Journal of Mathematics Research; Vol. 15, No. 6; December 2023
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

A Memoir on Pseudo-Variational Techniques for Parabolic PDE's Incorporating Boundary Value Constraints

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Received: August 29, 2023 Accepted: November 18, 2023 Online Published: November 28, 2023

doi:10.5539/jmr.v15n6p1 URL: https://doi.org/10.5539/jmr.v15n6p1

Abstract

To enrich existing literature and extend the reach of relevant apt theoretical techniques, an alternative for proxy pseudo-variational analysis of a fundamental class of parabolic Partial Differential Equations (PDE's) is brought to the fore in this paper. Classical results in Sobolev Space theory are extrapolated with the aid of more current, standard tools from developments in fractional calculus. The high potency of inference from data obtained via the trace boundary operator is proffered, with the simple heat equation in one spatial dimension as a case study.

Keywords: fractional calculus of variations, fractional lagrangians, adjoint differential operators, boundary value problems, symmetry invariant solutions, parabolic equations

1. Introduction

The question as to which PDE's can be formulated from Variational Problems has far reaching implications in the general study of these equations. To mention a few of these implications, one of them is in rigorous establishment of well-posed status of such equations in Boundary Value Problems (BVP's). For a class of PDE's (mostly the self-adjoint elliptic type), the technique of formulating the likely associated Variational Problem, then incorporating the imposed boundary value constraint(s) and attempting to eventually retrieve the original given PDE, is a common standard method of establishing that a BVP is well-posed. As another implication, being able to formulate a PDE from a Variational Problem provides us with possible one-parameter variational symmetries admitted by the equation, followed consequently by detection of conservation laws (or analogs of them) at play (Halder, Paliathanasis & Leach, 2018; Frederico & Torres, 2012). Hence, besides the pure mathematical discussions, we also have significant physical scientific undertones and adaptations of them which arise as an implication of successfully establishing links between PDE's and Variational Problems. Classically, however, the attempt to formulate PDE's from Variational Problems limits the possibility to just equations of the self-adjoint type; that is - equations in which all differential operators involved are self-adjoint. This initially works well with the theory on PDE's established by Sobolev early in the 20th century. It is noteworthy that a wide class of important PDE's were initially formulated (as multivariate Euler-Lagrange equations) from Variational Problems via the Calculus of Variations.

Now, the current resurgence of symmetry techniques in resolution of differential equations does not only present us with a contraption to integrate them exactly, but also with a viable means for in-depth analysis of those equations admitting them. For instance, one can make reference to the wieldy and ubiquitous theorem of Noether on symmetries of the variational kind and their associated conservation laws. Obviously, we are denied this particular vantage point into analysis of PDE's as long as these equations cannot be linked to Variational Problems. As a very recent trend, developments in fractional symmetry techniques permit us to forge standard tools for modification of Noether's theorem, extending it to classical PDE's which cannot be formulated from Variational Problems in the *usual* sense (Frederico & Torres, 2012). Hence, we have the crux of the motivation behind propositions of this paper: it is geared towards consolidation of variational techniques for a wider class of PDE's, to extend the utility of aforementioned analytical tools. Although snippets of this overarching concept are widely celebrated as trending journal publications, the need to consolidate these tools into agreeable, methodical and concise bodies of work is not yet achieved.

Consider the normalized heat equation (or diffusion equation), which is the fundamental, prototypical parabolic PDE, also employed here as the case study for computational purposes:

$$u_t = u_{xx}. \tag{1}$$

The differential operator on the right hand side of (1), that is $\frac{\partial^2}{\partial x^2}$, is self-adjoint on the Hilbert spaces $H^k(\Omega)$, or any of the embedded $C^k(\overline{\Omega})$ spaces, because

$$\langle u_{xx}, v\rangle = \langle v_{xx}, u\rangle$$

on the functional subspaces characterized by zero trace. However, the differential operator on the left hand side of (1), that is $\frac{\partial}{\partial t}$, is not self-adjoint:

$$\langle u_t, v\rangle = -\langle v_t, u\rangle\ ,$$

so that the adjoint equation of (1) is distinct from it. Equations that are not self-adjoint pose an immediate obstruction to the standard variational formulation techniques required from Green's integral formula and the Lax-Milgram theorem. In the works of Ibragimov (2005) and Cresson, Greff & Inizan (2011), accurate options for circumventing this obstruction are presented, albeit with their own hindrances to seamless analytical investigation of (1). A common development in both of these cited approaches is the doubling of the number of dependent variables in either case.

Ibragimov's approach circumvents the said obstruction to variational formulation for the heat equation by implementing the dependent variables for both (1) and its adjoint equation in a common Lagrangian. Not to undermine the novelty of this particular approach, it does not quite elicit the desired perspective on symmetry analysis of the equation, to be brought to the fore here in due course. Cresson's more current approach incorporates fractional calculus techniques with a doubling of the phase space for the dependent variable (u_+, u_-), with u_+ and u_- respectively representing evolutions of the solution to (1) to the future and the past; both implemented as dependent variables in a common Lagrangian. It was noted, however, that evolutions to the past are unlikely to have any practical signficance (there is no solution to (1) in reverse time). Cresson's approach actually conjures a more complicated version of the proxy Lagrangian considered for the study of (1) in this paper. To be engaged in this paper, the relevant portion of Cresson's fractional Lagrangian for the convection-diffusion equation, for which (1) has significant links to its Euler-Lagrange equation, is given as:

$$\mathcal{L}_0 = \int_{x_0}^{x_1}\int_0^b \left[(u_x)^2 - (D_-^{0.5}u)^2\right]\, dtdx. \tag{2}$$

Further details on relevant established results are to ensue in the body of this paper.

