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Origin Point Must Represent Critical Line as Location for Nontrivial Zeros of Riemann Zeta Function, and Set Prime Gaps With Subsets Odd Primes Are Arbitrarily Large in Number

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Abstract

We prove the Countably Infinite Subsets of odd primes have cardinality of Arbitrarily Large in Number. This is achieved by demonstrating the asymptotic law of distribution of prime numbers that involves natural logarithm function to be applicable to all these subsets of odd primes derived from every even Prime gaps which are again Arbitrarily Large in Number. We prove the location for Countably Infinite Set of nontrivial zeros computed using Dirichlet eta function (proxy function for Riemann zeta function) is the unique critical line. This is achieved by plotting the infinitely many colinear lines of Riemann zeta function using both critical line and non-critical lines.

Keywords: Gram's Law, nontrivial zeros, Polignac conjecture, prime-composite identifier grouping, prime-composite quotient, Riemann hypothesis, Riemann zeta function, Rosser's Rule, twin prime conjecture

Mathematics Subject Classification (2010): Primary 11M26, Secondary 11A41

Data Availability Statement: All data generated or analyzed during this study are included in this published article.

Conflict of interest: The corresponding author states there is no conflict of interest. Fields of interest from Doctor of Philosophy (PhD) viewpoint: Ageing, Dementia, Sleep, Learning, Memory and Number theory. An autistic savant is someone with autism who also has a single extraordinary area of knowledge or ability. Savant skills are typically linked to massive memory retention ability required for advance *Concrete Mathematics* on rapid calculations, memorizing whole phone book, etc. The author possesses average level of working, short-term and long-term memory, and Concrete Mathematics ability. While conducting active research which requires advance *Abstract Mathematics*, the author practices behavioral augmentation on his personal Stage 3 Deep Sleep which contributes to insightful thinking, creativity and memory, and Stage 4 REM Sleep which is essential to cognitive functions memory, learning and creativity.

1. Introduction

****Contents** of this research paper are displayed for convenience after the Appendices on Page 55 (last page).**

As two different but related infinite-length equations through analytic continuation, Hasse principle is satisfied by Riemann zeta function as a certain type of equation that generates all infinitely-many trivial zeros [given as s = (rational) negative even numbers –2, –4, –6, –8, –10... located *outside* the $0 < \sigma < 1$-critical strip] but this principle is not satisfied by its *proxy* Dirichlet eta function as a dissimilar type of equation that generates all infinitely-many nontrivial zeros [specified by t-valued (irrational) transcendental numbers located *inside* the $0 < \sigma < 1$-critical strip on the $\sigma = \frac{1}{2}$-critical line].

Regarding prime numbers and prime gaps. We prove the Countably Infinite Subsets odd primes derived from even Prime gaps 2, 4, 6, 8, 10... all have cardinality of Arbitrarily Large in Number. This is achieved by demonstrating the asymptotic law of distribution of prime numbers that involves natural logarithm function to be applicable to all these unique subsets generated by each and every even Prime gaps [which constitute a Countably Infinite Set also having cardinality of Arbitrarily Large in Number].

There are only finitely many odd Composite gap of 1 and even Composite gap of 2 that generate all composite numbers. We derive the [complementary] asymptotic law of distribution of composite numbers that involves natural exponential function to be applicable to two unique subsets of even and odd composite numbers derived from Composite gap 1, whereby these equally distributed in the alternating repetitive order of even Gap 1-composite numbers and odd Gap 1-

composite numbers are located in between two consecutive odd prime numbers with even Prime gaps 4, 6, 8, 10, 12, 14...; viz, the exception here is these even Gap 1-composite numbers and odd Gap 1-composite numbers do not occur for twin primes with even Prime gap 2. However the asymptotic law of distribution of prime numbers that involves natural logarithm function will also be applicable to the unique subset of even composite numbers derived from Composite gap 2, whereby these even Gap 2-composite numbers always precede every odd prime numbers but with exception that the first odd prime number 3 is preceded by the very first and only even prime number 2 with odd Prime gap 1.

Definition of Prime-Composite identifier grouping: Let E = even numbers, O = odd numbers, $\mathbb{P}$ = prime numbers, even Prime gap$_i$ = O-$\mathbb{P}_{i+1}$ – O-$\mathbb{P}_i$ = 2, 4, 6, 8, 10, 12..., Composite gap$_i$ = $\mathbb{C}_{i+1} - \mathbb{C}_i$ = 1, 2. For even Prime gaps 4, 6, 8, 10, 12..., we can generate the orderly consecutive numbers as sequence {Gap 2-E-$\mathbb{C}_1$, O-$\mathbb{P}_i$, Gap 1-E-$\mathbb{C}_2$, Gap 1-O-$\mathbb{C}_3$, Gap 1-E-$\mathbb{C}_4$, Gap 1-O-$\mathbb{C}_5$,..., Gap 1-E-$\mathbb{C}_{n-2}$, Gap 1-O-$\mathbb{C}_{n-1}$, Gap 2-E-$\mathbb{C}_n$, O-$\mathbb{P}_{i+1}$}. The cardinality of sub-sequence {Gap 1-E-$\mathbb{C}_2$, Gap 1-O-$\mathbb{C}_3$, Gap 1-E-$\mathbb{C}_4$, Gap 1-O-$\mathbb{C}_5$,..., Gap 1-E-$\mathbb{C}_{n-2}$, Gap 1-O-$\mathbb{C}_{n-1}$} = even Prime gap$_i$ – 2 = n – 2. However for twin primes; this sub-sequence [as an empty set or null set] do not exist with its cardinality = 0 since even Prime gap 2 – 2 = 0. With cardinality of this sub-sequence given by the involved even Prime gap minus 2; we conveniently define **$\mathbb{P}$-$\mathbb{C}$ identifier grouping** as Gap 2-E-$\mathbb{C}_1$, O-$\mathbb{P}_i$, Gap 1-E-$\mathbb{C}_2$, Gap 1-O-$\mathbb{C}_3$, Gap 1-E-$\mathbb{C}_4$, Gap 1-O-$\mathbb{C}_5$,..., Gap 1-E-$\mathbb{C}_{n-2}$, Gap 1-O-$\mathbb{C}_{n-1}$ for Arbitrarily Large Number of even Prime gaps 4, 6, 8, 10, 12... with caveat **$\mathbb{P}$-$\mathbb{C}$ identifier grouping** for even Prime gap 2 is an exception given by Gap 2-E-$\mathbb{C}_1$, O-$\mathbb{P}_i$. ***For n = 1, 2, 3, 4, 5...; [decelerating] size of equally distributed Gap 2n-O-$\mathbb{P}$ and Gap 2-E-$\mathbb{C}$ is inversely proportional to [accelerating] size of equally distributed Gap 1-E-$\mathbb{C}$ and Gap 1-O-$\mathbb{C}$***. Gap 2-E-$\mathbb{C}_n$ is now acting as the new Gap 2-E-$\mathbb{C}_1$ for O-$\mathbb{P}_{i+1}$ in the following perpetually repeating cycles of O-$\mathbb{P}_i$ to O-$\mathbb{P}_{i+1}$ with a [usually] different even Prime gap$_i$ [except for rare recurring cases of two or more consecutive O-$\mathbb{P}$ having two or more identical consecutive even Prime gaps involving 6 and multiples of 6].

Useful deductions regarding prime numbers: Let relevant cardinality be denoted by the abbreviated notations as defined in Lemma 1.1: CFS, CIS and UIS with three CIS subtypes as CIS-IM-accelerating, CIS-IM-linear and CIS-ALN-decelerating. For n = 1, 2, 3, 4, 5...; CIS-ALN-decelerating Gap 2n-O-$\mathbb{P}$ + CIS-ALN-decelerating Gap 2-E-$\mathbb{C}$ is inversely proportional to CIS-IM-accelerating Gap 1-E-$\mathbb{C}$ + CIS-IM-accelerating Gap 1-O-$\mathbb{C}$. Then the Arbitrarily Large Number of CIS-ALN-decelerating Gap 2-O-$\mathbb{P}$, CIS-ALN-decelerating Gap 4-O-$\mathbb{P}$, CIS-ALN-decelerating Gap 6-O-$\mathbb{P}$... must all constitute valid subsets of odd prime numbers. The law of continuity is a heuristic principle that "whatever succeeds for the finite, also succeeds for the infinite". By itself, this law aesthetically implies each and every Arbitrarily Large Number of even Prime gaps 2, 4, 6, 8, 10, 12... must exist [albeit not always appearing as first occurrences of the relevant associated odd prime numbers and thus not always complying with the prescribed naturally occurring ascending order for even numbers]. ***This is because there is zero probability that the equally distributed even Gap 1-composite numbers and odd Gap 1-composite numbers existing as (i) recurring sets with varying different cardinality of 2, 4, 6, 8, 10... that correspond to even Prime gaps 4, 6, 8, 10, 12... and (ii) recurring null sets with non-varying same cardinality of 0 that correspond to even Prime gap 2, will [discriminatorily] cease to exist for any particular even Prime gap***. The Incompletely Predictable complex Sieve-of-Eratosthenes algorithm [as Σ(Gap 2n-Sieve-of-Eratosthenes algorithms) = Gap 2-Sieve-of-Eratosthenes algorithm + Gap 4-Sieve-of-Eratosthenes algorithm + Gap 6-Sieve-of-Eratosthenes algorithm +... for n = 1, 2, 3, 4, 5...] will faithfully generate all Arbitrarily Large Number of odd prime numbers. Applying logical reasoning to theoretical situation of, for instance, Twin prime conjecture being false; one would [falsely] contends the Gap 2-Sieve-of-Eratosthenes algorithm [= Σ(Gap 2n-Sieve-of-Eratosthenes algorithms) – Gap 4-Sieve-of-Eratosthenes algorithm – Gap 6-Sieve-of-Eratosthenes algorithm –... for n = 1, 2, 3, 4, 5...] can only generate a CFS of twin primes. By default, this is strictly [and incorrectly] a simple Completely Predictable algorithm. Therefore, by contradiction, Twin prime conjecture is consequently true in that there must be an Arbitrarily Large Number of twin primes. *Ditto* for all other remaining Gap 2n-Sieve-of-Eratosthenes algorithms derived from n = 2, 3, 4, 5, 6....

Regarding nontrivial zeros of Riemann zeta function. We prove the location for Countably Infinite Set nontrivial zeros computed using Dirichlet eta function (*proxy* function for Riemann zeta function) is the critical line. This is achieved by demonstrating all colinear plots of Riemann zeta function as colinear lines to be non-discriminatorily and simultaneously specified by [solitary unique] critical line and [infinitely many non-unique] non-critical lines.

Useful deductions regarding nontrivial zeros: For a given function (equation) $y = f(x)$, there may be no geometrical symmetry in the given equation whereby this equation *may or may not* intercept the Origin point; or there may be one or more geometrical symmetry in the given equation about the X-axis, Y-axis, Diagonal, or Origin point whereby this equation *may or may not* intercept the Origin point. For a given equation, these types of symmetry can be correspondingly tested by replacing y with $-y$, x with $-x$, both y with x and x with y, or both x with $-x$ and y with $-y$.

With Dirichlet eta function acting as *proxy* function for Riemann zeta function, we aesthetically argue using first principle that the infinitely many ***t*-valued** Origin intercept points [which faithfully represents all ***t*-valued** nontrivial zeros] of Dirichlet eta function will only be generated when its parameter $\sigma = \frac{1}{2}$ [which represents the solitary critical line] but not when its parameter $\sigma \neq \frac{1}{2}$ [which represents the infinitely many non-critical lines]. This is notwithstanding the simple deduction that Dirichlet eta functions when endowed with any σ values between 0 and 1 [viz, in the $0 < \sigma < 1$ critical strip which is bisected by the $\sigma = \frac{1}{2}$ critical line into two regions $0 < \sigma < \frac{1}{2}$ and $\frac{1}{2} < \sigma < 1$] will always behave mathematically as different independent colinear functions [all *without geometrical symmetry* if we only consider the range of either $0 < t < +\infty$ or $-\infty < t < 0$] whereby they generate mutually exclusive colinear lines that geometrically never cross over one another.

Upon inspecting relevant equations from Riemann zeta function and related functions in this paper e.g. Eq. (1) that manifest *exact* Dimensional analysis homogeneity when $\sigma = \frac{1}{2}$ whereby Σ(all fractional exponents) = $2(-\sigma)$ = exact negative whole number of –1 [as opposed to e.g. Eq. (2) that manifest *inexact* Dimensional analysis homogeneity when $\sigma = \frac{2}{5}$ whereby Σ(all fractional exponents) = $2(-\sigma)$ = inexact negative fractional number of $-\frac{4}{5}$]; we deduce only Dirichlet eta function containing parameter $\sigma = \frac{1}{2}$ will mathematically depict the [optimal] "formula symmetry" on Σ(all fractional exponents) as an exact negative whole number. However, this formula symmetry is not equivalent to geometrical symmetry about the X-axis, Y-axis, Diagonal, or Origin point [which do not exist for any Dirichlet eta functions]. With full range of t variable being $-\infty < t < +\infty$ whereby we conventionally adopt the positive range $0 < t < +\infty$, we state this observation confirms only $\sigma = \frac{1}{2}$-Dirichlet eta function will perpetually and geometrically intercept Origin point as Origin intercept points (i.e. will perpetually and mathematically lie on critical line as nontrivial zeros) an infinite number of times.

