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Linear Maps Preserving Inverses of Tensor Products of Hermite Matrices

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Abstract

Let C be a complex field, $H_{m_1m_2}$ be a linear space of tensor products of Hermite matrices $H_{m_1}\otimes H_{m_2}$ over C , and suppose m_1, $m_2\geq 2$ are positive integers. A linear map $f : H_{m_1m_2} \to H_n$ is called a linear inverse preserver if $f(X_1 \otimes X_2)^{-1} = f\left((X_1 \otimes X_2)^{-1}\right)$ for arbitrary invertible matrix $X_1 \otimes X_2 \in H_{m_1m_2}$ .The aim of this paper is to characterize the linear maps preserving inverses of tensor products of Hermite matrices.

Keywords: linear preserver, inverses, tensor product

1. Introduction

Let I_n be the $n \times n$ identity matrix, G_n be a set of all $n \times n$ invertible matrices, P^T be the transpose matrix of P , and suppose P^* is the conjugate transpose matrix of P. We denote by M_n and H_n the algebra of all $n \times n$ matrices and Hermite matrices over C , and by $M_{m_1\cdots m_l}$ and $H_{m_1\cdots m_l}$ the linear space of tensor products of matrices $M_{m_1} \otimes \cdots \otimes M_{m_l}$ and Hermite matrices $H_{m_1} \otimes \cdots \otimes H_{m_l}$. Let P_n^2 and P_n^3 be subspace of M_n consisting of all idempotent and tripotent matrices.

In 2012, Li Zhiguang combined the linear preserver problems with tensor products of matrices in the context of quantum theory, which not only provided a new direction for the study of the preserver problems, but also had potential application value for the future development of quantum information science. As for the study of linear preserver problems of tensor products of matrices, referred to references (Lim, 2014; Huang, Shi &Sze, 2016; Duffner &da Cruz, 2016).

Changing the set of preserving functions is one of the basic ideas in the study of preserver problems. Zheng, Xu and Fošner in 2015 described the linear maps preserving idmpotence of tensor products of matrices (Zheng, Xu&Fošner, 2015). Deng, Zheng and Xu in 2021 changed the set of the linear maps from tensor products of matrices to the tensor products of symmetric matrices(Deng, Zheng&Xu , 2021). Yan in 2022 described the linear maps preserving inverses tensor products of matrices (Yan, 2022). Based on it, this paper changes the set of linear maps from the tensor products of matrices to the tensor products of Hermite matrices.

Generally speaking, linear preservers of inverses come down to linear preservers of idempotence. Xu in 2016 described a linear map that preserves the idempotence of the tensor products of Hermite matrices (Xu, 2016), and the result is as follows: A linear map $f : H_{m_1\cdots m_l} \to H_n$ $(m_1 \cdots m_l \geq n)$ is called an idempotent preserver of tensor products of matrices if and only if $f = 0$ or there exists a unitary matrix $P \in M_n$ and a canonical map π on $H_{m_1\cdots m_l}$ such that $f(X) = P\pi(X)P^*$ for any $X \in H_{m_1\cdots m_l}$ when $m_1 \cdots m_l = n$.

The purpose of this paper is to describe the linear maps preserving inverses of tensor products of Hermite matrices as well as the idempotent preserver of tensor products of Hermite matrices with the restriction of $m_1 \cdots m_l \geq n$ is removed compared with reference(Xu, 2016).It enriches the preservation problem in Hermite matrices space and provides a more perfect mathematical theory foundation for quantum information science.

Because of the complexity and arbitrariness of the matrices, We need to do a lot of calculations. In order to solve this problem, in the following proofs, we only consider the case of $l = 2$, the case of $l > 2$ are similar. In addition, the following concepts are needed for this article: A linear map π on $H_{m_1\cdots m_l}$ is called canonical, if it satisfies

$$\pi(X_1 \otimes \cdots \otimes X_l) = \tau_1(X_1) \otimes \cdots \otimes \tau_l(X_l),$$

with $\tau_i(X_i)\,(i = 1, \cdots, l) = X_i$ or X_i^T . A linear map is called generalized canonical, if it satisfies

$$\sigma(X_1 \otimes \cdots \otimes X_l) = \left(\bigoplus_{i=1}^{s} \pi_i(X_1 \otimes \cdots \otimes X_l) \otimes I_{p_i}\right) \oplus 0,$$

where π_i are the canonical maps, $p_i\,(i = 1, \cdots, s)$ are natural numbers, and I_0 means that there is not the item which corresponds to it in driect sum.