As the synopsis for this paper, we first outline the foundations of parabolic PDE formulations from variational problems, making mention of the utility of boundary-value analysis from the formulation. Next, we discuss the standard requisite tools from fractional calculus and how their techniques significantly overlap with classical study of the heat equation in a standard BVP. Finally, we have key discussions about properties of the proxy fractional Lagrangian (2) for the heat equation, followed by the extent to which properties of equation (1) may be inherited or inferred from this Lagrangian, mainly with considerations of (pseudo-)variational symmetries.

2. Classical Variational Formulation of PDE's

Consider a Lagrangian functional on some Sobolev space $W^{k,p}(\Omega)$ given as

$$\mathcal{L}(u) = \int_\Omega F(\mathbf{x}, u, u^{(\alpha)}(x_i)_{i=1}^n)dV, \tag{3}$$

for some open and bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. Points $\mathbf{x}$ in Ω are given by coordinates $\mathbf{x} = (x_1, x_2, \cdots x_n)$. In the above Lagrangian, we shall only admit differentials $u^{(\alpha)}(x_i)$ for which $|\alpha| \leq 1$ (possibly also fractional). Elements $\overline{u}$ in the kernel of the Frechet differential of $\mathcal{L}$ comprise a linear subspace of $W^{k,p}(\Omega)$, such that for any $\overline{u} \in Ker[\mathcal{L}']$, we have

$$\langle \mathcal{L}'(\overline{u}), v\rangle = 0 \quad \forall v \in \left(W^{k,p}(\Omega)\right)^*.$$

Provided that the stated Lagrangian is sufficiently regular, then for all $u \in W^{k,p}(\Omega)$, $\mathcal{L}'(u)$ can be described as a bounded linear functional on the dual space $\left(W^{k,p}(\Omega)\right)^*$ of $W^{k,p}(\Omega)$.

By implementing differential calculus in the Banach Space $W^{k,p}(\Omega)$ from 'the first principles', we are able to deduce that each element $\overline{u} \in Ker[\mathcal{L}']$ satisfies certain Euler-Lagrange equations by generic formulation. It is a statutory measure in the analysis of PDE's to attempt to link these equations as optimizing Euler-Lagrange equations of appropriate Lagrangians, which is the celebrated variational procedure. With the aid of standard optimization theorems on Banach spaces, one may then argue existence and uniqueness of solution to the PDE as an Euler-Lagrange equation in this setting, upon imposition of appropriate boundary constraints. Imposition of boundary constraints on Euler-Lagrange equations of $\mathcal{L}$ can be regarded as restricting $Ker[\mathcal{L}']$ to corresponding sections. Although $W^{k,p}(\Omega)$ and $W^{k,p}(\overline{\Omega})$ may be the same linear space, it is noteworthy that $\int_{\Omega} F(\mathbf{x}, u, u^{(\alpha)}(x_i)_{i=1}^{n})dV$ and $\int_{\overline{\Omega}} F(\mathbf{x}, u, u^{(\alpha)}(x_i)_{i=1}^{n})dV$ (which, for ease of reference, we shall sometimes refer to as the 'total energy Lagrangian') usually differ by a non-trivial integral over the set $\partial\Omega$. This difference, known as the trace portion of $\mathcal{L}$, deserves a closer inspection into its analytical implications on solutions to the associated Euler-Lagrange equation, as will be mentioned again in due course of this paper, following computational observations. Summarily, we should elucidate the role of those trace integrals which do not alter the Euler-Lagrange equations after addition to the Lagrangian in (3) that is just over the interior of Ω .

The aforementioned classical optimization theorems have attached criteria, which we mention as regularity, convexity and properness (Opara, 2020) of the functional $\mathcal{L}$ in (3). Critical points of (3) detected in the presence of all three criteria above, when unique, are known to be extrema. Nevertheless, where (local) convexity is absent, we may still speak of saddle points of a Lagrangian as its critical points, which is precisely a case encountered in the functional (2). Let us hereby reckon with notations to be employed henceforth for the following two operators.

(1.) The trace operator,

$$\gamma : W^{k,p}(\Omega) \to L^p(\partial\Omega);\ u \mapsto \gamma(u) := u|_{\partial\Omega} .$$

(2.) The pullback operator $[\Omega^* : W^{1,p}(\partial\Omega) \to L^p(\Omega)]$, which implements transfer of the hypersurface form $\gamma(u)dS$ from $\partial\Omega$ to the interior of Ω via Stokes' Theorem for Pseudo-Riemannian manifolds, so that for any $f \in W^{1,p}(\partial\Omega)$;

$$\Omega^*(f) = F \implies \int_{\partial\Omega} f dS = \int_{\Omega} d(f dS) = \int_{\Omega} F dV.$$

(Because $\partial\Omega$ is bounded, then $f \in L^p(\partial\Omega)$ is also integrable over this domain.) The composition $[\Omega^* \circ \gamma]$ is *not* an identity map because the trace operator is not injective, and the compatibility of this composition can be strictly justified by specification of a suitable range for γ in an adaptable version of the Trace Theorem, to be stated shortly. However, in the event that $\overline{u} \in Ker[\mathcal{L}']$ is a critical point of (3), then we can conclude also that

$$\Omega^*(\gamma(\overline{u})) \in Ker[\mathcal{L}'],$$

as evidenced by standard variational procedures of proving well-posed status of BVP's.

