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Hermite Finite Element Method for Variable Coefficient Damping Beam Vibration Problem

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Abstract

Beam is the most common component in mechanical equipment and construction. The heterogeneous beam and the variable cross-section beam both have good structural performance. In this study, a Hermite finite element method is proposed for variable coefficient damping beam vibration. The first-order time derivative is approximated by the Richardson format, the second-order time derivative is approximated by the central difference method, and the fully discrete scheme is obtained. The variable coefficients are processed, the error estimate is analyzed in detail. Finally, we verify the validity of the scheme by Matlab and observe influence of damping coefficient variation on beam vibration.

Keywords: hermite element, numerical simulation, variable cross-section beam, heterogeneous beam

1. Introduction

Consider the following heterogeneous beam vibration problem:

$$\begin{cases} \frac{\partial^2}{\partial x^2}(E(x)I\frac{\partial^2 u}{\partial x^2}) + \rho A\frac{\partial^2 u}{\partial t^2} + \mu\frac{\partial u}{\partial t} = f(x,t),\ x \in (0,L),\ t \in (0,T], \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ x \in [0,L], \\ u(0,t) = u(L,t) = 0,\ u_x(0,t) = u_x(L,t) = 0,\ t \in [0,T], \end{cases} \tag{1}$$

where $u(x,t)$ represents displacement, $E(x)I$ is the bending stiffness, $E(x)$ is Youngs modulus, and I is the area moment of inertia, ρ is density, A is the cross-sectional area, $\mu > 0$ is damping coefficient, L is constant, $\varphi(x), \psi(x)$ and $f(x,t)$ are smooth functions which are given known, $E(x)$ is a smooth function over $[0,L]$.

$$0 < e \le E(x) \le E < \infty$$

The variable cross-section beam vibration problem we consider is as follows:

$$\begin{cases} \frac{\partial^2}{\partial x^2}(EI(x)\frac{\partial^2 u}{\partial x^2}) + \rho A(x)\frac{\partial^2 u}{\partial t^2} + \mu\frac{\partial u}{\partial t} = f(x,t),\ x \in (0,L),\ t \in (0,T], \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ x \in [0,L], \\ u(0,t) = u(L,t) = 0,\ u_x(0,t) = u_x(L,t) = 0,\ t \in [0,T], \end{cases} \tag{2}$$

where $EI(x)$ is the bending stiffness, $I(x)$ is the area moment of inertia, $A(x)$ is the cross-sectional area, $I(x)$, $A(x)$ is a smooth function over $[0,L]$.

$$0 < I_1 \le I(x) \le I_0 < \infty$$

$$0 < A_1 \le A(x) \le A_0 < \infty$$

Beam is the most common structural form in engineering applications. Uniform beam can not meet the needs of practical engineering. The non-uniformity of the beam is caused by the continuous variation of the cross-section size or material properties along the length. Beam vibration will lead to the deterioration of its physical properties, which will change the elastic modulus. The heterogeneous beam and the variable cross-section beam are widely used in the engineering design and practical application (Zuo, Fang, Zhong & Guo, 2018; Liu, 2017; Hu, 2016). Gupta numerically calculated the vibration of a conical beam and studied the influence of taper on the solution accuracy (Gupta & Arvind, 1985). Lee developed the solution theory for the vibration of non-uniform Bernoulli-Euler beams with general elastic constrained boundary conditions (Lee, Ke & Kuo, 1990). The analytical solutions of non-uniformly damped beams under various boundary conditions are presented (Ece, Aydogdu & Taskin, 2007; Tan, Wang & Jiao, 2016; Ait Atmane, Tounsi, Meftah, & Belhadj, 2011; Sohani & Eipakchi, 2018). A high-precision three-layer, three-parameter implicit scheme for solving the beam vibration equation was established (Zhou, Chen, Chen, 2010). Friswell proposed a viscoelastic foundation beam model and analyzed dynamic characteristics of beams by finite element method (Friswell, Adhikari & Lei, 2007). Alshorbagy studied functionally gradient beams by finite element method and got a numerical approximation of the motion equation (Alshorbagy, Eltaher & Mahmoud, 2011). Rahman established a one-dimensional finite element model for vibration of functionally gradient materials (Khan, Naushad Alam & Wajid, 2016).

Mathews analyzed the classical beam theory for three different beams with variable cross-section (Toudehdehghan & Mathews, 2019). Tang and Yin used Hermite finite element to study the vibration of a class of viscoelastic beams and gave a numerical example (Tang & Yin, 2021).

The rest of this paper is as follows. In Section 2, we give the semi-discrete and fully discrete schemes for (1), and consider the error estimates of the schemes. In Section 3, equation (2) is treated similarly to the Section 2. Numerical experiments are performed in Section 4 to verify the validity of the schemes and explore the influence of variation of damping coefficient. Conclusions are given in Section 5.