2. Preliminary Result

Before proving the theorem, we need the following lemmas.

Lemma 1 *Let $U \in M_n$ be an invertible matrix, if $UX = XU$ for any $X \in H_n$, then $U = \lambda I$ with $\lambda \neq 0$.*

Proof. Take$X = E_{ii}\,(i = 1, \cdots, n)$. $UE_{ii} = E_{ii}U$ shows that U is a diagonal matrix. Then take $X = E_{ij} + E_{ji}$ and $X = \mathbf{i}\left(E_{ij} - E_{ji}\right)(1 \leq i < j \leq n)$, and then we get that $U\left(E_{ij} + E_{ji}\right) = \left(E_{ij} + E_{ji}\right)U$and $U\mathbf{i}\left(E_{ij} - E_{ji}\right) = \mathbf{i}\left(E_{ij} - E_{ji}\right)U$. The Lemma1 can be proved.

Lemma 2(Zhang&Cao, 2001) *If $A \in P_n^2$,there exists an invertible matrix Q and a non-negative integer r such that*

$$A = Qdiag\,(I_r, 0_{n-r})\,Q^{-1},$$

where $rankA = r$.

Lemma 3(Zhang&Cao, 2001) *If $A \in P_n^3$, there exists an invertible matrix Q and non-negative integers p, q, s such that*

$$A = Qdiag\left(I_p, -I_q, 0_s\right)Q^{-1},$$

where $rankA = p + q$, $p = rank\,(A + I_n) + rank\,(A) - n$ and $p + q + s = n$.

Lemma 4(Sheng&Tang, 2020) *Let F be a field of $Ch \neq 2$. A linear map $\varphi : M_{m_1\cdots m_l}(F) \to M_n(F)\,(m_1 \cdots m_l < n)$ preserves idempotence of tensor products of matrices if and only if there exists an invertible matrix $Q \in M_n(F)$ and a generalized canonical map σ on $M_{m_1\cdots m_l}(F)$ such that*

$$\varphi(X) = Q\sigma(X)\,Q^{-1}, \forall X \in M_{m_1\cdots m_l}(F).$$

Lemma 5 *A linear map $\varphi : H_{m_1\cdots m_l} \to H_n\,(m_1 \cdots m_l < n)$ preserves idempotence of tensor products of matrices if and only if there exists a unitary matrix $T \in M_n$ and a generalized canonical map σ on $H_{m_1\cdots m_l}$ such that*

$$\varphi(X) = T\sigma(X)\,T^{*}, \forall X \in H_{m_1\cdots m_l}.$$

Proof. First, we give the proof of the sufficiency part of the lemma : If $X^2 = X$ for any $X = X_1 \otimes \cdots \otimes X_l \in H_{m_1\cdots m_l}$, then

$$\varphi(X)^2 = T\sigma(X)\,T^{*}T\sigma(X)\,T^{*} = T\sigma(X)^2\,T^{*}.$$

Because

$$(X_1 \otimes \cdots \otimes X_l)^2 = X_1^2 \otimes \cdots \otimes X_l^2 = X_1 \otimes \cdots \otimes X_l,$$

we have

$$\left(\pi\,(X_1 \otimes \cdots \otimes X_l) \otimes I_{p_i}\right)^2 = \tau_1(X_1)^2 \otimes \cdots \otimes \tau_l(X_l)^2 \otimes I_{p_i}^2 = \tau_1(X_1) \otimes \cdots \otimes \tau_l(X_l) \otimes I_{p_i},$$

and by the definitions of the canonical maps and generalized canonical maps, we have $\sigma(X)^2 = \sigma(X)$,i.e., $\varphi(X)^2 = \varphi(X)$. Then the sufficiency part is established and the necessity part is proved below.

If $\varphi = 0$, the necessity is obvious.

