Journal of Mathematics Research; Vol. 15, No. 5; October 2023
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Volume and Surface of the Hypersphere

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Received: June 26, 2023 Accepted: October 4, 2023 Online Published: October 17, 2023

doi:10.5539/jmr.v15n5p69 URL: https://doi.org/10.5539/jmr.v15n5p69

Abstract

This paper presents a simply method to calculate the volume and the surface of a hypersphere, and is mainly addressed to students who study physics or engineering. The basic knowledges of geometry and mathematical analysis are sufficient to understand the material presented below.

Keywords: content, generalized surface, unit radius sphere, unit radius surface

1. Introduction

The content, or the generalized volume of a domain(body) in a Euclidean n-dimensional space, shows the amount of space occupied by this domain. For example, if the domain is a curve (one-dimensional), then the content is expressed by its length. The content of a surface (two-dimensional) is its area. For the three-dimensional subspaces the content is identical with the classical volume.

The generalized surface(frontier), or boundary of a body in n dimensional Euclidian space represents the (n-1)-dimensional domain, which separates the body from the rest of space.

Let us see as example the two-dimensional sphere (circle). See the figure 1, below:

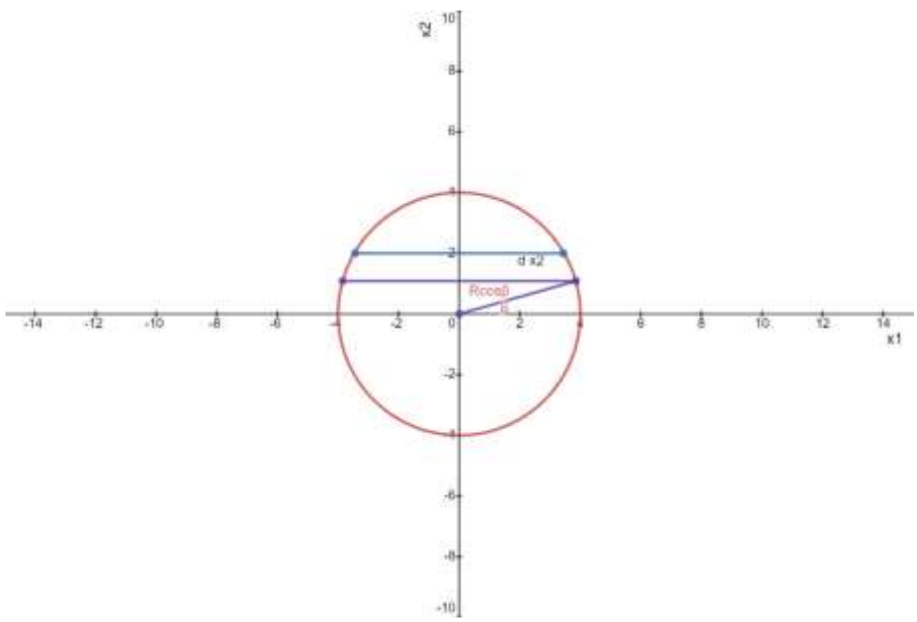

Figure 1. The two-dimensional sphere, of radius R=2 divisions

The differential volume is a rectangle with the length of 2Rcosβ, and width of dx_2 (the differential of the variable x_2).

$$dx_2 = d(R\sin\beta) = R\cos\beta d\beta$$

$$dV_2(R) = 2R^2\cos^2\beta\, d\beta \tag{1.1}$$

$$V_2(R) = 2R^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\beta d\beta = 4R^2\int_0^{\frac{\pi}{2}}\cos^2\beta d\beta \tag{1.2}$$

In general, we may write: $I_n = \int_0^{\frac{\pi}{2}} cos\pi^n \beta d\beta$

$$I_2 = \int_0^{\frac{\pi}{2}} cos^2\beta \, d\beta = \int_0^{\frac{\pi}{2}} \frac{cos2\beta+1}{2} d\beta = \frac{\pi}{4} \tag{1.3}$$

$$V_2(R) = \pi R^2 \text{, or } V_2 = \pi. \tag{1.4}$$

The expression V_2, which represents the “volume” for a unit radius, is a numerical constant. We may write:

$$V_2(R) = V_2 R^2 \tag{1.5}$$

The S_2 is the perimeter of the circle which separates the kernel of the “two-dimensional sphere” from the neighboring space.

$$dS_2(R) = \; Rd\beta \tag{1.6}$$

Integrating it obtains:

$$S_2(R) = R\int_0^{2\pi} d\beta = 2\pi R = S_2 R \tag{1.7}$$

The expression S_2 represents the “surface” for a “sphere” with unit radius.

2. The Volume of a N-Dimensional Sphere

The coordinates of a point P in an n-dimensional cartesian system can be written as:

$$x_i = R\cos\alpha_i \cos\beta \quad \text{, for i= 1, 2,…….n-1}$$
$$\sum_{i=0}^{n-1} cos^2\alpha_i = 1 \tag{2.1}$$
$$x_n = R\sin\beta$$

It is obvious that all points with the same radius(R) build a n-dimensional sphere in the corresponding space. Its content takes, generally speaking, the following form:

$V_n(R) = V_n R^n$, where V_n is a mathematical constant.

There is a network of (n-1)-dimensional spheres each having a radius of Rcosβ, which belongs to this n-dimensional sphere. It is like this n-dimensional sphere is built of slices with “bases” consisting of such (n-1)-dimensional spheres.