2. Finite Element Approximation

Let $I = [0, L]$, introducing the Sobolev space $H_0^2(I) = \{u|u \in H^2(I), u(0, t) = u(L, t) = 0, u_x(0, t) = u_x(L, t) = 0\}$. Take the inner product on both sides of equation (1) with $v(x) \in H_0^2(I)$.

$$(\frac{\partial^2}{\partial x^2}(E(x)I\frac{\partial^2 u}{\partial x^2}), v) + (\rho A\frac{\partial^2 u}{\partial t^2}, v) + (\mu\frac{\partial u}{\partial t}, v) = (f, v). \tag{3}$$

Based on (3), using Green's formula and initial boundary value conditions, a weak formulation for problem (1) can be obtained: find $u \in H_0^2(I)$, satisfying

$$\begin{cases} (E(x)Iu_{xx}, v_{xx}) + (\rho Au_{tt}, v) + (\mu u_t, v) = (f, v),\ \forall v \in H_0^2(I), \\ u(x, 0) = \varphi(x),\ u_t(x, 0) = \psi(x). \end{cases} \tag{4}$$

2.1 Error Estimation of Semi-discrete Finite Element Schemes

We consider $I_h : 0 = x_0 < x_1 < \cdots < x_M = L$ being a uniform partitioning of the interval $[0, L]$, $h = \frac{L}{M}$, $x_j = jh$, $j = 0, 1, 2, \cdots, M$. We define the finite element space V_h is:

$$V_h = \{v \in H_0^2(I) : v_h \in P_3\}$$

where P_3 denotes a piecewise cubic Hermite polynomial over I_h. The semi-discrete formulation of (3) is given as follows: find $u_h \in V_h$ satisfying

$$(E(x)Iu_{h,xx}, v_{h,xx}) + (\rho Au_{h,tt}, v_h) + (\mu u_{h,t}, v_h) = (f, v_h), \forall v_h \in V_h. \tag{5}$$

In order to estimate the error, firstly, we introduce the elliptic projection $R_h : H_0^2(I) \to V_h$ such that

$$(E(x)IR_hu_{xx}, v_{xx}) = (E(x)Iu_{xx}, v_{xx}),\ \forall v \in V_h. \tag{6}$$

The projection has the following estimate (Ciarlet, 1978).

Lemma 1 *We have, for any* $u \in H_0^2 \bigcap H^4$.

$$||u - R_hu|| + h||u - R_hu||_1 + h^2||u - R_hu||_2 \le Ch^4||u||_{k+1}. \tag{7}$$

Now, we prove the estimate for the error between the solution of the semi-discrete and exact solution.

Theorem 2 *Let u be the solutions of (1),* u_h *be the solutions of (5), and* $u \in H_0^2 \bigcap H^4$*. Then we have the following error estimate:*

$$||u - u_h|| \le Ch^4[||u||_4 + \int_0^t((\int_0^\tau(||u_{tt}||^2 + ||u_t||^2)ds)^{\frac{1}{2}})d\tau], \tag{8}$$

where C is a constant, independent of τ *and h.*

Proof. Based on Lemma 1, we obtain

$$||\eta|| = ||u - R_hu|| \le Ch^4||u||_4. \tag{9}$$

Subtracting (5) from (4), we get

$$(EI(u_{xx} - u_{h,xx}), v_{h,xx}) + (\rho A(u_{tt} - u_{h,tt}), v_h) + (\mu(u_t - u_{h,t}), v_h) = 0. \tag{10}$$

Now we decompose the error as

$$u - u_h = u - R_h u + R_h u - u_h = \eta + \theta$$

Where $\eta = u - R_h u$ and $\theta = R_h u - u_h$, (10) can be written as

$$(E(x)I\theta_{xx}, v_{h,xx}) + (\rho A\theta_{tt}, v_h) + (\mu\theta_t, v_h) = -(\rho A\eta_{tt}, v_h) - (\mu\eta_t, v_h). \tag{11}$$

Letting $v_h = \theta_t$ in (11), we have

$$(E(x)I\theta_{xx}, \theta_{t,xx}) + (\rho A\theta_{tt}, \theta_t) + (\mu\theta_t, \theta_t) = -(\rho A\eta_{tt}, \theta_t) - (\mu\eta_t, \theta_t). \tag{12}$$

By the Young inequality, we get

$$\frac{eI}{2}\frac{d}{dt}||\theta_{xx}||^2 + \frac{\rho A}{2}\frac{d}{dt}||\theta_t||^2 \le \frac{\rho^2 A^2}{2\mu}||\eta_{tt}||^2 + \frac{\mu}{2}||\eta_t||^2. \tag{13}$$

Multiplying both sides by 2, integrating (13) over $[0, t]$ and taking $||\theta_{xx}(0)|| = 0, ||\theta_t(0)|| = 0$

$$eI||\theta_{xx}||^2 + \rho A||\theta_t||^2 \le \int_0^t \frac{\rho^2 A^2}{\mu}||\eta_{tt}||^2 ds + \int_0^t \mu||\eta_t||^2 ds. \tag{14}$$

Based Lemma 1, the right-hand term of the (14) is

$$\int_0^t \frac{\rho^2 A^2}{\mu}||\eta_{tt}||^2 ds + \int_0^t \mu||\eta_t||^2 ds \le Ch^8(\int_0^t ||u_t||_4^2 ds + \int_0^t ||u_{tt}||_4^2 ds). \tag{15}$$

Considering $||\theta_{xx}||^2 > 0$ and $||\theta|| \le ||\theta(x, 0)|| + \int_0^t ||\theta_t||ds$, we get

$$||\theta|| \le Ch^4 \int_0^t (\int_0^\tau (||u_{tt}||_4^2 + ||u_t||_4^2)ds)^{\frac{1}{2}} d\tau. \tag{16}$$

Using triangle inequality, we deduce

$$||u - u_h|| = ||u - R_h u + R_h u - u_h|| \le ||\eta|| + ||\theta||. \tag{17}$$

The estimate (8) is proved.