If $\varphi \neq 0$, let φ be linearly extended into $\tilde{\varphi} : M_{m_1\cdots m_l} \to M_n$. Since $ChC = 0$, applying the Lemma 4, we know that there exists an invertible matrix $Q \in M_n$ and a generalized canonical map σ on $M_{m_1\cdots m_l}$, such that

$$\tilde{\varphi}(X) = Q\sigma(X)\,Q^{-1}, \forall X \in M_{m_1\cdots m_l}.$$

Since $\tilde{\varphi}(H_{m_1\cdots m_l}) = \varphi(H_{m_1\cdots m_l}) \in H_n$, we derive that

$$\left(Q\sigma(X)\,Q^{-1}\right)^{*} = Q\sigma(X)\,Q^{-1}, \forall X \in H_{m_1\cdots m_l},$$

that is

$$\sigma(X)\,Q^{*}Q = Q^{*}Q\sigma(X)\,, \forall X \in H_{m_1\cdots m_l}.$$

Since $\sigma(X) \in H_n$ and $Q^{*}Q \in G_n$,we have $Q^{*}Q = \lambda I$, $\lambda \neq 0$ according to Lemma 1. Because Q is an invertible matrix, we get $|Q^{*}Q| = |Q^{*}||Q| = \lambda^n$. Suppose $|Q^{*}| = a + b\mathbf{i}$, $|Q| = a - b\mathbf{i}$. Then $\lambda > 0$. Let $T = \lambda^{-\frac{1}{2}}Q$, then $T^{*}T = I$. So T is a unitary matrix, and we derive that

$$\tilde{\varphi}(X) = T\sigma(X)\,T^{*}, \forall X \in M_{m_1\cdots m_l},$$

$$\varphi(X) = T\sigma(X)T^*, \forall X \in H_{m_1\cdots m_l}.$$

Then Lemma 5 can be proved.

Lemma 6 *Assuming $f : H_{m_1m_2} \to H_n$ is a linear map preserving inverses of tensor products of Hermite matrices, then*

$$f(I_{m_1m_2}) f(E_{ii} \otimes E_{kk}) = f(E_{ii} \otimes E_{kk}) f(I_{m_1m_2}) = f(E_{ii} \otimes E_{kk})^2,$$

where $i \in [1, m_1]$ and $k \in [1, m_2]$.

Proof. Since $I_{m_1m_2}^{-1} = I_{m_1m_2}$, we have $f(I_{m_1m_2})^{-1} = f(I_{m_1m_2})$ and $f(I_{m_1m_2})^2 = I_n$. So for any $x \neq 0, x \neq 1$

$$(I_{m_1m_2} + (x-1)(E_{ii} \otimes E_{kk}))^{-1} = I_{m_1m_2} + \left(x^{-1} - 1\right)(E_{ii} \otimes E_{kk}),$$

then

$$f(I_{m_1m_2} + (x-1)(E_{ii} \otimes E_{kk})) f\left(I_{m_1m_2} + \left(x^{-1} - 1\right)(E_{ii} \otimes E_{kk})\right) = I_n, \tag{1}$$

$$f\left(I_{m_1m_2} + \left(x^{-1} - 1\right)(E_{ii} \otimes E_{kk})\right) f(I_{m_1m_2} + (x-1)(E_{ii} \otimes E_{kk})) = I_n, \tag{2}$$

combine equation (1) and (2)

$$\left(x^{-1} - 1\right)(f(I_{m_1m_2}) f(E_{ii} - E_{kk}) - f(E_{ii} - E_{kk}) f(I_{m_1m_2})) = 0.$$

Due to the arbitrariness of x , it follows that

$$f(I_{m_1m_2}) f(E_{ii} - E_{kk}) = f(E_{ii} - E_{kk}) f(I_{m_1m_2}), \tag{3}$$

and put (3) into equation (1) or (2)

$$\left(x^{-1} + x - 2\right)\left(f(I_{m_1m_2}) f(E_{ii} - E_{kk}) - f(E_{ii} - E_{kk})^2\right) = 0.$$

Because of the arbitrariness of x , the Lemma 6 can be proved.

