1\varepsilon})_\varepsilon$ is bounded in $L^2(\Sigma_1)$, it follows, the mixed Dirichlet-Neumann problem (27) being well posed in the Hadamard's sense, that the sequence $(y_1(v_0, w_{1\varepsilon}))_\varepsilon$ is bounded in $H^{2,2}(Q)$, and therefore again in $L^2(Q)$, by continuity of the canonical injection of $H^{2,2}(Q)$ in $L^2(Q)$. Then, we deduce that we can extract, from $(w_{1\varepsilon})_\varepsilon$ and $(y_1(v_0, w_{1\varepsilon}))_\varepsilon$ respectively, subsequences, again denote in the same way, which converge in $L^2(\Sigma_1)$ and $H^{2,2}(Q)$, respectively. Thus, there exist

$$w_1 \in L^2(\Sigma_1) \qquad \text{and} \qquad y_1 \in H^{2,2}(Q)$$

such that, when $\varepsilon \to 0$,

$$w_{1\varepsilon} \longrightarrow w_1 \quad \text{weakly in } L^2(\Sigma_1)\,,$$

$$y_1(v_0, w_{1\varepsilon}) \longrightarrow y_1 \quad \text{weakly in } H^{2,2}(Q)\,.$$

But then, we have on the one hand that, when $\varepsilon \to 0$,

$$\left\| \frac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) - v_1 \right\|_{L^2(\Sigma_0)} < \varepsilon \qquad \text{and} \qquad y_1(v_0, w_{1\varepsilon}) \longrightarrow y_1 \quad \text{weakly in } H^{2,2}(Q)\,,$$

involve, by continuity of the trace operator $\gamma_1 : L^2\left(0, T, H^{\frac{3}{2}}(\Omega)\right) \longrightarrow L^2(\Sigma)$ (see Lions 1968, p. 21), that

$$\frac{\partial y_1}{\partial \nu} = v_1 \quad \text{on } \Sigma_0. \tag{31}$$

On the other hand, for all $\varphi \in C^\infty\left(\overline{Q}\right)$, we have:

$$\frac{\partial^2 y_1}{\partial t^2}(v_0, w_{1\varepsilon}) - \Delta y_1(v_0, w_{1\varepsilon}) = 0 \quad \text{in } Q \implies \left(\frac{\partial^2 y_1}{\partial t^2} - \Delta y_1\,,\, \varphi \right)_{L^2(Q)} = 0,$$

so, denoting $\phi_{1\varepsilon} = y_1(v_0, w_{1\varepsilon})$, that

$$\left(\frac{\partial^2 \phi_{1\varepsilon}}{\partial t^2} - \Delta \phi_{1\varepsilon}\,,\, \varphi \right)_{L^2(Q)} = 0 \iff \left(\frac{\partial^2 \phi_{1\varepsilon}}{\partial t^2}\,,\, \varphi \right)_{L^2(Q)} - (\Delta \phi_{1\varepsilon}\,,\, \varphi)_{L^2(Q)} = 0\,;$$

that is to say

$$\begin{aligned}\left(\frac{\partial \phi_{1\varepsilon}}{\partial t}(T)\,,\,\varphi(T)\right)_{L^2(\Omega)} &- \left(\phi_{1\varepsilon}(T)\,,\,\frac{\partial \varphi}{\partial t}(T)\right)_{L^2(\Omega)} + \left(\phi_{1\varepsilon}\,,\,\frac{\partial^2 \varphi}{\partial t^2}\right)_{L^2(Q)} - (\phi_{1\varepsilon}\,,\,\Delta\varphi)_{L^2(Q)}\\ &- \left(\frac{\partial \phi_{1\varepsilon}}{\partial \nu}\,,\,\varphi\right)_{L^2(\Sigma_0)} - \left(\frac{\partial \phi_{1\varepsilon}}{\partial \nu}\,,\,\varphi\right)_{L^2(\Sigma_1)} + \left(\phi_{1\varepsilon}\,,\,\frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_0)}\\ &+ \left(\phi_{1\varepsilon}\,,\,\frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_1)} = 0,\end{aligned}$$

and so

$$\begin{aligned}\left(\frac{\partial \phi_{1\varepsilon}}{\partial t}(T)\,,\,\varphi(T)\right)_{L^2(\Omega)} &- \left(\phi_{1\varepsilon}(T)\,,\,\frac{\partial \varphi}{\partial t}(T)\right)_{L^2(\Omega)} + \left(\phi_{1\varepsilon}\,,\,\frac{\partial^2 \varphi}{\partial t^2}\right)_{L^2(Q)} - (\phi_{1\varepsilon}\,,\,\Delta\varphi)_{L^2(Q)}\\ &- \left(\frac{\partial \phi_{1\varepsilon}}{\partial \nu}\,,\,\varphi\right)_{L^2(\Sigma_0)} - (w_{1\varepsilon}\,,\,\varphi)_{L^2(\Sigma_1)} + \left(v_0\,,\,\frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_0)}\\ &+ \left(\phi_{1\varepsilon}\,,\,\frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_1)} = 0.\end{aligned}$$

By passing to the limit, it comes that

$$\begin{aligned}\left(\frac{\partial y_1}{\partial t}(T)\,,\,\varphi(T)\right)_{L^2(\Omega)} &- \left(y_1(T)\,,\,\frac{\partial \varphi}{\partial t}(T)\right)_{L^2(\Omega)} + \left(y_1\,,\,\frac{\partial^2 \varphi}{\partial t^2}\right)_{L^2(Q)} - (y_1\,,\,\Delta\varphi)_{L^2(Q)}\\ &- \left(\frac{\partial y_1}{\partial \nu}\,,\,\varphi\right)_{L^2(\Sigma_0)} - (w_1\,,\,\varphi)_{L^2(\Sigma_1)} + \left(v_0\,,\,\frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_0)}\\ &+ \left(y_1\,,\,\frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_1)} = 0,\end{aligned}$$

which is equivalent to

$$\begin{aligned}-\left(y_1(0)\,,\,\frac{\partial \varphi}{\partial t}(0)\right)_{L^2(\Omega)} &+ \left(\frac{\partial y_1}{\partial t}(0)\,,\,\varphi(0)\right)_{L^2(\Omega)} + \left(\frac{\partial^2 y_1}{\partial t^2}\,,\,\varphi\right)_{L^2(Q)} - (\Delta y_1\,,\,\varphi)_{L^2(Q)}\\ &+ \left(v_0 - y_1\,,\,\frac{\partial \varphi}{\partial \nu}\right)_{L^2(\Sigma_0)} + \left(\frac{\partial y_1}{\partial \nu} - w_1\,,\,\varphi\right)_{L^2(\Sigma_1)} = 0.\end{aligned}$$

