Volume and Surface of the Hypersphere

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Abstract
This paper presents a simply method to calculate the volume and the surface of a hypersphere, and is mainly addressed to students who study physics or engineering. The basic knowledges of geometry and mathematical analysis are sufficient to understand the material presented below.

Keywords: content, generalized surface, unit radius sphere, unit radius surface

1. Introduction
The content, or the generalized volume of a domain(body) in a Euclidean n-dimensional space, shows the amount of space occupied by this domain. For example, if the domain is a curve (one-dimensional), then the content is expressed by its length. The content of a surface (two-dimensional) is its area. For the three-dimensional subspaces the content is identical with the classical volume.

The generalized surface(frontier), or boundary of a body in n dimensional Euclidian space represents the (n-1)-dimensional domain, which separates the body from the rest of space.

Let us see as example the two-dimensional sphere (circle). See the figure 1, below:

![Figure 1. The two-dimensional sphere, of radius R=2 divisions](image)

The differential volume is a rectangle with the length of 2Rcosβ, and width of dx₂ (the differential of the variable x₂).

\[
dx_2 = d(R\sin\beta) = R\cos\beta d\beta \\
dV_2(R) = 2R^2 \cos^2 \beta \ d\beta \\
V_2(R) = 2R^2 \int_0^{\pi/2} \cos^2 \beta \ d\beta = 4R^2 \int_0^{\pi/2} \cos^2 \beta \ d\beta 
\]
In general, we may write: \( I_n = \int_0^\pi \cos^n \beta d\beta \)

\[
I_2 = \int_0^\pi \cos^2 \beta \; d\beta = \int_0^\pi \frac{\cos 2\beta + 1}{2} \; d\beta = \frac{\pi}{4} 
\]

(1.3)

The expression \( V_2 \), which represents the “volume” for a unit radius, is a numerical constant. We may write:

\[
V_2(R) = \pi R^2 , \text{ or } V_2 = \pi .
\]

(1.4)

The \( S_2 \) is the perimeter of the circle which separates the kernel of the “two-dimensional sphere” from the neighboring space.

\[
dS_2(R) = R d\beta
\]

(1.6)

Integrating it obtains:

\[
S_2(R) = R \int_0^{2\pi} d\beta = 2\pi R = S_2 R
\]

(1.7)

The expression \( S_2 \) represents the “surface” for a “sphere” with unit radius.

2. The Volume of a N-Dimensional Sphere

The coordinates of a point \( P \) in an \( n \)-dimensional cartesian system can be written as:

\[
x_i = R \cos \alpha_i \cos \beta, \text{ for } i=1,2,…….n-1
\]

\[
\sum_{i=0}^{n-1} \cos^2 \alpha_i = 1
\]

\[
x_n = R \sin \beta
\]

(2.1)

It is obvious that all points with the same radius \( R \) build a \( n \)-dimensional sphere in the corresponding space. Its content takes, generally speaking, the following form:

\[
V_n(R) = V_n R^n, \text{ where } V_n \text{ is a mathematical constant.}
\]

There is a network of \( (n-1) \)-dimensional spheres each having a radius of \( R \cos \beta \), which belongs to this \( n \)-dimensional sphere. It is like this \( n \)-dimensional sphere is built of slices with “bases” consisting of such \( (n-1) \)-dimensional spheres.

The volume element of this hyper-sphere is:

\[
dV_n(R) = V_{n-1} (R \cos \beta) d\beta = V_{n-1} R^n \cos^n \beta \; d\beta
\]

(2.2)

Integrating (2.2), it obtains the expression of the volume:

\[
V_n(R) = 2R^n V_{n-1} \int_0^\pi \cos^n \beta \; d\beta = 2R^n V_{n-1} I_n
\]

(2.3)

Integrating In by parts we obtain:

\[
\int_0^\pi \cos^n \beta \; d\beta = \frac{n-1}{n} \int_0^\pi \cos^{n-2} \beta \; d\beta \; , \text{ or } I_n = \frac{n-1}{n} I_{n-2}
\]

(2.4)

Generally speaking, we have:

\[
V_n(R) = V_n R^n , \text{ and according to (2.3), } V_n = 2V_{n-1} I_n
\]

(2.5)