As far as statutory establishment of existence and uniqueness of solution to (1) in a BVP, we make reference to J.L.Lions theorem (Brezis, 2011, pp.341), which for practical purposes, is the parabolic counterpart of the Lax-Milgram theorem. This procedure is instrumental in justifying further variational measures to be initiated in this paper. Hence, for the spatial interval $I = (x_0, x_1)$, we identify with the Hilbert spaces: $H = L^2(I),\ V = H^1(I)$ and the dense embeddings

$$V \subset H \subset V^* .$$

We also reckon with the symmetric, continuous bilinear form a on V given by

$$a(u, v) = \int_I \frac{du}{dx}\frac{dv}{dx} ,$$

which is clearly also characterized by

$$a(v, v) = ||v||_V^2 - ||v||_H^2$$

for any $u, v \in V$. As a simple application of J.L.Lions theorem, there exists a unique function $u \in L^2(0, b; V) \cap C([0, b]; H)$ such that

$$\left.\begin{array}{c} \frac{du}{dt} \in L^2(0, b; V^*), \\ \\ \langle \frac{du}{dt}, v\rangle + a(u, v) = \langle f(t), v\rangle \ \text{ for } a.e.\ t \in (0, b),\ \forall v \in V \\ \\ \text{and } \ u(0) = 0 \end{array}\right\} \tag{4}$$

The function $f(t)$ seen in the formulation (4) above is actually a summarization of joint boundary constraints on the spatial end-points x_0 and x_1 in the following manner. For any t fixed in $(0, b)$, we have via integration by parts:

$$a(u, v) = \int_I \frac{du(t)}{dx}\frac{dv}{dx} = \left[\frac{du(t)}{dx}v - \int v\frac{d^2u(t)}{dx^2}dx\right]_{x=x_0}^{x=x_1} .$$

To be compatible with (1), we must have that

$$\langle f(t), v\rangle = \left[\frac{du(t)}{dx}.v\right]_{x_0}^{x_1} .$$

The above weak formulation (4) is precisely the grounds we seek to build up further analysis of (1) here, except that it cannot be cast classically as an optimization problem for a Lagrangian because the operator $\frac{d}{dt}$ is not self-adjoint. However, existence of an implicit fractional Lagrangian associated to (1) is assumed in this article. At the crux of what ensues, the operator $\frac{d}{dt}$ is to be replaced with an alternative operator, exploring which properties of solutions to (1) may be preserved or inferred, and this exactly gives rise to the fractional Euler-Lagrange equation associated to (2). Before bringing up the established tools required here for fractional calculus, we shall give one more significant motivation for this venture by way of the Trace Theorem for the function space $[W^{1,2}([0,b]\times[x_0,x_1]) := H^1(\Omega)]$, which provides the weak solution to (1) guaranteed via J.L.Lions theorem.

Trace Theorem (Ding, 1996) Let $\Omega \subset \mathbb{R}^n$ be a special bounded Lipschitz domain and $\frac{1}{2} < s \leq 1$. Then the trace operator $\gamma|_{\partial\Omega} : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega)$ is linear and bounded.

Hence, we are involving the case $s = 1$ in this our study, giving the range fractional Sobolev space $H^{\frac{1}{2}}(\partial\Omega)$ focal attention. By the definition of this function space, $\frac{1}{2}-$ (fractional) derivatives of its elements have to be engaged to elicit nuances of its properties. Moreover, this particular space must be investigated in order to determine which properties of functions remain invariant under the operator $[\Omega^* \circ \gamma]$ that was identified previously. Observe that $L^2(\partial\Omega) \supseteq H^{\frac{1}{2}}(\partial\Omega) \supseteq H^1(\partial\Omega)$, thereby confirming compatibility of this particular operational composition acting on $H^1(\Omega)$. A noteworthy finding in the 'Results and Discussion' section shall provide further motivation to investigate this concept more keenly. Before then, we are hereby set to present crucial requisite tools from fractional calculus.

3. Analytical Tools of Fractional Calculus

The depth of our study of the evolution equation (1) requires a means of fractional derivation with respect to the time variable 't'. We thus refer to the standard Riemann-Liouville and Caputo fractional derivatives, as they relate to the Riemann-Liouville fractional Integral. The left Riemann-Liouville fractional integral of order α $(0 < \alpha < 1)$ of a function $u(t)$ with respect to t is given as:

$${}_aI_t^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}u(\tau)d\tau .$$

Γ denotes the special Gamma function, which is an extension of the usual factorial from the natural numbers to $\mathbb{R} - (\mathbb{Z}^- \cup \{0\})$, such that

$$\Gamma(n+1) = n!\ \ \forall n \in \mathbb{N}\cup\{0\}; \quad \Gamma(p+1) = p\Gamma(p)\ \ \forall p \in \mathbb{R} - (\mathbb{Z}^- \cup \{0\}) .$$

Hence, the left Riemann-Liouville fractional derivative of order α $(0 < \alpha < 1)$ of $u(t)$ with respect to t is given as:

$${}_aD_t^{\alpha}[u](t) = \frac{d}{dt}[{}_aI_t^{1-\alpha}u(t)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{u(\tau)}{(t-\tau)^{\alpha}}d\tau .$$

The operators ${}_aI_t^{1-\alpha}$ and $\frac{d}{dt}$ do not commute, and switching their order from the Riemann-Liouville definition to ${}_aI_t^{1-\alpha}\circ\frac{d}{dt}$ gives us the left Caputo fractional differential operator, denoted ${}_a^CD_t^{\alpha}$. We make the following remark pertaining to the above left derivatives:

$${}_a^CD_t^{\alpha}[u](t) = {}_aD_t^{\alpha}[u](t) - \frac{u(a)}{(t-a)^{\alpha}\Gamma(1-\alpha)} .$$

