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Bounds Problems in a Class of Quasi-conformal Maps

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Abstract

In this paper, an analogue of Bieberbach's distortion theorem has been proposed, in a class of quasi-conformal maps.

Keywords: Bieberbach's theorem, Cauchy's inequalities, Class S, quasi-conformal maps.

1. Introduction

The problems of bounds in the theory of functions of complex variable had its rise in the 19th century after the eminent result of Louis Debranges (in 1985) (he demonstrates the Bieberbach hypothesis) in class S, class of maps f, holomorphic and univalent, defined in $\Delta = \left\{z \in \mathbb{C} : |z| < 1\right\}$, standardized by $f(0) = 0$ et $f'(0) = 1$ and writing $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$,

$a_n \in \mathbb{C}-$ Taylor series coefficients maps f.

With the objective of extending these results in other classes of complex variable maps (class of quasi-conformal maps, harmonics, etc $\cdots$), there have been several attempts by eminent mathematicians as L. Ahlfors, J. Clunie Sherl-Small, P. Duren , V.V Starkoff, S.Y. Graf. For example, in class S_H, class of maps f, univalent harmonics, satisfying the conditions

$f(0) = 0$, $fz(0) = 1$, defined in $\Delta = \left\{z \in \mathbb{C} : |z| < 1\right\}$ and written $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$, $(a_n; b_n) \in \mathbb{C}^2$; the assumptions $\Big||a_n| - |b_n|\Big| \leq n$, $|a_n| < \frac{1}{3}(2n^2 + 1)$ and $|b_n| < \frac{1}{3}(2n^2 + 1)$ $n \geq 2$ remain without demonstration Clunie J, Sheil-Small T (1984); Duren P (2014)). And in class S_H^0- subclass of S_H, made up of maps, defined in $\Delta = \left\{z \in \mathbb{C} : |z| < 1\right\}$ such that, $f(0) = 0$, $f_z(0) = 1$ and $f_{\bar{z}}(0) = 0$ and written $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$, $(a_n; b_n) \in \mathbb{C}^2$, the assumptions $\Big||a_n| - |b_n|\Big| \leq n$, $|a_n| \leq \frac{1}{6}(2n + 1)(n + 1)$ and $|b_n| \leq \frac{1}{6}(2n - 1)(n - 1)$, $n \geq 2$ remain to be demonstrated.

2. The Class S

We call class S, class of univalent, conformal map f defined on $\Delta = \left\{z \in \mathbb{C} : |z| < 1\right\}$, normed such that $f(0) = 0$ and $f_z(0) = 1$.

The maps of this class are written as $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$.

It should also be noted that this class S is endowed with the properties of affine invariance and linear invariance.

2.1 Property of Linear Invariance

This property results in the fact that the map

$$L_\varphi(f)(z) = \frac{f\circ\varphi(z) - f\circ\varphi(0)}{f'(\varphi(0)).\varphi'(0)}$$

also belongs to S; where

$$\varphi(z) = e^{i\theta}\frac{z - z_0}{1 - \overline{z_0}.z}$$

with $\theta \in \mathbb{R}$, $z_0 \in \Delta$

(φ is conform automorphism in Δ)

2.2 Propriety of Affine Invariance

$$L_\varepsilon(f)(z) = \frac{f(z) + \varepsilon\overline{f(z)}}{1 + \varepsilon\overline{f_{\bar{z}}(0)}}$$

also belongs to S; with $f \in S$ and $\varepsilon \in \Delta$.

Based on some classical results in class S, we try to propose some bounds (estimates) in a class of quasi-conformal maps. These are the following results:

2.3 Conjoncture of Bieberbach 1918 (Louis de Branges 1984, Golusine G.M. 1966)

Let f be an injective holomorphic function on the unit disk Δ, whose expansion into an integer series is of the form:

$\forall z \in \Delta,\quad f(z) = z + a_2z^2 + a_3z^3 + \cdots$, with $a_2,\ a_3, \cdots \in \mathbb{C}$

Then : $\qquad \forall n \geq 2, \quad |a_n| \leq n$

2.4 Koëbe's Quarter Theorem

Let $f \in S$. Suppose that there exists $w_0 \in \mathbb{C}$ such that $w_0 \notin f(\Delta)$. Then $|w_0| \geq \dfrac{1}{4}$.

