Journal of Mathematics Research; Vol. 15, No. 5; October 2023
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Note on the Classification of the Unitary $SL_2(\mathbb{R})$ Representations

Amjad Alghamdi

Correspondence: Department of Mathematics, Al-Jumum University College, Umm Al-Qura University, Saudi Arabia. E-mail: asmghamdi@uqu.edu.sa

Received: August 8, 2023 Accepted: October 4, 2023 Online Published: October 16, 2023

doi:10.5539/jmr.v15n5p22 URL:https://doi.org/10.5539/jmr.v15n5p22

Abstract

The aim of this paper is to explain the classification of the unitary $SL_2(\mathbb{R})$ representations done by Gelfand [8] by using the induced representation technique. We induce the $SL_2(\mathbb{R})$ representation from the subgroup N. We get a representation constructed on a space of homogeneous functions in two variables. Then, we move to induce the $SL_2(\mathbb{R})$ representation in stages. Consequently, the representation of $SL_2(\mathbb{R})$ acts on a space of functions of one variable.

Keywords: $SL_2(\mathbb{R})$ group, representation, irreducible, unitary, induced representation, invariant

1. Introduction

The special linear group $SL_2(\mathbb{R})$ is the group of 2×2 matrices with real entries and a determinant equal to one. It is an interesting and important example of a locally compact real Lie group of three dimension. In 1947, Bargmann classified the irreducible unitary representation of $SL_2(\mathbb{R})$ [2]. His approach has been presented in different sources [25, 20, 15]. The main tool of Bargmann's classification is to work on the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. Gelfand studied the $SL_2(\mathbb{R})$ representations on the Lie group instead of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ [8, VII]. In this paper, we use the induced representation technique (in the sense of Mackey [21]) and the Gelfand method [8, VII] to review the classification of the irreducible unitary representations on the Lie group $SL_2(\mathbb{R})$.

The affine group is a subgroup of $SL_2(\mathbb{R})$, and it is often used to build wavelets. To study the induced representation of the group $SL_2(\mathbb{R})$, I start by considering the unitary representations of the affine group, which are due to Gelfand and Naimark [9] .

2. Preliminaries

In this section, we present some basic notions of representation theory that are needed for our study.

2.1 Representations of Groups

Definition 2.1. [3] Let G be a group with identity element e_G , and let V be a vector space. A representation π of G in V is a homomorphism of G into $GL(V)$ (the group of invertible, linear mappings that carry V to itself), that is

$$\pi : G \to GL(V), \quad g \mapsto \pi(g).$$

The representation operator $\pi(g) : V \to V, \quad g \in G$ satisfies the following properties:

$$\pi(g_1 g_2) = \pi(g_1)\pi(g_2), \quad \pi(e_G) = \mathbb{I}.$$

The repreaentation π is called linear if V is a linear space and the mappings $\pi(g)$ are linear operators. The space V is called the representation space of π.

Let π be a representation of a Lie group G on a Hilbert space $\mathcal{H}$. A strong continuity of π means that for any vector $u \in \mathcal{H}$ and for any convergent sequence $(g_j) \to g \in G$, we have [25, p.9]

$$\|\pi(g_j)u - \pi(g)u\| \to 0.$$

Definition 2.2. [25] A representation π of a Lie group G on a Hilbert space $\mathcal{H}$ is called a unitary representation if the operator $\pi(g)$ is unitary, that is

$$\pi(g)^* = \pi(g)^{-1} = \pi(g^{-1}), \quad g \in G.$$

There is a natural equivalence relation on the set of all representations of a group, which is defined by an intertwining property.

Definition 2.3. [5] Let π_1 and π_2 be unitary representations of a Lie group G in spaces $\mathcal{H}_{\pi_1}$ and $\mathcal{H}_{\pi_2}$, respectively. An operator $U : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ is called an intertwining operator between π_1 and π_2 if for every $g \in G$, we have

$$U\pi_1(g) = \pi_2(g)U.$$

The set of all intertwining operators is denoted by $C(\pi_1, \pi_2)$. The representations π_1 and π_2 are unitarily equivalent if $C(\pi_1, \pi_2)$ contains a unitary operator U so that $\pi_1(g) = U\pi_2(g)U^{-1}$. We shall write $C(\pi)$ for $C(\pi, \pi)$, which is the space of the bounded operators on $\mathcal{H}_\pi$ that commute with $\pi(g)$.

Definition 2.4. [20] Let π be a representation of a Lie group G on the vector space V. Define the subspace V^∞ to consist of functions $f \in V$ such that the map $g \mapsto \pi(g)f$ is infinitely differentiable for any $g \in G$. Then, the derived representation generated by an element X of the corresponding Lie algebra $\mathfrak{g}$ is the representation $d\pi(X)$ of $\mathfrak{g}$ given as follows:

$$d\pi(X)f := \frac{d}{dt}\pi(\exp tX)f\bigg|_{t=0}, \quad \text{where} \quad f \in V^\infty. \tag{2.1}$$

2.2 Decomposition of Representations

One of the main problems of the theory of representations is the problem of decomposing representations of a group G into the simplest possible components. In the following, we will provide some relevant notation.

Definition 2.5. [17] Let π be a linear representation of a Lie group G in a Hibert space $\mathcal{H}$. A linear subspace $L \subset \mathcal{H}$ is an invariant subspace for π if for any $x \in L$ and $g \in G$ the vector $\pi(g)x$ again belongs to L.

There are two trivial invariant subspaces, the null subspace and the entire space. All other invariant subspaces are non-trivial. Let π be a representation of a Lie group G on a Hilbert space $\mathcal{H}$. If there are only two trivial invariant subspaces, then π is an irreducible representation. Otherwise, we have a reducible representation.

Definition 2.6. [17] A representation on H is called decomposable if there are two non-trivial invariant subspaces H_1 and H_2 of H such that $H = H_1 \oplus H_2$.

Any unitary representation is either irreducible or decomposable. The irreducibility of representation is often established by Schur's lemma.

Lemma 2.7. (Schur's lemma)[5] Let G be a group and $C(\pi)$ be the set of all intertwining operators. Then

- A unitary representation π of G is irreducible if and only if $C(\pi)$ contains only scalar multiples of the identity.
- Suppose π_1 and π_2 are irreducible unitary representations of G. If π_1 and π_2 are equivalent,then $C(\pi_1, \pi_2)$; is one-dimensional otherwise, $C(\pi_1, \pi_2) = 0$.

Definition 2.8. [3] A character χ of an Abelian locally compact group G is a continuous function $\chi : G \to \mathbb{C}$, which satisfies

$$|\chi(g)| = 1, \quad \chi(g_1 g_2) = \chi(g_1)\chi(g_2),$$

and for all $g_1, g_2 \in G$. That is, a character χ is a one-dimensional continuous irreducible unitary representation of G.

2.3 Induced Representations

In this section, we describe the construction of induced representations [5, 16, 17]. Let G be a group H be a closed subgroup of G; then $X = G/H$ is the left coset space. For a character $\chi : H \to \mathbb{T}$, where $\chi(h_1h_2) = \chi(h_1)\chi(h_2)$ and $|\chi(h)| = 1$, let V_χ be the vector space of functions $F : G \to \mathbb{C}$ having the property:

$$F(gh) = \overline{\chi(h)}F(g), \quad \forall g \in G, h \in H. \tag{2.2}$$

The space V_χ is invariant under the left action of G, that is

$$\Lambda(g) : V_\chi \to V_\chi, \quad [\Lambda(g)F](g') = F(g^{-1}g'), \quad g, g' \in G. \tag{2.3}$$

The restriction of the left action of G on the space V_χ is called the induced representation.

An equivalent realisation of the above induced representation can be defined on the homogeneous space $X = G/H$. Let $\mathsf{s} : X \to G$, be a section map that is a right inverse of the natural projection map $\mathsf{p} : G \to X$, that is

$$\mathsf{p} \circ \mathsf{s} = \mathbb{I}_X.$$

Then the left action of G on the homogeneous space X is given by:

$$g \cdot x = \mathsf{p}(g\mathsf{s}(x)),$$

where $g \in G$ and $x \in X$. Any element $g \in G$ can be uniquely decomposed as $g = \mathsf{s}(\mathsf{p}(g))\mathsf{r}(g)$ where the map $\mathsf{r} : G \to H$ is given by $\mathsf{r}(g) = \mathsf{s}(\mathsf{p}(g))^{-1}g$.

Now, for a character χ of the subgroup H, introduce the lifting map $\mathcal{L}_\chi : W(X) \to V_\chi$, as follows:

$$[\mathcal{L}_\chi f](g) = \overline{\chi(\mathsf{r}(g))} f(\mathsf{p}(g)), \quad f \in W(X),$$

where $W(X) := \{f : X \to \mathbb{C}\}$ is the vector space of all complex functions on the homogeneous space $X = G/H$. Let the pulling map $\mathcal{P} : V_\chi \to W(X)$, given by:

$$[\mathcal{P}F](x) = F(\mathsf{s}(x)),$$

such that $\mathcal{P} \circ \mathcal{L}_\chi = \mathbb{I}_{W(X)}$. and $\mathcal{L}_\chi \circ \mathcal{P} = \mathbb{I}_{V_\chi}$.

Next, the operator $\pi_\chi(g)$ on $W(X)$ is given as follows:

$$\pi_\chi(g) := \mathcal{P} \circ \Lambda(g) \circ \mathcal{L}_\chi. \tag{2.4}$$

This can be represented by the following commutative diagram:

$$\begin{array}{ccc} V_\chi & \xrightarrow{\Lambda(g)} & V_\chi \\ {\scriptstyle \mathcal{L}_\chi}\uparrow & & \downarrow{\scriptstyle \mathcal{P}} \\ W(X) & \xrightarrow{\rho_\chi(g)} & W(X) \end{array}$$

Figure 1: Induced representation from a character of a subgroup

Thus, the representation π_χ acts on $W(X)$ via the following explicit formula:

$$[\pi_\chi(g) f](x) = \bar{\chi}(\mathsf{r}(g^{-1} * \mathsf{s}(x))) f(g^{-1} \cdot x). \tag{2.5}$$

3. The Group $\mathrm{SL}_2(\mathbb{R})$

The Lie group $\mathrm{SL}_2(\mathbb{R})$ consists of 2×2 matrices with real entries and a determinant equal to one

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, \ a, b, c, d \in \mathbb{R} \right\}.$$

It acts on the upper half-plane by Möbius transformation

$$g \cdot z = \frac{az + b}{cz + d},$$

where $g \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$.

The group $\mathrm{SL}_2(\mathbb{R})$ contains the following three subgroups:

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}, \tag{3.1}$$

$$A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha > 0 \right\}, \tag{3.2}$$

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}. \tag{3.3}$$

Hence,we have the Iwasawa decomposition $\mathrm{SL}_2(\mathbb{R}) = \mathrm{KAN}$. Therefore, every element $g \in \mathrm{SL}_2(\mathbb{R})$ has a unique representation as $g = kan$, where $k \in K$, $a \in A$ and $n \in N$. That is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \tag{3.4}$$

The values of parameters in the above decomposition are as follows:

$$\alpha = \sqrt{a^2 + c^2}, \quad x = \frac{ab + cd}{a^2 + c^2}, \quad \theta = \arctan\frac{-c}{a}.$$

Consequently, $\cos\theta = \frac{a}{\sqrt{a^2+c^2}}$ and $\sin\theta = \frac{-c}{\sqrt{a^2+c^2}}$.

Moreover, the affine group defined as follows:

$$\mathrm{Aff} = \left\{ \frac{1}{\sqrt{a}}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, a > 0, b \in \mathbb{R} \right\},$$

is a subgroup of the $SL_2(\mathbb{R})$ group. That is because we can decompose the affine group as a semi-direct product of the subgroups A and N i.e. $\mathrm{Aff} = A \ltimes N$.