A useful analogy for above confirmation statement: The mathematical equations (as colinear functions having geometrical symmetry) $y = a\cos(x)^b - c\cos(x)^d$ = zero [<=> solitary Origin intercept point at the Origin point] only when the unique solitary $a = c$ condition is met. This represents the Proposition: The solitary unique $\sigma = \frac{1}{2}$ condition will always produce nontrivial zeros as Origin intercept points. Then, $y = a\cos(x)^b - c\cos(x)^d \neq$ zero [<=> nil Origin intercept point at the Origin point] when the non-unique multiple $a \neq c$ conditions are met, can represent the Corollary: The infinitely many non-unique $\sigma \neq \frac{1}{2}$ conditions will never produce nontrivial zeros as Origin intercept points. We note it is immaterial whether the $b = d$ or $b \neq d$ conditions are met for both situations. We also deduce that occurrences of infinitely many violations (failures) of Gram's Law and Rosser's Rule resulting in altered appearances of Gram points [w.r.t. nontrivial zeros] in $\sigma = \frac{1}{2}$-Dirichlet eta function do not contradict this confirmation statement since possible solutions for $\sigma \neq \frac{1}{2}$-Dirichlet eta functions geometrically as true X-axis intercept points [w.r.t. true Origin intercept points] or mathematically as true Gram points [w.r.t. true nontrivial zeros] is a geometrical or mathematical impossibility. Here, Gram points denote Gram[y=0] points.

1.1 Completely and Incompletely Predictable entities in Sets and Tuples

Legend: CP = Completely Predictable, IP = Incompletely Predictable.

The sets of prime numbers, composite numbers, nontrivial zeros, etc are all morphologically constituted by pseudo-random numbers [<=> *our Incompletely Predictable numbers*] in the sense that these sets of numbers are actually NOT random but behave like one. The word *number* [singular noun] or *numbers* [plural noun] in reference to CP even and odd numbers, IP prime and composite numbers, IP nontrivial zeros and two types of Gram points can be interchanged with the word *entity* [singular noun] or *entities* [plural noun]. Gram points refer to Gram[x=0,y=0] points or nontrivial zeros, Gram[y=0] points and Gram[x=0] points when $\sigma = \frac{1}{2}$. Virtual Gram points refer to virtual Gram[y=0] points and virtual Gram[x=0] points when $\sigma \neq \frac{1}{2}$ whereby virtual Gram[x=0,y=0] points do not exist. For i = all integers ≥ 0 or i = all integers ≥ 1; both the i^{th} position of i^{th} CP numbers or entities and the i^{th} position of i^{th} IP numbers or entities are simply given by i. Apart from the very first Gram[y=0] point and the very first virtual Gram[y=0] point both occurring at t = 0, we note all types of Gram points and virtual Gram points will consist of t-valued transcendental numbers whose positions are IP with infinitely many digits after the decimal point in each transcendental number again being IP.

CP simple equation or algorithm generates CP numbers. A generated CP number is **locationally defined** as a number whose i^{th} position is *independently* determined by simple calculations without needing to know related positions of all preceding numbers. We supply the example using even and odd numbers.

E-O Pairing: For i = 1, 2, 3,..., ∞; let i^{th} Even and i^{th} Odd numbers = E_i and O_i, and i^{th} even and i^{th} odd number gaps = $eGap_i$ and $oGap_i$. We ignore $E_0 = 0$. The positions of E_i and O_i are CP, and are independent from each other.

E_i	2		4		6		8		10		12	.....
$eGap_i$		2		2		2		2		2		2

We can precisely, easily and independently calculate E_5 = (2 X 5) = 10 and O_5 = (2 X 5) – 1 = 9.

O_i	1		3		5		7		9		11	.....
$oGap_i$		2		2		2		2		2		2

IP complex equation or algorithm generates IP numbers. A generated IP number is **locationally defined** as a number whose ith position is *dependently* determined by complex calculations with needing to know related positions of all preceding numbers. We supply the example using prime and composite numbers (and note analogous examples can readily be created using nontrivial zeros, Gram[y=0] points and Gram[x=0] points when $\sigma = \frac{1}{2}$).

P-C Pairing: For i = 1, 2, 3,..., ∞; let ith Prime and ith Composite numbers = P_i and C_i, and ith prime and ith composite number gaps = pGap$_i$ and cGap$_i$. The positions of P_i and C_i are IP, and are dependent on each other.

P_i	2		3		5		7		11		13	.....
pGap$_i$		1		2		2		4		2		4

We can precisely, tediously and dependently compute C_6 = 12 and P_6 = 13: 2 is 1st prime, 3 is 2nd prime, 4 is 1st composite, 5 is 3rd prime, 6 is 2nd composite, 7 is 4th prime, 8 is 3rd composite, 9 is 4th composite, 10 is 5th composite, 11 is 5th prime, 12 is 6th composite, 13 is 6th prime, etc. Our desired integer 12 is the 6th composite and integer 13 is the 6th prime.

C_i	4		6		8		9		10		12	.....
cGap$_i$		2		2		1		1		2		2

In email correspondence from author of this paper dated 22 January 2022 to Prof. Maciej Radziejewski, Secretary of Acta Arithmetica; the following is appropriate 10 February 2022 email letter reply from Radziejewski on Subject Heading *Error on Page 250 in Acta Arithmetica 148 (2011), 225 – 256*. This issue concerns the first exception to Rosser's Rule being incorrectly stated by Prof. Timothy S. Trudgian on page 250 of his paper (Trudgian, 2011) *On the success and failure of Gram's Law and the Rosser Rule*, Acta Arithmetica, vol. 148 (2011), pp. 225 – 256, DOI: http://dx.doi.org/10.4064/aa148-3-2 as 13,999,825th Gram point. This should instead be 13,999,525th Gram point as stated by Prof. Richard P. Brent in Table 3 (page 1369) of his paper (Brent, 1979) *On the zeros of the Riemann zeta function in the critical strip*, Math. Comp., vol. 33 (1979), pp. 1361 – 1372, DOI: https://doi.org/10.1090/S0025-5718-1979-0537983-2, and also by Charles R. Greathouse IV, author of OEIS A216700 (Greathouse, 2012) dated 17 September 2012 *Violations of Rosser's rule: numbers n such that the Gram block [g(n), g(n+k)) contains fewer than k points t such that Z(t) = 0, where Z(t) is the Riemann-Siegel Z-function*.

Dear Professor Ting,
Thank you for your letter. I understand that you are confused by the fact that two research papers contradict each other by giving different values for the same quantity.

If you look at the first three sentences of Section 7.3 in T. Trudgian's paper, it is clear that the author is refering to known previous computations. He has not repeated these computations and he is not pointing to any error therein. Therefore the discrepancy that you mention is clearly due to a typo in T. Trudgian's paper. For your information, I contacted the author and he confirms that.

While it is regrettable that the value in that paper is incorrect, the Editor does not feel that it should lead to misunderstandings, unless someone uses that value for something. However, this should not be done without consulting the source works, precisely because such a typo is quite possible. For this reason we are not going to ask the author for an erratum.

Yours sincerely,
Maciej Radziejewski
Secretary of Acta Arithmetica

This reproduced email letter has provided incentive and inspiration for relevant parts of this paper. From Appendix A, we note Gram's Law is the tendency for nontrivial zeros of Riemann-Siegel function $Z(t)$ to alternate with Gram[y=0] points when $\sigma = \frac{1}{2}$. The first violation (failure) of Gram's Law occurs at n = 126. Rosser's Rule states that every Gram block contains the expected number of roots as Gram[y=0] points when $\sigma = \frac{1}{2}$. The first violation (failure) of Rosser's Rule occurs at the much larger n = 13999525. With assistance from Greathouse IV on 24 January 2022, we found one nontrivial zero between n = 13999825 as one Gram[y=0] point with t-value 6820050.984896... and n = 13999827 as another Gram[y=0] point with t-value 6820051.8891147...; and two nontrivial zeros in the Gram block between n = 13999824 as one Gram[y=0] point and n = 13999826 as another Gram[y=0] point. In any event, n = 13999825 mentioned by Professor Trudgian in his paper is [serendipitously] a *bad* Gram[y=0] point and a Gram's Law violation. Thus, n = 13999825 is not a valid choice for Rosser's Rule violation (failure).

A set is a collection of zero (viz, the empty set) or more elements (viz, a finite set with a finite number of elements or an infinite set with an infinite number of elements). A singleton refers to a finite set with a single element. A set can be any kind of mathematical objects: numbers, symbols, points in space, lines, other geometrical shapes, variables, or even other sets whereby these [mutable] non-repeating elements are not arranged in an unique order. A subset can be a [smaller] finite set derived from its [larger] parent finite set or its [larger] parent infinite set. A subset can also be a [smaller] infinite set derived from its [larger] parent infinite set. A tuple, which can potentially be subdivided into subtuples, is a finite ordered list (sequence) of elements whereby these [immutable] non-repeating elements are arranged in an unique order. Thus a tuple or a subtuple is regarded as a special type of finite set with the extra imposed restriction.

Lemma 1.1. We can validly classify countably infinite sets as accelerating, linear or decelerating subtypes; and concisely define Prevalence of prime and composite numbers [including logical deductions that support Polignac's and Twin prime conjectures to be true] and Prevalence of nontrivial zeros.

Proof. We provide the following required mathematical arguments.
Cardinality: With increasing size, arbitrary Set [or Subset] **X** can be countably finite set (**CFS**), countably infinite set (**CIS**) or uncountably infinite set (**UIS**). Denoted as ||**X**|| in this paper, the cardinality of Set **X** measures *number of elements* in Set **X**. For example, Set **negative Gram[y=0] point** as constituted by a [solitary] rational ($\mathbb{Q}$) t-value of 0 instead of a usual transcendental ($\mathbb{R} - \mathbb{A}$) t-value has CFS of negative Gram[y=0] point with this particular ||**negative Gram[y=0] point**|| = 1, Set even Prime number ($\mathbb{P}$) has CFS of solitary even $\mathbb{P}$ 2 with ||**even** $\mathbb{P}$|| = 1, Set Natural numbers ($\mathbb{N}$) has CIS of $\mathbb{N}$ with $||\mathbb{N}|| = \aleph_0$, and Set Real numbers ($\mathbb{R}$) has UIS of $\mathbb{R}$ with $||\mathbb{R}|| = \mathfrak{c}$ (cardinality of the continuum). Then with ||**CIS**|| = $\aleph_0$ = [countably] infinitely many elements; we provide a novel definitive classification for CIS based on its number of elements (cardinality) manifesting linear, accelerating or decelerating phenomena thus constituting three subtypes of CIS.
CIS-IM-accelerating: CIS with its cardinality given by ||**CIS-IM-accelerating**|| = $\aleph_0$-accelerating = [countably] infinitely many elements that will (overall) acceleratingly reach an *infinity value*. Examples: CP integers 0, 1, 4, 9, 16... generated by simple equation $y = x^2$ for x = 0, 1, 2, 3, 4... and CP values obtained from natural exponential function $y = e(x)$; and IP composite numbers 4, 6, 8, 9, 10... faithfully generated by complex Complement-Sieve-of-Eratosthenes algorithm [which is equivalent to simply discarding 0, 1, and all generated prime numbers via Sieve-of-Eratosthenes algorithm from the set of integers 0, 1, 2, 3, 4, 5...].
CIS-IM-linear: CIS with its cardinality given by ||**CIS-IM-linear**|| = $\aleph_0$-linear = [countably] infinitely many elements that will (overall) linearly reach an *infinity value*. Examples: CP entities 0, 1, 2, 3, 4, 5... [representing all positive integer numbers] generated by simple equation $y = x$ for x = 0, 1, 2, 3, 4...; CP entities 0, 2, 4, 6, 8, 10... [representing all positive even numbers] generated by simple equation $y = 2x$ for x = 0, 1, 2, 3, 4...; CP entities 1, 3, 5, 7, 9, 11... [representing all positive odd numbers] generated by simple equation $y = 2x - 1$ for x = 1, 2, 3, 4, 5...; and IP nontrivial zeros, Gram[y=0] points and Gram[x=0] points (all given as $\mathbb{R} - \mathbb{A}$ t-values) generated from complex equation Riemann zeta function via its *proxy* Dirichlet eta function. These IP entities will inevitably manifest IP perpetual repeating violations (failures) in Gram's Law occuring infinitely many times resulting in Set **negative Gram[y=0] points**. Thus CIS negative Gram[y=0] points is constituted by $\mathbb{R} - \mathbb{A}$ t-values and is classified as having ||**negative Gram[y=0] points**|| = ||**CIS-IM-linear**|| = $\aleph_0$-linear.
CIS-ALN-decelerating: CIS with its cardinality given by ||**CIS-ALN-decelerating**|| = $\aleph_0$-decelerating = [countably] arbitrarily large number of elements that will (overall) deceleratingly reach an *arbitrarily large number value*. Examples: CP entities 0, 1, $\sqrt{2}$, $\sqrt{3}$, 2, $\sqrt{5}$... generated by simple equation $y = \sqrt{x}$ for x = 0, 1, 2, 3, 4... and CP values obtained from natural logarithm function $y = ln(x)$; and IP prime numbers 2, 3, 5, 7, 11... faithfully generated by complex Sieve-of-Eratosthenes algorithm.

We analyze the data of all CIS-IM-linear computed nontrivial zeros (NTZ) when extrapolated out over a wide range of t ≥ 0 real number values. We can symbolically define nontrivial zeros counting function NTZ-π(t) = number of NTZ ≤ t with t assigned to having real number values which are conveniently designated by 10^n whereby n = 1, 2, 3, 4, 5.... **Prevalence of nontrivial zeros** = NTZ-π(t) / t = NTZ-π(t) / (10^n) when t = 0 to 10^n, whereby denominator t is also [artificially] regarded as having integer number values. We can conceptually define all consecutive NTZ gaps as i^{th} t-valued NTZ – $(i\text{-}1)^{th}$ t-valued NTZ. Thus there are CIS-IM-linear computed NTZ gaps. The numbers of NTZ between $10^0 - 10^1$ [interval = 9], $10^1 - 10^2$ [interval = 90], $10^2 - 10^3$ [interval = 900], $10^3 - 10^4$ [interval = 9000], $10^4 - 10^5$ [interval = 90000], $10^5 - 10^6$ [interval = 900000], $10^6 - 10^7$ [interval = 9000000], $10^7 - 10^8$ [interval = 90000000]... are 0, 29, 620, 9493, 127927, 1609077, 19388979, 226871900... with corresponding calculated "rolling" **Prevalence of nontrivial zeros** = 0, 0.322, 0.689, 1.055, 1.421, 1.788, 2.154, 2.521... $\Longrightarrow$ **Prevalence of nontrivial zeros** seems to overall fluctuatingly increase by around 0.366 in a "linear" manner. This limited observation alone would suggest Cardinality of nontrivial zeros = ||**CIS-IM-linear**|| = $\aleph_0$-linear.