2.5 Distortion Theorem Let $f \in S$ and $z_0 \in \Delta$, $|z_0| = r > 0$, we have:

a)

$$\frac{1-r}{(1+r)^3} \leq |f'(z_0)| \leq \frac{1+r}{(1-r)^3}$$

b)

$$\frac{r}{(1+r)^2} \leq |f(z_0)| \leq \frac{r}{(1-r)^2}$$

3. Near Compliant Maps

Let f be a map defined in Δ, and with values in $\mathbb{C}$.

f is said to be quasi-conformal if it verifies the Beltrami equation: $f_{\bar{z}} = \mu_f.f_z$

4. Complex Feature or Dilation

Let f be a map of Δ. The quotient $\mu_f = \dfrac{f_{\bar{z}}}{f_z}$ is called the complex dilation of f or the Beltrami coefficient of f.

Remark 1.

As $\quad |f_z| > |f_{\bar{z}}|$, then $\quad |\mu_f| = \left|\dfrac{f_{\bar{z}}}{f_z}\right| < 1$

5. Cauchy's Inequalities

Let f be a holomorphic function in a disk $D(0, R)$, then for any natural number n and for any $r \in]0, R[$, we have :

$$\left|a_n\right| = \frac{1}{n!}\left|f^{(n)}(0)\right| \leq \frac{\sup\{|f(z)|; |z| = r\}}{r^n}$$

We can now introduce our main result.

Theorem 1 Let h and g be two analytic and univalent maps, defined in Δ. For any quasi-conformal map f such that $f(z) = h(z) + \overline{g(z)}$, we have :

$$|f_z(z)| < \frac{1}{2}\sum_{n=0}^{\infty} n(|a_n| + |b_n|) \tag{1}$$

Proof.

h and g being analytic, then they are written as follows :

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \ \ and \ \ g(z) = \sum_{n=0}^{\infty} b_n z^n$$

Replacing h and g in the map f, we find :

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} \overline{b_n}\overline{z}^n \tag{2}$$

Deriving f with respect to z and with respect to $\overline{z}$ in relation (2) we have :

$$\forall z \in \Delta, \qquad f_z(z) = h_z(z) = \sum_{n=0}^{\infty} n a_n z^{n-1} \text{ and } f_{\overline{z}}(z) = g_z(z) = \sum_{n=0}^{\infty} n\overline{b_n}\overline{z}^{n-1}$$

Taking the modulus of each derivative and adding the two moduli, we get :

$$\left|f_z(z)\right| + \left|f_{\overline{z}}(z)\right| = \left|\sum_{n=0}^{\infty} n a_n z^{n-1}\right| + \left|\sum_{n=0}^{\infty} n\overline{b_n}\overline{z}^{n-1}\right|$$

Applying the triangle inequality to the previous line, it follows that :

$$|f_z(z)| + |f_{\overline{z}}(z)| \le \left|\sum_{n=0}^{\infty} n a_n z^{n-1}\right| + \left|\sum_{n=0}^{\infty} n\overline{b_n}\overline{z}^{n-1}\right|$$

$$|f_z(z)| + |f_{\overline{z}}(z)| \le \sum_{n=0}^{\infty} n|a_n||z^{n-1}| + \sum_{n=0}^{\infty} n|\overline{b_n}||\overline{z}^{n-1}$$

Let's put $|z| = r$ with $0 \le r < 1$, then :

$$|f_z(z)| + |f_{\overline{z}}(z)| < \sum_{n=0}^{\infty} n|a_n|r^{n-1} + \sum_{n=0}^{\infty} n|b_n|$$

$$|f_z(z)| + |f_{\overline{z}}(z)| < \sum_{n=0}^{\infty} n(|a_n| + |b_n|)$$

$$|f_z(z)|\left(1 + \frac{|f_{\overline{z}}(z)|}{|f_z(z)|}\right) < \sum_{n=0}^{\infty} n(|a_n| + |b_n|)$$

f being quasi-conform, so :

$$\frac{|f_{\overline{z}}(z)|}{|f_z(z)|} = \left|\mu_{f(z)}\right|$$

$$|f_z(z)|\left(1 + \left|\mu_{f(z)}\right|\right) < \sum_{n=0}^{\infty} n(|a_n| + |b_n|)$$

$$|f_z(z)| < \frac{1}{1 + \left|\mu_{f(z)}\right|}\sum_{n=0}^{\infty} n(|a_n| + |b_n|) \tag{3}$$