Theorem 3 *In Theorem 2, we get the L^2 norm error estimate for the semi-discrete approximation. Then, we obtain H^2 norm error estimate is*

$$||u - u_h||_2 \le Ch^2[||u||_4 + (\int_0^t (h^4||u_t||_4^2 + h^4||u_{tt}||_4^2)ds)^{\frac{1}{2}}], \tag{18}$$

where C is a constant, independent of τ and h.

Proof. On the basis of (14), drop $||\theta_t||^2$, we get

$$||\theta_{xx}||^2 \le \int_0^t \frac{\rho^2 A^2}{eI\mu}||\eta_{tt}||^2 ds + \int_0^t \frac{\mu}{eI}||\eta_t||^2 ds. \tag{19}$$

Based Lemma 1, the right-hand term of the (19) is

$$\int_0^t \frac{\rho^2 A^2}{eI\mu}||\eta_{tt}||^2 ds + \int_0^t \frac{\mu}{eI}||\eta_t||^2 ds \le C\int_0^t (h^8||u_t||_4^2 ds + h^8||u_{tt}||_4^2)ds. \tag{20}$$

Combining (19)and (20), we have

$$||\theta||_2 = ||\theta_{xx}|| \le Ch^2(\int_0^t (h^4||u_t||_4^2 ds + h^4||u_{tt}||_4^2)ds)^{\frac{1}{2}}. \tag{21}$$

Appling triangle inequality, we get (18).

2.2 Error Estimation of Fully Discrete Finite Element Schemes

We consider $0 = t_0 < t_1 < \cdots < t_N = T$ being an uniform partition of the interval $[0, T]$, in which $\tau = \frac{T}{N}$, $t_n = n\tau$, $n = 0, 1, 2, \cdots, N$. The first order time derivative in equation (4) can be approximated as Richardson format, the second order time derivative is approximated as central difference quotient. The fully discrete finite element discretization of (1) is given as follows: find $u_h^n \in V_h$ such that

$$(E(x)I\frac{u_{h,xx}^{n+1}+u_{h,xx}^{n-1}}{2}, v_{h,xx}) + (\rho A\frac{u_h^{n+1}-2u_h^n+u_h^{n-1}}{\tau^2}, v_h) + (\mu\frac{u_h^{n+1}-u_h^{n-1}}{2\tau}, v_h) = (f^n, v_h), \forall v_h \in V_h. \tag{22}$$

In this section, we will estimate the error between the solution of the fully discrete and exact solution.

Theorem 4 *Let u be the solutions of (3), u_h be the solutions of (22), and $u \in H_0^2 \bigcap H^4$. Then we have the following error estimate:*

$$\|u^n - u_h^n\| \leq C(\tau^2 + h^4), \tag{23}$$

where C is a constant, independent of τ and h.

Proof. Evaluating (3) at $t = t_n$, subtracting (22) from (3), we get

$$\begin{aligned}&(E(x)I(u_{xx}^n - \frac{u_{h,xx}^{n+1} + u_{h,xx}^{n-1}}{2}), v_{h,xx}) + (\rho A(u_{tt}^n - \frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\tau^2}), v_h)\\ &+ (\mu(u_t^n - \frac{u_h^{n+1} - u_h^{n-1}}{2\tau}), v_h) = 0.\end{aligned} \tag{24}$$

We decompose the error as

$$u^n - u_h^n = u^n - R_h u^n + R_h u^n - u_h^n = \eta^n + \theta^n$$

Where $\eta^n = u^n - R_h u^n$ and $\theta^n = R_h u^n - u_h^n$, then (24) can be written as

$$\begin{aligned}&(E(x)I\theta_{xx}^n, v_{h,xx}) + (\rho A\frac{\theta^{n+1} - 2\theta^n + \theta^{n-1}}{\tau^2}, v_h) + (\mu\frac{\theta^{n+1} - \theta^{n-1}}{2\tau}, v_h)\\ &= (\rho A(\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} - u_{tt}^n), v_h) - (\rho A\frac{\eta^{n+1} - 2\eta^n + \eta^{n-1}}{\tau^2}, v_h)\\ &+ (\mu(\frac{u^{n+1} - u^{n-1}}{2\tau} - u_t^n), v_h) - (\mu\frac{\eta^{n+1} - \eta^{n-1}}{2\tau}, v_h)\\ &- (E(x)I(u_{h,xx}^n - \frac{u_{h,xx}^{n+1} + u_{h,xx}^{n-1}}{2}), v_{h,xx}).\end{aligned} \tag{25}$$

Taking $v_h = \frac{\theta^{n+1}+\theta^{n-1}}{2}$ in (25), using $\varepsilon - Cauchy$ inequality and by

$$u_{h,xx}^n - \frac{u_{h,xx}^{n+1} + u_{h,xx}^{n-1}}{2} = \frac{\tau^2}{2}\frac{\partial^4 u}{\partial x^2 \partial t^2}$$

For $(\mu\frac{\theta^{n+1}-\theta^{n-1}}{2\tau}, \frac{\theta^{n+1}+\theta^{n-1}}{2})$, we deduce

$$\begin{aligned}&(\mu\frac{\theta^{n+1} - \theta^{n-1}}{2\tau}, \frac{\theta^{n+1} + \theta^{n-1}}{2})\\ &= \frac{\mu}{2}(\frac{\theta^{n+1} - \theta^{n-1}}{2\tau}, \frac{\theta^{n+1} + \theta^{n-1}}{2}) + \frac{\mu}{2}(\frac{\theta^{n+1} - \theta^{n-1}}{2\tau}, \frac{\theta^{n+1} + \theta^{n-1}}{2})\\ &= \frac{\mu}{8\tau}(\|\theta^{n+1}\|^2 - \|\theta^{n-1}\|^2) + \frac{\mu}{2\tau}\|\frac{\theta^{n+1} + \theta^{n-1}}{2}\|^2 - \frac{\mu}{4\tau}(\theta^{n-1}, \theta^{n+1} + \theta^{n-1}).\end{aligned} \tag{26}$$