This last equality being valid for all $\varphi \in C^\infty\left(\overline{Q}\right)$, it follows that

$$\left\{\begin{array}{l} \dfrac{\partial^2 y_1}{\partial t^2} - \Delta y_1 = 0 \quad \text{in } Q,\\[2ex] y_1|_{t=0} = 0, \quad \left.\dfrac{\partial y_1}{\partial t}\right|_{t=0} = 0 \quad \text{in } \Omega,\\[2ex] y_1 = v_0 \quad \text{on } \Sigma_0, \quad \dfrac{\partial y_1}{\partial \nu} = w_1 \quad \text{on } \Sigma_1. \end{array}\right. \tag{32}$$

Then, (31) and (32) give in particular that

$$\left\{\begin{array}{l} \dfrac{\partial^2 y_1}{\partial t^2} - \Delta y_1 = 0 \quad \text{in } Q,\\[2ex] y_1|_{t=0} = 0, \quad \left.\dfrac{\partial y_1}{\partial t}\right|_{t=0} = 0 \quad \text{in } \Omega,\\[2ex] y_1 = v_0, \quad \dfrac{\partial y_1}{\partial \nu} = v_1 \quad \text{on } \Sigma_0, \end{array}\right.$$

so that $y_1 \in H^{2,2}(Q)$ is solution of the ill-posed Cauchy problem (30), a regular solution, due to the well-posed nature of (27).

Symmetrically, the above shows that, assuming that $(w_{2\varepsilon})_\varepsilon$ is bounded in $L^2(\Sigma_1)$, we likewise obtain that there exist

$$w_2 \in L^2(\Sigma_1) \qquad \text{and} \qquad y_2 \in H^{2,2}(Q)\,,$$

such that, when $\varepsilon \to 0$,

$$w_{2\varepsilon} \longrightarrow w_2 \quad \text{weakly in } L^2(\Sigma_1)\,,$$

$$y_{2\varepsilon}(v_1, w_{2\varepsilon}) \longrightarrow y_2 \quad \text{weakly in } H^{2,2}(Q)\,,$$

where $y_2 \in H^{2,2}(Q)$ is also solution to the ill-posed Cauchy problem (30).

2. Now, we assume that the Cauchy problem (30) admits a solution $z \in H^{2,2}(Q)$.

Then, we have

$$z|_{\Sigma_1} \in L^2\Big(0, T; H^{3/2}(\Gamma_1)\Big) \subset L^2(\Sigma_1) \qquad \text{and} \qquad \frac{\partial z}{\partial \nu}\bigg|_{\Sigma_1} \in L^2\Big(0, T; H^{1/2}(\Gamma_1)\Big) \subset L^2(\Sigma_1)\,.$$

So that, for all $\varepsilon > 0$, we can easily define

$$w_{1\varepsilon} = \frac{\partial z}{\partial \nu}\bigg|_{\Sigma_1} \in L^2(\Sigma_1) \qquad \text{and} \qquad w_{2\varepsilon} = z|_{\Sigma_1} \in L^2(\Sigma_1)$$

to obtain existence of sequences

$$(w_{1\varepsilon})_\varepsilon,\ (w_{2\varepsilon})_\varepsilon \subset L^2(\Sigma_1)$$

bounded in $L^2(\Sigma_1)$ since constants; from where the result. □

From Theorem 1, it follows

Corollary 3. *$z \in H^{2,2}(Q)$ being a regular solution of the ill-posed Cauchy problem* (30)*, then*

$$y_1 = z = y_2.$$

3. The Optimal Control problem

Let us start by recalling that we are interested here in the control of the hyperbolic ill-posed Cauchy problem. Starting by the following problem

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} - \Delta z = 0 \quad \text{in } Q, \\[2ex] z|_{t=0} = 0, \quad \dfrac{\partial z}{\partial t}\bigg|_{t=0} = 0 \quad \text{in } \Omega, \\[2ex] z = v_0, \quad \dfrac{\partial z}{\partial \nu} = v_1 \quad \text{on } \Sigma_0, \end{cases} \tag{33}$$

we consider, for all control-state pair (v, z), the cost function

$$J(v, z) = \frac{1}{2}\|z - z_d\|^2_{L^2(Q)} + \frac{N_0}{2}\|v_0\|^2_{L^2(\Sigma_0)} + \frac{N_1}{2}\|v_1\|^2_{L^2(\Sigma_0)}, \tag{34}$$

being interested in the control problem

$$\inf\,\{J(v, z)\,;\ (v, z) \in \mathcal{A}\}\,. \tag{35}$$

More precisely, it is here about the characterization of the optimal control-state pair (u, y), via a singular strong and decoupled optimality system.

To do this, we propose in the rest of this section, a regularization method, called controllability method, based on the results of the previous section. This last, recently introduced in the stationary case (cf. Guel and Nakoulima 2023), approaches the initial control problem by a sequence of approached control problems relating to the well-posed problems (27) and (28). The control problems then considered being regular, the classical theory of optimal control easily apply to lead to the expected result.

3.1 The Controllability Method

Starting by the non-vacuity assumption $\mathcal{A} \neq \emptyset$ and within the framework of Remark 3, we have, for all

$$v = (v_0, v_1) \in \mathcal{U}_{ad} \qquad \text{and} \qquad \varepsilon > 0,$$

that there exist

$$w_{1\varepsilon},\ w_{2\varepsilon} \in L^2(\Sigma_1) \qquad \text{such that} \qquad y_1(v_0, w_{1\varepsilon}),\ y_2(v_1, w_{2\varepsilon}) \in H^{2,2}(Q)$$

satisfy

$$\begin{cases} \dfrac{\partial^2 y_1}{\partial t^2}(v_0, w_{1\varepsilon}) - \Delta y_1(v_0, w_{1\varepsilon}) = 0 \quad \text{in } Q, \\ y_1(v_0, w_{1\varepsilon})|_{t=0} = 0, \quad \dfrac{\partial y_1}{\partial t}(v_0, w_{1\varepsilon})\Big|_{t=0} = 0 \quad \text{in } \Omega, \\ y_1(v_0, w_{1\varepsilon}) = v_0 \quad \text{on } \Sigma_0, \quad \dfrac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) = w_{1\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{36}$$