As for the right fractional differential operators, we have the right Riemann-Liouville derivative of order α :

$${}_tD_b^{\alpha}[u](t) = -\frac{d}{dt}[{}_tI_b^{1-\alpha}u(t)] = \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b \frac{u(\tau)}{(\tau-t)^{\alpha}}d\tau ,$$

and the right Caputo derivative of order α:

$${}_t^CD_b^{\alpha}[u](t) = -{}_tI_b^{1-\alpha}\left[\frac{du}{dt}\right] = -{}_tD_b^{\alpha}[u](t) + \frac{u(b)}{(b-t)^{\alpha}\Gamma(1-\alpha)} .$$

We henceforth impose the boundary value constraint $[u(0,x) = 0]$ in (1), so that the left Caputo and Riemann-Liouville derivatives coincide. We shall hence employ the more convenient notations $[D_-^{\alpha}u]$ and $[D_+^{\alpha}u]$ to stand for $[{}_0^CD_t^{\alpha}[u](t) =$

${}_0D_t^{\alpha}[u](t)]$ and $[{}_tD_b^{\alpha}[u](t)]$ respectively, as all fractional derivatives to be employed shall be of the Riemann-Liouville type. The Riemann-Liouville derivative is compatible with the operator $\frac{d}{dt}$, due to the relation:

$$(D_-^{0.5} \circ D_-^{0.5})[u] = \frac{d}{dt}[u] \ , \text{ for almost every } \ u.$$

A rare exception to the above relation is the function $u(t) = \sqrt{t}$, yielding zero on the left, but not on the right. Since $(D_-^{0.5} \circ D_-^{0.5})$ is not self-adjoint, we aim to explore properties of (1) that remain invariant under replacement of $(D_-^{0.5} \circ D_-^{0.5})$ with the self-adjoint operator $(D_+^{0.5} \circ D_-^{0.5})$. The equation

$$(D_+^{0.5} \circ D_-^{0.5})[u] = u_{xx} \tag{5}$$

is actually the Euler-Lagrange equation for the fractional Lagrangian (2), as we find upon implementation of standard necessary and sufficient fractional Euler-Lagrange criteria (Agrawal, 2001; Bourdin, Cresson, Greff & Inizan, 2016). It is well-known that fundamental solutions of interest to (1) are of class C^{∞} within the interior of $\Omega := (0, b) \times (x_0, x_1)$, with the boundary $[t = 0]$ possibly limiting an extension of infinite differentiability to the closure of this domain. Nevertheless, the class of fractionally differentiable functions is significantly larger than that of classically differentiable functions. In any event, for a useful perspective on the fractional differential operators in use, we shall employ their Maclaurin Series expansions, initially overlooking the possible lack of sufficient regularity on $[t = 0]$. For almost every analytic function $u(t)$, we have locally that

$$(D_-^{0.5} \circ D_-^{0.5})[u] = \sum_{n=0}^{\infty} u^{(n+1)}(0)\frac{t^n}{n!}$$

and

$$(D_+^{0.5} \circ D_-^{0.5})[u] = \sum_{n=0}^{\infty} u^{(n+1)}(0)\frac{b^n}{n!}\sum_{k=0}^{\infty}\begin{pmatrix} n \\ -\frac{1}{2}+k \end{pmatrix}(-1)^k.\left(1-\frac{t}{b}\right)^{-\frac{1}{2}+k} .$$

Interestingly, the above two series representations are not so dissimilar. In this vein, inspect the tweak to the stated expansion for $(D_+^{0.5} \circ D_-^{0.5})[u]$ (almost everywhere) as given below:

$$\sum_{n=0}^{\infty} u^{(n+1)}(0)\frac{b^n}{n!}\sum_{k=0}^{\infty}\begin{pmatrix} n \\ k \end{pmatrix}(-1)^k.\left(1-\frac{t}{b}\right)^{k}$$

$$= \sum_{n=0}^{\infty} u^{(n+1)}(0)\frac{b^n}{n!}\left(1-\left(1-\frac{t}{b}\right)\right)^n = (D_-^{0.5} \circ D_-^{0.5})[u].$$

This suggests a platform for formulation of functional analytic parities between both operators. We do not expect appearances of the above series expansions to differ so dramatically, due to moving the center of expansion into the interior of the interval $(0, b)$. Findings to be revealed in the succeeding section show certain substantial connections through this switch of operators from $(D_-^{0.5} \circ D_-^{0.5})$ to $(D_+^{0.5} \circ D_-^{0.5})$, despite the identified alteration of functional form as they act on functions u.

Before proceeding to the next section, let us inspect frameworks for establishment of strict and pseudo-variational symmetries from Lagrangians. By perturbing each of the variables of $\mathcal{L}(t, x, u, D^{\alpha}u, u_x)$ in (3) by a common (infinitesimal) parameter, we are able to deduce criteria for invariance of optimality of a function $(\overline{u})$ despite variation by this parameter at zero (Frederico & Torres, 2012; Zhang & Zheng, 2021). To strictly satisfy these criteria, we must have

$$pr^{(\alpha,1)}\mathbf{v}[F] + F.div(\tau, \xi) \equiv 0 \ , \tag{6}$$

where

$$\mathbf{v} = \tau\partial_t + \xi\partial_x + \eta\partial_u$$

is the infinitesimal generator of the aforementioned perturbation. We refer to (6) as the strict infinitesimal variational symmetry criterion, and symmetries admitted by any Lagrangian are also known to be admitted by its Euler-Lagrange equations. Considering modern advances in symmetry analysis, the condition (6) was deemed too strict to be used to limit utility of variational symmetries and consequently, $\lambda-$ and $\mu-$ pseudo-variational symmetries were coined to relax this condition (Muriel et al., 2005; Cicogna & Gaeta, 2007). The concept of a $\mu-$ symmetry is hereby discussed summarily.