We analyze the data of all CIS-IM-accelerating computed composite numbers when extrapolated out over a wide range of x ≥ 4 integer values. We define composite counting function Composite-π(x) = number of composites ≤ x with x conveniently assigned to having odd number values of the form $10^n - 1$ whereby n = 1, 2, 3, 4, 5.... **Prevalence of all composite numbers** = Composite-π(x) / x = Composite-π(x) / ($10^n - 4$) when x = 4 to $10^n - 1$. CIS-IM-accelerating composite numbers in totality all have either odd composite gap 1 or even composite gap 2. All the odd integers which are not prime are odd composite numbers. The consecutive odd composite numbers are 9, 15, 21, 25, 27, 31.... All the even integers which are not prime are even composite numbers. The consecutive even composite numbers are 4, 6, 8, 10, 12, 14, 16.... We can create **Prevalence of even composite numbers** and **Prevalence of odd composite numbers** [with

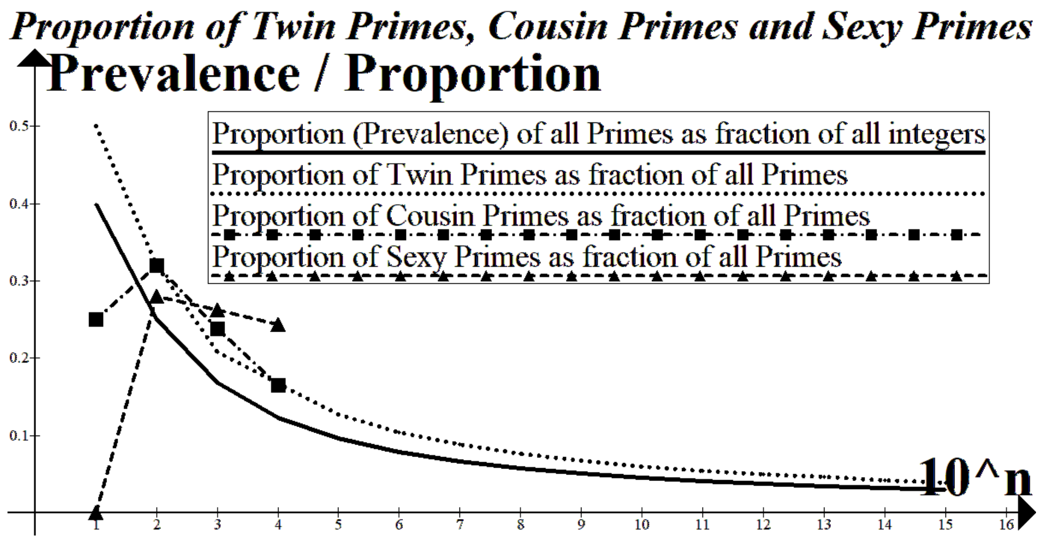

Figure 1. Proportion of Twin primes, Cousin primes [as partial calculations] and Sexy Primes [as partial calculations] with Proportion (Prevalence) of all Primes included that depict deceleratingly reaching an infinitesimal small number value $\frac{1}{\infty}$ just above 0 [but never reaches 0] as $n \to \infty$. For $n = 1, 2, 3, 4, 5, 6, 7, 8...$ used for the same horizontal or x axis defined by 10^n; $\text{Prev}_{allC} = 0.5, 0.75, 0.832, 0.8771, 0.9041, 0.9215, 0.9335, 0.9424...$ will fully reflect the statement Integers = {0, 1} + all Primes {2, 3, 5, 7, 11, 13...} + all Composites {4, 6, 8, 9, 10, 12...}. Then the Proportion (Prevalence) of all Composites will reciprocally depict acceleratingly reaching an infinitesimal small number value $\frac{1}{\infty}$ just below 1 [but never reaches 1] as $n \to \infty$.

the former always being larger than the later because, with exception of even prime number 2, all prime numbers are odd numbers].

We analyze the data of all CIS-ALN-decelerating computed prime numbers when extrapolated out over a wide range of x ≥ 2 integer values. We define prime counting function Prime-π(x) = number of primes $\leq$ x with x conveniently assigned to having odd number values of the form $10^n - 1$ whereby n = 1, 2, 3, 4, 5.... **Prevalence of all prime numbers** = Prime-π(x) / x = Prime-π(x) / (10^n − 2) when x = 2 to $10^n - 1$. Prime gaps for all odd prime numbers can only be constituted by CIS-ALN-decelerating even Prime gaps 2, 4, 6, 8, 10.... One could further create **Prevalence of twin primes with prime gap 2**, **Prevalence of cousin primes with prime gap 4**, **Prevalence of sexy primes with prime gap 6**, etc.

n	1	2	3	4	5	6	7
Prev_{allP}	0.4	0.25	0.168	0.1229	0.09592	0.078498	0.0664579
Prev_{Pgap2}	0.2	0.08	0.035	0.0205	0.01224	0.008169	0.0058980
Prev_{Pgap4}	0.1	0.08	0.040	0.0202	...	...	...
Prev_{Pgap6}	0.0	0.07	0.044	0.0299	...	...	...

n	8	9	10	11	...
Prev_{allP}	0.05761455	0.050847534	0.0455052511	0.04118054813	...
Prev_{Pgap2}	0.00440312	...	...	...	...

The terms *Prevalence* and *Proportion* are interchangeable. For n = 1, 2, 3, 4, 5... in expression x = 2 to $10^n - 1$, we obtain the above tabulated computations with captured manifestations of decelerating properties in regards to relationship Prevalence of all primes (Prev_{allP}) = Prevalence of twin primes having prime gap 2 (Prev_{Pgap2}) + Prevalence of cousin primes having prime gap 4 (Prev_{Pgap4}) + Prevalence of sexy primes having prime gap 6 (Prev_{Pgap6}) +....

n	1	2	3	4	5	6	7	8
$\frac{Prev_{Pgap2}}{Prev_{allP}}$	0.5	0.32	0.2083	0.1668	0.1276	0.1041	0.08875	0.07642
$\frac{Prev_{Pgap4}}{Prev_{allP}}$	0.25	0.32	0.2381	0.1644	...	...	...	...
$\frac{Prev_{Pgap6}}{Prev_{allP}}$	0.0	0.28	0.2619	0.2433	...	...	...	...

We next calculate in the above table, which are then graphically depicted in Figure 1, Proportion of Twin primes with prime gap 2, Cousin primes with prime gap 4, and Sexy primes with prime gap 6. These are respectively derived using relevant ratios $\frac{Prev_{Pgap2}}{Prev_{allP}}$, $\frac{Prev_{Pgap4}}{Prev_{allP}}$ and $\frac{Prev_{Pgap6}}{Prev_{allP}}$ with the Proportion of all Primes also depicted for comparison.

We can ignore the solitary even Prime number 2 [at x = 2] and simply regard all Primes here as odd primes. Then there is an arbitrarily large number of all Primes for x = 3 to $10^n - 1$. From subsection 1.3 and supported by Appendix G, we conclude the average prime gaps in relation to arbitrarily large number of all even Prime gaps 2, 4, 6, 8, 10, 12, 14... must overall and individually manifest the asymptotically zero behavior of natural logarithm. Proportion of all Primes is known to deceleratingly reach an infinitesimal small number value $\frac{1}{\infty}$ [but never 0] as $n \to \infty$. Then for the case of prime gap 2, we infer the following deduction based on the all-important condition *Proportion of Twin primes is coupled to Proportion of all Primes as two self-similar fractal objects* [displayed in Figure 1]: Whereas the Proportion of Twin Primes and Proportion of all Primes are coupled together as two self-similar *fractal objects* [that are never identical], so must both the well-defined Proportion of Twin Primes and Proportion of all Primes always manifest deceleratingly reaching an infinitesimal small number value $\frac{1}{\infty}$ [but never 0] as $n \to \infty$. We analogically create the same condition using Proportion of Prime numbers for remaining cases of prime gaps 4, 6, 8, 10, 12... coupled to Proportion of all Primes as arbitrarily large number of self-similar *fractal objects* [that are never identical]. With this extrapolation, we similarly infer the same deductions to that for twin primes. The (Modified) Polignac's and Twin prime conjectures are concluded to be true since the well-defined Proportion of Prime numbers with prime gaps 2, 4, 6, 8, 10, 12, 14... and Proportion of all Primes must logically manifest deceleratingly reaching an infinitesimal small number value $\frac{1}{\infty}$ [but never 0] as $n \to \infty$. *The proof is now complete for Lemma 1.1*□.

Remark 1.1. Incorporating the subtype classification of countably infinite sets, we outline the simple and complex properties manifested by Completely Predictable and Incompletely Predictable entities. As an example of simple property, x-axis intercept points for simple function *sin* n are Completely Predictable to "linearly" occur infinitely many times when n = all positive and negative multiples of π. Here are examples of complex properties: As stated by Gram's Law, x-axis intercept points for complex function Riemann-Siegel function $Z(t)$ or Riemann zeta function [via its proxy Dirichlet eta function] "linearly" occur infinitely many times as Incompletely Predictable t-values that represent usual positive Gram[y=0] points which tend to alternate with nontrivial zeros. As unique Incompletely Predictable events "linearly" occurring infinitely many times, there are intermittent observable *various geometric variants* of two consecutive (positive first and then negative) Gram[y=0] points that is alternatingly followed by two consecutive nontrivial zeros. These events denote violations (failures) of Gram's Law. Violations (failures) of Rosser's Rule refer to the much less frequent intermittently occurring Incompletely Predictable observable *various geometric variants* of reduction in expected number of t-values for certain x-axis intercept points. "Linearly" occurring infinitely many times, each of these events gives rise to two missing Gram[y=0] points or, equivalently, to two extra nontrivial zeros. *Plus Gap 2 Composite Number Continuous Law* and *Plus-Minus Gap 2 Composite Number Alternating Law* (Ting, 2020) outlined in section 3 are two overall Incompletely Predictable properties seen when we dependently combine [deceleratingly-occurring] primes and [acceleratingly-occurring] composites with associated prime gaps and composite gaps for critical analysis.

We validly ignore solitary even prime number 2 with odd prime gap 1 for most calculations in this paper and arbitrarily divide CIS-ALN-decelerating even prime gaps 2, 4, 6, 8, 10... into *small prime gaps* of 2 and 4, and *large prime gaps* of ≥ 6. Occurring over 2000 years ago (c. 300 BC), ancient Euclid's theorem on infinitude of prime numbers in totality predominantly by *reductio ad absurdum* (proof by contradiction) is earliest known but not the only proof for this simple problem. Since then dozens of proofs have been devised such as three chronologically listed: Goldbach's Proof using Fermat numbers (written in a letter to Swiss mathematician Leonhard Euler, July 1730), Furstenberg's Topological Proof (Furstenberg, 1955), and Filip Saidak's Proof (Saidak, 2006). The strangest candidate is Furstenberg's Topological Proof.

In 2013, Yitang Zhang proved a landmark result showing some unknown even number $N < 70$ million such that this condition holds: There are CIS-ALN-decelerating odd primes that differ by N between each other (Zhang, 2014). By optimizing Zhang's bound, subsequent Polymath Project collaborative efforts using a new refinement of GPY sieve in 2013 lowered N to 246; and assuming Elliott-Halberstam conjecture and its generalized form further lower N to 12 and 6, respectively. Intuitively, N has more than one valid values such that the same condition holds for each N value. We can at most lower N to 2 and 4 in regards to odd primes having small prime gaps 2 and 4 with each uniquely generating CIS-ALN-decelerating odd primes. We anticipate there are all the remaining prime gaps in regards to odd primes with large prime gaps denoted by $N \geq 6$ values whereby each large prime gap will generate its own unique CIS-ALN-decelerating odd primes.

1.2 Inverse functions of ln(x) with e(x) and li(x) with Ei(x)

We start with the conditional statement "If P, then Q" which is notated as P → Q. The converse of the conditional statement is "If Q, then P" which is notated as Q → P. The contrapositive of the conditional statement is "If not Q, then not P" which is notated as ~Q → ~P. The inverse of the conditional statement is "If not P then not Q" which is notated as ~P → ~Q. An inverse function (or anti-function) is a function that "reverses" another function: if the function f applied to an input x gives a result of y, then applying its inverse function g to y gives the result x, i.e., $g(y)$ = x if and only if $f(x)$ = y. Not all functions have an inverse. The inverse function of f is also denoted as f^{-1} and it exists if and only if f is bijective. Since a function is a special type of binary relation, many of the properties of an inverse function correspond to following three properties of converse relations:

(i) Uniqueness. If an inverse function exists for a given function f, then it is unique. This follows since the inverse function must be the converse relation, which is completely determined by f.

(ii) Symmetry. There is a symmetry between a function and its inverse. Specifically, if f is an invertible function with domain X and codomain Y, then its inverse f^{-1} has domain Y and image X, and the inverse of f^{-1} is the original function f. In symbols, for functions f: X → Y and f^{-1}: Y → X, $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$. This statement is a consequence of the implication that for f to be invertible it must be bijective.