$$\begin{cases} \dfrac{\partial^2 y_2}{\partial t^2}(v_1, w_{2\varepsilon}) - \Delta y_2(v_1, w_{2\varepsilon}) = 0 \quad \text{in } Q, \\ y_2(v_1, w_{2\varepsilon})|_{t=0} = 0, \quad \dfrac{\partial y_2}{\partial t}(v_1, w_{2\varepsilon})\Big|_{t=0} = 0 \quad \text{in } \Omega, \\ \dfrac{\partial y_2}{\partial \nu}(v_1, w_{2\varepsilon}) = v_1 \quad \text{on } \Sigma_0, \quad y_2(v_1, w_{2\varepsilon}) = w_{2\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{37}$$

$$\left\| \frac{\partial y_1}{\partial \nu}(v_0, w_{1\varepsilon}) - v_1 \right\|_{L^2(\Sigma_0)} < \varepsilon \qquad \text{and} \qquad \|y_2(v_1, w_{2\varepsilon}) - v_0\|_{L^2(\Sigma_0)} < \varepsilon. \tag{38}$$

Then, we consider, for $\theta_1, \theta_2 \in \mathbb{R}_+ :\ \theta_1 + \theta_2 = 1$, the functional

$$\begin{aligned} J_\varepsilon(v_0, v_1) = \frac{\theta_1}{2}\|y_1(v_0, w_{1\varepsilon}) - z_d\|^2_{L^2(Q)} + \frac{\theta_2}{2}\|y_2(v_1, w_{2\varepsilon}) - z_d\|^2_{L^2(Q)} \\ + \frac{N_0}{2}\|v_0\|^2_{L^2(\Sigma_0)} + \frac{N_1}{2}\|v_1\|^2_{L^2(\Sigma_0)}, \end{aligned} \tag{39}$$

being interested in the approached optimal control problem

$$\inf\left\{J_\varepsilon(v_0, v_1)\ ;\ v = (v_0, v_1) \in \mathcal{U}_{ad}\right\}, \tag{40}$$

for which we immediately have

Proposition 2. *For all $\varepsilon > 0$, the control problem* (40) *admits a unique solution, the approached optimal control $\overline{u}_\varepsilon = (\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon})$.*

Proof. The space $\mathcal{U}_{ad}$ is closed convex and the functional J_ε is coercive and strictly convex. From where the result. $\square$

Remark 8. *Let us note that, choosing $\theta_1, \theta_2 \in \mathbb{R}_+ : \theta_1 + \theta_2 = 1$ in* (39) *ensures that, by passing to the limit, since*

$$y_{1\varepsilon} = y_1(v_0, w_{1\varepsilon}) \longrightarrow y \quad and \quad y_{2\varepsilon} = y_2(v_1, w_{2\varepsilon}) \longrightarrow y,$$

the functional (39) *converges towards the functional* (34).

3.2 Convergence of the Method

Let $\varepsilon > 0$. Since we have existence and uniqueness of the approached optimal control $\overline{u}_\varepsilon = (\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon}) \in \mathcal{U}_{ad} \subset \left(L^2(\Sigma_0)\right)^2$, it follows, with what precedes, that there exist

$$\overline{w}_{1\varepsilon}, \overline{w}_{2\varepsilon} \in L^2(\Sigma_1) \qquad \text{and} \qquad \overline{y}_{1\varepsilon}, \overline{y}_{2\varepsilon} \in H^{2,2}(Q)$$

such that

$$\begin{cases} \dfrac{\partial^2 \overline{y}_{1\varepsilon}}{\partial t^2} - \Delta \overline{y}_{1\varepsilon} = 0 \quad \text{in } Q, \\ \overline{y}_{1\varepsilon}\big|_{t=0} = 0, \quad \dfrac{\partial \overline{y}_{1\varepsilon}}{\partial t}\bigg|_{t=0} = 0 \quad \text{in } \Omega, \\ \overline{y}_{1\varepsilon} = \overline{u}_{0\varepsilon} \quad \text{on } \Sigma_0, \quad \dfrac{\partial \overline{y}_{1\varepsilon}}{\partial \nu} = \overline{w}_{1\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{41}$$

$$\begin{cases} \dfrac{\partial^2 \overline{y}_{2\varepsilon}}{\partial t^2} - \Delta \overline{y}_{2\varepsilon} = 0 \quad \text{in } Q, \\ \overline{y}_{2\varepsilon}\big|_{t=0} = 0, \quad \dfrac{\partial \overline{y}_{2\varepsilon}}{\partial t}\bigg|_{t=0} = 0 \quad \text{in } \Omega, \\ \dfrac{\partial \overline{y}_{2\varepsilon}}{\partial \nu} = \overline{u}_{1\varepsilon} \quad \text{on } \Sigma_0, \quad \overline{y}_{2\varepsilon} = \overline{w}_{2\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{42}$$

$$\left\| \frac{\partial \overline{y}_{1\varepsilon}}{\partial \nu} - \overline{u}_{1\varepsilon} \right\|_{L^2(\Sigma_0)} < \varepsilon \qquad \text{and} \qquad \left\| \overline{y}_{2\varepsilon} - \overline{u}_{0\varepsilon} \right\|_{L^2(\Sigma_0)} < \varepsilon, \tag{43}$$

with, for all $v \in \mathcal{U}_{ad}$, $J_\varepsilon(\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon}) \leq J_\varepsilon(v_0, v_1)$. Therefore in particular

$$J_\varepsilon(\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon}) \leq J_\varepsilon(u_0, u_1), \tag{44}$$

where $u = (u_0, u_1)$ is the optimal control for (33)(34)(35).