Lemma 7 *Assuming $f : H_{m_1m_2} \to H_n$ is a linear map preserving inverses of tensor products of Hermite matrices, and let $E_{ii} = D_i$, $E_{kk} = D_k$, $E_{ij} + E_{ji} = D_{ij}^+$, $E_{ij} - E_{ji} = D_{ij}^-$, $E_{kl} + E_{lk} = D_{kl}^+$,$E_{kl} - E_{lk} = D_{kl}^-$, where $i, j \in [1, m_1]$, $k, l \in [1, m_2]$, $i \neq j$ and $k \neq l$, then we have that*

$$f(I_{m_1m_2}) f\left(\left(aD_i + b\mathbf{i}D_{ij}^-\right) \otimes \left(cD_k + dD_{kl}^+\right)\right) = f\left(\left(aD_i + b\mathbf{i}D_{ij}^-\right) \otimes \left(cD_k + dD_{kl}^+\right)\right) f(I_{m_1m_2}), \tag{4}$$

$$f(I_{m_1m_2}) f\left(\left(aD_i + bD_{ij}^+\right) \otimes \left(cD_k + d\mathbf{i}D_{kl}^-\right)\right) = f\left(\left(aD_i + bD_{ij}^+\right) \otimes \left(cD_k + d\mathbf{i}D_{kl}^-\right)\right) f(I_{m_1m_2}), \tag{5}$$

$$f(I_{m_1m_2}) f\left(\left(aD_{ij}^+ + b\mathbf{i}D_{ij}^-\right) \otimes \left(cD_{kl}^+ + d\mathbf{i}D_{kl}^-\right)\right) = f\left(\left(aD_{ij}^+ + b\mathbf{i}D_{ij}^-\right) \otimes \left(cD_{kl}^+ + d\mathbf{i}D_{kl}^-\right)\right) f(I_{m_1m_2}), \tag{6}$$

where $a, b, c, d \in \{0, 1\}$, $a \neq b$ and $c \neq d$.

Proof. First, we give the proof of the equation (4) . For any $x \neq \pm 1$,

$$\begin{aligned}&\left(I_{m_1m_2} + \frac{x}{1+x}\left(\left(aD_i + b\mathbf{i}D_{ij}^-\right) \otimes \left(cD_k + dD_{kl}^+\right)\right) - \frac{x}{1+x}\left(\left(D_i + bD_j\right) \otimes (D_k + dD_l)\right)\right)^{-1} \\ &= I_{m_1m_2} - \frac{x}{1-x}\left(\left(aD_i + b\mathbf{i}D_{ij}^-\right) \otimes \left(cD_k + dD_{kl}^+\right)\right) + \frac{x}{1-x}\left(\left(D_i + bD_j\right) \otimes (D_k + dD_l)\right),\end{aligned}$$

and suppose $f(I_{m_1m_2}) = A$, $f\left(\left(aD_i + b\mathbf{i}D_{ij}^-\right) \otimes \left(cD_k + dD_{kl}^+\right)\right) = C$, $f\left(\left(D_i + bD_j\right) \otimes (D_k + dD_l)\right) = B$, and then

$$\left(A + \frac{x}{1+x}C - \frac{x}{1+x}B\right)\left(A - \frac{x}{1-x}C + \frac{x}{1-x}B\right) = \left(A - \frac{x}{1-x}C + \frac{x}{1-x}B\right)\left(A + \frac{x}{1+x}C - \frac{x}{1+x}B\right).$$

According to Lemma 6, we get $AB = BA$. So it follows that $2x(AC - CA) = 0$. As the arbitrariness of x , the equation (4) is proved. Evidently

$$\begin{aligned}&\left(I_{m_1m_2} + \frac{x}{1+x}\left(\left(aD_i + bD_{ij}^+\right) \otimes \left(cD_k + d\mathbf{i}D_{kl}^-\right)\right) - \frac{x}{1+x}\left(\left(D_i + bD_j\right) \otimes (D_k + dD_l)\right)\right)^{-1} \\ &= I_{m_1m_2} - \frac{x}{1-x}\left(\left(aD_i + bD_{ij}^+\right) \otimes \left(cD_k + d\mathbf{i}D_{kl}^-\right)\right) + \frac{x}{1-x}\left(\left(D_i + bD_j\right) \otimes (D_k + dD_l)\right),\end{aligned}$$