Then, we get the following equation, where $\alpha \geq \frac{1}{2}$

$$\begin{aligned}&\frac{\rho A}{2}(\|\frac{\theta^{n+1} - \theta^n}{\tau}\|^2 - \|\frac{\theta^n - \theta^{n-1}}{\tau}\|^2) + \frac{\mu}{8\tau}(\|\theta^{n+1}\|^2 - \|\theta^{n-1}\|^2)\\ &\leq \frac{2\tau}{\mu}\|\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} - u_{tt}^n\|^2 + \frac{2\tau}{\mu}\|\frac{\eta^{n+1} - 2\eta^n + \eta^{n-1}}{\tau^2}\|^2 + 2\mu\tau\|\frac{u^{n+1} - u^{n-1}}{2\tau} - u_t^n\|^2\\ &+ 2\mu\tau\|\frac{\eta^{n+1} - \eta^{n-1}}{2\tau}\|^2 + \frac{E^2 I}{e}\|\frac{\tau^2}{2}\frac{\partial^4 u}{\partial x^2 \partial t^2}\|^2 + \frac{\mu}{4\tau}(\alpha\|\theta^{n-1}\|^2 + \frac{1}{4\alpha}\|\theta^{n+1}\|^2) + \frac{\mu}{4\tau}\|\theta^{n-1}\|^2.\end{aligned} \tag{27}$$

The right-hand terms of (27) will be estimated by the integral form of Taylors theorem, we have

$$|\frac{u^{n+1}-2u^n+u^{n-1}}{\tau^2}-u_{tt}^n|^2 \leq \frac{\tau^3}{126}\int_{t_{n-1}}^{t_{n+1}}|\frac{\partial^4 u}{\partial t^4}|^2 ds. \tag{28}$$

$$|\frac{\eta^{n+1}-2\eta^n+\eta^{n-1}}{\tau^2}|^2 = |\frac{1}{\tau^2}[\int_{t_n}^{t_{n+1}}\eta_{tt}(s)(t_{n+1}-s)ds+\int_{t_{n-1}}^{t_n}\eta_{tt}(s)(s-t_{n-1})ds]|^2. \tag{29}$$

$$|\frac{u^{n+1}-u^{n-1}}{2\tau}-u_t^n|^2 = \frac{\tau^3}{80}\int_{t_{n-1}}^{t_{n+1}}|\frac{\partial^3 u}{\partial t^3}|^2 ds. \tag{30}$$

$$|\frac{\eta^{n+1}-\eta^{n-1}}{2\tau}|^2 = |\frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}}\eta_t(s)ds|^2. \tag{31}$$

This is given by (28)

$$\|\frac{u^{n+1}-2u^n+u^{n-1}}{\tau^2}-u_{tt}^n\|^2 \leq \frac{\tau^3}{126}\int_{t_{n-1}}^{t_{n+1}}\|\frac{\partial^4 u}{\partial t^4}\|^2 ds. \tag{32}$$

Using Schwarz inequality

$$\|\int_a^b f(x,t)dt\|^2 \leq (b-a)\int_a^b \|f\|^2 dt.$$

And Lemma 2.1, combining (29)-(31), we have

$$\begin{aligned}
\|\frac{\eta^{n+1}-2\eta^n+\eta^{n-1}}{\tau^2}\|^2 &\leq \|\frac{1}{\tau^2}\int_{t_n}^{t_{n+1}}\eta_{tt}(s)(t_{n+1}-s)ds\|^2+\|\frac{1}{\tau^2}\int_{t_{n-1}}^{t_n}\eta_{tt}(s)(s-t_{n-1})ds\|^2\\
&\leq \frac{1}{\tau}\int_{t_n}^{t_{n+1}}\|\eta_{tt}(s)\|^2 ds+\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\|\eta_{tt}(s)\|^2 ds\\
&= \frac{1}{\tau}\int_{t_{n-1}}^{t_{n+1}}\|\eta_{tt}(s)\|^2 \leq \frac{1}{\tau}Ch^8\int_{t_{n-1}}^{t_{n+1}}\|u_{tt}\|_4^2 ds.
\end{aligned} \tag{33}$$

and

$$\|\frac{u^{n+1}-u^{n-1}}{2\tau}-u_t^n\|^2 \leq \frac{\tau^3}{80}\int_{t_{n-1}}^{t_{n+1}}\|\frac{\partial^3 u}{\partial t^3}\|^2 ds. \tag{34}$$

and

$$\begin{aligned}
\|\frac{\eta^{n+1}-\eta^{n-1}}{2\tau}\|^2 = \|\frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}}\rho_t(s)ds\|^2 &\leq \frac{1}{2\tau}\int_{t_{n-1}}^{t_{n+1}}\|\eta_t(s)\|^2 ds\\
&\leq \frac{1}{\tau}Ch^8\int_{t_{n-1}}^{t_{n+1}}\|u_t\|_4^2 ds.
\end{aligned} \tag{35}$$