But we check that $J_\varepsilon(u_0, u_1) = J(u, y)$ is independent of ε, and consequently that (44) becomes

$$J_\varepsilon(\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon}) \leq J_\varepsilon(u_0, u_1) = J(u, y). \tag{45}$$

From which it follows that there exist constants $C_i \in \mathbb{R}_+ \setminus \{0\}$, independent of ε, such that

$$\begin{cases} \left\| \overline{y}_{1\varepsilon} \right\|_{L^2(Q)} \leq C_1, \quad \left\| \overline{y}_{2\varepsilon} \right\|_{L^2(Q)} \leq C_2, \\ \| \overline{u}_{0\varepsilon} \|_{L^2(\Sigma_0)} \leq C_3, \quad \| \overline{u}_{1\varepsilon} \|_{L^2(\Sigma_0)} \leq C_4, \end{cases} \tag{46}$$

and therefore that there exist $\widehat{u_0}, \widehat{u_1} \in L^2(\Sigma_0)$ and $\widehat{y_1}, \widehat{y_2} \in L^2(Q)$ such that, when $\varepsilon \to 0$,

$$\begin{cases} \overline{u}_{0\varepsilon} \longrightarrow \widehat{u_0} \quad \text{weakly in } L^2(\Sigma_0) \\ \\ \overline{u}_{1\varepsilon} \longrightarrow \widehat{u_1} \quad \text{weakly in } L^2(\Sigma_0)\,, \end{cases} \tag{47}$$

and

$$\begin{cases} \overline{y}_{1\varepsilon} \longrightarrow \widehat{y_1} \quad \text{weakly in } L^2(Q)\,, \\ \\ \overline{y}_{2\varepsilon} \longrightarrow \widehat{y_2} \quad \text{weakly in } L^2(Q)\,, \end{cases} \tag{48}$$

the pair $(\widehat{u} = (\widehat{u_0}, \widehat{u_1})\,, \widehat{y})$, with $\widehat{y_1} = \widehat{y} = \widehat{y_2}$, being admissible. Thus, it comes on the one hand, by optimality of the optimal control-state pair (u, y), that

$$J(u, y) \leq J(\widehat{u}, \widehat{y}) \tag{49}$$

and on the other hand, passing to the limit in (45), that

$$J(\widehat{u}, \widehat{y}) \leq J(u, y), \tag{50}$$

so that

$$J(\widehat{u}, \widehat{y}) \leq J(u, y) \leq J(\widehat{u}, \widehat{y})\,.$$

That is to say, by uniqueness of the optimal control-state pair (u, y), that $(\widehat{u}, \widehat{y}) = (u, y)$. Thereby we have just proved the following result.

Proposition 3. *For all $\varepsilon > 0$, the approached optimal control $\overline{u}_\varepsilon = (\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon})$, solution of* (40)*, is such that, when $\varepsilon \to 0$, $(\overline{u}_\varepsilon, \overline{y}_\varepsilon)$ verifies*

$$\begin{cases} \overline{u}_{0\varepsilon} \longrightarrow u_0 \quad \textit{weakly in } L^2(\Sigma_0)\,, \\ \\ \overline{u}_{1\varepsilon} \longrightarrow u_1 \quad \textit{weakly in } L^2(\Sigma_0)\,, \\ \\ \overline{y}_{1\varepsilon} \longrightarrow y \quad \textit{weakly in } L^2(Q)\,, \\ \\ \overline{y}_{2\varepsilon} \longrightarrow y \quad \textit{weakly in } L^2(Q)\,, \end{cases} \tag{51}$$

where (u, y) is the optimal control-state pair for (33)(34)(35).

But we have even more as shown now.

Theorem 2. *The approached optimal control $\overline{u}_\varepsilon = (\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon})$ and the associate approached optimal state $\overline{y}_\varepsilon = \left(\overline{y}_{1\varepsilon}, \overline{y}_{2\varepsilon}\right)$ are such that, when $\varepsilon \to 0$,*

$$\begin{cases} \overline{u}_{0\varepsilon} \longrightarrow u_0 \quad \textit{strongly in } L^2(\Sigma_0)\,, \\ \\ \overline{u}_{1\varepsilon} \longrightarrow u_1 \quad \textit{strongly in } L^2(\Sigma_0) \end{cases} \tag{52}$$

and

$$\begin{cases} \overline{y}_{1\varepsilon} \longrightarrow y \quad \textit{strongly in } L^2(Q)\,, \\ \\ \overline{y}_{2\varepsilon} \longrightarrow y \quad \textit{strongly in } L^2(Q)\,. \end{cases} \tag{53}$$

Proof. From the previous results, we have, when $\varepsilon \to 0$,

$$\begin{cases} \overline{u}_{0\varepsilon} \longrightarrow u_0 & \text{weakly in } L^2(\Sigma_0)\,, \\ \overline{u}_{1\varepsilon} \longrightarrow u_1 & \text{weakly in } L^2(\Sigma_0)\,, \end{cases} \tag{54}$$

$$\begin{cases} \overline{y}_{1\varepsilon} \longrightarrow y & \text{weakly in } L^2(Q)\,, \\ \overline{y}_{2\varepsilon} \longrightarrow y & \text{weakly in } L^2(Q) \end{cases} \tag{55}$$

and

$$J(u, y) = \lim_{\varepsilon\to 0} J_\varepsilon(\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon})\,, \tag{56}$$

this last result being translated by

$$\begin{aligned} &\|y - z_d\|^2_{L^2(Q)} + N_0\|u_0\|^2_{L^2(\Sigma_0)} + N_1\|u_1\|^2_{L^2(\Sigma_0)} \\ &= \lim_{\varepsilon\to 0}\Big(\theta_1\big\|\overline{y}_{1\varepsilon} - z_d\big\|^2_{L^2(Q)} + \theta_2\big\|\overline{y}_{2\varepsilon} - z_d\big\|^2_{L^2(Q)} + N_0\|\overline{u}_{0\varepsilon}\|^2_{L^2(Q)} + N_1\|\overline{u}_{1\varepsilon}\|^2_{L^2(\Sigma_0)}\Big). \end{aligned} \tag{57}$$

But then, the norms being continuous, a fortiori weakly lower semi-continuous, it follows, with (54) and (55), that

$$\begin{cases} \|y - z_d\|^2_{L^2(Q)} \leq \liminf\limits_{\varepsilon\to 0}\Big(\theta_1\big\|\overline{y}_{1\varepsilon} - z_d\big\|^2_{L^2(Q)} + \theta_2\big\|\overline{y}_{2\varepsilon} - z_d\big\|^2_{L^2(Q)}\Big) \\ \|u_0\|^2_{L^2(\Sigma_0)} \leq \liminf\limits_{\varepsilon\to 0}\|\overline{u}_{0\varepsilon}\|^2_{L^2(\Sigma_0)}, \\ \|u_1\|^2_{L^2(\Sigma_0)} \leq \liminf\limits_{\varepsilon\to 0}\|\overline{u}_{1\varepsilon}\|^2_{L^2(\Sigma_0)}. \end{cases} \tag{58}$$