(iii) Self-inverses. If X is a set, then the identity function on X is its own inverse: ${\mathrm{id}_X}^{-1} = \mathrm{id}_X$. More generally, a function f : X → X is equal to its own inverse, if and only if the composition $f \circ f$ is equal to id_X. Such a function is called an involution. The involutory nature of the inverse can be concisely expressed by $\left(f^{-1}\right)^{-1} = f$. The inverse of $g \circ f$ is $\left(f^{-1}\right) \circ \left(g^{-1}\right)$. The inverse of a composition of functions is given by $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Notice that the order of g and f have been reversed; to undo f followed by g, we must first undo g, and then undo f. For a function $f : X \to Y$, its inverse $f^{-1} : Y \to X$ admits an explicit description: it sends each element $y \in Y$ to the unique element $x \in X$ such that $f(x) = y$.

As the base of natural logarithm, irrational (transcendental) number e is a mathematical constant approximately equal to 2.71828. It is the limit of $(1 + \frac{1}{n})^n$ as n approaches ∞ and can also be calculated as sum of infinite series $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots$. We outline the important properties of natural logarithm function and natural exponential function with their connections to logarithmic integral function and exponential integral function as inverse or pseudo-inverse functions. The natural logarithm function, if considered as a real-valued function of a positive real variable, is the inverse of exponential function, leading to the following identities: $\begin{array}{ll} e^{\ln x} = x & \text{if } x \text{ is strictly positive,} \\ \ln e^x = x & \text{if } x \text{ is any real number.} \end{array}$

As shown in Figure 2, the natural logarithm ln(x) has a vertical asymptote of x = 0 [y-axis] as x approaches 0 [with ln(x) becoming −∞]. Its inverse function e(x) has a horizontal asymptote of y = 0 [x-axis] as x approaches −∞ [with e(x) becoming 0]. With the slope of horizontal line being 0, and the slope of vertical line being an undefined value; we recognize the slope of ln(x) becomes an infinitesimal small number $(+\frac{1}{\infty})$ that approaches 0 as x grows towards ∞ but the slope of its inverse function e(x) becomes an infintely large number (+∞) that approaches an undefined value as x grows towards ∞. Extrapolations: **(1) Slope for Prevalence of all Primes as fraction of all integers.** We recognize the slope of ln(x) can symbolically denote, for instance, the (decelerating) slope for Prevalence of all Primes will approach 0 as an infinitesimal small number value [but never becomes 0] as x grows towards ∞. **(2) Slope for Prevalence of all Composites as fraction of all integers.** Similarly, the slope of e(x) can symbolically denote, for instance, the (accelerating) slope for Prevalence of all Composites will approach an undefined value as an infinite large number value [but never becomes an undefined value] as x grows towards ∞.

Like all logarithms, the natural logarithm maps multiplication of positive numbers into addition: $\ln(x \cdot y) = \ln x + \ln y$. Logarithms can be defined for any positive base other than 1, not only e. However, logarithms in other bases differ only by a constant multiplier from the natural logarithm, and can be defined in terms of the latter, $\log_b x = \ln x / \ln b = \ln x \cdot \log_b e$. Their properties are $\ln 1 = 0$; $\ln e = 1$; $\ln(xy) = \ln x + \ln y$ for $x > 0$ and $y > 0$; $\ln(x/y) = \ln x - \ln y$; $\ln(x^y) = y \ln x$ for $x > 0$; $\ln x < \ln y$ for $0 < x < y$; $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$; $\lim_{\alpha \to 0} \frac{x^\alpha - 1}{\alpha} = \ln x$ for $x > 0$; $\frac{x-1}{x} \leq \ln x \leq x - 1$ for $x > 0$; $\ln(1 + x^\alpha) \leq \alpha x$ for $x \geq 0$ and $\alpha \geq 1$.

The real natural exponential function $\exp : \mathbb{R} \to \mathbb{R}$ can be characterized in a variety of equivalent ways. It is commonly defined by Taylor series $\exp x := \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!} + \cdots$. By way of binomial theorem

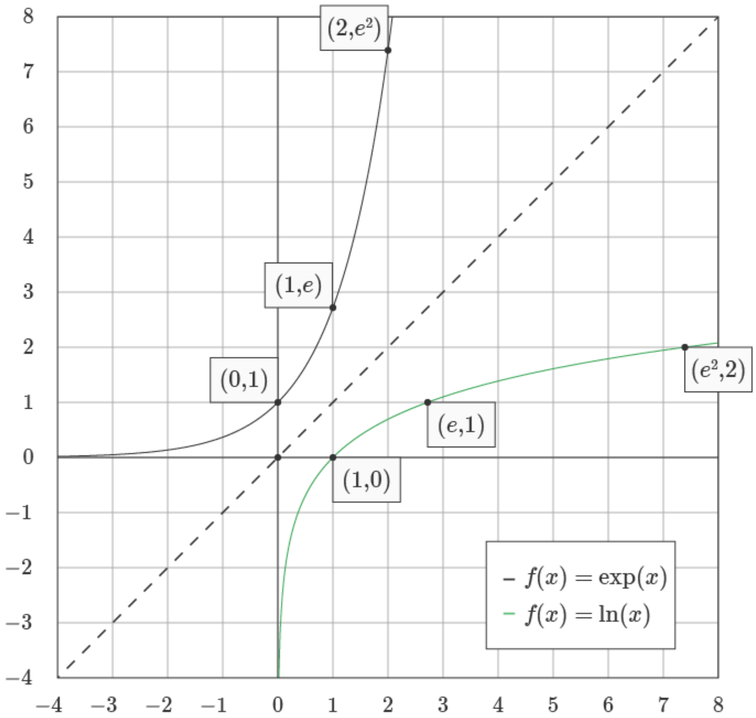

Figure 2. The natural logarithm function $log_e x$ or $ln(x)$ and the natural exponential function exp(x) or e^x. The graphs of $log_e x$ and its inverse e^x are symmetric with respect to line y = x thus geometrically denoting diagonal symmetry of these two functions.

and power series definition, the exponential function can also be defined as the limit: $\exp x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$. It can be shown that every continuous, nonzero solution of the functional equation $f(x+y) = f(x)f(y)$ is an exponential function, $f : \mathbb{R} \to \mathbb{R},\ x \mapsto e^{kx}$, with $k \in \mathbb{R}$. The exponential function satisfies the exponentiation identity $e^{x+y} = e^x e^y$ for all $x, y \in \mathbb{R}$, which, along with the definition $e = \exp(1)$, shows that factors $e^n = \underbrace{e \times \cdots \times e}_{n \text{ factors}}$ for positive integers n, and relates exponential function to the elementary notion of exponentiation. The base of ***natural*** exponential function, its value at 1, $e = \exp(1)$ is a ubiquitous mathematical constant called Euler's number approximately equal to 2.71828 – as mentioned above, this number also acts as base of ***natural*** logarithm function.

The [analogical] logarithmic integral function li(x) is defined as $\mathrm{li}(x) = \int_0^x \frac{dt}{\ln t}$. The function $1/(\ln t)$ has a singularity at t = 1, and the integral for x > 1 is interpreted as a Cauchy principal value, $\mathrm{li}(x) = \lim_{\varepsilon\to 0+}\left(\int_0^{1-\varepsilon}\frac{dt}{\ln t} + \int_{1+\varepsilon}^{x}\frac{dt}{\ln t}\right)$. The li (x) function is related to its inverse exponential integral function Ei(x) via equation li (x) = Ei (ln x), and is valid for x > 0. This identity provides a series representation of li(x) as $\mathrm{li}(e^u) = \mathrm{Ei}(u) = \gamma + \ln|u| + \sum_{n=1}^{\infty}\frac{u^n}{n \cdot n!}$ for $u \neq 0$, where $\gamma \approx 0.57721\ 56649\ 01532...$ is the Euler-Mascheroni constant. A more rapidly convergent series by Ramanujan is $\mathrm{li}(x) = \gamma + \ln\ln x + \sqrt{x}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(\ln x)^n}{n!\,2^{n-1}}\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\frac{1}{2k+1}$. The asymptotic behavior for x → ∞ is $\mathrm{li}(x) = O\left(\frac{x}{\ln x}\right)$ where O is big O notation. The full asymptotic expansion is $\mathrm{li}(x) \sim \frac{x}{\ln x}\sum_{k=0}^{\infty}\frac{k!}{(\ln x)^k}$ or $\frac{\mathrm{li}(x)}{x/\ln x} \sim 1 + \frac{1}{\ln x} + \frac{2}{(\ln x)^2} + \frac{6}{(\ln x)^3} + \cdots$. This gives more accurate asymptotic behaviour: $\mathrm{li}(x) - \frac{x}{\ln x} = O\left(\frac{x}{(\ln x)^2}\right)$.

1.3 Prime number theorem and Composite number theorem

All prime numbers generated by Sieve-of-Eratosthenes algorithm and all composite numbers generated by Complement-Sieve-of-Eratosthenes algorithm are mutually exclusive and complementary numbers when analyzed in the context of all integers 0, 1, 2, 3, 4, 5, 6.... These two algorithms will act as pseudo-inverse algorithms of each other whereby the integers 0 and 1 are neither prime nor composite.

We define Prime gaps as P–$g_n = P_{n+1} - P_n$ and Composite gaps as C–$g_n = C_{n+1} - C_n$. The former are constituted by CFS of P–g_n = 1 representing solitary even prime number {2}; and CIS-ALN-decelerating of P–g_n {2, 4, 6, 8, 10...} representing all CIS-ALN-decelerating odd prime numbers {3, 5, 7, 11, 13, 17, 19...}. The later are constituted by CFS of C–g_n = 1 representing all CIS-IM-accelerating odd composite numbers {9, 15, 21, 25, 27, 33, 35, 39...} and all CIS-IM-accelerating even composite numbers {8, 14, 20, 24, 26, 32, 34, 38, 44...} [that both occur together in between any two odd prime numbers specified by P–$g_n \geq 4$]; and C–g_n = 2 representing all CIS-ALN-decelerating even composite numbers [that precede all odd prime numbers] {4, 6, 10, 12, 16, 18, 22, 28, 30, 36, 40, 42...}. One notice three *useful* facts: (i) the [only] solitary even prime number 2 do not have a preceding composite number since 1 is neither prime nor composite, (ii) the [recurring] CIS-ALN-decelerating even composite numbers following all twin primes {4, 6, 12, 18, 30, 42, 60, 72...} having C–g_n = 2 always represent the next even composite number that will precede the following odd prime number [with thus complete absence of both even composite numbers having C–g_n = 1 and odd composite numbers having C–g_n = 1 in between the two odd primes that specify the involved twin primes], and (iii) the [only] consecutive twin primes [with both having P–g_n = 2] that can occur involves three consecutive odd prime numbers 3, 5 and 7; and are associated with the [only] two existing consecutive even composite numbers 4 and 6 [with both having C–g_n = 2].

Combined Completely Predictable Even-Odd formula:
CIS-IM-linear Gap 1-integers {0, 1, 2, 3, 4, 5, 6...}
= CIS-IM-linear Gap 2-even numbers {0, 2, 4, 6, 8, 10, 12...} + CIS-IM-linear Gap 2-odd numbers {1, 3, 5, 7, 11, 13...}

We deduce the independent functions (equations) $y = f(x) = 2x$ ***with its inverse function*** $y^{-1} = f^{-1}(x) = \frac{x}{2}$ ***that generate all Gap 2-even numbers and*** $y = f(x) = 2x - 1$ ***with its inverse function*** $y^{-1} = f^{-1}(x) = \frac{x+1}{2} = \frac{x}{2} + \frac{1}{2}$ ***that generate all Gap 2-odd numbers must act as two complementary and balanced functions [whereby these are pseudo-inverse functions of each other since the two inverse functions only differ by the constant*** $\frac{1}{2}$***].*** We compare this to the two dependent functions $y = e(x)$ and $y = ln(x)$ which are complementary and balanced inverse functions of each other.

Combined Incompletely Predictable Prime-Composite formulae:
CIS-IM-linear Gap 1-integers {0, 1, 2, 3, 4, 5, 6...}
= CFS integers {0, 1} + CFS even prime number {2} + CIS-ALN-decelerating odd prime numbers {3, 5, 7, 11, 13, 17, 19...} + CIS-IM-accelerating composite numbers {4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20...}.
We deduce the dependent algorithms that generate all odd prime numbers [which does not include solitary even prime number 2] and all composite numbers must act as two complementary and balanced pseudo-inverse algorithms.
= CFS integers {0, 1} + CFS Gap 1-even prime number {2} + even Prime Gaps 2, 4, 6, 8, 10...-CIS-ALN-decelerating odd prime numbers {3, 5, 7, 11, 13, 17, 19...} + CIS-ALN-decelerating even Gap 2-composite numbers [that precede all odd prime numbers except for odd prime number 3] {4, 6, 10, 12, 16, 18, 22, 28, 30, 36, 40, 42...} + CIS-IM-accelerating even Gap 1-composite numbers {8, 14, 20, 24, 26, 32, 34, 38, 44...} [in between two odd prime numbers that are not twin primes] + CIS-IM-accelerating odd Gap 1-composite numbers {9, 15, 21, 25, 27, 33, 35, 39...} [in between two odd prime numbers that are not twin primes].
We deduce the dependent paired algorithms that generate (i) {odd prime numbers [but which does not include solitary even prime number 2] + even Gap 2-composite numbers [but which does not precede solitary odd prime number 3]} and (ii) {even Gap 1-composite numbers + odd Gap 1-composite numbers} must also act as two complementary and balanced paired pseudo-inverse algorithms.