The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is the set of all 2×2 real matrices of trace zero. It is a three-dimensional Lie algebra so we can choose a basis$\{Z, A, B\}$ of $\mathfrak{sl}_2(\mathbb{R})$ by setting

$$Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \quad B = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.5}$$

Note that

$$[Z, A] = 2B, \quad [Z, B] = -2A, \quad [A, B] = -\frac{1}{2}Z. \tag{3.6}$$

The exponential map of each matrix Z, A and B forms a one-dimensional subgroup of the group $SL_2(\mathbb{R})$ given as follows:

$$\exp(\theta Z) \in \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}, \tag{3.7}$$

$$\exp(\theta A) \in \left\{ \begin{pmatrix} e^{\frac{-\theta}{2}} & 0 \\ 0 & e^{\frac{\theta}{2}} \end{pmatrix} : \theta \in \mathbb{R} \right\}, \tag{3.8}$$

$$\exp(\theta B) \in \left\{ \begin{pmatrix} \cosh\frac{\theta}{2} & \sinh\frac{\theta}{2} \\ \sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} \end{pmatrix} : \theta \in \mathbb{R} \right\}. \tag{3.9}$$

4. Irreducible Unitary Representations of $\mathrm{SL}_2(\mathbb{R})$

The irreducible unitary strongly continuous representation of $\mathrm{SL}_2(\mathbb{R})$ was classified by Bargmann in 1947 [2], and his approach has been used in different sources , such as [20, 25]. Suppose that ρ is an irreducible unitary strongly continuous representation of $\mathrm{SL}_2(\mathbb{R})$ on a Hilbert space $\mathcal{H}$. The classification steps are as follows:

Step 1: Set the Gårding space[6] for ρ,

$$\mathcal{G}(\rho) = \{\rho(f)u : u \in \mathcal{H}, f \in C_0^\infty(G)\},$$

where $G = \mathrm{SL}_2(\mathbb{R})$. Denote the derived representations of the matrices Z, A, and B (3.5) by

$$d\rho(Z) = E, \; d\rho(A) = A_1, \text{ and } d\rho(B) = B_1.$$

From (3.6), we find that

$$[E, A_1] = 2B_1, \;\; [E, B_1] = -2A_1, \;\; \text{and} \quad [A_1, B_1] = -\frac{1}{2}E. \tag{4.1}$$

Step 2: Consider the ladder operators

$$L_+ = A_1 - iB_1, \quad \text{and} \quad L_- = A_1 + iB_1. \tag{4.2}$$

Since ρ is unitary, then A_1 and B_1 are skew-symmetric. This implies that

$$L_+^* = -L_-.$$

From the commutator relation in (4.1), we have

$$[E, L_\pm] = \pm 2iL_\pm, \quad [L_+, L_-] = -iE. \tag{4.3}$$

Step 3: The Casimir operator given by $C := Z^2 - 4A^2 - 4B^2$, is an element of the centre of the universal enveloping algebra for the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. Therefore, by Schur's lemma (2.7), it acts as a scalar for the irreducible unitary representation ρ,

$$d\rho(C) = \lambda I. \tag{4.4}$$

Step 4: The decomposition into the irreducible subspace of the representation $\rho(K)$ on the Hilbert space $\mathcal{H}$ leads to the orthogonal sum, since K is a compact subgroup,

$$\mathcal{H} = \overline{\bigoplus_{k\in\mathbb{Z}} V_k}.$$

The unitary irreducible representation on the subgroup K is the character e^{iks}

$$\rho(\exp sZ) = \mathrm{e}^{iks} I \ \text{ on } V_k.$$

Thus,

$$\begin{aligned} E = d\rho(Z) &= \frac{d}{ds}\rho(\mathrm{e}^{sZ})|_{s=0} \\ &= \frac{d}{ds}\mathrm{e}^{iks}|_{s=0} \\ &= ik \ \text{ on } V_k. \end{aligned} \tag{4.5}$$

Moreover, for the Casimir operator $C := Z^2 - 4A^2 - 4B^2$, we have

$$\begin{aligned} d\rho(C) &:= d\rho(Z)^2 - 4d\rho(A)^2 - 4d\rho(B)^2 \\ &= E^2 - 4A_1^2 - 4B_1^2 \\ &= E^2 - 2(L_+L_- + L_-L_+). \end{aligned}$$

From(4.3), we have

$$\begin{aligned} 4L_+L_- &= E^2 - 2iE - \lambda, \\ 4L_-L_+ &= E^2 + 2iE - \lambda. \end{aligned}$$

Then by (4.5),

$$\begin{aligned} -4L_+L_- &= k^2 - 2k + \lambda, \\ -4L_-L_+ &= k^2 + 2k + \lambda. \end{aligned}$$

Since $L_+^* = -L_-$, then

$$\begin{aligned} \|L_-\|_{\mathcal{L}(V_k, V_{k-2})} &= \frac{1}{2}[(k-1)^2 + \lambda - 1]^{\frac{1}{2}}, \\ \|L_+\|_{\mathcal{L}(V_k, V_{k+2})} &= \frac{1}{2}[(k+1)^2 + \lambda - 1]^{\frac{1}{2}}. \end{aligned}$$

From the commutator relation (4.3), we have

$$[E, L_\pm] = \pm 2iL_\pm \Leftrightarrow EL_\pm = L_\pm E \pm 2iL_\pm.$$

Then by (4.5), for $v \in V_k$,

$$E(L_\pm v) = L_\pm(Ev) \pm 2iL_\pm v = (k \pm 2)i(L_\pm v).$$

Therefore, the ladder operators $L_\pm$ act as

$$L_\pm : V_k \to V_{k\pm 2}.$$

Step 5: We have the commutator relation $[L_+, L_-] = -iE$. Then, for each vector $v_k \in V_k$, where $k \in spec(1/i)E$, the collection of vectors

$$v_{k+2n} := (L_+)^n v_k,$$
$$v_{k-2n} := (L_-)^n v_k, \quad n \in \mathbb{Z}^+,$$

is invariant under the operators $L_+,\ L_-,\ E$. Therefore, V_k is a one-dimensional space.

Step 6: The ladder operators act on the vector spaces V_k where $k \in spec(1/i)E$. There are only four possibilities for the spectrum of the operator $(1/i)E$. First, if the ladder operators are two-sided infinite operators, given that the representation ρ is irreducible, the spectrum is either in the even or odd integer set. That is,

$$spec(1/i)E = \{.......-4, -2, 0, 2, 4......\},$$
$$spec(1/i)E = \{.....-5, -3, 1, -1, 3, 5, ...\}.$$

Second, if the ladder operators are one-sided infinite operators, then for $V_k \neq 0$, we have the following sets of spectrum:

- For $L_+ = 0$ on V_k, $\quad spec(1/i)E = \{.....n-4, n-2, n\}, \quad n \in \mathbb{Z}^+$.
- For $L_- = 0$ on V_k, $\quad spec(1/i)E = \{n, n+2, n+4, ...\}, \quad n \in \mathbb{Z}^+$.

Step 7: In each case above select a unit vector $v_k \in V_k$, $k \in spec(\frac{1}{i})E$. We have $L_+ v_k = \alpha_k v_{k+2}$. The absolute value of α_k is

$$|\alpha_k| = \frac{1}{2}[(k+1)^2 + \lambda - 1]^{\frac{1}{2}}. \tag{4.6}$$

The action of L_- on v_{k+2} is given as follows:

$$L_- v_{k+2} = \beta_k v_k, \quad \text{where} \quad \beta_k = -\overline{\alpha_k}.$$

Therefore, the type of the spectrum together with the value of $d\rho(C) = \lambda I$, fully determines the unitary irreducible representation of $\mathrm{SL}_2(\mathbb{R})$. This stated in the following theorem.

Theorem 4.1. [25] Any nontrivial irreducible unitary representation of $\mathrm{SL}_2(\mathbb{R})$ is unitary equivalent to one of the following types:

- Members of the holomorphic discrete series, denoted by ρ_n^+ such that
$$d\rho_n^+(C) = 1 - (n-1)^2,\ n \in \mathbb{Z}^+,$$
when $spec(1/i)E = \{n, n+2, ...\}$.
- Members of the anti-holomorphic discrete series, denoted by ρ_{-n}^- such that
$$d\rho_{-n}^-(C) = 1 - (n-1)^2,\ n \in \mathbb{Z}^+,$$
when $spec(1/i)E = \{....., n-4, n-2, n\}$.
- Mock discrete series ρ_1^+,ρ_{-1}^-, for $n = 1$.
- A member of the first principal series, denoted by ρ_{is}^e such that
$$d\rho_{is}^e(C) = 1 + s^2,\ s \in \mathbb{R},$$
when $spec(1/i)E = \{......., -4, -2, 0, 2, 4, ...\}$.
- A member of the complementary series, denoted by ρ_s^e such that
$$d\rho_s^e(C) = 1 - s^2,\ s \in (-1, 1)\backslash\{0\},$$
when $spec(1/i)E = \{......., -4, -2, 0, 2, 4, ...\}$.
- A member of the second principal series, denoted by ρ_{is}^o such that
$$d\rho_{is}^o(C) = 1 + s^2,\ s \in \mathbb{R}\backslash\{0\},$$
when $spec(1/i)E = \{..., -5, -3, -1, 1, 3, 5, ...\}$.

5. Induced Representation of the Group $\mathrm{SL}_2(\mathbb{R})$

In this section, we induce a representation of the group $\mathrm{SL}_2(\mathbb{R})$ from a trivial character of the subgroup N. We get a representation on a space of functions with two variables. Then, we can have this representation on a space of functions with one variable by using inducing in stages technique. That is, first induce a representation for the affine group from a trivial character of the subgroup N. We get an affine group representation that can be decomposed into a one-dimensional representation which is a complex character. Then, we induce a representation for the group $\mathrm{SL}_2(\mathbb{R})$ from a complex character of the affine group.

5.1 The $\mathrm{SL}_2(\mathbb{R})$ Induced Representation from the Subgroup N

Let $\chi_e : N \to \mathbb{T}$ be a trivial character of the subgroup N. The character χ_e induces a linear representation of $\mathrm{SL}_2(\mathbb{R})$. This induced representation is constructed in the vector space V, which consists of the functions $F_e : \mathrm{SL}_2(\mathbb{R}) \mapsto \mathbb{C}$ with the property

$$F_e\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi_e\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} F\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}.$$

The space V is invariant under the left shift of the group $\mathrm{SL}_2(\mathbb{R})$. The restriction of the left shift on V is the left regular representation of the group $\mathrm{SL}_2(\mathbb{R})$, which is given by

$$[\Lambda(g)F_e](g') = F_e(g^{-1} * g'). \tag{5.1}$$

In the following, we obtain an equivalent induced representation constructed in the left homogeneous space $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{N}$.