Cicogna and Gaeta (2007) are heralded among prominent pioneers of the concept of $\mu-$symmetries: a handy tool yet not fully tapped in terms of formally established utility. Here, we shall simply extract some relevant applications of this concept in pseudo-variational symmetry analysis. Given a differential one-form $[\mu = \lambda_{(1,t)}dt + \lambda_{(1,x)}dx]$ on the tangent bundle TM of the $(t, x, u)-$solution space that satisfies the horizontal Maurer-Cartan equation:

$$D\mu + \frac{1}{2}[\mu, \mu] = 0,$$

the $\mu-$prolonged vector field of the section X of TM $(X = \tau\partial_t + \xi\partial_x + \eta\partial_u)$ acting in the Jet bundle $J^{(r)}M$ is denoted $X^{(r)}_{(\mu)}$, and defined by

$$X^{(r)}_{(\mu)} = X + \Psi^J \frac{\partial}{\partial u_J},$$

where the coefficients Ψ^J differ from coefficients η^J of the conventional prolongation

$$X^{(r)} = X + \eta^J \frac{\partial}{\partial u_J}$$

in the manner specified below.

$$\Psi^J = \eta^J + F_J,$$

where the difference terms F_J are characterized by the recursive relation

$$F_{J,I} = (D_I + \lambda_I)F_J + \lambda_I D_J Q.$$

Observe that $F_0 = 0$ and that $[Q := \eta - u_t\tau - u_x\xi]$ is the characteristic of the vector field X. Pertaining to the one-form μ, its coefficients $\lambda_{(1,t)}$ and $\lambda_{(1,x)}$ are C^∞ functions on $(J^{(1)}M)$, and if the Maurer-Cartan form μ is *locally* exact (that is, if $\mu = Df$ locally for some analytic function $f : M \to \mathbb{R}$), then a $\mu-$symmetry is gauge-equivalent to some other non-local symmetry of the admitting equation.

Vector fields considered in the 'Results and Discussion' section below are non-fractional, and the nature of their prolonged action on the Lagrangian integrand in (2) can permit us to circumvent most considerations of fractional vector calculus, in the cases to be discussed. Specifically, we shall discuss cases of infinitesimal generators X for which

$$pr^{(0.5,1)}X(F) = k.F, \quad \text{for constant } k.$$

Since $|\alpha| \leq 1$ for the orders α of all differential operators in F, then we would only be necessitated to execute low order iterations of the above recursive relation at most twice, from this approach. The infinitesimal variational $\mu-$symmetry criterion (Muriel et al., 2005) is given as

$$X^{(r)}_{(\mu)}(F) + F(div(\tau, \xi) + \mu[(\tau, \xi)]) = 0.^1 \tag{7}$$

Let X be a section of TM, which as a $\mu-$symmetry of (2) satisfies the criterion $pr^{(0.5,1)}X(F) = k.F$. We then have

$$X(F) = 0 \implies pr^{(0.5,1)}X(F) = \left[\eta^{0.5,t}\frac{\partial}{\partial u_t^{0.5}} + \eta^x\frac{\partial}{\partial u_x}\right](F) = k.F\ .$$

But also,

$$pr^{(0.5,1)}_{(\mu)}X(F) = \left[\Psi^{0.5,t}\frac{\partial}{\partial u_t^{0.5}} + \Psi^x\frac{\partial}{\partial u_x}\right](F) = -F(div(\tau, \xi) + \mu[(\tau, \xi)]).$$

Taking the difference between the above two lines, we have:

$$\left[\lambda_{(0.5,t)}.Q\frac{\partial}{\partial u_t^{0.5}} + \lambda_{(1,x)}.Q\frac{\partial}{\partial u_x}\right](F) = -F(div(\tau, \xi) + \mu[(\tau, \xi)]) - k.F.$$

For such cases, the coefficients $\lambda_{(0.5,t)}$ and $\lambda_{(1,x)}$ are relatively easily computed. To determine $\lambda_{(1,t)}$, we then apply the recursive relation for differences F_J :

$$F_{(1,t)} = F_{0.5,(0.5,t)} = (D_t^{0.5} + \lambda_{(0.5,t)})F_{(0.5,t)} + \lambda_{(0.5,t)}D_t^{0.5}Q.$$

The fractional operator $D_t^{0.5}$ consulted above is the left Riemann-Liouville half differential operator, which is chosen to be consistent with the first iteration of the recursive difference formula used for $\lambda_{(0.5,t)}$. Recall the requisite identified compatibility with the usual differential operator $\frac{d}{dt}$ (almost everywhere) for this operator: $(D_-^{0.5} \circ D_-^{0.5})[u] = \frac{du}{dt}$, *a.e. u.*

[1]Degeneracy of the pseudo-variational technique to that of $\lambda-$symmetries for these cases is deemed to occur, and the authors hereby adopted the infinitesimal criterion of Muriel et. al. (2005) instead of Cicogna & Gaeta (2007). Formal reconciliation of the criteria from both manuscripts is required to establish a credible platform for further developments pertaining to this approach.