Prime-Composite quotient: Based on Prime-$\pi(x)$ and Composite-$\pi(x)$ as $x \to \infty$, the Prime-Composite quotient is derived from Conservation of Set (total odd primes generated by all even Prime gaps) and its corresponding Subsets (odd primes generated by each and every even Prime gaps): Whereas the overall algorithm that generate all odd prime numbers from even Prime gaps 2, 4, 6, 8, 10... is classified as CIS-ALN-decelerating; so must each and every sub-algorithms that generate Gap 2-odd primes (twin primes) from even Prime gap 2, Gap 4-odd primes (cousin primes) from even Prime gap 4, Gap 6-odd primes (sexy primes) from even Prime gap 6, etc be also classified as CIS-ALN-decelerating [and not be classified as CIS-IM-accelerating nor as CFS]. This deduction then allows the balanced conservation of CIS-IM-linear (all integers) = CFS (neither prime nor composite 1, 2) + CFS (even prime 2) + CIS-ALN-decelerating (odd primes 3, 5, 7, 11, 13, 17, 19...) + CIS-IM-accelerating (even and odd composites 4, 6, 8, 9, 10, 12, 14, 15, 16, 18...) whereby (i) CIS-ALN-decelerating (odd primes 3, 5, 7, 11, 13, 17, 19...) = CIS-ALN-decelerating (twin primes 3, 5, 11, 17, 29, 41...) + CIS-ALN-decelerating (cousin primes 7, 13, 19, 37, 43, 67...) + CIS-ALN-decelerating (sexy primes 23, 31, 47, 53, 61, 73...) +... in an *ad infinitum* manner and (ii) for n = 1, 2, 3, 4, 5..., the limit of [combined] **Prime-Composite quotient** as x increases without bound is 0; namely,

$$\lim_{x\to\infty} \frac{\text{CIS-ALN-decelerating (Gap 2n-odd primes)} + \text{CIS-ALN-decelerating (Gap 2-even composites)}}{\text{CIS-IM-accelerating (Gap 1-even composites)} + \text{CIS-IM-accelerating (Gap 1-odd composites)}} = 0$$

Using asymptotic notation, this [inversely proportional] quotient result can be restated as:

$$\text{CIS-ALN-decelerating (Gap 2n-odd primes)} + \text{CIS-ALN-decelerating (Gap 2-even composites)}$$
$$\sim \frac{1}{\text{CIS-IM-accelerating (Gap 1-even composites)} + \text{CIS-IM-accelerating (Gap 1-odd composites)}}$$

We reiterate again that Gap 1-even composites and Gap 1-odd composites are missing between all twin primes. Strictly, CIS-ALN-decelerating (Gap 2n-odd primes) = CIS-ALN-decelerating (Gap 2-even composites) + 1 whereby the even number 2 that precede first odd prime number 3 is prime and thus not a Gap 2-even composite. Since CIS-ALN-decelerating (Gap 2n-odd primes) = CIS-ALN-decelerating (Gap 2-even composites) and CIS-IM-accelerating (Gap 1-even composites) = CIS-IM-accelerating (Gap 1-odd composites) is a sufficiently accurate and valid statement, then the following are also valid statements:

$$\text{CIS-ALN-decelerating (Gap 2n-odd primes) or CIS-ALN-decelerating (Gap 2-even composites)}$$
$$\sim \frac{1}{2 \cdot \text{CIS-IM-accelerating (Gap 1-even composites)} + 2 \cdot \text{CIS-IM-accelerating (Gap 1-odd composites)}}$$
$$\sim \frac{1}{4 \cdot \text{CIS-IM-accelerating (Gap 1-even composites)}}$$
$$\sim \frac{1}{4 \cdot \text{CIS-IM-accelerating (Gap 1-odd composites)}}$$

Thus two randomly selected consecutive odd prime numbers can systematically be further classified [non-overlappingly] according to their even Prime gaps as shown below:
P–g_n = 2 as Gap 2-O-$\mathbb{P}$ or twin primes: 3, 5, 11, 17, 29, 41, 59, 71, 101, 107, 137, 149, 179, 191, 197, 227, 239, 269,... as CIS-ALN-decelerating
P–g_n = 4 as Gap 4-O-$\mathbb{P}$ or cousin primes: 7, 13, 19, 37, 43, 67, 79, 97, 103, 109, 127, 163, 193, 2223, 229, 277, 307, 313, 349,... as CIS-ALN-decelerating
P–g_n = 6 as Gap 6-O-$\mathbb{P}$ or sexy primes: 23, 31, 47, 53, 61, 73, 83, 131, 151, 157, 167, 173, 233, 251, 257, 263, 271,... as CIS-ALN-decelerating
...and involving all other even prime gaps in an *ad infinitum* manner...

Traditionally, Twin prime conjecture refers to the proposal even Prime gap 2 will generate CIS of all twin primes. It is simply a subset of Polignac's conjecture that refers to the proposal on CIS of all even Prime gaps 2, 4, 6, 8, 10... whereby each even Prime gap will generate its own unique CIS of all odd prime numbers as twin primes, cousin primes, sexy primes, etc. Our Modified Polignac's and Twin prime conjectures simply refer to these same concepts except that the mentioned terms of CIS must now be replaced by CIS-ALN-decelerating.

The proofs for Modified Polignac's and Twin prime conjectures are thus obtained by demonstrating the asymptotic law of distribution of prime numbers [that involves the natural logarithm function] to be applicable to Set (all Incompletely Predictable odd prime numbers) derived from all even Prime gaps 2, 4, 6, 8, 10... and simultaneously also applicable to corresponding Incompletely Predictable Subsets (odd prime numbers derived from each and every even Prime gaps). Here, we can ignore the Completely Predictable first and only even prime number 2 as an exception. The (complementary) asymptotic law of distribution of composite numbers [that involves the natural exponential function] must then be applicable to Set (all Incompletely Predictable odd composite numbers and even composite numbers 8, 9, 14, 15, 20, 21, 23, 24, 25, 26, 27... having odd Composite gap 1) – these particular composite numbers only involve all non-twin primes with even Prime gaps 4, 6, 8, 10, 12... but not twin primes with even Prime gap 2 as an exception since twin primes are never associated with either odd Gap 1-composite numbers or even Gap 1-composite numbers. We again note the exception that all the Incompletely Predictable even composite numbers 4, 6, 10, 12, 16, 18, 22, 28... having even Composite gap 2 that precede every odd prime numbers [apart from the very first odd prime number 3 which is preceded by even prime number

2] will instead manifest asymptotic law of distribution of prime numbers [that involves natural logarithm function].

Let Prime-$\pi(x)$ be prime-counting function defined to be the number of primes less than or equal to x, for any real number x. Let Composite-$\pi(x)$ be composite-counting function defined to be the number of composites less than or equal to x, for any real number x. Prime number theorem (and Composite number theorem) describes the asymptotic distribution of prime numbers (and asymptotic distribution of composite numbers) among the positive integers. Respectively, they formalize the intuitive idea that primes (and composites) become deceleratingly less (and acceleratingly more) common as they become larger by precisely quantifying the rate at which this occurs. Prime number theorem is concluded to be heuristically true and was proved independently by Jacques Hadamard (1896) and Charles Jean de la Vallee Poussin (1896) using ideas introduced by Bernhard Riemann (in particular, Riemann zeta function). This theorem has also been rigorously proven as the elementary proofs of Atle Selberg (1949) and Paul Erdos (1949), and as the non-elementary proof by Donald J. Newman (1980) in the sense that he used Cauchy's integral theorem from complex analysis in his proof.

The asymptotic law of distribution of prime numbers is given as $\lim_{x\to\infty} \frac{\text{Prime-}\pi(x)}{\left[\frac{x}{\ln(x)}\right]} = 1$. Using asymptotic notation, this result can be restated as Prime-$\pi(x) \sim \frac{x}{\ln(x)}$, thus representing Prime number theorem. Prime number theorem is also equivalent to the statement that the nth prime number P$_n$ satisfies $P_n \sim n\ln(n)$. The [pseudo-inverse] asymptotic law of distribution of composite numbers is given as $\lim_{x\to\infty} \frac{\text{Composite-}\pi(x)}{\left[\frac{x}{e(x)}\right]} = 1$. Using asymptotic notation, this result can be restated as Composite-$\pi(x) \sim \frac{x}{e(x)}$, thus representing Composite number theorem. Composite number theorem is also equivalent to the statement that the nth composite number C$_n$ satisfies $C_n \sim ne(n)$. The following asymptotic relations for prime numbers [and composite numbers] are logically equivalent:

$\lim_{x\to\infty} \frac{\text{Prime-}\pi(x)\ln(x)}{x} = 1$, and $\lim_{x\to\infty} \frac{\text{Prime-}\pi(x)\ln(\text{Prime-}\pi(x))}{x} = 1$.

$\lim_{x\to\infty} \frac{\text{Composite-}\pi(x)e(x)}{x} = 1$, and $\lim_{x\to\infty} \frac{\text{Composite-}\pi(x)e(\text{Composite-}\pi(x))}{x} = 1$.

Prime number theorem is also equivalent to $\lim_{x\to\infty} \frac{\vartheta(x)}{x} = \lim_{x\to\infty} \frac{\psi(x)}{x} = 1$, where ϑ and ψ are the first and the second Chebyshev functions respectively, and to $\lim_{x\to\infty} \frac{M(x)}{x} = 0$, where $M(x) = \sum_{n\le x} \mu(n)$ is the Mertens function. Here, the most common generalized counting function is Chebyshev function $\psi(x)$ defined by $\psi(x) = \sum_{\substack{p^k\le x,\\ p \text{ is prime}}} \ln p$. This is sometimes written as $\psi(x) = \sum_{n\le x} \Lambda(n)$, where Λ(n) is von Mangoldt function, namely $\Lambda(n) = \begin{cases} \ln p & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \ge 1, \\ 0 & \text{otherwise.} \end{cases}$.

The logarithmic integral function li(x) is defined by $\text{li}(x) = \int_0^x \frac{dt}{\ln t}$. An even better approximation to Prime-$\pi(x)$ is given by the offset logarithmic integral function Li(x) which is defined by $\text{Li}(x) = \int_2^x \frac{dt}{\ln t} = \text{li}(x) - \text{li}(2)$; or equivalently, $\text{li}(x) = \int_0^x \frac{dt}{\ln t} = \text{Li}(x) + \text{li}(2)$. Also, Li(x) = $Ei(\ln(x)) - Ei(\ln(2))$ since logarithmic integral function li(x) is related to inverse exponential function Ei(x) via equation li(x) = $Ei(\ln(x))$, valid for x > 0. Both li(x) and Li(x) strongly support the notion that density of prime numbers around t should be $\frac{1}{\ln(t)}$; and is related to natural logarithm by the asymptotic expansion $\text{Li}(x) \sim \frac{x}{\ln x}\sum_{k=0}^{\infty} \frac{k!}{(\ln x)^k} = \frac{x}{\ln x} + \frac{x}{(\ln x)^2} + \frac{2x}{(\ln x)^3} + \cdots$. With inverse $\text{Ei}(x) = \text{Li}\,(e(x)) - \text{Li}\,(e(2))$, we conclude $Ei(x) - Ei(2)$ or $Ei(x)$ will both strongly support the notion that density of composite numbers around t should be $\frac{1}{e(t)}$.

The asymptotic law of distribution of prime numbers [and composite numbers] is given as $\lim_{x\to\infty} \frac{\text{Prime-}\pi(x)}{\text{Li}(x)} = 1$ or $\lim_{x\to\infty} \frac{\text{Prime-}\pi(x)}{\text{li}(x)} = 1$ [and $\lim_{x\to\infty} \frac{\text{Composite-}\pi(x)}{\text{Ei}(x)-\text{Ei}(2)} = 1$ or $\lim_{x\to\infty} \frac{\text{Composite-}\pi(x)}{\text{Ei}(x)} = 1$]. Using asymptotic notation, this result is correspondingly restated as Prime-$\pi(x) \sim \text{Li}(x)$ or Prime-$\pi(x) \sim \text{li}(x)$ [and Composite-$\pi(x) \sim \text{Ei}(x) - \text{Ei}(2)$ or Composite-$\pi(x) \sim \text{Ei}(x)$], thus equivalently representing Prime number theorem [and complementary pseudo-inverse

Composite number theorem]. In 1899, de la Vallee Poussin proved the estimate Prime-$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-a\sqrt{\ln x}}\right)$ as $x \to \infty$ is valid for some positive constant a, where $O(...)$ is big O notation. The statements Prime-$\pi(x) = \mathrm{Li}(x) + O\left(\sqrt{x}\ln x\right)$ and $|\operatorname{li}(x) - \text{Prime-}\pi(x)| = O(x^{1/2+a})$ for any $a > 0$ are equivalent to Riemann hypothesis.

Riemann's prime-power counting function, usually denoted as $\Pi_0(x)$ or $J_0(x)$, has jumps of $\frac{1}{n}$ at prime powers p^n and takes a value halfway between the two sides at the discontinuities of $\pi(x)$. That added detail is used because the function may then be defined by an inverse Mellin transform. We formally define $\Pi_0(x)$ by $\Pi_0(x) = \frac{1}{2}\left(\sum_{p^n<x}\frac{1}{n} + \sum_{p^n\le x}\frac{1}{n}\right)$ where the variable p in each sum ranges over all primes within the specified limits. We may also write $\Pi_0(x) = \sum_{n=2}^{x}\frac{\Lambda(n)}{\ln n} - \frac{\Lambda(x)}{2\ln x}$ $= \sum_{n=1}^{\infty}\frac{1}{n}\pi_0(x^{1/n})$ where $\Lambda(n)$ is the von Mangoldt function and $\pi_0(x) = \lim_{\varepsilon\to 0}\frac{\pi(x-\varepsilon)+\pi(x+\varepsilon)}{2}$. The Mobius inversion formula then gives $\pi_0(x) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\Pi_0(x^{1/n})$, where $\mu(n)$ is the Mobius function. Using Perron formula and the relationship between logarithm of Riemann zeta function and von Mangoldt function Λ; we have $\ln\zeta(s) = s\int_0^{\infty}\Pi_0(x)x^{-s-1}\,\mathrm{d}x$.