The Iwasawa decomposition $\mathrm{SL}_2(\mathbb{R}) = \mathrm{KAN}$ implies that the homogeneous space $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{N}$ topologically identifies to $KA \simeq \mathbb{T} \times \mathbb{R}_+ \simeq \mathbb{R}^2 \backslash \{0\}$. Hence we can choose the section map to be given by

$$\begin{aligned} \mathsf{s} &: X \to \mathrm{SL}_2(\mathbb{R}), \\ &: (u, v) \mapsto \begin{pmatrix} u & 0 \\ v & u^{-1} \end{pmatrix}, \quad u > 0. \end{aligned}$$

The natural projection map will be

$$\begin{aligned} \mathsf{p} &: \mathrm{SL}_2(\mathbb{R}) \to \mathrm{X}, \\ &: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, c), \end{aligned}$$

such that s is the right inverse of p. Therefore, the unique decomposition of $g \in \mathrm{SL}_2(\mathbb{R})$ is of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}.$$

The map $\mathsf{r} : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{N}$ is given by

$$\mathsf{r}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}. \tag{5.2}$$

The $\mathrm{SL}_2(\mathbb{R})$ action on the space $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{N}$ can be expressed in terms of p and s as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot w = \mathsf{p}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot \mathsf{s}(u, v)\right) = (du - bv, av - cu).$$

Let W be a vector space of function f on the homogeneous space X. The lifting map for the subgroup N and its character χ_e is given by:

$$\begin{aligned} [\mathcal{L}_{\chi_e} f]\begin{pmatrix} a & b \\ c & d \end{pmatrix} &:= \overline{\chi_e}\left(\mathsf{r}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f\left(\mathsf{p}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\ &= f(a, c). \end{aligned} \tag{5.3}$$

Then, the pulling map $\mathcal{P} : V \to W$, which is the right inverse of the lifting map, is given by

$$[\mathcal{P}F](u, v) := F(\mathsf{s}(u, v)).$$

Therefore, the representation $U : W \to W$, which is induced by the character χ_e, is

$$U(g) = \mathcal{P} \circ \Lambda(g) \circ \mathcal{L}_{\chi_e}. \tag{5.4}$$

To calculate the explicit form of $U(g)$, take the left action of the lifting map

$$[\Lambda(g)\mathcal{L}_{\chi_e} f](g') = [\mathcal{L}_{\chi_e} f](g^{-1} * g') = f(da' - bc', ac' - a'c) = F_e(g'). \tag{5.5}$$

Then, apply pulling for the function F_e

$$\begin{aligned} [\mathcal{P}F_e](u, v) &= F_e(\mathsf{s}(u, v)) \\ &= F_e \begin{pmatrix} u & 0 \\ v & u^{-1} \end{pmatrix} \\ &= f(du - bv, av - cu). \end{aligned} \tag{5.6}$$

Hence, from(5.4), we obtain the following formula:

$$[U(g)f](u, v) = f(du - bv, av - cu), \tag{5.7}$$

where $(u, v) \in X$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

5.2 Affine Group Representation Induced From a Trivial Character

For the trivial character χ_e, the induced representation of the subgroup N is $\rho^+_{\chi_e} : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$ and is expressed as

$$[\rho^+_{\chi_e}(a, b)f](x) = \sqrt{a} f(ax), \quad f \in L_2(\mathbb{R}_+). \tag{5.8}$$

It is a reducible unitary representation. To decompose it into irreducible components, we will find the eigenfunction of the operator $\rho^+_{\chi_e}(a, b)f$ as follows:

$$[\rho^+_{\chi_e}(a, b)f](t) = \lambda_{a,b} f(t) \quad \Rightarrow \quad \sqrt{a} f(at) = \lambda_{a,b} f(t).$$

Let $f(t) = t^\alpha$, where $\alpha \in \mathbb{C}$. Then, we obtain

$$[\rho^+_{\chi_e}(a, b)](t^\alpha) = \sqrt{a}(at)^\alpha = a^{\alpha+\frac{1}{2}} t^\alpha.$$

Hence, the eigenfunction of $\rho^+_{\chi_e}(a, b)$ is t^α. Let the inverse Mellin transform be given by

$$[\mathsf{M}^{-1}\tilde{f}](t) = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^{-\frac{1}{2}+is} \tilde{f}(s) ds, \quad t \in \mathbb{R}_+, \tag{5.9}$$

where $\alpha = -\frac{1}{2} + is$. The function $\tilde{f}(s)$ is the Mellin transform $\tilde{f}(s) = [\mathsf{M}f](x) = \int_0^\infty x^s f(x) \frac{dx}{x}$. Therefore, we obtain

$$\begin{aligned} [\rho^+_{\chi_e}(a, b)f](t) &= \sqrt{a} f(at) \\ &= \sqrt{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(s)(at)^{-\frac{1}{2}+is} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a^{is} \tilde{f}(s) t^{-\frac{1}{2}+is} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_s(a, b) \tilde{f}(s) t^{-\frac{1}{2}+is} ds, \end{aligned} \tag{5.10}$$

where $\chi_s(a, b) = a^{is}$ is a complex character of the affine group. Hence, the irreducible component of the representation $\rho^+_{\chi_e}$ (5.8) is the character χ_s.

5.3 Induction in Stages

Let P be the subgroup of $\mathrm{SL}_2(\mathbb{R})$, which is defined as follows:

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}\backslash\{0\}, b \in \mathbb{R} \right\}.$$

There exists a homomorphism $T : P \to \mathrm{Aff}$ such that $T^{-1}(a, b)$ has two elements, one for $a > 0$ and the other for $a < 0$. The $SL_2(\mathbb{R})$ representations (5.7) can be obtained by induction in stages. That is

$$\mathrm{Ind}_P^{\mathrm{SL}_2(\mathbb{R})}[\mathrm{Ind}_N^P \chi_e] = \mathrm{Ind}_N^{\mathrm{SL}_2(\mathbb{R})}[\chi_e].$$

First induce the trivial character χ_e of the subgroup N to the affine group. We will obtain the co-adjoint representation $\rho_\chi^+ : \mathsf{U} \to \mathsf{U}$, which is given as follows:

$$\left[\rho_{\chi_e}^+ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} g\right](t) = \sqrt{a} g(at).$$

The vector space U consists of all functions on the homogeneous space $\mathrm{Aff}/N = A$. It is reducible, and from subsection 5.2 we can decompose it into irreducible component which is the following the character:

$$\chi_\alpha \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^\alpha, \quad \alpha \in \mathbb{C}.$$

Therefore, for the subgroup $P = AN$, the character is given by

$$\chi_s \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = |a|^s \mathrm{sgn}^\epsilon(a), \quad \epsilon = \{0, 1\}, s \in \mathbb{C}.$$

Next, the character χ_s induces a representation of the group $SL_2(\mathbb{R})$. This representation is constructed on the vector space V, which consist of the functions $F_s : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ with the following property:

$$F_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi_s \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} F \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix}.$$

This vector space is invariant under the left shift of the group $\mathrm{SL}_2(\mathbb{R})$. The restriction of the left shift on this space is an induced representation.

An equivalent form of the induced representation can be constructed on the homogeneous space $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{P}$. The space of the left cosets $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{P}$ can be defined by the following equivalence relation: $g \sim g'$ if and only if there exists $x \in P$ such that $g = g'x$. Then, the equivalence class for all $g \in \mathrm{SL}_2(\mathbb{R})$ is given by the following:

$$[g] = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = [c : a] = \begin{cases} [\frac{c}{a} : 1], & a \neq 0 \\ [1 : 0], & a = 0 \end{cases}.$$

Thus, we can identify the space $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{P}$ by the real projective line $\mathbb{P}(\mathbb{R})$.

Next, let $\mathsf{s} : \mathbb{P}(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ be the section map given by

$$\mathsf{s}(w) = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}. \tag{5.11}$$

The natural projection map will be

$$\begin{aligned} \mathsf{p} &: \mathrm{SL}_2(\mathbb{R}) \to \mathbb{P}(\mathbb{R}) \\ &: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{c}{a}, \end{aligned} \tag{5.12}$$

where $a \neq 0$, and $\mathsf{p} \circ \mathsf{s} = \mathbb{I}_{\mathbb{P}(\mathbb{R})}$. The unique decomposition of any $g \in \mathrm{SL}_2(\mathbb{R})$ defined by s is of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

Hence, the map $\mathsf{r} : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{P}$ is given by

$$\mathsf{r} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}. \tag{5.13}$$

The $\mathrm{SL}_2(\mathbb{R})$ action on the left homogeneous space $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{P} \cong \mathbb{P}(\mathbb{R})$ is the Möbius transformation and we can express it in terms of p and s as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot w = \mathsf{p}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot \mathsf{s}(x)\right) = \frac{ax - c}{d - bx}, \tag{5.14}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $x \in \mathbb{P}(\mathbb{R})$ and $\cdot$ is the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{P}(\mathbb{R})$ from the left.

Let W be the vector space of all functions on the homogeneous space $X = \mathbb{P}(\mathbb{R})$. The lifting map $\mathcal{L}_{\chi_s} : \mathsf{W} \to \mathsf{V}$ for the subgroup P and its character χ_s associates each function f on the projective line $\mathbb{P}(\mathbb{R})$ with a function F on the $\mathrm{SL}_2(\mathbb{R})$ group. That is

$$\begin{aligned} \left[\mathcal{L}_{\chi_s} f\right]\begin{pmatrix} a & b \\ c & d \end{pmatrix} &:= \overline{\chi_s}\left(\mathsf{r}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f\left(\mathsf{p}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\ &= |a|^s \mathrm{sgn}^\epsilon(a) f\left(\frac{c}{a}\right), \end{aligned} \tag{5.15}$$

where $a \neq 0$. Then, the pulling map $\mathcal{P} : \mathsf{V} \to \mathsf{W}$, which is the right inverse of the lifting map, is given as follows:

$$[\mathcal{P}F](x) := F(\mathsf{s}(x)).$$

Therefore, the representation $T : \mathsf{W} \to \mathsf{W}$ that induced by the character χ_s is given as follows:

$$T(g) = \mathcal{P} \circ \Lambda(g) \circ \mathcal{L}_{\chi_s}. \tag{5.16}$$

The explicit formula of $T(g)$ is calculated as follows. First, take the left action of the lifting map

$$\begin{aligned} [\Lambda(g)\mathcal{L}_{\chi_s} f](g') &= [\mathcal{L}_{\chi_s} f](g^{-1}g') \\ &= |da' - bc'|^s \mathrm{sgn}^\epsilon(da' - bc') f\left(\frac{ac' - ca'}{da' - bc'}\right) = F_s(g'). \end{aligned} \tag{5.17}$$

Then, apply pulling to the function F_s

$$\begin{aligned} [\mathcal{P}F_s](x) &= F_s(\mathsf{s}(x)) \\ &= F_s\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \\ &= |d - bw|^s \mathrm{sgn}^\epsilon(d - bx) f\left(\frac{ax - c}{d - bx}\right). \end{aligned} \tag{5.18}$$

Hence, by (5.17) and (5.18) from(5.16), we obtain the formula

$$[T_s(g)f](x) = |d - bx|^s \mathrm{sgn}^\epsilon(d - bx) f\left(\frac{ax - c}{d - bx}\right), \tag{5.19}$$

where $f \in \mathsf{W}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

6. Gelfand Method to Classify the Group $\mathrm{SL}_2(\mathbb{R})$ Representation

In section 4, we present Bargmann's classification for the $\mathrm{SL}_2(\mathbb{R})$ representations which used the derived representation and find the vector modules of the representations on the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. In [8, chapter VII], the representations for the group $\mathrm{SL}_2(\mathbb{R})$ have been classified by working on the Lie group instead of the Lie algebra. The method is based on studying the invariance of bilinear functional on a normed space. Then, we move to study the invariance of the inner product on a Hilbert space. The following sections explain the Gelfand method in details.

7. Invariant Bilinear Functionals

In section 5, the $\mathrm{SL}_2(\mathbb{R})$ representations are constructed on the vector space of functions W_t on the homogeneous space $X = \mathrm{SL}_2(\mathbb{R})/N = KAN/N$. The space X can be topologically identified as follows:

$$X = KA \simeq \mathbb{T} \times \mathbb{R}_+ \simeq \mathbb{R}^2 \backslash \{0\}.$$

Definition 7.1. Consider pairs of numbers $t = (s, \epsilon)$, where s is any complex number and $\epsilon = 0$ or 1. Then associate each such pair with the space W_t that consists of functions $f(x_1, x_2)$ with the following properties:

- Every function $f(x_1, x_2) \in W_t$ is homogeneous of degree $s - 1$, and it has even parity if $\epsilon = 0$ and odd parity if $\epsilon = 1$. This means that for $a \neq 0$

$$f(ax_1, ax_1) = |a|^{s-1} \mathrm{sgn}^\epsilon(a) f(x_1, x_2).$$

- The function $f(x_1, x_2)$ is infinitely differentiable for every x_1 and x_2 except at the point $(0, 0)$.