$$\left(I_{m_1m_2} + \frac{x}{1+x}\left(\left(aD^+_{ij} + b\mathbf{i}D^-_{ij}\right)\otimes\left(cD^+_{kl} + d\mathbf{i}D^-_{kl}\right)\right) - \frac{x}{1+x}\left(\left(D_i + D_j\right)\otimes(D_k + D_l)\right)\right)^{-1}$$
$$= I_{m_1m_2} - \frac{x}{1-x}\left(\left(aD^+_{ij} + b\mathbf{i}D^-_{ij}\right)\otimes\left(cD^+_{kl} + d\mathbf{i}D^-_{kl}\right)\right) + \frac{x}{1-x}\left(\left(D_i + D_j\right)\otimes(D_k + D_l)\right),$$

and then the above formulas can be used to prove the validity of equations(5) and (6) through direct calculation.

Lemma 8 *Assuming $f : H_{m_1m_2} \to H_n$ is a linear map preserving inverses of tensor products of Hermite matrices, then*

$$f(E_{ii}\otimes E_{kk}) f\left(E_{jj}\otimes E_{ll}\right) = f\left(E_{jj}\otimes E_{ll}\right) f(E_{ii}\otimes E_{kk}) = 0,$$

where $i, j \in [1, m_1]$, $k, l \in [1, m_2]$, $i \neq j$ and $k \neq l$.

Proof. For any $x, y \neq 0$, $x, y \neq 1$

$$\left(I_{m_1m_2} + (x-1)(E_{ii}\otimes E_{kk}) + (y-1)\left(E_{jj}\otimes E_{ll}\right)\right)^{-1} = I_{m_1m_2} + \left(x^{-1}-1\right)(E_{ii}\otimes E_{kk}) + \left(y^{-1}-1\right)\left(E_{jj}\otimes E_{ll}\right),$$

and suppose $f(I_{m_1m_2}) = A$, $f(E_{ii}\otimes E_{kk}) = B_1$ and $f\left(E_{jj}\otimes E_{ll}\right) = B_2$ then

$$(A + (x-1)B_1 + (y-1)B_2)\left(A + \left(x^{-1}-1\right)B_1 + \left(y^{-1}-1\right)B_2\right) = I_n, \tag{7}$$

$$\left(A + \left(x^{-1}-1\right)B_1 + \left(y^{-1}-1\right)B_2\right)(A + (x-1)B_1 + (y-1)B_2) = I_n, \tag{8}$$

combine the above equations

$$\left(yx^{-1} - y - x^{-1} - xy^{-1} + x + y^{-1}\right)(B_1B_2 - B_2B_1) = 0.$$

Because of the arbitrariness of x, y , we get $B_1B_2 = B_2B_1$. With the equation (7) or (8) , it can imply that

$$\left(x + x^{-1} - 2\right)AB_1 + \left(y + y^{-1} - 2\right)AB_2 + \left(2 - x - x^{-1}\right)B_1^2 + \left(2 - y - y^{-1}\right)B_2^2$$
$$+\left(yx^{-1} - y^{-1} - x + 1 + xy^{-1} - x^{-1} - y + 1\right)B_1B_2 = 0.$$

According to Lemma 6, we get that $B_1^2 = AB_1$ and $B_2^2 = AB_2$. Then $\left(yx^{-1} - y^{-1} - x + 1 + xy^{-1} - x^{-1} - y + 1\right)B_1B_2 = 0.$ Because of the arbitrariness of x, y , we have $B_1B_2 = 0$. And the proof is completed.

3. Results

Theorem 1 *A linear map $f : H_{m_1m_2} \to H_n$ preserves inverses of tensor products of Hermite matrices if and only if f is one of the following two forms:*
(1) When $m_1m_2 = n$, there exists a unitary matrix $P \in M_n$, a natural number $\lambda \in \{-1, 1\}$ and a canonical map π on $H_{m_1m_2}$ such that $f(X) = \lambda P\pi(X)P^*$, $\forall X \in H_{m_1m_2}$;
(2) When $m_1m_2 < n$, there exist natural numbers $p_i, q_i\,(i = 1, 2, 3, 4)$, a unitary matrix $P \in M_n$ and canonical maps $\pi_i\,(i = 1, 2, 3, 4)$ on $H_{m_1m_2}$ such that

$$f(X) = P\left(\left(\bigoplus_{i=1}^{4}\pi_i(X)\otimes I_{p_i}\right)\oplus\left(\bigoplus_{i=1}^{4}\pi_i(X)\otimes\left(-I_{q_i}\right)\right)\right)P^*, \forall X \in H_{m_1m_2},$$

where I_0 means that there is not the item which corresponds to it in driect sum.