Finally, the exact form of Prime-$\pi(x)$ was provided by Bernhard Riemann (1826 – 1866). For x > 1 let $\pi_0(x) = \pi(x) - 1/2$ when x is a prime number, and $\pi_0(x) = \pi(x)$ otherwise. It is proved that $\pi_0(x) = \mathrm{R}(x) - \sum_{\rho}\mathrm{R}(x^{\rho})$, where $\mathrm{R}(x) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\operatorname{li}(x^{1/n})$, μ(n) is the Möbius function, $li(x)$ is the logarithmic integral function, ρ indexes every zero of Riemann zeta function, and $\operatorname{li}(x^{\rho/n})$ is not evaluated with a branch cut but instead considered as $\mathrm{Ei}(\frac{\rho}{n}\ln x)$ where Ei(x) is the exponential integral.

1.4 Intersection of Dirichlet eta function and Prime counting function

The [alternating] harmonic series Dirichlet eta function (equation) must act as *proxy* function for the [non-alternating] harmonic series Riemann zeta function (equation). Faithfully generated by Dirichlet eta function when parameter $\sigma = \frac{1}{2}$, [mathematical] nontrivial zeros (or Gram[x=0,y=0] points) and closely-related two types of Gram points (as Gram[y=0] points and Gram[x=0] points) form corresponding [geometrical] Origin intercept points, x-axis intercept points and y-axis intercept points whereby they constitute three complementary, mutually exclusive and dependent countably infinite sets which are Incompletely Predictable. Proofs for Riemann hypothesis regarding location of nontrivial zeros, and Polignac's and Twin prime conjectures regarding cardinality of prime gaps and prime numbers can, respectively, be dubbed Equation-type and Algorithm-type proofs. When deriving these proofs, we recognize above-mentioned entities together with classified Prime k-tuplets and Prime k-tuples in Table 1 exist as Incompletely Predictable entities contained in well-defined sets, subsets, tuples and subtuples. They will all inevitably manifest their associated Incompletely Predictable properties. To this end, we employ simple mathematical tools such as coprime numbers and basic arithmetic operations (in Appendix J), modular arithmetic, set theory and probability theory.

Incorporating relevant major historical achievements in Number theory, our main objective in this paper is to obtain correct and complete mathematical arguments to fully validate the [combined] statement ***Origin point must represent Critical line as location for Nontrivial zeros of Riemann zeta function, and Set Prime gaps with Subsets Odd primes are Arbitrarily Large in Number***. Although some of the arguments in this paper depend on visuals of limited experiments using computations and graphs, these experiments were arbitrarily designed with deduced proofs that are always meaningful and *perpetually* applicable to all positive integers, primes, composites and nontrivial zeros. Our actions culminate in rigorous proofs for Riemann hypothesis, (Modified) Polignac's and Twin prime conjectures. In so doing, we have to crucially apply infinitesimal numbers in two places: Proposition 1.4 *In the limit of never reaching a nonexisting zero* conceptually seen as prevalence of prime gaps with associated prime numbers never becoming zero whereby arbitrarily large number of different even Prime gaps that uniquely accompany all odd prime numbers in totality will never stop recurring. Proposition 1.5 *In the limit of reaching an existing zero* conceptually seen as trajectory of Dirichlet eta function, proxy function for Riemann zeta function, touching the zero-dimensional Origin point only when its parameter $\sigma = \frac{1}{2}$ whereby all nontrivial zeros [mathematically] located on one-dimensional $\sigma = \frac{1}{2}$-critical line will [geometrically] declare themselves in totality as Origin intercept points. We outline vital concepts from Completely and Incompletely Predictable entities and classify countably infinite sets in Lemma 1.1 as accelerating, linear or decelerating subtypes that,

Admissible Prime k-tuples: Subtypes and Varieties
Subtype I Admissible Prime k-tuplets (Sub I Adm P k-tuplets)
Subtype II Admissible Prime k-tuples (Sub II Adm P k-tuples)
First variety of Subtype II Admissible Prime k-tuples (1st V Sub II Adm P k-tuples)
Second variety of Subtype II Admissible Prime k-tuples (2nd V Sub II Adm P k-tuples)
Inadmissible Prime k-tuples: Subtypes and Varieties
Subtype I Inadmissible Prime k-tuples (Sub I Inadm P k-tuples)
Subtype II Inadmissible Prime k-tuples (Sub II Inadm P k-tuples)
First variety of Subtype II Inadmissible Prime k-tuples (1st V Sub II Inadm P k-tuples)
Second variety of Subtype II Inadmissible Prime k-tuples (2nd V Sub II Inadm P k-tuples)

Table 1. Classification of two main types of Prime k-tuples with abbreviations. The [virtual] 1st V Sub II Inadm P k-tuples that do not mathematically exist are directly related to 1st V Sub II Adm P k-tuples which cater for large(r) prime numbers with large(r) prime gaps. The 2nd V Sub II Inadm P k-tuples that mathematically exist just once are directly related to 2nd V Sub II Adm P k-tuples which cater for small(er) prime numbers with small(er) prime gaps.

respectively, manifest acceleratingly reaching an infinity value, linearly reaching an infinity value or deceleratingly reaching an arbitrarily large number value. This allows optimal description of countably infinite sets in many frontier branches of mathematics – we congratulate Ukrainian mathematician Maryna Viazovska in winning the prestigious Fields Medal on 5 July 2022 for solving sphere packing problem in eight dimensions, especially under the difficult circumstances that her home country Ukraine is invaded by Russia without provocation. We regard irrational numbers to exist in Isolated countably finitely-sized group or Connected countably infinitely-sized group (in Appendix K).

In mathematics, logarithmic integral function or integral logarithm li(x) is a special function. Relevant to problems of physics with number theoretic significance, it occurs in Prime number theorem as an estimate of Prime-π(x). Prime-π(x) is defined so that li(2) = 0; viz, offset logarithmic integral function Li(x) $\equiv \int_2^x \frac{du}{\ln u}$ = li(x) – li(2). Here, ln denotes the natural logarithm. There are less accurate ways of estimating Prime-π(x) such as conjectured by Gauss and Legendre at end of 18th century. This is approximately $\frac{x}{\ln x}$ in the sense $\lim_{x\to\infty} \frac{\text{Prime-}\pi(x)}{x/\ln x} = 1$. Skewes' number is any of several extremely large numbers used by South African mathematician Stanley Skewes as upper bounds for smallest natural number x for which Prime-π(x) > li(x). These bounds have since been improved by others: there is a crossing near $e^{727.95133}$ (<1.397 X 10^{316}) but it is not known whether this is the smallest. (On the other hand, Li(x) is smaller than Prime-π(x) already for x = 2; indeed, Li(2) = 0, while π(2) = 1.) John Edensor Littlewood who was Skewes' research supervisor proved that there is such a [first] number; and found the difference Prime-$\pi(x)$ – li(x) changes its sign infinitely many times (Littlewood, 1914). This refute all prior numerical evidence that suggested li(x) was always > Prime-$\pi(x)$. Key point is [100% accurate] perfect Prime-$\pi(x)$ stepped-mathematical function being *wrapped around* by [less-than-100% accurate] approximate Li(x) smooth-mathematical function infinitely many times via this *sign of difference* changes implies Li(x) is the most efficient approximate mathematical function. Contrast this with *crude* [less-than-100% accurate] approximate $\frac{x}{\ln x}$ smooth-mathematical function whereby studied values diverge away from Prime-$\pi(x)$ at increasingly greater rate for larger range of prime numbers.

1.5 Admissible Prime k-tuples, Inadmissible Prime k-tuples and Nontrivial zeros

For k ≥ 2, a Prime k-tuple [that can be subdivided into available subtuples for sufficiently large k values] is a repeatable pattern of finite k consecutive primes {p_1, p_2,..., p_k} [viz, a finite collection with $p_1 < p_2 <...< p_k$] having diameter d defined as difference between its largest and smallest elements [viz, diameter d = p_k – p_1]. There are two main types of Prime k-tuples as per our classification in Table 1 based on comparative same k values (that will insightfully depict overlapping mathematical landscape of Prime k-tuples): (I) [repeating] Admissible Prime k-tuples as two subtypes with ***each subtype and relevant associated varieties deceleratingly reaching an arbitrarily large number value***, and (II) [non-repeating] Inadmissible Prime k-tuples as two subtypes with ***each subtype and relevant associated varieties deceleratingly reaching an arbitrarily large number value***.

The principles behind primorial are outlined in the caption of Table 2. As part of group theory with notation that read as (Z/nZ)*, concepts behind multiplicative group of integers modulo n are important for theory of prime k-tuples or constellations. It contains a subset of integers from 1 to n-1. The elements of (Z/nZ)* are integers from 1 to n-1 that are relatively prime to n. If n is a prime number, then (Z/nZ)* contains all integers from 1 to n-1. If n has many divisors, then (Z/nZ)* will contain fewer elements. To find Sub I P Adm k-tuplets, we need to consider the multiplicative group of

k	p_k	k#
1	2	2
2	3	6
3	5	30
4	7	210
5	11	2310
6	13	30,030
7	17	510,510
8	19	9,699,690
9	23	223,092,870
10	29	6,469,693,230

Table 2. Tabulated data of k primorial for k = 1 to 10. Let p_k be the k^{th} prime with k = 1, 2, 3, 4, 5.... Then k primorial (k#) is product of first k primes whereby [even] numbers in third column are product of primes in second column. It is a well-defined Incompletely Predictable function acceleratingly reaching an infinity value.

integers mod k primorial. This group contains set of integers less than k primorial that are relatively prime to k primorial with its significances described below.

The multiplicative group mod 6 [2#] has two elements; viz, (Z/6Z)* = {1, 5}. Then all primes greater than 3 have the form 6*n ± 1. To search for the smaller of twin prime pairs [Sub I P Adm 2-tuplets], one should look at [odd] numbers of the form 6*n + 5. The multiplicative group mod 30 [3#] has 8 elements; viz, (Z/30Z)* = {1, 7, 11, 13, 17, 19, 23, 29}. By looking at the differences between adjacent elements in this set, we see Sub I P Adm 3-tuplets as pattern (p, p+2, p+6) is found only in the expressions 30*n + 11 and 30*n + 17. The ordered set (Z/30Z)* = {1, 7, 11, 13, 17, 19, 23, 29} can be manipulated by taking the differences between adjacent elements; viz, d30 = [6, 4, 2, 4, 2, 4, 6] $\Longrightarrow$ the particular pattern (p, p+2, p+6, p+8) which has differences [2, 4, 2] is found inside ordered set d30. Thus we see Sub I P Adm 4-tuplets having pattern (p, p+2, p+6, p+8) must have the form 30*n + 11.

Proposition 1.2. Let consecutive primes {p_1, p_2,..., p_k} represent Subtype I Admissible Prime k-tuplets linked to Subtype I Inadmissible Prime k-tuples and/or first variety of Subtype II Admissible Prime k-tuples linked to [nonexisting] first variety of Subtype II Inadmissible Prime k-tuples and/or second variety of Subtype II Admissible Prime k-tuples linked to second variety of Subtype II Inadmissible Prime k-tuples. Then except for p_1 = 2 case [having the empty set of Admissible Prime k-tuple] and for each p_1 commencing from p_1 = 2, 3, 5, 7, 11, 13...; we can generate a finite number of Admissible Prime k-tuplets/k-tuples denoted by their k values and an associated arbitrarily large number of Inadmissible Prime k-tuples denoted by their larger and different [complementary] k values whereby both types of tuples will fully comply with relevant corresponding admissibility and inadmissibility criteria.

Proof. Suppose one is given a k_0-tuple $\mathcal{H} = (h_1,\ldots, h_{k_0})$ of k_0 distinct integers for some $k_0 \geq 1$, arranged in increasing order. We anticipate finding an arbitrarily large number of translates $n + \mathcal{H} = (n+h_1,\ldots, n+h_{k_0})$ of $\mathcal{H}$ which consist entirely of consecutive primes will prove Polignac's and Twin prime conjectures to be true. The case $k_0 = 1$ is just Euclid's theorem on the infinitude of primes. The case $k_0 = 2$ [as subset of $k_0 \geq 2$] with $\mathcal{H} = (0, 2)$ correspond to twin prime conjecture that non-overlappingly deals with prime gap = 2. The arbitrarily large number of cases $k_0 \geq 2$ [as full set] in their entirety correspond to Polignac's conjecture that [additionally] involve all other remaining cases such as k_0 = 3 with $\mathcal{H} = (0, 2, 6)$ as pattern-1 or (0, 4, 6) as pattern-2, $k_0 = 4$ with $\mathcal{H} = (0, 2, 6, 8)$ as solitary pattern, etc. Thus we have [overlappingly] dealt with all even prime gaps = 2, 4, 6, 8, 10....

More generally, if there is a prime p_1 such that $\mathcal{H}$ meets each of the p_1 residue classes $0 \mod p_1, 1 \mod p_1, \ldots, p_1\text{-}1 \mod p_1$, then every translate of $\mathcal{H}$ contains at least one multiple of p_1. Since p_1 is the only multiple of p_1 that is prime, this shows that there are only finitely many translates of $\mathcal{H}$ that consist entirely of consecutive primes.

A k_0-tuple $\mathcal{H}$ is admissible if it avoids at least one residue class mod p for each prime p. It is easy to check for admissibility in practice, since a k_0-tuple is automatically admissible in every prime p larger than k_0, so one only needs to check a finite number of primes in order to decide on the admissibility of a given tuple. We can now succinctly state the **first Hardy-Littlewood conjecture** or **Prime tuples conjecture** in its qualitative form: If $\mathcal{H}$ is an admissible k_0-tuple, then there exists an arbitrarily large number of translates of $\mathcal{H}$ that consist entirely of consecutive primes.