In subsection 5.3, the $\mathrm{SL}_2(\mathbb{R})$ representations (5.19) have been constructed on the vector space of functions W_t on the real projective line $\mathbb{P}(\mathbb{R})$. We can realise the space W_t as the space of one variable by associating a function $f(x_1, x_2) \in W_t$ with a function $\varphi(x) \in \mathsf{W}_t$ as follows:

$$f(x_1, x_2) = |x_2|^{s-1}\mathrm{sgn}^{\epsilon}(x_2)\varphi\left(\frac{x_1}{x_2}\right). \tag{7.1}$$

Definition 7.2. From the relation (7.1), every function $\varphi(x) \in \mathsf{W}_t$ is given by $\varphi(x) = f(x, 1)$. Then, the function $\varphi(x)$ has the following properties:

- $\varphi(x)$ is infinitely differentiable.
- The function $\tilde{\varphi}(x) = f(1, x) = |x|^{s-1}\mathrm{sgn}^{\epsilon}(x)\varphi\left(\frac{1}{x}\right)$, is infinitely differentiable. Then, we obtain

$$\varphi(x) = |x|^{s-1}\mathrm{sgn}^{\epsilon}(x)\tilde{\varphi}\left(\frac{1}{x}\right) = |x|^{s-1}\mathrm{sgn}^{\epsilon}(x)f\left(1, \frac{1}{x}\right).$$

  As $|x| \to \infty$, we have $\varphi(x) \sim |x|^{s-1}\mathrm{sgn}^{\epsilon}(x)f(1, 0)$.

  This condition shows the behaviour of $\varphi(x)$ for large $|x|$. In particular, it implies that asymptotically as $|x| \to \infty$, the function $\varphi(x)$ goes as

$$\varphi(x) \sim C|x|^{s-1}\mathrm{sgn}^{\epsilon}(x).$$

In this section, we will study the case of the $\mathrm{SL}_2(\mathbb{R})$ representations (5.19) possessing an invariant bilinear functional. Associate the pairs of numbers $t_1 = (s_1, \epsilon_1)$ and $t_2 = (s_2, \epsilon_2)$ with the spaces W_{t_1} and W_{t_2},respectively. Then, consider the following two representations of $\mathrm{SL}_2(\mathbb{R})$:

$$[T_{s_1}(g)\varphi](x) = |d - bx|^{s_1-1}\mathrm{sgn}^{\epsilon_1}(d - bx)\varphi\left(\frac{ax - c}{d - bx}\right), \tag{7.2}$$

$$[T_{s_2}(g)\psi](x) = |d - bx|^{s_2-1}\mathrm{sgn}^{\epsilon_2}(d - bx)\psi\left(\frac{ax - c}{d - bx}\right), \tag{7.3}$$

acting on the spaces W_{t_1} and W_{t_2}, respectively.

A bilinear functional $(\cdot, \cdot) : \mathsf{W}_{t_1} \times \mathsf{W}_{t_2} \to \mathbb{R}$, is called invariant if

$$(T_{s_1}(g)\varphi, T_{s_2}(g)\psi) = (\varphi, \psi), \tag{7.4}$$

for all $g \in SL_2(\mathbb{R})$,$\varphi \in \mathsf{W}_{t_1}$ and $\psi \in \mathsf{W}_{t_2}$.

By the Iwasawa decomposition $\mathrm{SL}_2(\mathbb{R}) = \mathrm{KAN}$, every matrix $g \in \mathrm{SL}_2(\mathbb{R})$ can be written as a product of the following three matrices:

$$g_1 = \begin{pmatrix} 1 & 0 \\ x_0 & 1 \end{pmatrix} \in N, \quad g_2 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in A, \quad g_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in K. \tag{7.5}$$

Hence, the linear fractional transformation (5.14) can be obtained by combining the following three types of transformation:

- Translation: $x \to g_1^{-1} \cdot x = x - x_0$.
- Dilation: $x \to g_2^{-1} \cdot x = \alpha^2 x$.
- Inversion: $x \to g_3^{-1} \cdot x = \frac{-1}{x}$.

Therefore, in determining whether a bilinear functional is invariant, it is sufficient to consider the operators corresponding to the three matrices g_1, g_2 and g_3.

7.1 Invariance Under Translation

For the matrix g_1, the representations (7.2) and (7.3) are given as follows :

$$[T_{s_1}(g_1)\varphi](x) = \varphi(x - x_0), \tag{7.6}$$

$$[T_{s_2}(g_1)\psi](x) = \psi(x - x_0). \tag{7.7}$$

We want to find a bilinear functional (φ, ψ) that satisfies the following condition :

$$(T_{s_1}(g_1)\varphi, T_{s_2}(g_2)\psi) = (\varphi, \psi).$$

We shall restrict our considerations to the infinitely differentiable functions with bounded support in the spaces W_{t_1} and W_{t_2}. Then, by the kernel theorem A.5 we can define an integral transform as follows:

$$L_k : \varphi \to L_k(\varphi) \quad \text{such that} \quad [L_k\varphi](x_2) = \int k(x_1, x_2)\varphi(x_1)dx_1.$$

Hence, we obtain

$$(L_k(\varphi), \psi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x_1, x_2)\varphi(x_1)\psi(x_2)dx_1dx_2,$$

where $x_1, x_2 \in \mathbb{R}$ and $k(x_1, x_2)$ is the kernel of the integral. We can consider

$$(\varphi, \psi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x_1, x_2)\varphi(x_1)\psi(x_2)dx_1dx_2. \tag{7.8}$$

Then, by using (7.6)and (7.7), we have

$$\begin{aligned}(T_{s_1}(g_1)\varphi, T_{s_2}(g_2)\psi) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x_1 - x_0, x_2 - x_0)\varphi(x_1 - x_0)\psi(x_2 - x_0)dx_1dx_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(x_1', x_2')\varphi(x_1')\psi(x_2')dx_1'dx_2' \\ &= (\varphi, \psi)\end{aligned}$$

where $x_1' = x_1 - x_0$, and $x_2' = x_2 - x_0$.

Therefore, the kernel is invariant under translation. We may associate $k(x_1, x_2)$ with a function of a single variable that is

$$k(x_1, x_2) = k(x_1 - x_2, 0) = k_0(x_1 - x_2).$$

Hence, every bilinear functional (φ, ψ) (7.8) invariant with respect to translation is of the form

$$(\varphi, \psi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_0(x_1 - x_2)\varphi(x_1)\psi(x_2)dx_2dx_1. \tag{7.9}$$

7.2 Invariance Under Dilation

Now, we wish to further that (φ, ψ) be invariant under the representations (7.2) and (7.3) for g_2. These operators are given as follows:

$$[T_{s_1}(g_2)\varphi](x) = |\alpha|^{-s_1+1}\text{sgn}^{\epsilon_1}(\alpha)\varphi(\alpha^2 x),$$

$$[T_{s_2}(g_2)]\psi(x) = |\alpha|^{-s_2+1}\text{sgn}^{\epsilon_2}(\alpha)\psi(\alpha^2 x).$$

The condition that (φ, ψ) be invariant under these operators may consequently be written as

$$(\varphi, \psi) = |\alpha|^{-s_1-s_2+2}\text{sgn}^{\epsilon_1+\epsilon_2}(\alpha)(\varphi(\alpha^2 x), \psi(\alpha^2 x)). \tag{7.10}$$

First, note that this requires that $\epsilon_1 = \epsilon_2$.

Let $x = x_1 - x_2$ in the integral (7.9). The bilinear functional will be given as follows:

$$(\varphi, \psi) = \int_{-\infty}^{\infty} k_0(x)\int_{-\infty}^{\infty} \varphi(x_1)\psi(x_1 - x)dx_1dx = (k_0, \omega) \tag{7.11}$$

where $\omega(x) = \int_{-\infty}^{\infty} \varphi(x_1)\psi(x_1 - x)dx_1$.

Next, substitute (k_0, ω) for (φ, ψ) in (7.10) considering that

$$\alpha^{-2}\omega(\alpha^2 x) = \int_{-\infty}^{\infty} \varphi(\alpha^2 x_1)\psi(\alpha^2[x_1 - x])dx_1.$$

We get

$$(k_0, \omega) = |\alpha|^{-s_1-s_2}(k_0, \omega(\alpha^2 x)).$$

Let $\alpha > 0$ and replace α by $\alpha^{\frac{1}{2}}$; then, the above equation becomes

$$(k_0, \omega) = \alpha^{-\frac{1}{2}(s_1+s_2)}(k_0, \omega(\alpha x)),$$

which shows that k_0 is a homogeneous generalized function of degree $\lambda = -\frac{1}{2}(s_1 + s_2) - 1$.

Recall one of the basic properties of homogeneous generalized functions of a single variable [10]. For every complex number λ, there exists one even and one odd homogeneous generalized function of degree λ and every other homogeneous generalized function of this degree is a linear combination of these. Hence, $k_0(x)$ is given by one of the two following forms:

- If $\frac{1}{2}(s_1 + s_2) \neq 0, 1, 2....., n..$, where $n \in \mathbb{Z}$, then

$$k_0(x) = C_1|x|^{-\frac{1}{2}(s_1+s_2)-1} + C_2|x|^{-\frac{1}{2}(s_1+s_2)-1}\text{sgn}(x). \tag{7.12}$$

- If $\frac{1}{2}(s_1 + s_2) = 0, 1, 2, 3, ...n...$ is a non-negative integer, then

$$k_0(x) = C_1\delta^{\frac{1}{2}(s_1+s_2)}(x) + C_2 x^{-\frac{1}{2}(s_1+s_2)-1}. \tag{7.13}$$

The function $\delta^{\frac{1}{2}(s_1+s_2)}(x)$ is the derivative of the delta function. It is defined by

$$\int \varphi(x_1)\delta^{\frac{1}{2}(s_1+s_2)}(x_1 - x_2) = \varphi^{\frac{1}{2}(s_1+s_2)}(x_2).$$

We established that an invariant bilinear functional (φ, ψ) can exist only if $\epsilon_1 = \epsilon_2$ for the representations (7.2) (7.3).

7.3 Invariance Under Inversion

Let us now use the condition of invariance under inversion in addition to the invariance under translation and dilation . The operators $T_{s_1}(g)$ and $T_{s_2}(g)$ for the matrix g_3 are given as follows:

$$[T_{s_1}(g_3)\varphi](x) = |x|^{s_1-1}\text{sgn}^{\epsilon}(x)\varphi\left(\frac{-1}{x}\right),$$
$$[T_{s_2}(g_3)\psi](x) = |x|^{s_2-1}\text{sgn}^{\epsilon}(x)\psi\left(\frac{-1}{x}\right).$$

The invariant condition of bilinear functional (7.4) under $T_{s_1}(g_3)$ and $T_{s_2}(g_3)$ become

$$(T_{s_1}(g_3)\varphi, T_{s_2}(g_3)\psi) = (\varphi, \psi).$$

Then,by using (7.9) and changing the variable, we get

$$\int\int k_0(x_1 - x_2)\varphi(x_1)\psi(x_2)dx_1dx_2 = \int\int k_0\left(\frac{x_1 - x_2}{x_1x_2}\right)|x_1|^{-s_1-1}|x_2|^{-s_2-1}\text{sgn}^{\epsilon}(x_1x_2)\varphi(x_1)\psi(x_2)dx_1dx_2. \tag{7.14}$$

To find the value of s_1 and s_2 for which (7.14) is valid, we will consider the different forms of $k_0(x)$, which are given by (7.12) and (7.13).