So that (57) and (58) bring

$$\begin{cases} \|y - z_d\|^2_{L^2(Q)} = \lim\limits_{\varepsilon\to 0}\Big(\theta_1\big\|\overline{y}_{1\varepsilon} - z_d\big\|^2_{L^2(Q)} + \theta_2\big\|\overline{y}_{2\varepsilon} - z_d\big\|^2_{L^2(Q)}\Big) \\ \|u_0\|^2_{L^2(\Sigma_0)} = \lim\limits_{\varepsilon\to 0}\|\overline{u}_{0\varepsilon}\|^2_{L^2(\Sigma_0)} \\ \|u_1\|^2_{L^2(\Sigma_0)} = \lim\limits_{\varepsilon\to 0}\|\overline{u}_{1\varepsilon}\|^2_{L^2(\Sigma_0)}. \end{cases} \tag{59}$$

So therefore, since

$$\begin{aligned} \|\overline{u}_{0\varepsilon} - u_0\|^2_{L^2(\Sigma_0)} + \|\overline{u}_{1\varepsilon} - u_1\|^2_{L^2(\Sigma_0)} &= \|\overline{u}_{0\varepsilon}\|^2_{L^2(\Sigma_0)} + \|u_0\|^2_{L^2(\Sigma_0)} + \|\overline{u}_{1\varepsilon}\|^2_{L^2(\Sigma_0)} + \|u_1\|^2_{L^2(\Sigma_0)} \\ &\quad - 2(\overline{u}_{0\varepsilon}\,,\, u_0)_{L^2(\Sigma_0)} - 2(\overline{u}_{1\varepsilon}\,,\, u_1)_{L^2(\Sigma_0)} \end{aligned}$$

we obtain, by passing to the limit with (54), $(59)_2$ and $(59)_3$, that

$$\lim_{\varepsilon\to 0}\Big(\|\overline{u}_{0\varepsilon} - u_0\|^2_{L^2(\Sigma_0)} + \|\overline{u}_{1\varepsilon} - u_1\|^2_{L^2(\Sigma_0)}\Big) = 0,$$

which leads to

$$\begin{cases} \overline{u}_{0\varepsilon} \longrightarrow u_0 & \text{strongly in } L^2(\Sigma_0)\,, \\ \overline{u}_{1\varepsilon} \longrightarrow u_1 & \text{strongly in } L^2(\Sigma_0)\,. \end{cases} \tag{60}$$

Moreover, let us begin by noting that we can take, in $(59)_1$, successively

$$(\theta_1 = 1,\ \theta_2 = 0) \qquad \text{then} \qquad (\theta_1 = 0,\ \theta_2 = 1)$$

to obtain

$$\lim_{\varepsilon\to 0} \left\|\overline{y}_{1\varepsilon} - z_d\right\|^2_{L^2(Q)} = \|y - z_d\|^2_{L^2(Q)} = \lim_{\varepsilon\to 0} \left\|\overline{y}_{2\varepsilon} - z_d\right\|^2_{L^2(Q)}. \tag{61}$$

So, from

$$\left\|\overline{y}_{1\varepsilon} - y\right\|^2_{L^2(Q)} = \left\|\overline{y}_{1\varepsilon} - z_d\right\|^2_{L^2(Q)} + \|y - z_d\|^2_{L^2(Q)} - 2\left(\overline{y}_{1\varepsilon} - z_d\,,\, y - z_d\right)_{L^2(Q)}$$

and

$$\left\|\overline{y}_{2\varepsilon} - y\right\|^2_{L^2(Q)} = \left\|\overline{y}_{2\varepsilon} - z_d\right\|^2_{L^2(Q)} + \|y - z_d\|^2_{L^2(Q)} - 2\left(\overline{y}_{2\varepsilon} - z_d\,,\, y - z_d\right)_{L^2(Q)}$$

it follows, with (61) and (55), that

$$\begin{cases} \overline{y}_{1\varepsilon} \longrightarrow y \quad \text{strongly in } L^2(Q)\,, \\ \\ \overline{y}_{2\varepsilon} \longrightarrow y \quad \text{strongly in } L^2(Q)\,. \end{cases} \tag{62}$$

So we end up proving the result. □

Let us now establish, with these results of strong convergence, the approached and singular optimality systems, for the approached optimal control $\overline{u}_\varepsilon$ and the optimal control-state pair (u, y), respectively.

3.3 Approached Optimality System

Let $\varepsilon > 0$. Let us start by recalling that, for the approached optimal control $\overline{u}_\varepsilon = (\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon}) \in \mathcal{U}_{ad}$, there exist $\overline{w}_{1\varepsilon}$, $\overline{w}_{2\varepsilon} \in L^2(\Sigma_1)$ and $\overline{y}_{1\varepsilon}$, $\overline{y}_{2\varepsilon} \in H^{2,2}(Q)$, such that

$$\begin{cases} \dfrac{\partial^2 \overline{y}_{1\varepsilon}}{\partial t^2} - \Delta \overline{y}_{1\varepsilon} = 0 \quad \text{in } Q, \\ \\ \overline{y}_{1\varepsilon}\Big|_{t=0} = 0, \quad \dfrac{\partial \overline{y}_{1\varepsilon}}{\partial t}\Big|_{t=0} = 0 \quad \text{in } \Omega, \\ \\ \overline{y}_{1\varepsilon} = \overline{u}_{0\varepsilon} \quad \text{on } \Sigma_0, \quad \dfrac{\partial \overline{y}_{1\varepsilon}}{\partial \nu} = \overline{w}_{1\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{63}$$

$$\begin{cases} \dfrac{\partial^2 \overline{y}_{2\varepsilon}}{\partial t^2} - \Delta \overline{y}_{2\varepsilon} = 0 \quad \text{in } Q, \\ \\ \overline{y}_{2\varepsilon}\Big|_{t=0} = 0, \quad \dfrac{\partial \overline{y}_{2\varepsilon}}{\partial t}\Big|_{t=0} = 0 \quad \text{in } \Omega, \\ \\ \dfrac{\partial \overline{y}_{2\varepsilon}}{\partial \nu} = \overline{u}_{1\varepsilon} \quad \text{on } \Sigma_0, \quad \overline{y}_{2\varepsilon} = \overline{w}_{2\varepsilon} \quad \text{on } \Sigma_1, \end{cases} \tag{64}$$

$$\left\|\frac{\partial \overline{y}_{1\varepsilon}}{\partial \nu} - \overline{u}_{1\varepsilon}\right\|_{L^2(\Sigma_0)} < \varepsilon \qquad \text{and} \qquad \left\|\overline{y}_{2\varepsilon} - \overline{u}_{0\varepsilon}\right\|_{L^2(\Sigma_0)} < \varepsilon. \tag{65}$$