Substitute (32)-(35) into (27), we get

$$\begin{aligned}
&\frac{\rho A}{2}(\|\frac{\theta^{n+1}-\theta^n}{\tau}\|^2-\|\frac{\theta^n-\theta^{n-1}}{\tau}\|^2)+(\frac{\mu}{8\tau}-\frac{\mu}{16\alpha\tau})(\|\theta^{n+1}\|^2-\|\theta^{n-1}\|^2)\\
&\leq \frac{\tau^4}{63\mu}\int_{t_{n-1}}^{t_{n+1}}\|\frac{\partial^4 u}{\partial t^4}\|^2 ds+Ch^8\int_{t_{n-1}}^{t_{n+1}}\|u_{tt}\|_4^2 ds+\frac{\mu}{40}\tau^4\int_{t_{n-1}}^{t_{n+1}}\|u_{tt}\|^2 ds\\
&+Ch^8\int_{t_{n-1}}^{t_{n+1}}\|u_t\|_4^2 ds+\frac{E^2 I}{e}\|\frac{\tau^2}{2}\frac{\partial^4 u}{\partial x^2\partial t^2}\|^2+\frac{C}{\tau}\|\theta^{n-1}\|^2.
\end{aligned} \tag{36}$$

Summing from 1 to n, we obtain

$$\begin{aligned}
&\frac{\rho A}{2}\|\frac{\theta^{n+1}-\theta^n}{\tau}\|^2+(\frac{\mu}{8\tau}-\frac{\mu}{16\tau\alpha})(\|\theta^{n+1}\|^2+\|\theta^n\|^2) \leq \frac{\rho A}{2}\|\frac{\theta^1-\theta^0}{\tau}\|^2\\
&+(\frac{\mu}{8\tau}-\frac{\mu}{16\tau\alpha})(\|\theta^1\|^2+\|\theta^0\|^2)+\frac{\tau^4}{63\mu}\int_0^{t_{n+1}}\|\frac{\partial^4 u}{\partial t^4}\|^2 ds+Ch^8\int_0^{t_{n+1}}\|u_{tt}\|_4^2 ds\\
&+\frac{\mu}{40}\tau^4\int_0^{t_{n+1}}\|u_{tt}\|^2 ds+Ch^8\int_0^{t_{n+1}}\|u_t\|_4^2 ds+\frac{E^2 I}{e}\tau^4\sum_{i=1}^{n}\|\frac{\partial^4 u}{\partial x^2\partial t^2}\|^2+\frac{C}{\tau}\sum_{i=0}^{n-1}\|\theta^i\|^2.
\end{aligned} \tag{37}$$

By discrete Gronwall inequality, we get

$$\begin{aligned}
&\frac{\rho A}{2}\|\frac{\theta^{n+1}-\theta^{n}}{\tau}\|^2+(\frac{\mu}{8\tau}-\frac{\mu}{16\tau\alpha})(\|\theta^{n+1}\|^2+\|\theta^{n}\|^2)\\
&\leq \frac{\rho A}{2}\|\frac{\theta^{1}-\theta^{0}}{\tau}\|^2+(\frac{\mu}{8\tau}-\frac{\mu}{16\alpha})(\|\theta^{1}\|^2+\|\theta^{0}\|^2)+\frac{\tau^4}{63\mu}\int_0^{t_{n+1}}\|\frac{\partial^4 u}{\partial t^4}\|^2 ds\\
&+Ch^8\int_0^{t_{n+1}}\|u_{tt}\|_4^2 ds+\frac{\mu}{40}\tau^4\int_0^{t_{n+1}}\|u_{tt}\|^2 ds+Ch^8\int_0^{t_{n+1}}\|u_t\|_4^2 ds.
\end{aligned} \tag{38}$$

Based on (37), we have

$$\begin{aligned}
\|\theta_t^n\| = \|\theta_t^n-\frac{\theta^{n+1}-\theta^{n}}{\tau}+\frac{\theta^{n+1}-\theta^{n}}{\tau}\| &\leq \|\theta_t^n-\frac{\theta^{n+1}-\theta^{n}}{\tau}\|+\|\frac{\theta^{n+1}-\theta^{n}}{\tau}\|\\
&\leq C(\tau^2+h^4).
\end{aligned} \tag{39}$$

We get the conclusion that

$$\|u^n-u_h^n\| \leq \|\eta^n\|+\|\theta^n\| = \|\eta^n\|+\|\theta^0\|+\int_0^t \|\theta_t^n\| \leq C(\tau^2+h^4). \tag{40}$$

This completes the proof.

3. Damping Beam With Variable Cross-section

Let $I=[0,L]$, introducing the Sobolev space $H_0^2(I)$. Take the inner product of both sides of equation (2) with $v(x)\in H_0^2(I)$.

$$(\frac{\partial^2}{\partial x^2}(EI(x)\frac{\partial^2 u}{\partial x^2}),v)+(\rho A(x)\frac{\partial^2 u}{\partial t^2},v)+(\mu\frac{\partial u}{\partial t},v)=(f,v). \tag{41}$$

Based on (41), using Green's formula and initial boundary value conditions, a weak formulation for problem (2) can be obtained: Find $u\in H_0^2(I)$, satisfying

$$\begin{cases} (EI(x)u_{xx},v_{xx})+(\rho A(x)u_{tt},v)+(\mu u_t,v)=(f,v),\ \forall v\in H_0^2(I),\\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x). \end{cases} \tag{42}$$

3.1 Error Estimation of Semi-discrete Finite Element Schemes

The semi-discrete formulation of (2) is given as follows: find $u_h\in V_h$ satisfying

$$(EI(x)u_{h,xx},v_{h,xx})+(\rho A(x)u_{h,tt},v_h)+(\mu u_{h,t},v_h)=(f,v_h),\forall v_h\in V_h. \tag{43}$$

We prove the estimate for the error between the solution of the semi-discrete and exact solution.