Our p_1 commencing values as constituted from the entire CIS-ALN-decelerating prime numbers 2, 3, 5, 7, 11, 13... will act as reference points to orderly include all possible subtypes and varieties of Admissible Prime k-tuplets/k-tuples and Inadmissible Prime k-tuples whereby these k-tuplets and k-tuples are constituted by k consecutive prime numbers starting from p_1. We invoke multiplicative group of integers modulo p_1 that, via brute force algorithm, must result in a subset of consecutive integers as residues from 0 to p_1-2 and p_1-1 whereby some of these integers that represent corresponding residues will inevitably repeat more than once. For instance at p_1 commencing value = 11, the sequence of integers that mechanically represent corresponding residues from mod prime 11 as iteratively computed using all available prime gaps are 0, 2, 6, 8, 1, 7, 9, 4, 8, 10, 3, 9, 4, 6, 1 [Admissible] and 5 [Inadmissible] whereby [non-comprehensive] integers 1, 4, 6, 8 and 9 are overlappingly depicted more than once and two [uniquely nominated] integers 0 and 5 must always be non-overlappingly depicted just once but with the [solitary] integer 5 being (firstly) absent when the involved k-tuple is admissible and (secondly) present when the involved (k+1)-tuple is inadmissible. The p_1 commencing value = 11 has thus provided us with (i) Sub I Adm P 15-tuplet as consecutive primes (11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67) that is mechanically ≡ progressive prime gaps (0, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6) ≡ cummulative prime gaps (0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50, 56) and (ii) 2nd V Sub II Inadm P 16-tuple as consecutive primes (11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71) that is mechanically ≡ progressive prime gaps (0, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6, 4) ≡ cummulative prime gaps (0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50, 56, 60). We note all the involved [consecutive] prime gaps of 2, 4 and 6 are each overlappingly depicted more than once for both the involved admissible 15-tuplet and inadmissible 16-tuple.

However, we easily deduce and confirm there will be an arbitrarily large number of exceptions as scattered counter-examples of Admissible k-tuplets/k-tuples and/or certain patterns of Admissible k-tuplets and/or Inadmissible k-tuples being affected whereby (i) greater than the two uniquely nominated integers that represent corresponding residues do not repeat more than once e.g. at p_1 commencing value = 47 when all relevant integers 0, 1, 2, 3,..., 46 [having cardinality of 47] for 2nd V Sub II Inadm P 79-tuples simply cannot be repeated more than once [because k = 79 is clearly not at least equal to (47–2)*2 + 2 = 92 when the two uniquely nominated integers are taken into account]; (ii) one of the prime gaps do not repeat more than once e.g. prime gap = 6 do not repeat in Sub I Adm P 7-tuplet as pattern-1 (0, 2, 6, 8, 12, 18, 20) ≡ progressive prime gaps (0, 2, 4, 2, 4, 6, 2) and as pattern-2 (0, 2, 8, 12, 14, 18, 20) ≡ progressive prime gaps (0, 2, 6, 4, 2, 4, 2); and (iii) one of the prime gaps is totally missing [skipped] e.g. prime gap 8 is missing in Sub I Adm P 25-tuplet for Pattern-9 cummulative prime gaps (0, 6, 8, 14, 20, 24, 30, 36, 38, 44, 50, 54, 56, 66, 68, 78, 80, 84, 86, 90, 96, 98, 104, 108, 110) ≡ progressive prime gaps (0, 6, 2, 6, 6, 4, 6, 6, 2, 6, 6, 4, 2, 10, 2, 10, 2, 4, 2, 4, 6, 2, 6, 4, 2) as shown in Appendix C. Frequency of patterns [that are progressively increasing by 8, 4 and 2 as related by 2^{-1}] NOT containing prime gap 8 = 2/18, prime gap 10 = 10/18, prime gap 12 = 14/18, and prime gap 14 = 16/18. Thus there is zero probability of prime gaps 8, 10, 12 or 14 NOT appearing amongst the available 18 patterns of Sub I Adm P 25-tuplet.

We need only consider the case of modulo p_1 for each p_1 commencing value since the very first result $p_1 \equiv 0 \pmod{p_1}$ is the solitary and unique result that can arise as part of the iterative computation to obtain the desired complete set of results $p_1 \equiv 0 \pmod{p_1}$, $p_1 \equiv 1 \pmod{p_1}$, $p_1 \equiv 2 \pmod{p_1}$, $p_1 \equiv 3 \pmod{p_1}$,..., $p_1 \equiv p_1\text{-}2 \pmod{p_1}$ [that will *maximally* conform to the admissibility criterion], $p_1 \equiv p_1\text{-}1 \pmod{p_1}$ [that will *minimally* conform to the inadmissibility criterion]. For each and every p_1 commencing value iteratively computed in Box 1 [in Appendix D] using first 15 p_1 commencing values [akin to a longitudinal study over the selected p_1 commencing values], we always obtain the allowable set of k-valued Admissible Prime k-tuplets/k-tuples with its cardinality being a finite number and the associated allowable set of k-valued Inadmissible Prime k-tuples with its cardinality being an arbitrarily large number. Progressively larger sets of finite k-valued Admissible Prime k-tuplets/k-tuples associated with the first 15 p_1 commencing values of 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47 have corresponding cardinality of 0, 1, 4, 5, 14, 20, 19, 34, 38, 58, 94, 72, 93, 76, 77.

In relation to all p_1 commencing values, the cardinality of associated Admissible Prime k-tuplets/k-tuples must be countably finite in number [except for p_1 commencing value of 2 having nil Admissible Prime k-tuple]; and the cardinality of associated Inadmissible Prime k-tuples must be countably arbitrarily large in number. Proposed to be predominantly applicable to selected Sub I Adm P k-tuplets and 2nd V Sub II Inadm P (k+1)-tuples, there are three potential (pseudo-)default mechanisms: (i) all eligible residues should be self-replicating for every p_1 commencing values [apart from the first-occurring Inadmissible Prime k-tuples with p_1 commencing values of 2 as two consecutive primes (2, 3); of 3 as three consecutive primes (3, 5, 7); and of 7 as seven consecutive primes (7, 11, 13, 17, 19, 23, 29) whose residues were all not self-replicating], (ii) all eligible even Prime gaps should be self-replicating for every p_1 commencing values [apart from relevant Prime k-tuplets/k-tuples with p_1 commencing values of 2 having odd prime gap = 1 which is not an even Prime gap and will never self-replicate], and (iii) an eligible even Prime gap should not be missing or skipped for every p_1 commencing values [apart from Sub I Inadm P 2-tuple with p_1 commencing value of 2 as consecutive primes (2, 3) having once-occurring odd Prime gap 1 which is not an even Prime gap and is permanently missing or skipped in all other Prime k-tuplets/k-tuples that do not contain even prime number 2].

Remark 1.2. All above three (pseudo-)default mechanisms are simply not ubiquitous mechanisms because they will not be applicable for an arbitrarily large number of Prime k-tuplets/k-tuples. Applying logical deductive reasoning to the last two (pseudo-)default mechanisms for relevant Admissible Prime k-tuplets/k-tuples and Inadmissible Prime k-tuples [as classically confirmed using the 18 patterns of Subtype I Admissible Prime 25-tuplets in Appendix C], we observe the *sine qua non* phenomenon of eligible even Prime gaps that are [temporarily] not self-replicating and/or are [temporarily] missing or skipped in these tuples do not [permanently] persist over the entire sequence of consecutive odd primes. Otherwise, these two [known non-perpetuating] discriminatory processes involving certain even Prime gaps will cause affected even Prime gaps to inappropriately and abruptly terminate or disappear.

In short summary, there must be at least p_1-1 consecutive integers representing residues 0, 1, 2, 3,..., p_1-2 that cater for longest possible Admissible Prime k-tuple [and at least p_1 consecutive integers representing residues 0, 1, 2, 3,..., p_1-1 that cater for shortest possible Inadmissible Prime (k+1)-tuple]. Apart from first four cardinality depicted above that are smaller than or equal to p_1-1, all subsequent cardinality must not be smaller than their corresponding p_1-1 with the all-important implication that we can always derive arbitrarily long Admissible Prime k-tuples with maximal k values that must be at least equal to [but are usually always larger than] p_1-1. We recognize the incidental Incompletely Predictable cyclical nature of computed data in Box 1 [in Appendix D] having, in general, ever larger p_1 commencing values [overall] associated with ever larger k-valued Admissible Prime k-tuplets/k-tuples that characteristically have ever larger zenith diameter d and zenith average gaps. *The proof is now complete for Proposition 1.2*□.

For every appropriately paired Admissible Prime k-tuplet patterns endowed with same modulo number, there exists a counterpart. For instance, Sub I Adm Prime 7-tuplet pattern-1 (0, 2, 6, 8, 12, 18, 20) has its p_1 congruent to 11 (modulo 210) and Sub I Adm Prime 7-tuplet pattern-2 (0, 2, 8, 12, 14, 18, 20) has its p_1 congruent to 179 (modulo 210). We see that 11 + 179 (viz, the counterpart) + 20 (viz, the diameter d) = 210 (viz, the modulo number). The *offset and multiplier* containing variable n is related to *p_1 congruent to p (modular q)* for Admissible Prime k-tuplets as explained here using various examples.

Example 1: For Sub I Adm P 7-tuplet with pattern-1 given as cummulative prime gaps (0, 2, 6, 8, 12, 18, 20) ≡ consecutive prime numbers (11, 13, 17, 19, 23, 29, 31) [as based on first-occurring p_1 = 11]; the p_1 congruent to 11 (modulo 210) is equivalent to offset and multiplier 11 + 210*n. This is given by A022009 *Initial members of prime septuplets (p, p+2, p+6, p+8, p+12, p+18, p+20).* (Roonguthai, n.d. 1) having values 11, 165701, 1068701, 11900501, 15760091, 18504371, 21036131, 25658441, 39431921, 45002591, 67816361, 86818211, 93625991, 124716071, 136261241, 140117051, 154635191, 162189101, 182403491, 186484211, 187029371, 190514321, 198453371... which is cross linked to A182387 *Numbers n such that 210*n+11, 13, 17, 19, 23, 29, 31 are 7 consecutive primes.* (Seidov, 2012) having values 0, 789, 5089, 56669, 75048, 88116, 100172, 122183, 187771, 214298, 322935, 413420, 445838, 593886, 648863, 667224, 736358, 772329, 868588, 888020, 890616, 907211, 945016, 1052954, 1078331, 1106177, 1146724, 1223888, 1432230, 1452437, 1458355, 1509878, 1535216....

Example 2: For Sub I Adm P 7-tuplet with pattern-2 given as cummulative prime gaps (0, 2, 8, 12, 14, 18, 20) ≡ consecutive prime numbers (5639, 5641, 5647, 5651, 5653, 5657, 5659) [as based on first-occurring p_1 = 5639]; the p_1 congruent to 179 (modulo 210) is equivalent to offset and multiplier 179 + 210*n. This is given by A022010 *Initial members of prime septuplets (p, p+2, p+8, p+12, p+14, p+18, p+20).* (Roonguthai, n.d. 2) having values 5639, 88799, 284729, 626609, 855719, 1146779, 6560999, 7540439, 8573429, 17843459, 19089599, 24001709, 42981929, 43534019, 69156539, 74266259, 79208399, 80427029, 84104549, 87988709, 124066079, 128469149, 144214319, 157131419, 208729049, 218033729... which is cross linked to A357889 *a(n) = (A022010(n) – 179)/210.* (Pfoertner, 2022) having values 26, 422, 1355, 2983, 4074, 5460, 31242, 35906, 40825, 84968, 90902, 114293, 204675, 207304, 329316, 353648, 377182, 382985, 400497, 418993, 590790, 611757, 686734, 748244, 993947, 1038255, 1181931, 1246060, 1310026, 1347976, 1354707, 1440679, 1477788, 1559980, 1720425, 1915719, 1989590....

Example 3: For Sub I Adm P 38-tuplet there are six possible patterns with pattern-4 given as cummulative prime gaps (0, 6, 8, 14, 18, 20, 24, 30, 36, 38, 44, 48, 50, 56, 60, 66, 74, 78, 80, 84, 86, 90, 104, 108, 114, 116, 126, 128, 134, 140, 144, 150, 156, 158, 168, 170, 174, 176) ≡ consecutive prime numbers (23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199) [as based on first-occurring p_1 = 23]; the p_1 congruent to 2541318803 (modulo 6469693230) which is equivalent to offset and multiplier 2541318803 + 6469693230*n is also applicable in a similar manner to previous two examples.

Named after him, Norman Luhn first noted on *circa* 9 February 1999 the prime number 229 belong to a special class of prime numbers 229, 239, 241, 257, 269, 271, 277, 281, 439, 443, 463, 467, 479... that is defined by A061783 *Luhn primes: primes p such that p + (p reversed) is also a prime* (Murthy, 2001). On 26 November 2022 in relation to his other prime number research, Luhn generously supplied his tiny freebasic program on patterns of Sub I Adm P k-tuplets that correctly work from k = 2 to 50 – see Appendix B whereby A083409 *Number of prime k-tuplet constellations, i.e., patterns with minimal diameter A008407* (Ellermann, 2003) is relevant. Computed for k = 2, 3, 4, 5, 6...; the number of possible patterns are 1, 2, 1, 2, 1, 2, 3, 4, 2, 2, 2, 6, 2, 4, 2, 4, 2, 4, 2, 2, 4, 2, 4, 18, 2, 8, 10, 2, 2, 2, 4, 14, 20, 2, 2, 2, 6, 26, 26, 8, 2, 6, 18, 4, 4, 4, 2, 2, 22, 22, 2, 2, 26, 6, 6, 2, 2, 4, 2, 2, 6, 2, 2, 2, 2, 18, 2, 20, 2, 2, 2, 10, 2, 14, 14, 40, 8, 2, 14, 14, 16, 4, 2, 2, 60, 50, 2, 2, 2, 16, 2, 18, 12....