So, for all $v = (v_0, v_1) \in \mathcal{U}_{ad}$ and $\lambda \in \mathbb{R} \setminus \{0\}$, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} J_\varepsilon(\overline{u}_{0\varepsilon} + \lambda\,(v_0 - \overline{u}_{0\varepsilon})\,, \overline{u}_{1\varepsilon})\Big|_{\lambda=0} = \theta_1\left(\overline{y}_{1\varepsilon} - z_d\,,\, \phi_{1\varepsilon}\right)_{L^2(Q)} + N_0(\overline{u}_{0\varepsilon}\,,\, v_0 - \overline{u}_{0\varepsilon})_{L^2(\Sigma_0)}, \tag{66}$$

where $\phi_{1\varepsilon} = y_1(v_0 - \overline{u}_{0\varepsilon}, \overline{w}_{1\varepsilon}) - y_1(0, \overline{w}_{1\varepsilon})$ is given by

$$\begin{cases} \dfrac{\partial^2 \phi_{1\varepsilon}}{\partial t^2} - \Delta\phi_{1\varepsilon} = 0 \quad \text{in } Q, \\ \phi_{1\varepsilon}\Big|_{t=0} = 0, \quad \dfrac{\partial \phi_{1\varepsilon}}{\partial t}\Big|_{t=0} = 0 \quad \text{in } \Omega, \\ \phi_{1\varepsilon} = v_0 - \overline{u}_{0\varepsilon} \quad \text{on } \Sigma_0, \quad \dfrac{\partial \phi_{1\varepsilon}}{\partial \nu} = 0 \quad \text{on } \Sigma_1. \end{cases} \tag{67}$$

Similarly, we get that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} J_\varepsilon(\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon} + \lambda(v_1 - \overline{u}_{1\varepsilon}))\Big|_{\lambda=0} = \theta_2\Big(\overline{y}_{2\varepsilon} - z_d\,,\, \phi_{2\varepsilon}\Big)_{L^2(Q)} + N_1(\overline{u}_{1\varepsilon}\,,\, v_1 - \overline{u}_{1\varepsilon})_{L^2(\Sigma_0)}, \tag{68}$$

with $\phi_{2\varepsilon} = y_2(v_1 - \overline{u}_{1\varepsilon}, \overline{w}_{2\varepsilon}) - y_2(0, \overline{w}_{2\varepsilon})$ defined by

$$\begin{cases} \dfrac{\partial^2 \phi_{2\varepsilon}}{\partial t^2} - \Delta\phi_{2\varepsilon} = 0 \quad \text{in } Q, \\ \phi_{2\varepsilon}\Big|_{t=0} = 0, \quad \dfrac{\partial \phi_{2\varepsilon}}{\partial t}\Big|_{t=0} = 0 \quad \text{in } \Omega, \\ \dfrac{\partial \phi_{2\varepsilon}}{\partial \nu} = v_1 - \overline{u}_{1\varepsilon} \quad \text{on } \Sigma_0, \quad \phi_{2\varepsilon} = 0 \quad \text{on } \Sigma_1. \end{cases} \tag{69}$$

Which gives, with the first order Euler-Lagrange conditions, that the approached optimal control $\overline{u}_\varepsilon = (\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon})$ is the unique element of $\mathcal{U}_{ad}$ satisfying

$$\begin{cases} \forall\, v = (v_0, v_1) \in \mathcal{U}_{ad}, \\ \theta_1\Big(\overline{y}_{1\varepsilon} - z_d\,,\, \phi_{1\varepsilon}\Big)_{L^2(Q)} + N_0(\overline{u}_{0\varepsilon}\,,\, v_0 - \overline{u}_{0\varepsilon})_{L^2(\Sigma_0)} \quad \geq 0, \\ \theta_2\Big(\overline{y}_{2\varepsilon} - z_d\,,\, \phi_{2\varepsilon}\Big)_{L^2(Q)} + N_1(\overline{u}_{1\varepsilon}\,,\, v_1 - \overline{u}_{1\varepsilon})_{L^2(\Sigma_0)} \quad \geq 0. \end{cases} \tag{70}$$

Then, we introduce the adjunct states $p_{1\varepsilon}$ and $p_{2\varepsilon}$ respectively defined by

$$\begin{cases} \dfrac{\partial^2 p_{1\varepsilon}}{\partial t^2} - \Delta p_{1\varepsilon} = \theta_1\left(\overline{y}_{1\varepsilon} - z_d\right) \quad \text{in } Q, \\ p_{1\varepsilon}\Big|_{t=T} = 0, \quad \dfrac{\partial p_{1\varepsilon}}{\partial t}\Big|_{t=T} = 0 \quad \text{in } \Omega, \\ p_{1\varepsilon} = 0 \quad \text{on } \Sigma_0, \quad \dfrac{\partial p_{1\varepsilon}}{\partial \nu} = 0 \quad \text{on } \Sigma_1, \end{cases} \tag{71}$$

and

$$\begin{cases} \dfrac{\partial^2 p_{2\varepsilon}}{\partial t^2} - \Delta p_{2\varepsilon} = \theta_2\left(\overline{y}_{2\varepsilon} - z_d\right) \quad \text{in } Q, \\ p_{2\varepsilon}\Big|_{t=T} = 0, \quad \dfrac{\partial p_{2\varepsilon}}{\partial t}\Big|_{t=T} = 0 \quad \text{in } \Omega, \\ \dfrac{\partial p_{2\varepsilon}}{\partial \nu} = 0 \quad \text{on } \Sigma_0, \quad p_{2\varepsilon} = 0 \quad \text{on } \Sigma_1. \end{cases} \tag{72}$$