Theorem 5 *Let u be the solutions of (41), u_h be the solutions of (42), and $u\in H_0^2\bigcap H^4$. Then we have the following error estimate:*

$$\|u-u_h\|\leq Ch^4[\|u\|_4+\int_0^t((\int_0^\tau(\|u_{tt}\|^2+\|u_t\|^2)ds)^{\frac{1}{2}})d\tau], \tag{44}$$

Proof. Similar with the proof of Theorem 2.

3.2 Error Estimation of Fully Discrete Finite Element Schemes

We get a full-discrete finite element scheme of problem (2): find $u_h^n\in V_h$ such that

$$(EI(x)\frac{u_{h,xx}^{n+1}+u_{h,xx}^{n-1}}{2},v_{h,xx})+(\rho A(x)\frac{u_h^{n+1}-2u_h^n+u_h^{n-1}}{\tau^2},v_h)+(\mu\frac{u_h^{n+1}-u_h^{n-1}}{2\tau},v_h)=(f^n,v_h),\forall v_h\in V_h. \tag{45}$$

The error between the solution of the fully discrete and exact solution have the following estimate.

Theorem 6 *Let u be the solutions of (41), u_h be the solutions of (44), and $u\in H_0^2\bigcap H^4$. Then we have the following error estimate:*

$$\|u^n-u_h^n\|\leq C(\tau^2+h^4). \tag{46}$$

where C is a constant, independent of τ and h.

Proof. Similar with the proof of Theorem 4.

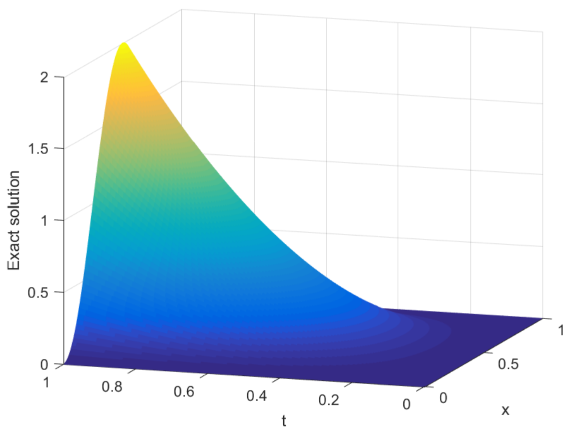

Figure 1. Exact solution with $h = \frac{1}{2^7}$

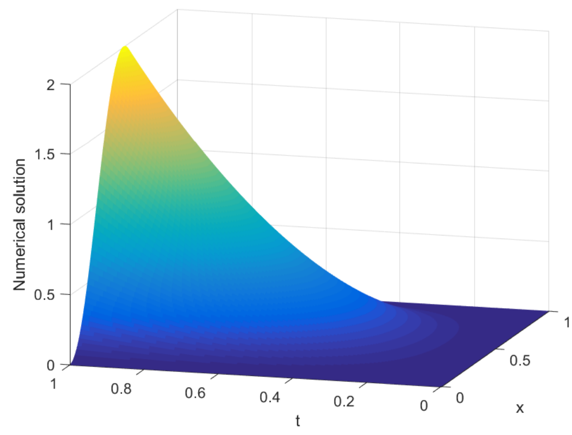

Figure 2. Solution of fully discrete finite element schemes with $h = \frac{1}{2^7}$

4. Numerical Experiments

In this part, we present four examples: Example 1 and Example 2 verify and illustrate Theorem 4, Theorem 6; Example 3 and Example 4 explore the influence of damping coefficient variation on beam vibration.

4.1 Example 1

We take $E(x) = 2 - x$, let ρ, A, μ, I, L and T in problem (1) be equal to 1. The exact solution are selected as $u(x,t) = t^2(1 - cos(2\pi x))$. Consider the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial^2}{\partial x^2}((2-x)\dfrac{\partial^2 u}{\partial x^2} + u_{tt} + u_t = t^2(-16\pi^4(2-x)cos(2\pi x) + 16\pi^3 sin(2\pi x)) \\ + 2(1 - cos(2\pi x)) + 2t(1 - cos(2\pi x)),\ x \in (0,1),\ t \in (0,1], \\ u(x,0) = 0,\ u_t(x,0) = 0,\ x \in [0,1], \\ u(0,t) = u(1,t) = 0,\ u_x(0,t) = u_x(1,t) = 0,\ t \in [0,1], \end{cases} \tag{47}$$

Table 1. Errors and spatial convergence orders of Example 1

τ	h	L^2-error	order	H^2-error	order
$\frac{1}{2^8}$	$\frac{1}{2^4}$	1.902e-05	–	4.974e-04	–
$\frac{1}{2^{10}}$	$\frac{1}{2^5}$	1.230e-06	3.951	3.199e-05	3.959
$\frac{1}{2^{12}}$	$\frac{1}{2^6}$	7.738e-08	3.991	2.077e-06	3.945
$\frac{1}{2^{14}}$	$\frac{1}{2^7}$	5.105e-09	3.922	1.361e-07	3.932