This inequality remains true for all $\rho \in]0;1[$, with $\rho = \left|\mu_{f(z)}\right|$. Thus the relation (3) becomes :

$$\left|f_z(z)\right| < \left[\inf_{\rho}\left(\frac{1}{1+\rho}\right)\right]\sum_{n=0}^{\infty} n(|a_n| + |b_n|) \tag{4}$$

Let's look at the function $\psi(\rho) = \dfrac{1}{1+\rho}$ on $]0;1[$.

$\psi'(\rho) = -\dfrac{1}{(1+\rho)^2}$, for any $\rho \in]0;1[$, $\psi'(\rho) < 0$ then ψ is strictly decreasing on $]0;1[$; so ψ reaches its infinimum at the point $\rho = 1$. Thus $\inf_{\rho} \psi = \dfrac{1}{2}$

This shows that, $$\inf_{\rho}\left(\frac{1}{1+\rho}\right) = \frac{1}{2}$$

Therefore

$$\boxed{|f_z(z)| < \frac{1}{2}\sum_{n=0}^{\infty} n(|a_n| + |b_n|)} \tag{5}$$

Lemma 1.

For any f-quasi-conformal such that $f(z) = h(z) + \overline{g(z)}$, with h and g two analytic and univalent maps, defined in Δ and for all $|z| = r \in]0,1[$ we have :

a)

$$\left|f_z(z)\right| \leq \frac{r^2}{(1-r)^4}\left|f_z(0)\right|$$

b)

$$\left|f(z)\right| \leq |f(0)| + \frac{1}{3}\left(\frac{r}{1-r}\right)^3\left|f_z(0)\right|$$

Proof.

Starting by proving the first inequality using the previous theorem. Let's put $\tilde{h}(z) = \dfrac{h(z) - h(0)}{h_z(0)}$ and $\tilde{g}(z) = \dfrac{g(z) - g(0)}{g_z(0)}$.

It is easy to check that $\tilde{g}$ and $\tilde{h}$ are in class S.

Let $\tilde{f}(z) = \tilde{h}(z) + \overline{\tilde{g}(z)}$, then the map $\tilde{f}$ satisfies the relation (5) of the previous theorem, that is:

$$\left|\tilde{f}_z(z)\right| < \frac{1}{2}\sum_{n=0}^{\infty} n(|\tilde{a}_n| + |\tilde{b}_n|) \tag{6}$$

Or $$\tilde{a}_n = \frac{1}{n!}\tilde{h}^{(n)}(0) \text{ et } \tilde{b}_n = \frac{1}{n!}\tilde{g}^{(n)}(0).$$

Using Cauchy's inequalities, we have:

$$\left|\tilde{a}_n\right| \leq \frac{1}{r^n}\sup\left\{\left|\tilde{h}(z)\right|, |z| = r\right\} \ \ and \ \ \left|\tilde{b}_n\right| \leq \frac{1}{r^n}\sup\left\{\left|\tilde{g}(z)\right|, |z| = r\right\}$$

Thus the relation (6) becomes :

$$\left|\tilde{f}_z(z)\right| \leq \frac{1}{2}\sum_{n=0}^{\infty}\frac{n}{r^n}\left(\sup\left\{\left|\tilde{h}(z)\right|, |z| = r\right\} + \sup\left\{\left|\tilde{g}(z)\right|, |z| = r\right\}\right) \tag{7}$$

$\tilde{h}$ and $\tilde{g}$ are in class S, then according to the Distortion Theorem, we have :

$$\frac{r}{(1+r)^2} \leq \left|\tilde{h}(z)\right| \leq \frac{r}{(1-r)^2} \quad et \quad \frac{r}{(1+r)^2} \leq \left|\tilde{g}(z)\right| \leq \frac{r}{(1-r)^2}$$

Let $z \in \Delta$ and $r < 1$ such that $|z| < r$

Let's look at the function $\varphi(x) = \dfrac{x}{(1-x)^2}$ on $[-r; r]$

$\varphi'(x) = \dfrac{1+x}{(1-x)^3} > 0$, so φ is increasing.