Proof. The sufficiency is obvious, we only need to prove the necessity.

If $m_1m_2 > n$, we can conclude that $f = 0$, which is contradicted with f being a linear map preserving inverses. Hence, $m_1m_2 \leq n$.

When $m_1m_2 \leq n$, obviously we obtain that $f(I_{m_1m_2})^{-1} = f\left(I^{-1}_{m_1m_2}\right) = f(I_{m_1m_2})$,i.e., $f(I_{m_1m_2})^3 = f(I_{m_1m_2})$. Applying $f(I_{m_1m_2}) \in H_n$ and Lemma 3, there is a unitary matrix $P \in M_n$ and two natural numbers t_1, t_2 , such that$f(I_{m_1m_2}) = Pdiag(I_{t_1}, -I_{t_2})P^{-1}$ and $t_1 + t_2 = n$. Let $f(E_{ii}\otimes E_{kk}) = P\begin{bmatrix}A_{ij} & C_{ij}\\ D_{ij} & B_{ij}\end{bmatrix}P^{-1}$, where $A_{ij} \in H_{t_1}$, $B_{ij} \in H_{t_2}$. Using Lemma 6, we have $f(E_{ii}\otimes E_{kk}) = P\begin{bmatrix}A_{ij} & 0\\ 0 & B_{ij}\end{bmatrix}P^{-1}$, $A_{ij} = A_{ij}^2$ and $-B_{ij} = B_{ij}^2$. According to Lemma 7, let

$$f\left(\left(aD_i + b\mathbf{i}D^-_{ij}\right)\otimes\left(cD_k + dD^+_{kl}\right)\right) = P\begin{bmatrix}C^1_{ijkl} & 0\\ 0 & D^1_{ijkl}\end{bmatrix}P^{-1},$$

$$f\left(\left(aD_i + bD_{ij}^{+}\right)\otimes\left(cD_k + d\mathbf{i}D_{kl}^{-}\right)\right) = P\begin{bmatrix} C_{ijkl}^{2} & 0 \\ 0 & D_{ijkl}^{2}\end{bmatrix}P^{-1},$$

$$f\left(\left(aD_{ij}^{+} + b\mathbf{i}D_{ij}^{-}\right)\otimes\left(cD_{kl}^{+} + d\mathbf{i}D_{kl}^{-}\right)\right) = P\begin{bmatrix} C_{ijkl}^{3} & 0 \\ 0 & D_{ijkl}^{3}\end{bmatrix}P^{-1}.$$

According to the properties of f , we can construct two maps $f_1 : H_{m_1m_2} \to H_{t_1}$ and $f_2 : H_{m_1m_2} \to H_{t_2}$. For any $X_1 \otimes X_2 \in H_{m_1m_2}$, suppose

$$f_1(X_1\otimes X_2) = \sum x_{ijkl}^{1}C_{ijkl}^{1} + \sum x_{ijkl}^{2}C_{ijkl}^{2} + \sum x_{ijkl}^{3}C_{ijkl}^{3},$$

$$-f_2(X_1\otimes X_2) = \sum x_{ijkl}^{1}D_{ijkl}^{1} + \sum x_{ijkl}^{2}D_{ijkl}^{2} + \sum x_{ijkl}^{3}D_{ijkl}^{3}.$$