In the first case ,(7.12) $k_0(x)$ is invariant if C_1 or C_2 is zero. Hence, we get

$$k_0(x) = |x|^{-\frac{1}{2}(s_1+s_2)-1}\text{sgn}^{\nu}(x), \quad \nu = 0 \quad \text{or} \quad 1.$$

Then, we substitute $|x|^{-\frac{1}{2}(s_1+s_2)-1}\mathrm{sgn}^{\nu}(x)$ for $k_0(x)$ in (7.14). We obtain that the bilinear functional is invariant if $s_1 = s_2 \neq 0, 1, 2.., n, , .$ In this case the invariant bilinear functional is given as follows:

$$(\varphi, \psi) = \int_{-\infty}^{\infty} |x_1 - x_2|^{-s_1-1}\mathrm{sgn}^{\epsilon}(x_1 - x_2)\varphi(x_1)\psi(x_2)dx_1dx_2. \tag{7.15}$$

Similar, for (7.13) , $k_0(x)$ is invariant if C_1 or C_2 is zero. Then, we obtain

$$k_0(x) = \delta^{\frac{1}{2}(s_1+s_2)}(x), \quad \text{or} \quad k_0(x) = x^{-\frac{1}{2}(s_1+s_2)-1}.$$

We substitute $\delta^{\frac{1}{2}(s_1+s_2)}(x)$ for $k_0(x)$ in (7.14). We get the following invariant bilinear functionals:

- if $s_1 = s_2$ is an integer but the representation is not holomorphic, we have

$$(\varphi, \psi) = \int_{-\infty}^{\infty} \varphi^{s_1}(x)\psi(x)dx, \tag{7.16}$$

- if $s_1 = -s_2$, we have

$$(\varphi, \psi) = \int_{-\infty}^{\infty} \varphi(x_1)\psi(x_2)dx_1dx_2. \tag{7.17}$$

For $k_0(x) = x^{-\frac{1}{2}(s_1+s_2)-1}$, the invariant bilinear functional is given as follows:

$$(\varphi, \psi) = \int_{-\infty}^{\infty} (x_1 - x_2)^{-s_1-1}\varphi(x_1)\psi(x_2)dx_1dx_2, \tag{7.18}$$

where $s_1 = s_2 \in \mathbb{Z}$ and the representation is holomorphic.

To conclude, the $SL_2(\mathbb{R})$ group representations T_{t_1} and T_{t_2} given by (7.2), (7.3) have an invariant bilinear functional if and only if $\epsilon_1 = \epsilon_2 = \{0, 1\}$ and either $s_1 = s_2$ or $s_1 = -s_2$, where $s_1, s_2 \in \mathbb{C}$.

8. Invariant Bilinear Functionals for Holomorphic Representations

In section 7, the bilinear functional (7.18) was invariant if $s_1 = s_2 = n \in \mathbb{Z}$. In this case, the representation operator is given by

$$[T_n(g)\varphi](x) = (d - bx)^{n-1}\varphi\left(\frac{ax - c}{d - bx}\right). \tag{8.1}$$

In this section, we illustrated the invariant subspaces of the $SL_2(\mathbb{R})$ representation T_n. The representation T_n is called holomorphic because it is constructed in a space of holomorphic functions. This is explained in the following text.

Let $\rho : H_2(\mathbb{R}) \to H_2(\mathbb{R})$ be the quasi-regular representation of the affine group given by

$$[\rho(a, b)f](x) = a^{\frac{-1}{2}} f\left(\frac{x - a}{b}\right).$$

Let the mother wavelet be $c(x) := \frac{1}{i\pi}\frac{1}{i\pm x}$, and let the operator $F_{\pm} : L_2(\mathbb{R}) \to \mathbb{C}$ be defined by

$$F_{\pm}(f) = \langle f, c\rangle = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(x)}{i \pm x}dx.$$

Then, from the Definition B.1, the covariant transform $\mathcal{W}^{\rho}_{F} : L_2(\mathbb{R}) \to H_2(\mathbb{R})$ becomes

$$[\mathcal{W}^{\rho}_{F_+} f](b + ai) = F_+(\rho(a, b)^{-1} f(t)) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)}{t - (b + ai)}dt.$$

The image space for this covariant transform consists of the null solution of the Cauchy-Riemann equation $\partial_{\bar{z}}$ in the upper half-plane. This has been explained in example B.5.

Also, for the affine group, consider the contravariant transform (see subsection C) $\mathcal{M} : H_2(\mathbb{R}) \to L_2(\mathbb{R})$, which is given by

$$[\mathcal{M}f](t) = \lim_{a\to 0} f(a, t).$$

Therefore, the composition $\mathcal{M} \circ \mathcal{W}^{\rho}_{F_+} : H_2(\mathbb{R}) \to H_2(\mathbb{R})$ is given as follows:

$$[\mathcal{M} \circ \mathcal{W}^{\rho}_{F_+} f](t) = \lim_{a\to 0} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-(b+ia)} dt. \tag{8.2}$$

This shows that at $a = 0$, we get the boundary value of the Cauchy integral $[Cf](b + ia)$, and the vector space of functions $[Cf](b + i0)$ is the Hardy space on the real line.

Now, for nonnegative integer n, let D_n be the space with the invariant bilinear functional

$$(\varphi, \psi) = \int_{-\infty}^{\infty} (x_1 - x_2)^{-n-1} \varphi(x_1)\psi(x_2) dx_1 dx_2. \tag{8.3}$$

To find the invariant subspaces of D_n, we choose the kernels $k_0(x) = (x - i0)^{-n-1}$ and $k_0(x) = (x + i0)^{-n-1}$. From (7.9), the functionals corresponding to them are

$$(\varphi, \psi)_+ = \int_{-\infty}^{\infty} (x_1 - x_2 - i0)^{-n-1} \varphi(x_1)\psi(x_2) dx_1 dx_2, \tag{8.4}$$

$$(\varphi, \psi)_- = \int_{-\infty}^{\infty} (x_1 - x_2 + i0)^{-n-1} \varphi(x_1)\psi(x_2) dx_1 dx_2, \tag{8.5}$$

where $\varphi(x)$ and $\psi(x) \in D_n$. From (8.2), we associate every $\varphi(x)$ with the following two bounded support functions:

$$\varphi_+(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(x_1)}{x_1 - x - i0} dx_1, \tag{8.6}$$

$$\varphi_-(x) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(x_1)}{x_1 - x + i0} dx_1. \tag{8.7}$$

These functions are in the Hardy space on the upper and lower half planes, respectively, and we have $\varphi(x) = \varphi_+(x)+\varphi_-(x)$.

Then, the bilinear functional on the upper and lower half planes, respectively, are given by the following:

$$(\varphi, \psi)_+ = \frac{2\pi i}{n} \int_{-\infty}^{\infty} \varphi_+^{(n)}(x)\psi(x) dx, \tag{8.8}$$

$$(\varphi, \psi)_- = \frac{2\pi i}{-n} \int_{-\infty}^{\infty} \varphi_-^{(n)}(x)\psi(x) dx. \tag{8.9}$$

The functions $\varphi_+^{(n)}(x)$ and $\varphi_-^{(n)}(x)$ are the nth derivative of $\varphi_+(x)$ and $\varphi_-(x)$, respectively, and are given as follows:

$$\varphi_+^{(n)}(x) = \frac{n}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(x_1)}{(x_1 - x - i0)^{n+1}} dx_1, \tag{8.10}$$

$$\varphi_-^{(n)}(x) = \frac{-n}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi(x_1)}{(x_1 - x + i0)^{n+1}} dx_1. \tag{8.11}$$

Theorem 8.1. [8, p.410] The integrals $(\varphi, \psi)_+$ (8.8) and $(\varphi, \psi)_-$ (8.9) converge for arbitrary φ and $\psi \in D_n$, and hence, we define invariant bilinear functionals on all of D_n.

Let $D_n^- \subset D_n$ be a subspace of $\varphi(x)$ functions such that $(\varphi, \psi)_+ = 0$ for every $\psi \in D_n$. Equation (8.8) shows that D_n^- contains all $\varphi(x)$ functions such that $\varphi_+^{(n)}(x) = 0$. Hence, we obtain $\varphi^n(x) = \varphi_-^{(n)}(x)$ on the space D_n^-. Thus, $\varphi(x)$ is the boundary value of a holomorphic function in the lower half-plane.

Similarly, $(\varphi, \psi)_- = 0$ on a subspace $D_n^+ \subset D_n$ of the function $\varphi(x)$, which is the boundary value of a holomorphic function in the upper half-plane.

The intersection of D_n^+ and D_n^- is the finite dimensional subspace E_n of all polynomials of degree $n - 1$ and less. To conclude, the space D_n of analytic representation contains three invariant subspaces: one finite dimensional and two infinite dimensional. In Lemma 9.4, we show that the quotient space D_n/E_n is the direct sum of the invariant subspaces D_n^+/E_n and D_n^-/E_n.

For $-n \in \mathbb{Z}_-$, let F_{-n} be the space where the invariant bilinear functional given by (8.3) is equal to zero . Hence, F_{-n} consists of functions $\varphi(x)$ that satisfy

$$\int_{-\infty}^{\infty} x^k \varphi(x) dx = 0, \quad k = 0, ..., -n-1. \tag{8.12}$$

Remark 8.2. For the homogeneous function $k_0(x) = x^{-n-1}$, let $k_1(x) = x^{-n-1}\ln|x|$ be an associated homogeneous function That is

$$\begin{aligned} k_1(\alpha x) &= (\alpha x)^{-n-1}\ln|\alpha x| \\ &= \alpha^{-n-1}x^{-n-1}[\ln|\alpha| + \ln|x|] \\ &= \alpha^{-n-1}[x^{-n-1}\ln|x| + \ln|\alpha|x^{-n-1}] \\ &= \alpha^{-n-1}[k_1(x) + \ln|\alpha|k_0(x)]. \end{aligned}$$

The bilinear functional of $k_1(x) = x^{-n-1}\ln|x|$ is defined on the space F_{-n} and is given by

$$(\varphi,\psi)_1 = \int_{-\infty}^{\infty}(x_1 - x_2)^{-n-1}\ln|x_1 - x_2|\varphi(x_1)\psi(x_2)dx_1dx_2. \tag{8.13}$$

By simple calculation, for g_2 (7.5) and T_n (8.1), we have

$$(T_n(g_2)\varphi, T_n(g_2)\psi))_1 = \left[(\varphi,\psi)_1 + \ln|\alpha^{-2}|(\varphi,\psi)\right], \tag{8.14}$$

where (φ,ψ) is given by (8.3). On the space F_{-n}, the invariant bilinear functional is $(\varphi,\psi) = 0$. Hence, we obtain

$$(T_n(g_2)\varphi, T_n(g_2)\psi)_1 = (\varphi,\psi)_1.$$

Therefore, the bilinear functional $(\varphi,\psi)_1$ is invariant under dilation on the space F_{-n}.

Also, by direct calculation, $(\varphi,\psi)_1$ is invariant under inversion on F_{-n}, that is,

$$(T_n(g_3)\varphi, T_n(g_3)\psi)_1 = (\varphi,\psi)_1,$$

where g_3 is given in (7.5) and T_n is (8.1). Hence, $(\varphi,\psi)_1$ is an invariant bilinear functional on F_{-n}.

Next, for $k_1(x) = x^{-n-1}\ln|x|$, there exists the following kernels:

$$k_1^+(x) = \lim_{y\to+0} x^{-n-1}\ln|x - iy| = x^{-n-1}\ln|x - i0|, \tag{8.15}$$

$$k_1^-(x) = \lim_{y\to-0} x^{-n-1}\ln|x + iy| = x^{-n-1}\ln|x + i0|. \tag{8.16}$$

The functionals corresponding to $k_1^+(x)$ and $k_1^-(x)$ are

$$(\varphi,\psi)_1^+ = \int_{-\infty}^{\infty}(x_1 - x_2)^{-n-1}\ln(x_1 - x_2 - i0)\varphi(x_1)\psi(x_2)dx_1dx_2,$$

$$(\varphi,\psi)_1^- = \int_{-\infty}^{\infty}(x_1 - x_2)^{-n-1}\ln(x_1 + x_2 - i0)\varphi(x_1)\psi(x_2)dx_1dx_2.$$

Hence, F_{-n} is an invariant space and contains two invariant subspaces:

- The subspace F_{-n}^+ is the subspace of functions in F_{-n}, which are the boundary values of the function in the upper half plane, where $(\varphi,\psi)_1^- = 0$.
- The subspace F_{-n}^- is the subspace of functions in F_{-n}, which are the boundary values of function in the lower half plane, where $(\varphi,\psi)_1^+ = 0$.