For an even \( n \) we obtain

\[
I_{2j} = \frac{1.3.5.\ldots.(2j-1)}{2.4.6.\ldots.2j} I_0 = \frac{1.3.5.\ldots.(2j-1)\pi}{2.4.6.\ldots.2j} = \frac{1.3.5.\ldots.(2j-1)\pi}{2.4.6.\ldots.2j} = \frac{1.3.5.\ldots.(2j-1)\pi}{2.4.6.\ldots.2j}
\]

(2.6)

For an odd \( n \) we get:

\[
I_{2j-1} = \frac{2.4.6.\ldots.(2j-2)}{1.3.5.\ldots.(2j-1)} I_1 = \frac{2.4.6.\ldots.(2j-2)}{1.3.5.\ldots.(2j-1)} I_1
\]

(2.7)

Let us develop expression (2.5) for \( n = 2k \).

\[
V_{2k} = 2V_{2k-1} I_{2k} = 2^2 V_{2k-2} I_{2k-1} I_{2k} = \cdots = 2^k I_2 (I_3 I_4) (I_{2k-1} I_{2k})
\]

(2.8)

As we see there are \( k-1 \) products \( (I_{2j-1} I_{2j}) \). for \( j=2 \) trough \( k \). Using (2.6) and (2.7) we get:
Considering that $I_2 = \frac{\pi}{4}$ we obtain finally:

$$V_{2k} = \frac{n^k}{k!}$$ (2.9)

Now let us do the same for $n=2k+1$

$$V_{2k+1}(1) = 2^{2k+1}(I_2 I_3) (I_4 I_5) \ldots (I_{2k} I_{2k+1})$$ (2.10)

There are $k$ products $(I_{2j} I_{2j+1})$, for $j=1$ through $k$, and:

$$I_{2j} I_{2j+1} = \frac{1}{2j+1} \frac{\pi}{2}$$ (2.11)

$$V_{2k+1}(1) = 2^{2k+1} \frac{n^k}{2^k \prod_{j=1}^{k} (2j+1)} = 2^{k+1} \frac{n^k}{(2k+1)!!}$$ (2.12)

Where $n!!$, denotes the double factorial, i.e., $11!! = 11.9.7.5.3.1$.

In my paper published in 2015, http://article.sapub.org/10.5923.j.ijtmp.20150502.03.html, can be found the deduction of the volume of the four-dimensional sphere, using the hypercomplex number representation (see paragraph 2.11, equation 2.57).

3. The Surface of the N-Dimensional Hypersphere

As was mentioned in the introduction, a hyper-sphere is separated from the complementary n-dimensional space by a boundary, which is called “surface” by extension of the familiar surface of a three-dimensional body.

The corresponding surface of an n-dimensional sphere is an $n-1$ domain with a content proportional with $R^{n-1}$. Using the same method as for the volume element we get:

$$dS_n(R) = S_{n-1}(R \cos \beta)Rd\beta = S_{n-1}(R)R \cos^{n-2} \beta \ d\beta$$ (3.1)

Integrating (3.1) it obtains:

$$S_n(R) = 2R S_{n-1}(R) I_{n-2}$$ (3.2)

Using the formula (2.4) it obtains:

$$S_n(R) = 2R S_{n-1}(R) I_{n-2}$$ (3.3)

If $R$ is the unit radius, then we may write:

$$S_n = 2S_{n-1} I_{n-1}$$ (3.4)

Further it will be used the same procedure as by calculation of the volume of the hypersphere. Finally, we obtain the following formula:

$$S_n = nV_n$$ or,

$$S_n(R) = \frac{dV_n(R)}{dR}$$ (3.5)

The next table shows the volume and the surface of a hyper-sphere for different $n$.

Table 1. The volume and the surface of a n-dimensional sphere

<table>
<thead>
<tr>
<th>Dimension</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume</td>
<td>$2R$</td>
<td>$\pi R^2$</td>
<td>$\frac{4}{3} \pi R^3$</td>
<td>$\frac{1}{2} \pi^2 R^4$</td>
<td>$\frac{8}{15} \pi^2 R^5$</td>
<td>$\frac{1}{6} \pi^3 R^6$</td>
</tr>
<tr>
<td>Surface</td>
<td>2</td>
<td>$2\pi R$</td>
<td>$4\pi R^2$</td>
<td>$2\pi^2 R^3$</td>
<td>$\frac{8}{3} \pi^2 R^4$</td>
<td>$\pi^3 R^5$</td>
</tr>
</tbody>
</table>
References

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