Let $F(X_1\otimes X_2) = P^{-1}f(X_1\otimes X_2)P$, then

$$F(X_1\otimes X_2) = f_1(X_1\otimes X_2)\oplus(-f_2(X_1\otimes X_2)), \tag{9}$$

and for any invertible $X_1 \otimes X_2 \in H_{m_1m_2}$, we have

$$f(X_1\otimes X_2)^{-1} = P\left(f_1(X_1\otimes X_2)^{-1}\oplus(-f_2(X_1\otimes X_2))^{-1}\right)P^{-1},$$

and

$$f\left((X_1\otimes X_2)^{-1}\right) = P\left(f_1\left((X_1\otimes X_2)^{-1}\right)\oplus\left(-f_2(X_1\otimes X_2)^{-1}\right)\right)P^{-1}.$$

So f_1 and f_2 preserve inverses of tensor products of Hermite matrices as f . Let $X_1 \otimes X_2 = I_{m_1m_2}$,then $F(I_{m_1m_2}) = diag(I_{t_1}, -I_{t_2}) = f_1(I_{m_1m_2}) \oplus (-f_2(I_{m_1m_2}))$. The following proof shows that f_1 and f_2 are linear maps preserving idmpotence of tensor products of matrices.

For any idempotent matrices $X_3 \otimes X_4 \in H_{m_1m_2}$, there is a unitary matrix T such that $X_3 \otimes X_4 = T\begin{bmatrix} I_v & 0 \\ 0 & 0\end{bmatrix}T^{-1}$. Suppose $g_1(X_3\otimes X_4) = f_1\left(T(X_3\otimes X_4)T^{-1}\right)$ .Then we derive that g_1 is a linear map preserving inverses of tensor products of matrices and $g_1(I_{m_1m_2}) = f_1\left(TI_{m_1m_2}T^{-1}\right) = f_1(I_{m_1m_2}) = I_{t_1}$. According to Lemma 6, we have that $g_1(I_{m_1m_2})g_1(E_{ii}\otimes E_{kk}) = g_1(E_{ii}\otimes E_{kk})^2$,i.e., $g_1(E_{ii}\otimes E_{kk}) = g_1(E_{ii}\otimes E_{kk})^2$. With the Lemma 8, it can imply that

$$g_1\begin{bmatrix} I_v & 0 \\ 0 & 0\end{bmatrix} = g_1^2\begin{bmatrix} I_v & 0 \\ 0 & 0\end{bmatrix},$$

and then we have that $f_1^2(X_3\otimes X_4) = f_1(X_3\otimes X_4)$, which is said that f_1 is a linear map preserving idmpotence and satisfies $f_1(I_{m_1m_2}) = I_{t_1}$. Using Lemma 5, we derive that f_1 is one of the following two forms for any $X \in H_{m_1m_2}$:

(1)When $m_1m_2 = t_1$, there exists a unitary matrix $P_1 \in M_{t_1}$ and a canonical map π_1 on $H_{m_1m_2}$ such that $f_1(X) = P_1\pi_1(X)P_1^{*}$;

(2)When $m_1m_2 < t_1$, there exist natural numbers $p_i\,(i = 1, 2, 3, 4)$, a unitary matrix $P_1 \in M_{t_1}$ and canonical maps $\pi_i\,(i = 1, 2, 3, 4)$ on $H_{m_1m_2}$ such that

$$f_1(X) = P_1\left(\overset{4}{\underset{i=1}{\oplus}}\pi_i(X)\otimes I_{p_i}\right)P_1^{*},$$

where I_0 means that there is not the item which corresponds to it in driect sum.

With the same proof method as above ,we derive that f_2 is also the linear map preserving idmpotence, and has one of the following two forms for any $X \in H_{m_1m_2}$.

(1)When $m_1m_2 = t_2$, there exists a unitary matrix $P_2 \in M_{t_2}$ and a canonical map π_2 on $H_{m_1m_2}$ such that $f_2(X) = P_2\pi_2(X)P_2^{*}$;

(2)When $m_1m_2 < t_2$, there exist natural numbers $q_i\,(i = 1, 2, 3, 4)$, a unitary matrix $P_2 \in M_{t_2}$ and canonical maps $\pi_i\,(i = 1, 2, 3, 4)$ on $H_{m_1m_2}$ such that

$$f_2(X) = P_2\left(\overset{4}{\underset{i=1}{\oplus}}\pi_i(X)\otimes I_{q_i}\right)P_2^{*},$$

where I_0 means that there is not the item which corresponds to it in driect sum.

Combining with (9) , the Theorem1 can be proved.

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