Next, we want to show that the subspaces F_{-n}^+ and F_{-n}^- consist of the boundary values of holomorphic functions in the upper and lower half-planes, respectively.

For $\varphi(z)$, a holomorphic function in the upper half-plane, we have $\lim_{y_+\to 0}\varphi(z) = \varphi(x)$, where $z = x + iy$. Then, $\varphi(x)$ is the boundary value for $\varphi(z)$.

Let $\hat{\varphi}(\zeta) = \int_{-\infty}^{\infty}\varphi(x)e^{-2\pi ix\zeta}$ be the Fourier transform of $\varphi(x)$. Then, we obtain

$$\int_{-\infty}^{\infty} x^k\varphi(x)dx = (-2\pi i)^k\hat{\varphi}^k(0). \tag{8.17}$$

By Cauchy's integral theorem for the function $\varphi(z)$, which is holomorphic in the upper half-plane, we get $\hat{\varphi}(\zeta) = 0$, $\zeta > 0$. Hence, $\hat{\varphi}^k(0) = 0$, and (8.17) is equal to zero. This implies that $\varphi(x) \in F_{-n}^+$.

The same is noted for the boundary value of holomorphic function in the lower half-plane.

9. Equivalence of the $\mathrm{SL}_2(\mathbb{R})$ Representations

In this section, we study under which conditions the $\mathrm{SL}_2(\mathbb{R})$ representations T_{t_1} (7.2) and T_{t_2} (7.3) are equivalent.

Definition 9.1. For the representations T_{t_1} and T_{t_2}, an intertwining operator A is a continuous mapping of the space W_{t_1} onto the space W_{t_2}, that is,

$$AT_{t_1}(g) = T_{t_2}(g)A.$$

The representations T_{t_1} and T_{t_2} are equivalent if there exists an intertwining operator A which is one-to-one continuous mapping with the continuous inverse A^{-1} such that:

$$T_{t_1}(g) = AT_{t_2}(g)A^{-1}.$$

To obtain the conditions for the existence of an intertwining operator A, we establish a relation between the operator A and the bilinear functional (φ, ψ). Let W_{-t_2} be the space of the representation T_{-t_2} acting on. The space W_{-t_2} is associated with the pair of number $-t_2 = (-s_2, \epsilon_2)$. Then let $B(\psi, \varphi)$ be an invariant bilinear functional on the spaces W_{-t_2} and W_{t_1}. It is shown in section 7 that if $s_1 = -s_2$ then the invariant bilinear functional is given by the following:

$$B(\psi, \varphi) = \int_{-\infty}^{\infty} \psi(x)\varphi(x)dx. \tag{9.1}$$

Let $A : \mathsf{W}_{t_1} \to \mathsf{W}_{t_2}$ be a linear operator. Then, we associate with A the bilinear functional (φ, ψ) on the spaces W_{t_1} and W_{-t_2} as expressed by the following:

$$(\varphi, \psi) = B(\psi, A\varphi) = \int_{-\infty}^{\infty} \psi(x)A\varphi(x)dx, \tag{9.2}$$

where $\varphi \in \mathsf{W}_{t_1}$, $\psi \in \mathsf{W}_{-t_2}$.

Lemma 9.2. The linear operator $A : \mathsf{W}_{t_1} \to \mathsf{W}_{t_2}$ intertwines with the representations T_{t_1} and T_{t_2} if and only if $(\varphi, \psi) = B(\psi, A\varphi)$ invariant under T_{t_1} and T_{-t_2}.

Proof. From equation (9.1), we obtain the following:

$$B(T_{-t_2}(g)\psi, AT_{t_1}(g)\varphi) = B(T_{-t_2}(g)\psi, T_{t_2}(g)A\varphi),$$

where $\varphi \in \mathsf{W}_{t_1}$ and $\psi \in \mathsf{W}_{-t_2}$. The invariance of the bilinear functional $B(\psi, \varphi)$ implies that

$$B(T_{-t_2}(g)\psi, T_{t_2}(g)A\varphi) = B(\psi, A\varphi),$$

for all ψ and φ. Then, we have

$$B(T_{-t_2}(g)\psi, AT_{t_1}(g)\varphi) = B(\psi, A\varphi) = (\varphi, \psi).$$

Therefore, (φ, ψ) is invariant under $T_{t_1}(g)$ and T_{-t_2}. □

In section 7, we found the conditions under which the invariant bilinear functionals (φ, ψ) exist. By substituting $-s_2$ for s_2 in these conditions, we get that the $SL_2(\mathbb{R})$ representations T_{t_1} and T_{t_2} have an intertwining operator A, which maps W_{t_1} continuously into W_{t_2} if and only if $\epsilon_1 = \epsilon_2 = \{0, 1\}$ and either $s_1 = s_2$ or $s_1 = -s_2$, where $s_1, s_2 \in \mathbb{C}$.

To obtain the expression of such an operator A, first consider the case $s_1 = s_2$, the invariant bilinear functional is given by

$$(\varphi, \psi) = \lambda \int_{-\infty}^{\infty} \varphi(x)\psi(x)dx.$$

Comparing this with (9.2), we conclude that every operator A on W_{t_2} follows the condition that $AT_{t_1}(g) = T_{t_2}(g)A$ is a multiplier of the unit operator. This implies that $A = \lambda I$, where λ is constant. Therefore, by Schur's lemma (2.7), all the representations T_{t_1} and T_{t_2} except the holomorphic representation are irreducible.

Next, for the case $s_1 = -s_2$, we have two invariant bilinear functionals (7.15) and (7.16). For the functional given by (7.15), the operator A is expressed as follows:

$$A\varphi(x) = \lambda \int_{-\infty}^{\infty} |x_1 - x|^{-s_1-1}\mathrm{sgn}^{\epsilon}(x_1 - x)\varphi(x_1)dx_1.$$

For (7.16), the operator A is given as follows:

$$A\varphi(x) = \varphi^{(s)}(x).$$

Theorem 9.3. [8, p.416] Consider the representation operators $T_{t_1}(g)$ and $T_{t_2}(g)$ given by (7.2) and (7.3), respectively, possessing an intertwining operator A maps W_{t_1} continuously into W_{t_2}. Then, A is a one-to-one map, and $T_{t_1}(g)$,$T_{t_2}(g)$ are equivalent.

9.1 Equivalence of the Holomorphic Representation of $\mathrm{SL}_2(\mathbb{R})$

Consider the analytic representations T_n and T_{-n} given by (8.1) for $n \in \mathbb{Z}^+$. From section 8, the bilinear invariant functional is expressed as follows:

$$(\varphi, \psi) = \int_{-\infty}^{\infty} [\lambda_1 \varphi_+^{(n)}(x) + \lambda_2 \varphi_-^{(n)}(x)]\psi(x)dx,$$

where λ_1 and λ_2 are arbitrary constants. The functions $\varphi_+{}^{(n)}(x)$ and $\varphi_-{}^{(n)}(x)$ are given by (8.10) and (8.11),respectively.

Hence, any operator intertwining with the holomorphic representations (8.1) is of the form

$$A'\varphi(x) = \frac{\lambda_1}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi^{(n)}(x_1)}{x_1 - x - i0}dx_1 - \frac{\lambda_2}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi^{(n)}(x_1)}{x_1 - x + i0}dx_1.$$

This shows that the holomorphic representations T_n and T_{-n} are inequivalent.

Let us illustrate the relations between the analytic representations. As mentioned in section 8 that for the analytic representations T_n and T_{-n} acting on D_n and D_{-n}, respectively, where $n \in \mathbb{Z}^+$, we have established the following:

- The space D_n contains three invariant subspaces:
  - E_n, the space of all polynomials of degree $n-1$ and less,
  - D_n^+ ,the subspace of all functions $\varphi(x)$ that are boundary values of holomorphic functions on the upper half plane such that $A_-\varphi(x) = 0$, and
  - D_n^-, the subspace of all functions $\varphi(x)$ that are boundary values of holomorphic functions on the lower half plane such that $A_+\varphi(x) = 0$. The intersection of D_n^+ and D_n^- is E_n, and their sum is the entire space D_n.

  Here, A_+ and A_- maps D_n onto D_{-n} and are defined by

$$A_+\varphi(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi^{(n)}(x_1)dx_1}{x_1 - x - i0}, \tag{9.3}$$

$$A_-\varphi(x) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi^{(n)}(x_1)dx_1}{x_1 - x + i0}. \tag{9.4}$$

- The space D_{-n} contains three subspaces:
  - F_n, the space of all $\varphi(x)$ such that

$$\int_{-\infty}^{\infty} x^k \varphi(x)dx = 0, \quad k = 0, ... - n - 1, \tag{9.5}$$

  - F_{-n}^+, the subspace of functions that are boundary values of holomorphic functions on the upper half plane, and
  - F_{-n}^-, the subspace of function that are the boundary values of holomorphic functions on the lower half plane.

Lemma 9.4. [8] The $SL_2(\mathbb{R})$ representations on the subspaces D_n/E_n and F_{-n} are reducible. Also, D_n/E_n and F_{-n} are direct sums of two invariant subspaces.

Proof. The quotient space D_n/E_n is the space of functions in D_n defined only up to the polynomial of degree $n-1$ and less. Consider the intertwining operators A_+ (9.3) and A_- (9.4) that maps the spaces D_n onto D_{-n}. The operators A_+ and A_- satisfy the following:

$$A_+T_n(g) = T_{-n}(g)A_+, \quad \text{and} \quad A_-T_n(g) = T_{-n}(g)A_-.$$

Every other intertwining operator for $T_n(g)$ and $T_{-n}(g)$ is a linear combination of A_+ and A_-.

Let $\varphi(x)$ be a function in the space D_n. In subsection 8, we show that

$$\varphi(x) = \varphi_+(x) + \varphi_-(x),$$

where the functions $\varphi_+(x)$ and $\varphi_-(x)$ are the boundary values of some holomorphic functions in the upper and lower half-planes, respectivly. That is $\varphi_+(x) \in D_n^+$ and $\varphi_-(x) \in D_n^-$. The above implies that space D_n/E_n is a direct sum of the form

$$D_n/E_n = D_n^+/E_n \oplus D_n^-/E_n.$$

Hence, the representation on the space D_n/E_n is reducible.

Next, let the subspaces F^+_{-n} and F^-_{-n} be the images of the subspaces D^+_n and D^-_n under the covariant transforms A_+ and A_-, respectively. The subspaces F^+_{-n} and F^-_{-n} are invariant under T_{-n} and $F^+_{-n} \cap F^-_{-n} = \{\phi\}$, respectivly. Thus, we have the direct sum

$$F_{-n} = F^+_{-n} \oplus F^-_{-n}.$$

□

Remark 9.5. Since we have shown that the $SL_2(\mathbb{R})$ representations on the subspaces D^+_n/E_n and F^+_{-n} are equivalent under the covariant transforms A_+, we can realise the representation in the upper half plane $\varphi(z)$. Then, the $SL_2(\mathbb{R})$ representation on $D^+_n/E_s \cong F^+_{-n}$ is given by

$$[T_n(g)\varphi](z) = (d - bz)^{n-1}\varphi\left(\frac{az - c}{d - bz}\right). \tag{9.6}$$

However, the subspace $D^+_n/E_n \cong F^+_{-n}$ does not consist of all analytic functions $\varphi(z)$ in the upper half-plane. The function $\varphi(z)$ must be infinitely differentiable together with $\tilde{\varphi}(z) = z^{n-1}\varphi(\frac{-1}{z})$ in the closed upper half-plane. The same is noted, for the $SL_2(\mathbb{R})$ representations on the subspaces $D^-_n/E_n \cong F^-_{-n}$.

Lemma 9.6. [8] The equivalence of the holomorphic representations T_n, T_{-n} in the following pairs of subspaces:

- E_n and D_{-n}/F_{-n}, where the intertwining operator is given by

$$A\varphi(x) = \int_{-\infty}^{\infty} (x_1 - x)^{n-1}\varphi(x_1)dx_1.$$

- D_n/E_n and F_{-n}, where A is the differential operator d^n/dx^n.
- D^+_n/E_n and F^+_{-n} or D^-_n/E_n and F^-_{-n}, where the intertwining operator is A_+(9.3) or A_- (9.4).