Since φ is increasing, then $\left|\tilde{h}(z)\right| \leq \sup\limits_{|z|=r}\left|\tilde{h}(z)\right|$ and $\left|\tilde{g}(z)\right| \leq \sup\limits_{|z|=r}\left|\tilde{g}(z)\right|$

So

$$\sup_{|z|=r}\left|\tilde{g}(z)\right| = \sup_{|z|=r}\left|\tilde{h}(z)\right| = \frac{r}{(1-r)^2}$$

Therefore, the relation (7) becomes :

$$\left|\tilde{f}_z(z)\right| \leq \frac{r}{(1-r)^2}\sum_{n=0}^{\infty}\frac{n}{r^n} \tag{8}$$

Or

$$\sum_{n=0}^{\infty}\frac{n}{r^n} = \frac{r}{(1-r)^2}$$

The relation (8) becomes :

$$\left|\tilde{f}_z(z)\right| \leq \frac{r^2}{(1-r)^4} \tag{9}$$

Note that

$$\tilde{h}_z(z) = \tilde{f}_z(z) = \frac{f_z(z)}{f_z(0)} \tag{10}$$

Hence by replacing relation (10) in relation (9), which leads us to the following result :

$$\boxed{\left|f_z(z)\right| \leq \frac{r^2}{(1-r)^4}\left|f_z(0)\right|} \tag{11}$$

To prove the second inequality, let us use the previous relation.

Let's put $z = re^{i\theta}$, $r > 0$.

$$f(z) - f(0) = \int_0^z f_z(\varepsilon)d\varepsilon = \int_0^r f_z(te^{i\theta})e^{i\theta}dt$$

$$\left|f(z)\right| = \left|\int_0^r f_z(te^{i\theta})e^{i\theta}dt + f(0)\right|$$

$$\left|f(z)\right| \leq \left|\int_0^r f_z(te^{i\theta})e^{i\theta}\right|dt + \left|f(0)\right|$$

According to the relation (11), $\left|f_z(z)\right| \leq \dfrac{r^2}{(1-r)^4}\left|f_z(0)\right|$

So

$$\left|f(z)\right| \le \int_0^r \frac{t^2}{(1-t)^4}\left|f_z(0)\right|dt + \left|f(0)\right|$$

$$\left|f(z)\right| \le \left|f_z(0)\right| \int_0^r \frac{t^2}{(1-t)^4}dt + \left|f(0)\right|$$

Or

$$\frac{t^2}{(1-t)^4} = \frac{1}{(1-t)^2} - \frac{2}{(1-t)^3} + \frac{1}{(1-t)^4}$$

$$\left|f(z)\right| \le \left|f_z(0)\right| \int_0^r \left[\frac{1}{(1-t)^2} - \frac{2}{(1-t)^3} + \frac{1}{(1-t)^4}\right]dt + \left|f(0)\right|$$

$$\left|f(z)\right| \le \left|f_z(0)\right|\left[\frac{1}{1-t} - \frac{1}{(1-t)^2} + \frac{\frac{1}{3}}{(1-t)^3}\right]_0^r + \left|f(0)\right|$$

$$\left|f(z)\right| \le \left|f_z(0)\right|\left[\frac{1}{1-r} - \frac{1}{(1-r)^2} + \frac{1}{3(1-r)^3} - \frac{1}{3}\right] + \left|f(0)\right|$$

Therefore

$$\boxed{\left|f(z)\right| \le \left|f(0)\right| + \frac{1}{3}\left(\frac{r}{1-r}\right)^3\left|f_z(0)\right|} \tag{12}$$

6. Conclusion

In this paper, we have found an analogue of the Koebe theorem and the Bieberbach theorem for a particular class of quasi-conformal maps, based on some known properties in the class S and on Cauchy's inequalities.

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