The volume element of this hyper-sphere is:

$$dV_n(R) = V_{n-1}(R\cos\beta)dx_n = V_{n-1} R^n cos^n \beta \, d\beta \tag{2.2}$$

Integrating (2.2), it obtains the expression of the volume:

$$V_n(R) = 2R^n V_{n-1} \int_0^{\frac{\pi}{2}} cos^n \beta \, d\beta = 2R^n V_{n-1} I_n \tag{2.3}$$

Integrating In by parts we obtain:

$$\int_0^{\frac{\pi}{2}} cos^n \beta \, d\beta = \frac{n-1}{n} \int_0^{\frac{\pi}{2}} cos^{n-2} \beta \, d\beta \qquad \text{, or } I_n = \frac{n-1}{n} I_{n-2} \quad . \tag{2.4}$$

Generally speaking, we have:

$$V_n(R) = V_n R^n \quad , \quad \text{and according to (2.3),} \quad V_n = 2V_{n-1} I_n \tag{2.5}$$

For an even n we obtain

$$I_{2j} = \frac{1.3.5....(2j-1)}{2.4.6.........2j} I_0 = \frac{1.3.5.......(2j-1)}{2.4.6.........2j} \frac{\pi}{2} \tag{2.6}$$

For an odd n we get:

$$I_{2j-1} = \frac{2.4.6........(2j-2)}{1.3.5.....(2j-1)} I_1 = \frac{2.4.6........(2j-2)}{1.3.5.....(2j-1)} \tag{2.7}$$

Let us develop expression (2.5) for $n = 2k$,

$$V_{2k} = 2V_{2k-1} I_{2k} = 2^2 V_{2k-2} I_{2k-1} I_{2k} = \cdots = 2^{2k} I_2 (I_3 I_4) \ldots . . (I_{2k-1} I_{2k}) \tag{2.8}$$

As we see there are k-1 products $(I_{2j-1} I_{2j})$, for j=2 trough k. Using (2.6) and (2.7) we get:

$I_{2j-1}I_{2j} = \frac{\pi}{2}\frac{1}{2j}$ Considering that $I_2 = \frac{\pi}{4}$ we obtain finnaly:

$$V_{2k} = \frac{\pi^k}{k!} \tag{2.9}$$

Now let us do the same for n=2k+1

$$V_{2k+1}(1) = 2^{2k+1}(I_2 I_3)(I_4 I_5)\ldots\ldots(I_{2k}I_{2k+1}) \tag{2.10}$$

There are k products $(I_{2j}I_{2j+1})$, for j=1 through k, and:

$$I_{2j}I_{2j+1} = \frac{1}{2j+1}\frac{\pi}{2} \tag{2.11}$$

$$V_{2k+1}(1) = 2^{2k+1}\frac{\pi^k}{2^k}\frac{1}{\prod_{j=1}^{k}(2j+1)} = 2^{k+1}\frac{\pi^k}{(2k+1)!!} \tag{2.12}$$

Where n!!, denotes the double factorial, i.e., 11!!= 11.9.7.5.3.1.

In my paper published in 2015, http://article.sapub.org/10.5923.j.ijtmp.20150502.03.html , can be found the deduction of the volume of the four-dimensional sphere, using the hypercomplex number representation (see paragraph 2.11, equation 2.57).

3. The Surface of the N-Dimensional Hypersphere

As was mentioned in the introduction, a hyper-sphere is separated from the complementary n-dimensional space by a boundary, which is called "surface" by extension of the familiar surface of a three-dimensional body.

The corresponding surface of an n-dimensional sphere is an n-1 domain with a content proportional withR^{n-1}. Using the same method as for the volume element we get

$$dS_n(R) = S_{n-1}(R\cos\beta)Rd\beta = S_{n-1}(R)R\cos^{n-2}\beta\, d\beta \tag{3.1}$$

Integrating (3.1) it obtains:

$$S_n(R) = 2RS_{n-1}(R)I_{n-2} \tag{3.2}$$

Using the formula (2.4) it obtains:

$$S_n(R) = 2RS_{n-1}(R)I_n\frac{n}{n-1} \tag{3.3}$$

If R is the unit radius, then we may write:

$$S_n = 2S_{n-1}I_n\frac{n}{n-1} \tag{3.4}$$

Further it will be used the same procedure as by calculation of the volume of the hypersphere. Finally, we obtain the following formula:

$$S_n = nV_n \quad \text{or,}$$

$$S_n(R) = \frac{dV_n(R)}{dR} \tag{3.5}$$

The next table shows the volume and the surface of a hyper-sphere for different n.

Table 1. The volume and the surface of a n-dimensional sphere

Dimension	1	2	3	4	5	6
Volume	$2R$	πR^2	$\frac{4}{3}\pi R^3$	$\frac{1}{2}\pi^2 R^4$	$\frac{8}{15}\pi^2 R^5$	$\frac{1}{6}\pi^3 R^6$
Surface	2	$2\pi R$	$4\pi R^2$	$2\pi^2 R^3$	$\frac{8}{3}\pi^2 R^4$	$\pi^3 R^5$

References

Collins, G. P. (2004). The shapes of space. *Scientific American*, *291*(1), 94-103. https://doi.org/10.1038/scientificamerican0704-94

Conway, J. H., & Sloane, N. J. A. (2013). *Sphere packings, lattices and groups* (Vol. 290). Springer Science & Business Media.

Mayrhofer, K. (1952). Inhalt und Mass (Content and measure), Springer-Verlag, MR 0053185. https://doi.org/10.1007/978-3-7091-7806-5

Vlad, L. N. (2015). Hyper-Complex Numbers in Physics. *International Journal of Theoretical and Mathematical Physics, 5*(2), 28-43. http://article.sapub.org/10.5923.j.ijtmp.20150502.03.html

Weisstein, E. W. (2014). Hypersphere. From mathworld–A Wolfram web resource. *URL http://mathworld. wolfram. com/Hypersphere. html*.

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