10. Unitary Representations of the Group $\mathrm{SL}_2(\mathbb{R})$

Unitary representation is a representation on a Hilbert space with an invariant inner product. Hence we need to find the conditions under which it is possible to define an invariant inner product under the $\mathrm{SL}_2(\mathbb{R})$ representation. Recall that an inner product is a positive definite non-degenerate Hermitian functional. Hence, we start by studying the invariance of the Hermitian functional.

10.1 The Existence of an Invariant Hermitian Functional

Let W_t be the space of the representation T_t (7.2) associated with the pair of numbers $t = (s, \epsilon)$, $s \in \mathbb{C}$. Then, for $\bar{t} = (\bar{s}, \epsilon)$, we have the space $\mathsf{W}_{\bar{t}}$ of the representation $T_{\bar{t}}$, which is given as follows:

$$[T_{\bar{t}}(g)\psi](x) = |d - bx|^{\bar{s}-1}\mathrm{sgn}^{\epsilon}(d - bx)\psi\left(\frac{ax - c}{d - bx}\right). \tag{10.1}$$

The Hermitian functional is defined as $\langle\varphi, \psi\rangle : \mathsf{W}_t \times \mathsf{W}_{\bar{t}} \to \mathbb{R}$. The goal of this subsection is to find the conditions under which this functional is invariant, that is

$$\langle\varphi, \psi\rangle = \langle T_t(g)\varphi, T_{\bar{t}}(g)\psi\rangle.$$

From section 7, the bilinear functional (φ, ψ) is invariant if and only if $s_1 = s_2$ or $s_1 = -s_2$. Let the number s_2 be the complex conjugate of s_1. Then, the bilinear functional (φ, ψ) will be converted to the Hermitian functional $\langle\varphi, \psi\rangle$. Therefore, the Hermitian functional $\langle\varphi, \psi\rangle$ is invariant if and only if $s = \bar{s}$ or $s = -\bar{s}$.

The expressions of the invariant Hermitian functional will be as follows:

- For $s = -\bar{s}$, i.e. s is pure imaginary, we have:

$$\langle\varphi, \psi\rangle = \int_{-\infty}^{\infty} \varphi(x)\overline{\psi}(x)dx. \tag{10.2}$$

- For $s = \bar{s}$, i.e. if s is real, we have:

$$\langle\varphi, \psi\rangle = \int_{-\infty}^{\infty} |x_1 - x_2|^{-s_1-1}\mathrm{sgn}^{\epsilon}(x_1 - x_2)\varphi(x_1)\overline{\psi}(x_2)dx_1dx_2. \tag{10.3}$$

Also, if s is a nonnegative integer and the representation is not holomorphic, the invariant Hermitian functional is

$$\langle\varphi,\psi\rangle = \int_{-\infty}^{\infty} \varphi^{(s)}(x)\overline{\psi}(x)dx.$$

- For the holomorphic representation (8.1) every invariant Hermitian functional is a linear combination of:

$$\langle\varphi,\psi\rangle_+ = \int_{-\infty}^{\infty} \varphi_+^{(n)}(x)\overline{\psi}(x)dx, \tag{10.4}$$

$$\langle\varphi,\psi\rangle_- = \int_{-\infty}^{\infty} \varphi_-^{(n)}(x)\overline{\psi}(x)dx. \tag{10.5}$$

where $\varphi_+^{(n)}(x)$ and $\varphi_-^{(n)}(x)$ are given by (8.10) and (8.11), respectively.

10.2 Positive Definite Invariant Hermitian Functional

The invariant Hermitian bilinear functional given by (10.2), is positive definite for pure imaginary number s. The invariant Hermitian bilinear functional given by (10.3), is positive definite if $\epsilon = 0$ and $|s| < 1$ [8, p.427].

Next, for the holomorphic representation, every invariant Hermitian bilinear functional is a linear combination of (10.4) and (10.5). Consider $\langle\varphi,\psi\rangle_+ \neq 0$ as a Hermitian functional on the subspace D_n^+/E_n. We will show that $\langle\varphi,\psi\rangle_+$ is positive definite.

The Fourier transform of $\varphi^{(n)}(x)$ is given by $\mathcal{F}[\varphi^{(n)}(\zeta)] = (-i)^n\zeta^n\hat{\varphi}(\zeta)$, where $\hat{\varphi}(\zeta) = \int_{-\infty}^{\infty}\varphi(x)e^{i\zeta x}dx$.

Note that since $\varphi_+(x)$ is the boundary value of a holomorphic function on the upper half-plane, then the Fourier transform of $\varphi_+(x)$ is supported on $-\infty < \zeta < 0$. Then, the Plancherel theorem implies that

$$\langle\varphi,\psi\rangle_+ = i^{-n}\int_{-\infty}^{\infty} \varphi_+^{(n)}(x)\overline{\psi}(x)dx = \frac{1}{2\pi}\int_{-\infty}^{0} |\zeta|^n\hat{\varphi}(\zeta)\overline{\hat{\psi}}(\zeta)d\zeta.$$

Thus, $\langle\varphi,\psi\rangle_+$ is positive definite on D_n^+/E_n.

Similarly, the invariant Hermitian functional $\langle\varphi,\psi\rangle_-$ is positive definite on the subspace D_n^-/E_n since we have

$$\langle\varphi,\psi\rangle_- = i^{-n}\int_{-\infty}^{\infty} \varphi_-^{(n)}(x)\overline{\psi}(x)dx = \frac{1}{2\pi}\int_{0}^{\infty} \zeta^n\hat{\varphi}(\zeta)\overline{\hat{\psi}}(\zeta)d\zeta.$$

For the case that n is a negative integer, we have shown in the proof of Theorem 9.4 that the subspaces D_n^+/E_n and D_n^-/E_n map to the subspaces F_{-n}^+ and F_{-n}^- by the intertwining operator A_+ and A_-, respectively. Hence, the invariant Hermitian functionals on F_{-n}^+ and F_{-n}^- are positive definite.

Recall in Remark 9.5 that we can realise F_{-n}^+ as the space of holomorphic function in the upper half-plane. The representation in this case is defined by

$$[T_n(g)\varphi](z) = (d-bz)^{n-1}\varphi\left(\frac{az-c}{d-bz}\right), \quad \text{where} \quad z = x+iy. \tag{10.6}$$

The expression of the positive invariant Hermitian functionals for this model is of the form

$$\langle\varphi,\psi\rangle = \int_{\mathrm{Im}z>0} \varphi(z)\overline{\psi}(z)\omega(z)dzd\bar{z},$$

where $\omega(z)$ is a positive function. To find the form of $\omega(z)$, we apply $T_n(g)$ (10.6) to $\varphi(z)$ and $\psi(z)$. Then, by direct calculation, the invariance condition is given by $\langle T_n(g)\varphi, T_n(g)\psi\rangle = \langle\varphi,\psi\rangle$, which is valid if and only if $\omega(z) = (\mathrm{Im}z)^{-n-1} = y^{-n-1}$.

10.3 Representations of $\mathrm{SL}_2(\mathbb{R})$ *on the Hilbert Space*

We found in subsection 10.2 the condition under which there exists a positive definite Hermitian functional $\langle\varphi,\psi\rangle$ invariant under $T_t(g)$, that is

$$\langle T_t(g)\varphi, T_t(g)\psi\rangle = \langle\varphi,\psi\rangle.$$

We can consider such a Hermitian functional as an inner product in the space W_t. Then, if W_t is completed with respect to the norm

$$\|\varphi\|^2 = \langle\varphi,\varphi\rangle,$$

we obtain a Hilbert space $\mathcal{H}$.

The operators $T_t(g)$ on W_t can be extended uniquely to unitary operators on $\mathcal{H}$. We denote these unitary operators, as before, by $T_t(g)$ such that they also satisfy the representation group property:

$$T_t(g_1 g_2) = T_t(g_1)T_t(g_2).$$

Hence, these unitary operators form a representation of $\mathrm{SL}_2(\mathbb{R})$.

Lemma 10.1. [8]. For every representation T_t that possesses a positive definite Hermitian functional, a corresponding representation of $\mathrm{SL}_2(\mathbb{R})$ by unitary operators on the Hilbert space exists. In this correspondence, equivalent representations correspond to unitary equivalent representations and inequivalent representations correspond to inequivalent ones.

Next, we wish to classify the unitary representation of the $\mathrm{SL}_2(\mathbb{R})$ group.

- Representations of the principal (continuous) series:
  For $s = i\rho$ where $\rho \in \mathbb{R}$ and $\epsilon = 0$ or 1, the representations are defined by

$$T_{i\rho}(g)\varphi(x) = |d - bx|^{i\rho-1}\mathrm{sgn}^{\epsilon}(d - bx)\varphi\left(\frac{ax - c}{d - bx}\right). \tag{10.7}$$

  From subsection 10.2 the inner product in this case is as follows:

$$\langle\varphi, \varphi\rangle = \int_{-\infty}^{\infty} \varphi(x)\overline{\varphi}(x) < \infty.$$

- Representations of the complementary series:
  These representations are defined by a real parameter $s \neq 0$ in the interval $-1 < s < 1$. The inner product is given by

$$\langle\varphi, \varphi\rangle = \int_{-\infty}^{\infty} |x_1 - x_2|^{s_1-1}\varphi(x_1)\overline{\varphi}(x_2)dx_1dx_2.$$

  The representation is defined by

$$T_s(g)\varphi(x) = |d - bx|^{s-1}\varphi\left(\frac{ax - c}{d - bx}\right). \tag{10.8}$$

- Representations of the discrete series:

  For each integer number n, the inner product on the space of holomorphic functions in the upper half plane is given by

$$\langle\varphi, \varphi\rangle = \int_{y>0}\int_{\mathbb{R}} |\varphi(x + iy)|^2 y^{-n-1}dxdy < \infty. \tag{10.9}$$

  The representation is identified by

$$T_n(g)\varphi(z) = (d - bz)^{n-1}\varphi\left(\frac{az - c}{d - bz}\right), \quad n \in \mathbb{Z}. \tag{10.10}$$

Acknowledgements

I would like to thank Dr.Vladimir Kisil for all his guidance, valuable advices and comments. Also, my deep gratitude to my family for their continuous love, help and support. I would like to thank my husband for his patience and kindness.Thanks for my kids who have made me stronger and filled me with happiness.

References

Astengo, F., Cowling, M. G., & Di Blasio, B. (2019). Uniformly bounded representations of sl(2,r). *Journal of Functional Analysis, 276*(1), 127–147.

Bargmann, V. (1947). Irreducible unitary representations of the Lorentz group. *Ann. of Math., 48(2)*,568–640.

Berndt, R. (2007). *Representations of linear groups*. Vieweg, Wiesbaden. An introduction based on examples from physics and number theory.

Elmabrok, A. S., & Hutník, O. (2012). Induced representations of the affine group and intertwining operators: I. Analytical approach. *J. Phys. A, 45*(24), 244017, 15.

Folland, G. B. (1995). *A course in abstract harmonic analysis*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL.

Gårding, L. (1947). Note on continuous representations of lie groups. *Proceedings of the National Academy of Sciences, 33*(11), 331–332.

Gelfand, I. M., Graev, M. I., & Pyatetskii-Shapiro, I. I. (1990). *Generalized functions. Vol. 6: Representation theory and automorphic functions*. Academic Press, Inc., Boston, MA. Translated from the Russian by K. A. Hirsch, Reprint of the 1969 edition.

Gelfand, I. M., Graev, M. I., & Vilenkin, N. Y. (1966). *Generalized functions. Vol. 5: Integral geometry and representation theory*. Translated from the Russian by Eugene Saletan. Academic Press, New York-London.

Gelfand, I. M., & Naimark, M. A. (1947). *Unitary representations of the group of linear transformations of the straight line*. Dokl. Akad. Nauk SSSR 55, 567C570. also p. 18C21 in Gelfands Collected Papers, vol II, Springer-Verlag, Berlin, 1988.