So it comes, from (67) et (71), that

$$\theta_1\left(\overline{y}_{1\varepsilon} - z_d\,,\,\phi_{1\varepsilon}\right)_{L^2(Q)} = \left(\frac{\partial^2 p_{1\varepsilon}}{\partial t^2}\,,\,\phi_{1\varepsilon}\right)_{L^2(Q)} - \left(\Delta p_{1\varepsilon}\,,\,\phi_{1\varepsilon}\right)_{L^2(Q)}$$

$$= \left(p_{1\varepsilon}\,,\,\frac{\partial^2 \phi_{1\varepsilon}}{\partial t^2}\right)_{L^2(Q)} - \left(p_{1\varepsilon}\,,\,\Delta\phi_{1\varepsilon}\right)_{L^2(Q)} - \left(\frac{\partial p_{1\varepsilon}}{\partial \nu}\,,\,v_0 - \overline{u}_{0\varepsilon}\right)_{L^2(\Sigma_0)}$$

$$= -\left(\frac{\partial p_{1\varepsilon}}{\partial \nu}\,,\,v_0 - \overline{u}_{0\varepsilon}\right)_{L^2(\Sigma_0)},$$

and, from (69) et (72), that

$$\theta_2\left(\overline{y}_{2\varepsilon} - z_d\,,\,\phi_{2\varepsilon}\right)_{L^2(Q)} = \left(\frac{\partial^2 p_{2\varepsilon}}{\partial t^2}\,,\,\phi_{2\varepsilon}\right)_{L^2(Q)} - \left(\Delta p_{2\varepsilon}\,,\,\phi_{2\varepsilon}\right)_{L^2(Q)}$$

$$= \left(p_{2\varepsilon}\,,\,\frac{\partial^2 \phi_{2\varepsilon}}{\partial t^2}\right)_{L^2(Q)} - \left(p_{2\varepsilon}\,,\,\Delta\phi_{2\varepsilon}\right)_{L^2(Q)} - \left(\frac{\partial p_{2\varepsilon}}{\partial \nu}\,,\,\phi_{2\varepsilon}\right)_{L^2(\Sigma_0)} - \left(\frac{\partial p_{2\varepsilon}}{\partial \nu}\,,\,\phi_{2\varepsilon}\right)_{L^2(\Sigma_1)} + \left(p_{2\varepsilon}\,,\,\frac{\partial \phi_{2\varepsilon}}{\partial \nu}\right)_{L^2(\Sigma_0)} + \left(p_{2\varepsilon}\,,\,\frac{\partial \phi_{2\varepsilon}}{\partial \nu}\right)_{L^2(\Sigma_1)}$$

$$= \left(p_{2\varepsilon}\,,\,\frac{\partial^2 \phi_{2\varepsilon}}{\partial t^2}\right)_{L^2(Q)} - \left(p_{2\varepsilon}\,,\,\Delta\phi_{2\varepsilon}\right)_{L^2(Q)} + \left(p_{2\varepsilon}\,,\,v_1 - \overline{u}_{1\varepsilon}\right)_{L^2(\Sigma_0)}$$

$$= \left(p_{2\varepsilon}\,,\,v_1 - \overline{u}_{1\varepsilon}\right)_{L^2(\Sigma_0)}.$$

Thus, the optimality condition (70) is rewritten

$$\begin{cases} \forall\, v = (v_0, v_1) \in \mathcal{U}_{ad}, \\ \left(N_0 \overline{u}_{0\varepsilon} - \dfrac{\partial p_{1\varepsilon}}{\partial \nu}\,,\,v_0 - \overline{u}_{0\varepsilon}\right)_{L^2(\Sigma_0)} \geq 0, \\ \left(N_1 \overline{u}_{1\varepsilon} + p_{2\varepsilon}\,,\,v_1 - \overline{u}_{1\varepsilon}\right)_{L^2(\Sigma_0)} \geq 0. \end{cases} \tag{73}$$

Which ends up proving the following theorem, characterizing the approached optimal control $\overline{u}_\varepsilon$.

Theorem 3. *Let $\varepsilon > 0$. The approached optimal control $\overline{u}_\varepsilon = (\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon})$ is unique solution to* (40) *if and only if there exist*

$$\overline{w}_{1\varepsilon}, \overline{w}_{2\varepsilon} \in L^2(\Sigma_1)\,, \qquad \overline{y}_{1\varepsilon}, \overline{y}_{2\varepsilon} \in H^{2,2}(Q) \qquad \textit{and} \qquad p_{1\varepsilon}, p_{2\varepsilon} \in H^{2,2}(Q),$$

such that the quadruplet $\left\{(\overline{u}_{0\varepsilon}, \overline{u}_{1\varepsilon})\,, (\overline{w}_{1\varepsilon}, \overline{w}_{2\varepsilon})\,, \left(\overline{y}_{1\varepsilon}, \overline{y}_{2\varepsilon}\right), (p_{1\varepsilon}, p_{2\varepsilon})\right\}$ is solution of the approached optimality system de-

fined by the partial differential systems

$$\begin{cases} \dfrac{\partial^2 \overline{y}_{1\varepsilon}}{\partial t^2} - \Delta \overline{y}_{1\varepsilon} = 0 \quad in\ Q, \\ \overline{y}_{1\varepsilon}\Big|_{t=0} = 0, \quad \dfrac{\partial \overline{y}_{1\varepsilon}}{\partial t}\Big|_{t=0} = 0 \quad in\ \Omega, \\ \overline{y}_{1\varepsilon} = \overline{u}_{0\varepsilon} \quad on\ \Sigma_0, \quad \dfrac{\partial \overline{y}_{1\varepsilon}}{\partial \nu} = \overline{w}_{1\varepsilon} \quad on\ \Sigma_1, \end{cases} \tag{74}$$

$$\begin{cases} \dfrac{\partial^2 \overline{y}_{2\varepsilon}}{\partial t^2} - \Delta \overline{y}_{2\varepsilon} = 0 \quad in\ Q, \\ \overline{y}_{2\varepsilon}\Big|_{t=0} = 0, \quad \dfrac{\partial \overline{y}_{2\varepsilon}}{\partial t}\Big|_{t=0} = 0 \quad in\ \Omega, \\ \dfrac{\partial \overline{y}_{2\varepsilon}}{\partial \nu} = \overline{u}_{1\varepsilon} \quad on\ \Sigma_0, \quad \overline{y}_{2\varepsilon} = \overline{w}_{2\varepsilon} \quad on\ \Sigma_1, \end{cases} \tag{75}$$

$$\begin{cases} \dfrac{\partial^2 p_{1\varepsilon}}{\partial t^2} - \Delta p_{1\varepsilon} = \theta_1 \left(\overline{y}_{1\varepsilon} - z_d\right) \quad in\ Q, \\ p_{1\varepsilon}\Big|_{t=T} = 0, \quad \dfrac{\partial p_{1\varepsilon}}{\partial t}\Big|_{t=T} = 0 \quad in\ \Omega, \\ p_{1\varepsilon} = 0 \quad on\ \Sigma_0, \quad \dfrac{\partial p_{1\varepsilon}}{\partial \nu} = 0 \quad on\ \Sigma_1, \end{cases} \tag{76}$$