4. Results and Discussion

The finite dimensional Lie Algebra for point symmetries admitted by (1) is spanned by the following six vector fields (Olver, 1993, pp. 118):

$$\mathbf{v}_1 = \partial_x,\ \mathbf{v}_2 = \partial_t,\ \mathbf{v}_3 = u\partial_u,\ \mathbf{v}_4 = x\partial_x + 2t\partial_t,\ \mathbf{v}_5 = 2t\partial_x - xu\partial_u,$$
$$\mathbf{v}_6 = 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u\ ;$$

and here we have the strict fractional infinitesimal variational symmetry criterion:

$$\left(\tau.\frac{\partial}{\partial t} + \xi.\frac{\partial}{\partial x} + \eta.\frac{\partial}{\partial u} + \eta^{\alpha,t}.\frac{\partial}{\partial u_t^{\alpha}} + \eta^{x}.\frac{\partial}{\partial u_x}\right) \circ F + F.div(\tau, \xi) \equiv 0\ .$$

The explicit expression of the fractional prolongation coefficient in the above vector field $[\eta^{\alpha,t}]$ relevant in our formulation for $[\alpha = 0.5]$ is hereby given as follows (Zhang & Zheng, 2021) :

$$\begin{aligned}\eta^{0.5,t} =\ & \partial_t^{0.5}\eta + (\eta_u - \tfrac{1}{2}D_t\tau)\partial_t^{0.5}u - u\partial_t^{0.5}(\eta_u) + \mu - \sum_{k=1}^{\infty}\begin{pmatrix} 0.5 \\ k \end{pmatrix} D_t^k\xi.\partial_t^{0.5-k}(u_x) \\ & + \sum_{k=1}^{\infty}\left[\begin{pmatrix} 0.5 \\ k \end{pmatrix}\partial_t^k(\eta_u) - \begin{pmatrix} 0.5 \\ k+1 \end{pmatrix} D_t^{k+1}\tau\right]\partial_t^{0.5-k}u\ ,\end{aligned}$$

where μ in the above expression is given as

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\begin{pmatrix} 0.5 \\ n \end{pmatrix}\begin{pmatrix} n \\ m \end{pmatrix}\begin{pmatrix} k \\ r \end{pmatrix}\frac{1}{k!}.\frac{t^{n-0.5}(-u)^r}{\Gamma(n+0.5)}.\frac{\partial^m(u^{k-r})}{\partial t^m}.\frac{\partial^{n-m}}{\partial t^{n-m}}\left(\frac{\partial^k\eta}{\partial u^k}\right)\ .$$

The explicit expression of the spatial prolongation coefficient $[\eta^x]$ is given as (Olver, 1993, pp. 114)

$$\eta^x = \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t.$$

Let the above vector field $\left(\tau.\frac{\partial}{\partial t} + \xi.\frac{\partial}{\partial x} + \eta.\frac{\partial}{\partial u} + \eta^{0.5,t}.\frac{\partial}{\partial u_t^{0.5}} + \eta^{x}.\frac{\partial}{\partial u_x}\right)$ be the fractional prolongation $[pr^{(0.5,1)}\mathbf{v}]$ of an infinitesimal symmetry $[\mathbf{v} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}]$ admitted by critical points of the proxy Lagrangian in (2). Apart from the translations ∂_t and ∂_x, none of the *individual* vector fields $\mathbf{v}_1$ to $\mathbf{v}_6$ satisfy this strict criterion. However, the scaling symmetries $\mathbf{v}_3$ and $\mathbf{v}_4$ can be shown to be admissible as $\mu-$symmetries of (2) with little difficulty, for specific Maurer-Cartan one-forms μ. To begin with, the Euler-Lagrange equation of (2), that is (5), admits $\mathbf{v}_3 = u\partial_u$ because this fractional P.D.E is linear. $\mathbf{v}_3$ satisfies the identified infinitesimal $\mu-$symmetry criterion (7) of (2) by way of the differential form $[\mu = -\frac{u_t}{u}dt - \frac{u_x}{u}dx]$. Moreover, the linear combination $[\mathbf{v}_4 - \frac{1}{2}\mathbf{v}_3]$ does satisfy the strict variational symmetry criterion for (2), meaning that it is admitted by (5) as well. We can thereby deduce that the infinitesimal generator

$$2\left(\mathbf{v}_4 - \frac{1}{2}\mathbf{v}_3\right) + \mathbf{v}_3 = 2\mathbf{v}_4$$

is also admitted by (5). This is not an indication that $\mathbf{v}_4$ is a $\mu-$variational symmetry for (2), but we find that $\mathbf{v}_4$ satisfies our identified infinitesimal $\mu-$symmetry criterion of (2) by way of the differential form

$$\mu = \frac{-xu_x}{4t(xu_x + 2tu_t)}dt + \frac{u_x}{2(xu_x + 2tu_t)}dx\ .$$

(Observe that $pr^{(0.5,1)}\mathbf{v}_3(F) = 2F$ and $pr^{(0.5,1)}\mathbf{v}_4(F) = -2F$.)

It is almost certain that more overlaps of symmetries between (1) and (5) besides the translations and scalings can be deduced using an assortment of techniques. A broader variety of shared symmetries between both equations would imply more invariants shared between them, and so a higher chance of common (or similar) group invariant solutions. Perhaps, the prime deterrent to explicit computations involved with symmetry analysis of (5) is intricacy of the fractional chain and Leibniz rules (Tarasov, 2015). Fortunately, the modern surge in fractional calculus techniques has been followed closely in chronological order by advances in computerized computational tools to alleviate these identified intricacies. Let us elucidate one more relevant and noteworthy result pertaining to trace theory analysis linking (1) and (5) before conclusive discussions.