Gelfand, I. M., & Shilov, G. E. (1964). *Generalized functions. Vol. I: Properties and operations*. Translated by Eugene Saletan. Academic Press, New York-London.

Gelfand, I. M., & Vilenkin, N. Y. (1964). *Generalized functions. Vol. 4: Applications of harmonic analysis*. Translated by Amiel Feinstein. Academic Press, New York - London, 1964.

Hall, B. C. (2003). *Lie groups, Lie algebras, and representations : an elementary introduction*. Graduate texts in mathematics ; 222. Springer, New York.

Harish-Chandra (1953). Representations of a semisimple Lie group on a Banach space. I. *Trans. Amer. Math. Soc.*, 75:185–243.

Hörmander, L. (2015). *The analysis of linear partial differential operators I: Distribution theory and Fourier analysis*. Springer.

Howe, R., & Tan, E. (1992). *Non-abelian harmonic analysis : applications of SL(2,R)*. Universitext. Springer-Verlag, New York.

Kaniuth, E., & Taylor, K. F. (2013). *Induced representations of locally compact groups*, volume 197. Cambridge university press.

Kirillov, A. A. (1976). *Elements of the theory of representations*. Springer-Verlag, Berlin-New York. Translated from the Russian by Edwin Hewitt, Grundlehren der Mathematischen Wissenschaften, Band 220.

Kisil, V. V. (2014). The real and complex techniques in harmonic analysis from the point of view of covariant transform. *Eurasian Math. J., 5*, 95–121. 1209.5072. http://emj.enu.kz/images/pdf/2014/5-1-4.pdfOn-line.

Knapp, A. W. (1986). *Representation theory of semisimple groups : an overview based on examples*. Princeton mathematical series ; 36. Princeton University Press, Princeton, N.J.

Lang, S. (1985). $SL_2(\mathbb{R})$, volume 105 of graduate texts in mathematics.

Mackey, G. W. (1952). Induced representations of locally compact groups i. *Annals of mathematics, 55*(1), 101–139.

Sally, P. J. (1970). Intertwining operators and the representations of sl(2, r). *Journal of Functional Analysis, 6*(3), 441–453.

Strichartz, R. S. (2003). *A guide to distribution theory and Fourier transforms*. World Scientific Publishing Co., Inc., River Edge, NJ. Reprint of the 1994 original [CRC, Boca Raton; MR1276724 (95f:42001)].

Sugiura, M. (1990). *Unitary representations and harmonic analysis: an introduction*, volume 44. Elsevier.

Taylor, M. E. (1986). *Noncommutative harmonic analysis*, volume 22 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI.

Vladimirov, V. S. (2002). *Methods of the theory of generalized functions*. Analytical methods and special functions ; v. 6. Taylor and Francis, London.

Appendices

A. Generalised Functions

A generalised function (also called a distribution) is a generalisation of the classical notion of a function. In the following, we provide basic definitions. For more information, refer to [23, 10, 26].

Definition A.1. [23] Let Ω be an open subset of $\mathbb{R}^n$. The set of test functions $\mathcal{D}(\Omega)$ consists of all real functions $\varphi(x)$ defined in Ω vanishing outside a bounded subset of Ω that stays away from the boundary of Ω, such that all partial derivatives of all order of φ are continuous.

Remark A.2. [14] We can define the test function to be the elements of the space $C_0^\infty(\Omega)$.

Definition A.3. [23] The set of all continuous linear functional on $\mathcal{D}$ is denoted by $\mathcal{D}'$, and its elements are called generalised functions. By functional, we mean the real or complex valued function on $\mathcal{D}$ written (f, φ) where $\varphi \in \mathcal{D}$.

A generalized function f is a linear functional if it satisfies the identity:

$$a_1(f, \varphi_1) + a_2(f, \varphi_2) = (f, a_1\varphi_1 + a_2\varphi_2).$$

By continuous, we mean that if φ_1 is close enough to φ, then (f, φ_1) is close to (f, φ).

Remark A.4. [23] If f is a function such that the integral $\int f(x)\varphi(x)dx$ exists for every test function ϕ, then:

$$(f, \varphi) = \int f(x)\varphi(x)dx$$

defines a generalized function.

Theorem A.5. (The Kernel Theorem) [11, p.18]

Every bilinear functional (φ, ψ) on the space $\mathcal{D}$ of all infinitely differentiable functions that have bounded supports and which is continuous in each of the arguments φ and ψ has the form:

$$(\varphi, \psi) = (k, \varphi(x) \otimes \psi(y)),$$

where k is a continuous linear functional on the space $\mathcal{D}(X \times Y)$ of infinitely differentiable functions of two variables having bounded supports.

Definition A.6. A function $f(x)$ is called a homogeneous function of degree λ if:

$$f(\alpha x) = \alpha^\lambda f(x), \quad \alpha \neq 0.$$

A function $f_1(x)$ is called an associated homogeneous function of degree λ if:

$$f_1(\alpha x) = \alpha^\lambda[f_1(x) + \ln|\alpha| f_0(x)], \quad \alpha \neq 0.$$

$f_0(x)$ is a homogeneous function of of degree λ.

B. Covariant Transform

Definition B.1. [18] Let ρ be a representation of a group G in a space V and F be an operator acting from V to a space U. We define a covariant transform $\mathcal{W}_F^\rho$ acting from V to the space $L(G, U)$ of U-valued functions on G by the formula:

$$\mathcal{W}_F^\rho : v \mapsto \hat{v}(g) = F(\rho(g^{-1})v), \quad v \in V, g \in G. \tag{B.1}$$

The operator F is called a fiducial operator.

Example B.2. [18] Let V be a Hilbert space with an inner product $\langle ., . \rangle$ and ρ be a unitary representation of a group G in the space V. Let $F : V \to \mathbb{C}$ be the functional $v \mapsto \langle v, v_0 \rangle$ defined by a vector $v_0 \in V$. The vector v_0 is called the mother wavelet. In the set-up, transformation (B.1) is the well-known expression for a wavelet transform

$$\mathcal{W} : v \mapsto \tilde{v}(g) = \langle \rho(g^{-1})v, v_0 \rangle = \langle v, \rho(g)v_0 \rangle, \quad v \in V, g \in G. \tag{B.2}$$

The family of the vectors $v_g = \rho(g)v_0$ is called wavelets or coherent states. The image of (B.2) consists of scalar valued functions on G.

Proposition B.3. [18] Let G be a Lie group and ρ be a representation of G in a space V. Let $[\mathcal{W}f](g) = F(\rho(g^{-1}f)$ be a covariant transform defined by a fiducial operator $F : V \to U$. Then the right shift $[\mathcal{W}f](gg')$ by g' is the covariant transform

$$[\mathcal{W}'f](g) = F'(\rho(g^{-1})f),$$

defined by the fiducial operator $F' = F \circ \rho(g^{-1})$. In other words, the covariant transform intertwines right shifts $R(g) : f(h) \to f(gh)$ on the group G with the associated action

$$\rho_B(g) : F \mapsto F \circ \rho(g^{-1}),$$

on fiducial operators

$$R(g) \circ \mathcal{W}_F = \mathcal{W}_{\rho_B(g)F}, \quad g \in G.$$

Corollary B.4. [18] Let a fiducial operator F be a null solution for the operator $A = \sum_j a_j d\rho_B^{X_j}$, where $X_j \in \mathfrak{g}$ and a_j are constants. Then the covariant transform $[\mathcal{W}_F](g) = F(\rho(g^{-1})f)$ for any f satisfies

$$D(\mathcal{W}_F f) = 0 \quad where \quad D = \sum_j \bar{a}_j \mathfrak{L}^{X_j}.$$

Here, $\mathfrak{L}^{X_j}$ are the left invariant fields (Lie derivatives) on G corresponding to X_j.

Example B.5. Consider the representation

$$[\pi_p(a,b)f](x) = a^{\frac{-1}{p}} f\left(\frac{x-b}{a}\right), \tag{B.3}$$

of the affine group on the space $L_p(\mathbb{R})$ with $p = 1$.

Let $X_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $X_N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ be the basis of the Lie algebra $\mathfrak{g}$ of the affine group. They generate one-parameter subgroups of $\mathfrak{g}$

$$a(t) = \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad n(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

then the derived representations are

$$[d\pi(X_A)f](x) = -f(x) - xf'(x),$$

$$[d\pi(X_N)f](x) = -f'(x).$$

The corresponding left invariant vector fields on the affine group are

$$\mathfrak{L}^{X_A} = a\partial_a, \quad \mathfrak{L}^{X_N} = a\partial_b.$$

The mother wavelet $\frac{1}{x+i}$ is a null solution of the operator

$$-d\pi(X_A) - id\pi(X_N) = I + (x+i)\frac{d}{dx}.$$

Therefore, the image of the covariant transform with the fiducial operator

$$F_+(f) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(x)}{i-x}dx,$$

consists of the null solutions to the operator

$$-\mathfrak{L}^{X_A} + i\mathfrak{L}^{X_N} = ia(\partial_b + i\partial_a),$$

that is essence of the Cauchy-Riemann operator $\partial_{\bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ in the upper half-plane.

C. The Contravariant Transform

Define the left action Λ of a group G on a space of functions over G by

$$\Lambda(g) : f(h) \to f(g^{-1}h).$$

An object invariant under the left action Λ is called left invariant. In particular, let L and L' be two left invariant spaces of functions on G. We say that a pairing $\langle .,. \rangle : L \times L' \to \mathbb{C}$ is a left invariant if

$$\langle \Lambda(g) f, \Lambda(g) f' \rangle = \langle f, f' \rangle,$$

for all $f \in L$, $f' \in L'$.

Definition C.1. [18] Let $\langle .,. \rangle$ be a left pairing on $L \times L'$ as above, let ρ be a representation of G in a space V, we define the function $w(g) = \rho(g) w_0$ for $w_0 \in V$ such that $w(g) \in L'$ in a suitable sense. The contravariant transform $\mathcal{M}^{\rho}_{w_0}$ is a map $L \to V$ defined by the pairing

$$\mathcal{M}^{\rho}_{w_0} : f \to \langle f, w \rangle, \quad \text{where} f \in L.$$

Definition C.2. Let $\widetilde{H}^p(\mathbb{R}^2_+)$, $1 < p < \infty$, be the space of all holomorphic functions f which satisfy the following norm:

$$\|f\|_{\widetilde{H}^p} = \lim_{a \to 0} \frac{1}{a} \left(\int_{-\infty}^{\infty} |f(a,b)|^p db \right)^{\frac{1}{p}}.$$

Example C.3. [18] Let G be the affine group with measure $d_\mu(a,b) = \frac{db}{a}$ and the representation π_p (B.3). The following invariant pairing on G is called Hardy pairing:

$$\langle f_1, f_2 \rangle = \lim_{a \to 0} \int_{-\infty}^{\infty} f_1(a,b) f_2(a,b) \frac{db}{a},$$

where $f_1 \in \widetilde{H}^p(\mathbb{R}^2_+)$ and $f_2 \in \widetilde{H}^q(\mathbb{R}^2_+)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
In this case, we can choose the function $v_0(x) = \frac{1}{i\pi} \frac{1}{x+i} \in L_p(\mathbb{R})$. Then, the contravariant transform is

$$\begin{aligned}
[\mathcal{M}_{v_0} f](x) &= \langle f, \pi_p(a,b) v_0 \rangle \\
&= \lim_{a \to 0} \int_{-\infty}^{\infty} f(a,b) \frac{a^{\frac{-1}{p}+1}}{\pi i (x + ia - b)} db \\
&= \lim_{a \to 0} \frac{a^{\frac{-1}{p}+1}}{\pi i} \int_{-\infty}^{\infty} \frac{f(a,b) db}{b - (x + ia)}.
\end{aligned} \tag{C.1}$$

The contravariant transform (C.1) is the boundary value of the the Cauchy integral as $a \to 0$.