Table 2. Errors and time convergence orders of Example 1

h	τ	L^∞-error	order	L^2-error	order
$\frac{1}{1000}$	$\frac{1}{20}$	3.000e-03	–	1.700e-03	–
$\frac{1}{1000}$	$\frac{1}{80}$	2.157e-04	1.896	1.362e-04	1.839
$\frac{1}{1000}$	$\frac{1}{160}$	6.934e-05	1.638	4.327e-05	1.652
$\frac{1}{1000}$	$\frac{1}{320}$	1.632e-05	2.009	9.724e-06	2.154

Table 1 shows the L^2 errors, H^2 errors at $t = t^n$ and spatial convergence orders of the exact and finite element solutions. Table 2 shows the L^∞ errors, L^2 errors at $t = t^n$ and time convergence orders of the exact and finite element solutions. From the two tables, we can see that the convergence orders is $O(\tau^2 + h^4)$. Surface plots of the exact solution and the numerical solution are presented in Figure 1, Figure 2.

4.2 Example 2

In this example, we consider exact solution $u(x,t) = x(1-x)sin(\pi x)t^2$. Let the variable $I(x) = \frac{\pi}{64}(1 - \frac{x}{2})^4$ and $A(x) = \frac{\pi}{4}(1 - \frac{x}{2})^2$, still assume the variables E, ρ, μ, L and T in problem (2) be equal to 1.

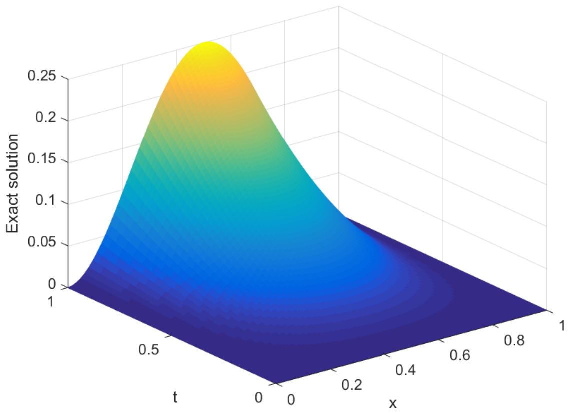

Figure 3. Exact solution with $h = \frac{1}{2^7}$

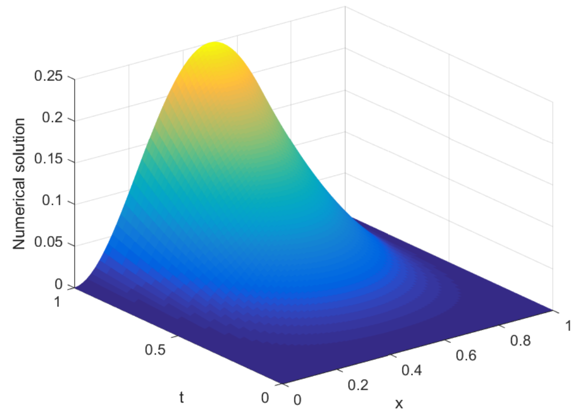

Figure 4. Solution of fully discrete finite element schemes with $h = \frac{1}{2^7}$

Table 3. Errors and spatial convergence orders of Example 2

τ	h	L^2-error	order	H^2-error	order
$\frac{1}{2^6}$	$\frac{1}{2^3}$	3.363e-05	–	9.983e-04	–
$\frac{1}{2^8}$	$\frac{1}{2^4}$	2.058e-06	4.030	7.080e-05	3.818
$\frac{1}{2^{10}}$	$\frac{1}{2^5}$	1.289e-07	4.007	4.730e-06	3.904
$\frac{1}{2^{12}}$	$\frac{1}{2^6}$	7.994e-09	4.011	3.057e-07	3.952

Table 4. Errors and time convergence orders of Example 2

h	τ	L_∞-error	order	L^2-error	order
$\frac{1}{1000}$	$\frac{1}{20}$	4.100e-03	–	2.500e-03	–
$\frac{1}{1000}$	$\frac{1}{40}$	9.363e-04	2.129	5.768e-04	2.098
$\frac{1}{1000}$	$\frac{1}{80}$	2.378e-04	1.977	1.408e-04	2.034
$\frac{1}{1000}$	$\frac{1}{160}$	5.827e-05	2.029	3.480e-05	2.017

4.3 Example 3

We set $I(x) = 5\pi(1 - \frac{x}{2})^4$ and $A(x) = \frac{\pi}{4}(1 - \frac{x}{2})^2$, initial problem value $u(x, 0) = sin(\pi x)$ and assume that the beam is in a free vibration state. Take the midpoint of the beam and draw a curve graph, we get Figure 5. It can be seen from Figure 5 that the amplitude of the beam are decreasing with increasing damping coefficient .

4.4 Example 4

Under the condition of Example 3, when the damping coefficient takes different values, the displacement u change image with x are drawn at different times. The changes of displacement with x at 0.3 seconds, 0.5 seconds, 0.6 seconds, 0.9 seconds are shown in Figure 6-9.