$$\begin{cases} \dfrac{\partial^2 p_{2\varepsilon}}{\partial t^2} - \Delta p_{2\varepsilon} = \theta_2 \left(\overline{y}_{2\varepsilon} - z_d\right) \quad in\ Q, \\ p_{2\varepsilon}\Big|_{t=T} = 0, \quad \dfrac{\partial p_{2\varepsilon}}{\partial t}\Big|_{t=T} = 0 \quad in\ \Omega, \\ \dfrac{\partial p_{2\varepsilon}}{\partial \nu} = 0 \quad on\ \Sigma_0, \quad p_{2\varepsilon} = 0 \quad on\ \Sigma_1, \end{cases} \tag{77}$$

the estimates

$$\left\| \frac{\partial \overline{y}_{1\varepsilon}}{\partial \nu} - \overline{u}_{1\varepsilon} \right\|_{L^2(\Sigma_0)} < \varepsilon \qquad and \qquad \left\| \overline{y}_{2\varepsilon} - \overline{u}_{0\varepsilon} \right\|_{L^2(\Sigma_0)} < \varepsilon, \tag{78}$$

and the variational inequality system

$$\begin{cases} \forall\, v = (v_0, v_1) \in \mathcal{U}_{ad}, \\ \left(N_0 \overline{u}_{0\varepsilon} - \dfrac{\partial p_{1\varepsilon}}{\partial \nu}\, ,\ v_0 - \overline{u}_{0\varepsilon} \right)_{L^2(\Sigma_0)} \geq 0, \\ \left(N_1 \overline{u}_{1\varepsilon} + p_{2\varepsilon}\, ,\ v_1 - \overline{u}_{1\varepsilon} \right)_{L^2(\Sigma_0)} \geq 0. \end{cases} \tag{79}$$

3.4 Singular Optimality System

From the results of Section 3.2, we have, when $\varepsilon \to 0$,

$$\begin{cases} \overline{u}_{0\varepsilon} \longrightarrow u_0 & \text{strongly in } L^2(\Sigma_0)\,, \\ \\ \overline{u}_{1\varepsilon} \longrightarrow u_1 & \text{strongly in } L^2(\Sigma_0) \end{cases} \qquad \text{and} \qquad \begin{cases} \overline{y}_{1\varepsilon} \longrightarrow y & \text{strongly in } L^2(Q)\,, \\ \\ \overline{y}_{2\varepsilon} \longrightarrow y & \text{strongly in } L^2(Q)\,, \end{cases}$$

where (u, y) is the optimal control-state pair for (35).

Then, systems (76) and (77) being well-posed in the sense of Hadamard, it follows that there exist

$$p_1, p_2 \in H^{2,2}(Q), \tag{80}$$

with, when $\varepsilon \to 0$,

$$\begin{cases} p_{1\varepsilon} \longrightarrow p_1 & \text{strongly in } L^2(Q)\,, \\ \\ p_{2\varepsilon} \longrightarrow p_2 & \text{strongly in } L^2(Q)\,. \end{cases} \tag{81}$$

Thus, the singular optimality system, for the optimal control-state pair (u, y) of the initial control problem (35), is as specified by the following theorem, easily deriving, with the strong convergence results recalled above, from the previous Theorem 3.

Theorem 4. *The control-state pair* (u, y) *is unique solution of* (33)(34)(35) *if and only if the triple* $\{u, y, p\}$*, with*

$$p = (p_1, p_2) \in \left(H^{2,2}(Q)\right)^2$$

as given above by (80) *and* (81)*, is solution of the singular optimality system defined by the partial differential systems*

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \Delta y = 0 \quad \mathit{in}\ Q, \\ \\ y|_{t=0} = 0, \quad \left.\dfrac{\partial y}{\partial t}\right|_{t=0} = 0 \quad \mathit{in}\ \Omega, \\ \\ y = u_0, \quad \dfrac{\partial y}{\partial \nu} = u_1 \quad \mathit{on}\ \Sigma_0, \end{cases} \tag{82}$$

$$\begin{cases} \dfrac{\partial^2 p_1}{\partial t^2} - \Delta p_1 = \theta_1\,(y - z_d) \quad \mathit{in}\ Q, \\ \\ p_1|_{t=T} = 0, \quad \left.\dfrac{\partial p_1}{\partial t}\right|_{t=T} = 0 \quad \mathit{in}\ \Omega, \\ \\ p_1 = 0 \quad \mathit{on}\ \Sigma_0, \quad \dfrac{\partial p_1}{\partial \nu} = 0 \quad \mathit{on}\ \Sigma_1, \end{cases} \tag{83}$$

$$\begin{cases} \dfrac{\partial^2 p_2}{\partial t^2} - \Delta p_2 = \theta_2\,(y - z_d) \quad \mathit{in}\ Q \\ \\ p_2|_{t=T} = 0, \quad \left.\dfrac{\partial p_2}{\partial t}\right|_{t=T} = 0 \quad \mathit{in}\ \Omega, \\ \\ \dfrac{\partial p_2}{\partial \nu} = 0 \quad \mathit{on}\ \Sigma_0, \quad p_2 = 0 \quad \mathit{on}\ \Sigma_1, \end{cases} \tag{84}$$

and the variational inequalities system

$$
\begin{cases}
\forall\, v = (v_0, v_1) \in \mathcal{U}_{ad}, \\
\left(N_0 u_0 - \dfrac{\partial p_1}{\partial \nu}\,,\, v_0 - u_0\right)_{L^2(\Sigma_0)} \geq 0, \\
(N_1 u_1 + p_2\,,\, v_1 - u_1)_{L^2(\Sigma_0)} \geq 0.
\end{cases}
\tag{85}
$$

As we indicated earlier, the present analysis addresses well the question of the control of the hyperbolic Cauchy system without using any other assumption than the sufficient ones of non-vacuity, convexity and closure of the sets of admissible controls. The density results obtained by the interpretation made of the initial problem being enough to achieve the strong convergence of the process.

4. Conclusion

In this work, we succeed in characterizing the optimal control-state pair of the control problem for the hyperbolic ill-posed Cauchy problem, using the controllability concept. The method consists in interpreting the initial problem as a system of inverse problems and therefore a system of controllability problems. An approach that allows us to obtain, in the general case with constraints on the control, a strong and decoupled singular optimality system. And that, without using any additional assumption, such as that of non-vacuity of the interior of the sets of admissible controls, a Slater-type assumption that many analyses have had to use.

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