Let $\Omega := (0, b) \times (x_0, x_1)$, $v \in H^1(\Omega)$, N the outward unit normal on $\partial\Omega$ and $h = u|_{\partial\Omega}$. The weak formulation for (5) is:

$$\langle (D_+^{0.5} \circ D_-^{0.5})[u], v\rangle = \langle u_{xx}, v\rangle$$

$$\Leftrightarrow \int_\Omega D_-^{0.5} u D_-^{0.5} v dt dx - \int_{\partial\Omega} D_-^{0.5} h. {}_tI_b^{0.5} v dx = \int_\Omega u_x v_x dt dx - \int_{\partial\Omega} \langle \nabla h, N\rangle v dS$$

$$\Leftrightarrow \int_\Omega (D_-^{0.5} u D_-^{0.5} v - u_x.v_x) dt dx = \int_{\partial\Omega} (D_-^{0.5} h. {}_tI_b^{0.5} v dx - \langle \nabla h, N\rangle v dS)\ . \tag{8}$$

Importantly, we have implemented both Green's classical formula (Brezis, 2011, pp.316) and Green's fractional formula (Odzijewicz, Malinowska & Torres, 2012) for bivariate integrals in the above result. A solution corresponding to weak formulation in (4) is

$$u(t, x) = k.erf\left(\frac{x}{2\sqrt{t}}\right) - k\ , \tag{9}$$

for a non-zero constant k. However, we must note that (9) is not a solution to the fractional equation (5). Observe in equation (8) that we have a continuous and symmetric bilinear form $[(u, v) \mapsto B(u, v)]$ on $H = H^1(\Omega)$ on the left, and a continuous linear form $[v \mapsto \varphi(v)]$ on the right. By differentiating in the Hilbert space of formulation, similar to an exact setting with the Lax-Milgram theorem (Brezis, 2011, pp.140), u is characterized by

$$\frac{1}{2}B(u, u) - \langle\varphi, u\rangle = \min_{v\in H}\left\{\frac{1}{2}B(v, v) - \langle\varphi, v\rangle\right\},$$

which gives the formulation for the 'total energy Lagrangian' incorporating boundary constraints. Instead of solving (5) explicitly, we shall instead incorporate the boundary constraints of classical formulation (4) to establish a noteworthy observation. That is to say, we shall use a mismatched boundary constraint obtained by restricting the solution (9) to $\partial\Omega$ in (8). Following this, the total energy Lagrangian via reference to Lax-Milgram theorem (replacing *min* with 'saddle point') is:

$$\frac{1}{2}\int_\Omega \left(v_x^2 - (D_-^{0.5} v)^2\right) dt dx - \int_{\partial\Omega}\left(\frac{\partial h}{\partial N} v d\sigma - D_-^{0.5} h. {}_tI_b^{0.5} v dx\right).$$

Let $\underline{h}$ be the restriction of (9) to $\partial\Omega$. With little difficulty, we compute the left Caputo (coinciding with the Riemann-Liouville due to our specitifications) half-derivative of $\underline{h}$, that is $D_-^{0.5}\underline{h}$, to be

$$[{}_0I_t^{0.5} \circ \partial_t]\underline{h} = \frac{kx}{2\sqrt{\pi}}\frac{1}{\Gamma(\frac{1}{2})}\int_0^t \frac{1}{\sqrt{\tau^3(t-\tau)}} exp\left(\frac{-x^2}{4\tau}\right) d\tau = \frac{k}{\sqrt{\pi^3 t}} exp\left(\frac{-x^2}{4t}\right).$$

Engaging Stokes' Theorem on the manifold $\overline{\Omega} = [0, b] \times [x_0, x_1]$, we re-evaluate the total energy Lagrangian used here to be

$$\int_\Omega \left[\frac{1}{2}\left(v_x^2 - (D_-^{0.5} v)^2\right) - d\left(\frac{\partial \underline{h}}{\partial N} v d\sigma - D_-^{0.5}\underline{h}. {}_tI_b^{0.5} v dx\right)\right] =$$
$$\frac{1}{2}\int_\Omega \left[v_x^2 - (D_-^{0.5} v)^2 + \Delta\left(\frac{-xv}{t} + 2v_x + \frac{3xv}{2t^2} - \frac{x^3 v}{4t^3} - \frac{x v_t}{t} + \frac{x^2 - 2t}{2\pi t^2} {}_tI_b^{0.5} v - \frac{2}{\pi} D_+^{0.5} v\right)\right],$$

where $\Delta = \dfrac{k}{\sqrt{\pi t}} exp\left(\dfrac{-x^2}{4t}\right)$, and we have used $\underline{h}$ instead of the actual functional restriction h of an appropriate solution of (5) to $\partial\Omega$. Upon implementation of the fractional Euler-Lagrange equation formulation (Agrawal, 2001) in the above Lagrangian, we interestingly recover the original associated fractional P.D.E (5). This is to say, for the proxy Lagrangian (2) of (1),

$$\Omega^*(\underline{h}) \in Ker(\mathcal{L}_0')$$

although $\underline{h}$ is obtained from (1), and not from a solution to the Euler-Lagrange equation (5) associated to (2). This particular result clearly should elicit investigation into key properties that remain invariant under the operator

$$\Omega^* \circ \gamma : H^1(\Omega) \to L^2(\Omega),$$

as was identified previously. As a crucial reminder, the bilinear form

$$(u, v) \ \mapsto \ \int_\Omega (D_-^{0.5} u D_-^{0.5} v - u_x.v_x) dt dx$$

in (8) is not coercive, meaning that we have a mere *ad hoc* adaptation of the Lax-Milgram theorem in this case, without guarantee of existence or uniqueness of a solution u to (5) corresponding to each sufficiently regular boundary specification h. The essential correspondence being put forth is that between h in (8) and $f(t)$ in the formulation (4) via J.L.Lions theorem. In a fairly detailed analysis by way of Trace Theory and Symmetry techniques, the results above therefore elucidate some snippets motivating adoption of (2) as a proxy Lagrangian for (1).