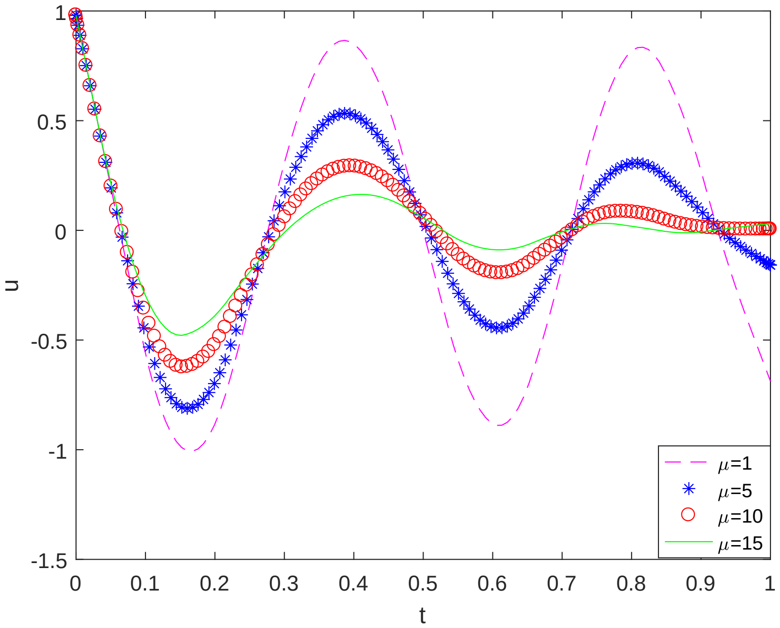

Figure 5. The comparison image of variable cross-section beam vibration amplitude when μ takes different values

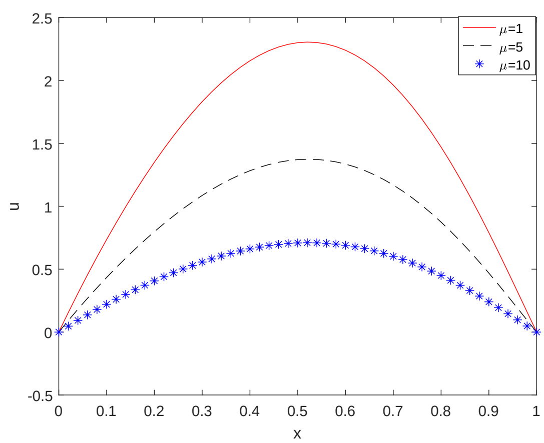

Figure 6. The image of displacement u at $0.3s$ when μ takes different values

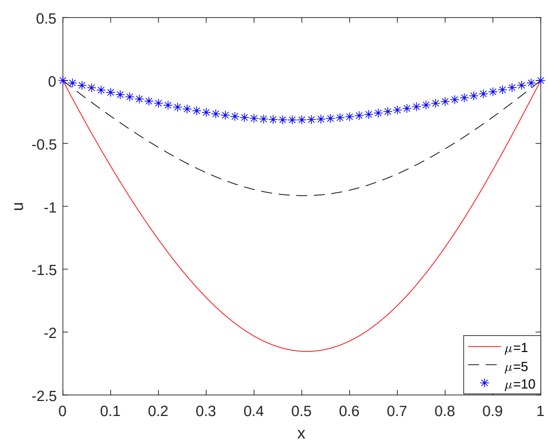

Figure 7. The image of displacement u at $0.5s$ when μ takes different values

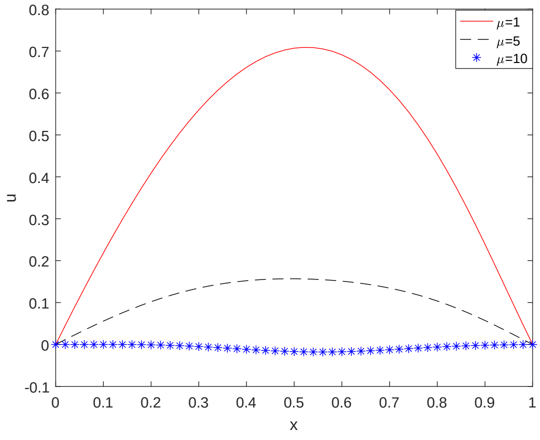

Figure 8. The image of displacement u at $0.6s$ when μ takes different values

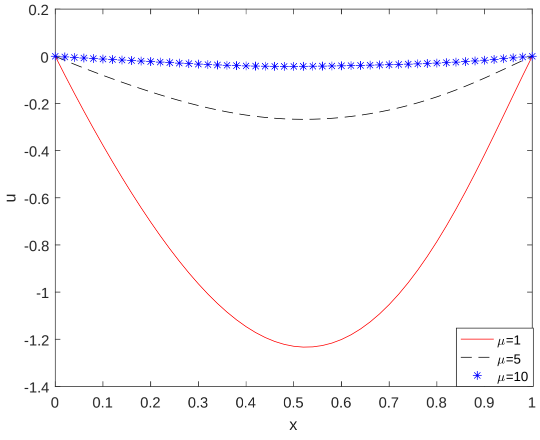

Figure 9. The image of displacement u at $0.9s$ when μ takes different values

4. Conclusion

In this paper, we have developed and analyzed numerical methods for vibration equation of the heterogeneous and the variable cross-section damping beam. By using the Richardson format and central difference quotient of the first and the second time derivative, we construct the Hermite finite element scheme of two kinds of beam equation. Next, we provide theoretical analysis to error estimates by means of elliptic operator. Several numerical experiments were presented to verify the order of finite element scheme is $O(\tau^2 + h^4)$ and demonstrate the influence of damping coefficient variation on beam vibration amplitude.

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