Now, we shall make a few conclusive remarks to round off discussions for this article. Despite the numerous techniques successfully adopted in analysis and solution of (1), it is hopefully made clear that fractional calculus techniques would further enrich the possible standard approaches available for this venture. This is in part, motivated by the substantial collection of heat-type equations that have been identified with deep theoretical and practical applications, such as Ricci Flow equations (Brezis, 2011, pp. 344) and Huxley & Fisher equations for gene propagation (Broadridge, Bradshaw, Fulford & Aldis, 2000), just to mention a few. It should also be brought up that proxy variational symmetry techniques are inadvertently unavoidable, even from the genesis of variational symmetry approaches. To back up this claim, consider the foundational Dirichlet problem, to minimize the Lagrangian

$$\mathcal{L}_1(u) = \int_\Omega \|u\|^2 dV$$

over specified constraints. Well-known fundamental solutions to this problem that are invariant under infinitesimal rotations have singularities; where the optimizing function $\overline{u}$ is not even classically differentiable (Opara, 2020). For this reason, strictly speaking, the actual Lagrangian associated to the Dirichlet problem ought to differ from $\mathcal{L}_1$ by a Radon (Dirac-Delta) measure. This was a major part of the reasoning behind formulation of Sobolev Spaces for analysis of P.D.E's in the first place. Differences in properties between (2) and the actual fractional Lagrangian associated to (1), which we implicitly assume to exist here, emanate from contrasts between the operators $(D_-^{0.5} \circ D_-^{0.5})$ and $(D_+^{0.5} \circ D_-^{0.5})$. Nevertheless, for now, the latter fractional differential operator linked to (5) is compatible with standard adopted variational tools and is being suggested for closer comparisons with the former operator, as is linked to (1). This would quickly contribute to proxy pseudo-variational analysis of substantive heat-type equations, for propounding of old results and development of new ones.

Acknowledgements

Not Applicable

Authors Contributions

Philip O. Mate[3] came up with the concept for this article, by comparing classical and fractional results in his personal study. *Festus I. Arunaye*[2] was instrumental in the Literature Review process, sifting out reference materials used by relevance and credibility. *Uchechukwu M. Opara*[1] was responsible for collating reference materials into the coherent text following the Literature Review, and for affirming computational results. The final draft of this manuscript was proof-read by all three authors prior to final submission.

Funding

This work was funded by the article authors.

Conflict of Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

Agrawal, O. P. (2001). Formulation of Euler-Lagrange Equations for Fractional Variational Problems. *Journal of Mathematical Analysis and Applications, 272*, 368 - 379.

Bourdin, L., Cresson, J., Greff, I., & Inizan, P. (2016). Variational Integrator for Fractional Euler-Lagrange Equations. *Applied Numerical Mathematics, Elsevier, 71*, 14-23.

Brezis, H. (2011). *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer Science + Business Media.

Broadridge, P., Bradshaw, B. H., Fulford, G. R., & Aldis, G. K. (2000). Huxley and Fisher Equations for Gene Propagation: An Exact Solution. *ANZIAM Journal, 44*, 11-20.

Cicogna, G., & Gaeta, G. (2007). *Noether Theorem for μ-symmetries*. Cornell University e-print Archive - arXiV:0708.3144v1 [math-ph].

Ibragimov, N. H. (2005). Integrating Factors, Adjoint Equations and Lagrangians. *Journal of Mathematical Analysis and Applications, 318*, 742-757.

Cresson, J., Greff, I., & Inizan, P. (2011). Lagrangian for the Convection-Diffusion Equation. *Mathematical Methods in the Applied Sciences, Wiley, 35*, 1885-1895.

Ding, Z. (1996). A Proof of the Trace Theorem of Sobolev Spaces on Lipschitz Domains. *Proceedings of the American Mathematical Society, 124*, 2.

Frederico, G., & Torres, D. (2012). *Fractional Noether's Theorem with Classical and Riemann-Liouville Derivatives*. Cornell University e-print Archive - arXiV:1209.1330v1 [math.OC].

Halder, A. K., Paliathanasis, A., & Leach, P. G. L. (2018). Noether's Theorem and Symmetry. *Symmetry, 10*, 744.

Muriel, C., Romero, J. L., & Olver, P. J. (2005). Variational C^{∞}-Symmetries and Euler-Lagrange Equations. *Journal of Differential Equations, 222*, 164 - 184.

Odzijewicz, T., Malinowska, A. B., & Torres, D. F. M. (2012). *Green's Theorem for Generalized Fractional Derivatives*. Cornell University e-print Archive - arXiV:1205.4851v2 [math.CA].

Olver, P. (1993). *Applications of Lie Groups to Differential Equations (Second Edition)*. Springer - Verlag New York, Inc.

Opara, U. (2020). Partial Differential Equation Formulations from Variational Problems. *American Journal of Applied Mathematics,* 1-8.

Tarasov, V. (2015). On Chain Rule for Fractional Derivatives. *Commun Nonlinear Sci Numer Simulat, Elsevier, 30*, 1-4.

Zhang, Z., & Zheng, J. (2021). *Symmetry Structure of Multi-Dimensional Time-Fractional Partial Differential Equations*. Cornell University e-print Archive - arXiV:1912.08602v2 [math.AP].