Professor Neil J. A. Sloane is the Chairman, On-Line Encyclopedia of Integer Sequences Foundation. As chronologically alluded to by the author of this paper during email chain correspondences between himself, Sloane [7 January 2022 – 8 January 2022] and Luhn [24 November 2022 – 27 November 2022]; the former main type Admissible Prime k-tuples when given in comparatively same k values consist of two subtypes (i) [repeating] Sub I Adm P k-tuplets, often called Prime k-tuplets or Prime constellations, which are simply Admissible Prime k-tuples with their diameter d being the smallest possible diameter and (ii) [repeating] Sub II Adm P k-tuples, divided into two different varieties [viz, 1st V Sub II Adm P k-tuples and 2nd V Sub II Adm P k-tuples] as outlined in Part I Analysis and Part II Analysis below, with their diameter d being much larger than and slightly larger than the corresponding smallest possible diameter of Sub I Adm P k-tuplets. As per Part I Analysis which fully comply with Proposition 1.2, 1st V Sub II Adm P k-tuples tend to generate large(r) prime numbers with large(r) prime gaps. As per Part II Analysis which fully comply with Proposition 1.4, 2nd V Sub II Adm P k-tuples tend to generate small(er) prime numbers with small(er) prime gaps.

Two subtypes of the later main type Inadmissible Prime k-tuples when given in comparatively same k values are fully discussed in Part I Analysis and Part II Analysis below: (i) [non-repeating] Sub I Inadm P k-tuples belong to the group with their diameter d being the smallest, and (ii) [non-repeating] Sub II Inadm P k-tuples belong to the other group which is again divided into two different varieties. As per Part I Analysis, the [nonexisting] 1st V Sub II Inadm P k-tuples with diameter d being much larger than this smallest possible diameter of Sub I Adm P k-tuplets are the Forbidden Inadmissible Prime k-tuples. As per Part II Analysis, the [existing-just-once] 2nd V Sub II Inadm P k-tuples have their varying diameter d being slightly smaller than, equal to or larger than the smallest possible diameter of Sub I Adm P k-tuplets.

A prime k-tuple is *admissible* in its sequence of consecutive primes $\{p_1, p_2, ..., p_k\}$ such that for every prime $q \le k$, not all the residues modulo q are represented by $p_1, p_2, ..., p_k$. Simpliest Sub I Adm P k-tuplets and Sub II Adm P k-tuples using k = 2 value include all twin primes as Sub I Adm P 2-tuplet with smallest possible diameter d (prime gap) = 2; all cousin primes as Sub II Adm P 2-tuple with larger diameter d (prime gap) = 4; all sexy primes as Sub II Adm P 2-tuple with larger diameter d (prime gap) = 6; etc. We note in next paragraph that Sub II Adm P 2-tuples could be [arbitrarily] classified as either 1st V Sub II Adm P k-tuples or 2nd V Sub II Adm P k-tuples. The Sub I Adm P 3-tuplet pattern-1 (p+0, p+2, p+6) and pattern-2 (p+0, p+4, p+6) have their respective first occurrences at consecutive prime numbers (5, 7, 11) and (7, 11, 13). Then an example of 1st V Sub II Adm P 3-tuple pattern-1 (p+0, p+2, p+8) is given by consecutive prime numbers (5639, 5641, 5647). We duly note this particular 1st V Sub II Adm P 3-tuple is also a subtuple forming part of Sub I Adm P 7-tuplet pattern-2 (p+0, p+2, p+8, p+12, p+14, p+18, p+20) given by first occurrence consecutive prime numbers (5639, 5641, 5647, 5651, 5653, 5657, 5659).

Both the Sub I Adm P 2-tuplets as two consecutive primes (p_1, p_k) with diameter d or prime gap = $p_k - p_1 = 2$ and the Sub II Adm P 2-tuples as two consecutive primes (p_1, p_k) with diameter d or prime gap = $p_k - p_1 \ge 4$ can match an arbitrarily large number of positions in the sequence of prime numbers. The Sub II Adm P 2-tuples could arbitrarily be regarded as belonging to either the 1st V Sub II Adm P k-tuples conforming to criterion $p_k - p_1 \gg p_1 - p_{k-2}$ or 2nd V Sub II Adm P k-tuples conforming to criterion $p_k - p_1 > p_1 - p_{k-2}$. These two varieties of Sub II Adm P k-tuples could in principle also form [bridging] smaller subtuples of accelerating primes in Sub I Adm P k-tuplets or 1st V and 2nd V Sub II Adm P k-tuples or Sub I Inadm P k-tuples or 2nd V Sub II Inadm P k-tuples when k ≥ 3. For n = 1, 2, 3, 4, 5...; the rarely occurring but nevertheless arbitrarily large number of Sub II Adm P 2-tuples conforming to criterion $p_k - p_1 = p_1 - p_{k-2}$ and manifesting as two identical consecutive prime gaps (6n, 6n) = (6, 6), (12, 12), (18, 18), etc could arbitrarily be regarded as belonging to either 1st V or 2nd V Sub II Adm P k-tuples. They could in principle also form [bridging] smaller subtuples of steady primes in Sub I Adm P k-tuplets or 1st V and 2nd V Sub II Adm P k-tuples or Sub I Inadm P k-tuples or 2nd V Sub II Inadm P k-tuples when k ≥ 3. The criterion $p_k - p_1 < p_1 - p_{k-2}$ will be conformed to by an arbitrarily large number of 1st V or 2nd V Sub II Adm P 2-tuples whereby they could in principle also form [bridging] smaller subtuples of decelerating primes in Sub I Adm P k-tuplets or 1st V and 2nd V Sub II Adm P k-tuples or Sub I Inadm P k-tuples or 2nd V Sub II Inadm P k-tuples when k ≥ 3. Thus subtuples of accelerating, steady and decelerating primes [as further elaborated upon in Proposition 2.3] essentially form eternal repeated groupings of small and/or large prime numbers and gaps.

Remark 1.3. At ever larger range of x ≥ 4 integer values manifesting progressively less prime numbers, we intuitively expect an overall slowly increasing prevalence of first variety of Subtype II Admissible Prime k-tuples that cater for large(r) prime numbers and gaps which is reciprocally and simultaneously associated with an overall slowly decreasing prevalence of second variety of Subtype II Admissible Prime k-tuples that cater for small(er) prime numbers and gaps. When Subtype I Admissible Prime 2-tuplets as two consecutive primes (p_1, p_k) with diameter d or prime gap = $p_k - p_1 = 2$ are combined with Subtype II Admissible Prime 2-tuples as two consecutive primes (p_1, p_k) with diameter d or prime gap = $p_k - p_1 \ge 4$, they will be able to [uniquely] represent every known odd prime numbers in an non-overlapping manner.

A Prime k-tuple is *inadmissible* in its sequence of consecutive primes $\{p_1, p_2, ..., p_k\}$ such that for some of the prime q ≤ k – example, for one of the prime q ≤ k when k ≥ 3 or for two of the prime q ≤ k if $p_1 = 2$ forms part of a Prime k-tuple when k ≥ 4; all the residues modulo q are represented by $p_1, p_2, ..., p_k$. All [non-repeating] Sub I Inadm P k-tuples only match one finite position in the sequence of prime numbers and are defined by their diameter d being the shortest. An arbitrarily large number of examples with one all-prime solution for this subtype include Prime 2-tuple (p+0, p+1) as prime numbers (2, 3) with d = 1; Prime 3-tuple (p+0, p+1, p+3) as prime numbers (2, 3, 5) with d = 3; Prime 3-tuple (p+0, p+2, p+4) as prime numbers (3, 5, 7) with d = 4; Prime 4-tuple (p+0, p+1, p+3, p+5) as prime numbers (2, 3, 5, 7) with d = 5; Prime 4-tuple (p+0, p+2, p+4, p+8) as prime numbers (3, 5, 7, 11) with d = 8; etc.

Modular arithmetic: a (mod n) is a/n ≡ r whereby a = dividend, n = divisor and r = remainder [round up to the next integer]. Therefore, a (mod n) ≡ a – (r ∗ n). With abbreviation *n* denoting numbers, we analyze the Completely Predictable even *n* and odd *n*. For i = 0, 1, 2, 3, 4, 5...; congruence $n \equiv 0 \pmod 2$ holds for even $n = E_i = 2*i = 0, 2, 4, 6, 8, 10...$ and for i = 1, 2, 3, 4, 5, 6...; congruence $n \equiv 1 \pmod 2$ holds for odd $n = O_i = (2*i)-1 = 1, 3, 5, 7, 9, 11...$ whereby 0 is the zeroth even *n* when we only consider all (non-negative) positive even *n* and odd *n* obtained from whole i = 0, 1, 2, 3, 4, 5.... We analyze the Incompletely Predictable prime numbers collectively grouped as k-tuples. For the worked example of modular arithmetic applied to test for admissibility on Sub I Inadm P 4-tuple (p+0, p+1, p+3, p+5) ≡ cummulative prime gaps (0, 1, 3, 5) with earliest and only candidate as consecutive prime numbers (2, 3, 5, 7) having progressive prime gaps (0, 1, 2, 2); we can use either [I] cummulative prime gaps: congruence 0, 1, 3, 5 ≡ 0, 1, 1, 1 (mod prime 2) and congruence 0, 1, 3, 5 ≡ 0, 1, 0, 2 (mod prime 3) or [II] consecutive prime numbers: congruence 2, 3, 5, 7 ≡ 0, 1, 1, 1 (mod prime 2) and congruence 2, 3, 5, 7 ≡ 2, 0, 2, 1 (mod prime 3). There are two failures at [firstly] mod prime 2 on second term = 1 (as prime gap) or 3 (as prime number) and [secondly] mod prime 3 on last term = 5 (as prime gap) or 7 (as prime number) $\implies$ this Sub I Inadm P 4-tuple is now confirmed to be inadmissible. Since twin prime (3, 5) is a Sub I Adm P 2-tuplet when first element p = 3, we can redundantly generate a complete all-inclusive countably arbitrarily large number of [non-repeating] Sub I Inadm P k-tuples using progressively longer k ≥ 3 values that should have the shortest diameter when first element p = 3. We can also redundantly generate a complete all-inclusive countably arbitrarily large number of [non-repeating] Sub I Inadm P k-tuples using progressively longer k ≥ 2 values that should have the shortest diameter when first element p = 2.

We hereby explain the example of [nonexisting] 1st V Sub II Inadm P k-tuple which is linked to Sub I Adm P 3-tuplet (p+0, p+2, p+6) pattern-1 having diameter d = 6 that first appear as consecutive primes (5, 7, 11). This Sub I Adm P 3-tuplet is associated with 1st V Sub II Inadm P 3-tuples with failure at mod prime 3 (last term = 10, 16, 22, 28...) and must fully conform with the [hidden] forbidden condition as stated here. Just as two consecutive twin primes given by Prime 3-tuple (p+0, p+2, p+4+6n) cannot exist at all apart from the solitary Sub I Inadm P 3-tuple occurring as consecutive primes (3, 5, 7) when n = 0, then so must all two consecutive *twin-related* primes given by Prime 3-tuple (p+0, p+2, p+4+6n) cannot exist at all when n = 1, 2, 3, 4... [since at least one of the three primes is divisible by 3]. Two other [hidden] forbidden conditions that must be conformed to by all Prime k-tuplets and Prime k-tuples including 1st V Sub II Inadm P k-tuples are:
(1) Apart from the solitary [single-digit] odd prime number 5 with its last and only digit also ending in odd number 5, all other larger [multiple-digit] odd prime numbers cannot have their last digit ending in odd number 5 and, consequently, these forbidden numbers can never belong to any Prime k-tuplets and Prime k-tuples. Thus, apart from the solitary odd prime number 5, it is an established mathematical fact that all odd prime numbers must have their last digit ending in odd numbers 1, 3, 7 or 9.
(2) The arbitrarily large number of Sub I Adm P 4-tuplets (p+0, p+2, p+6, p+8) with smallest possible diameter d = 8 is first given by consecutive primes (5, 7, 11, 13) whereby this must be differentiated from the totally different [solitary] Sub I Inadm P 4-tuple (p+0, p+2, p+4, p+8) given by consecutive primes (3, 5, 7, 11) with [same-valued] smallest diameter d = 8. All the arbitrarily large number of ≥ 2-digit primes in Sub I Adm P 4-tuplets commencing sequentially as (11, 13, 17, 19), (101, 103, 107, 109), (191, 193, 197, 199), (821, 823, 827, 829)... must always occur in the same ten-block. Hence it is an established mathematical fact that there must be exactly one with each of these unit digits 1, 3, 7 and 9 in all ≥ 2-digit primes from Sub I Adm P 4-tuplets. Except for the first term $p_1 = 5$ in Sub I Adm P 4-tuplet (5, 7, 11, 13), all other terms are congruent to 11 (mod 30). Thus all Sub I Adm P 4-tuplets except when first term $p_1 = 5$ are of the form (15k-4, 15k-2, 15k+2, 15k+4) with k ≥ 1, and so are centered on 15k.

Part I Analysis: Subtype I Admissible Prime k-tuplets + First variety of Subtype II Admissible Prime k-tuples + First variety of Subtype II Inadmissible Prime k-tuples. For the comparative same k value [akin to a cross-sectional study at specific k values], the countably arbitrarily large number of 1st V Sub II Adm P k-tuples are defined as having diameter d being [much] larger than the corresponding diameter d allocated for Sub I Adm P k-tuplets.

k	2-pat 1	3-pat 1	4-pat 1	5-pat 1	6-pat 1	7-pat 1	8-pat 1
Adm. k-Tuplet's p_i / d	3 / 2	5 / 6	5 / 8	5 / 12	7 / 16	11 / 20	11 / 26
Inadm. k-Tuple's p_{i-1} / d	2 / 1	3 / 4	3 / 8	3 / 10	5 / 14	7 / 22	7 / 24
Failure at p_{i-1} mod q (term p)	mod 2(3)	mod 3(7)	mod 3(7)	mod 3(7)	mod 5(19)	mod 7(29)	mod 7(29)

9-pat 1	10-pat 1	11-pat 1	12-pat 1	13-pat 1	14-pat 1	15-pat 1	16-pat 1	17-pat 1
11 / 30	11 / 32	11 / 36	11 / 42	11 / 48	11 / 50	11 / 56	13 / 60	13 / 66
7 / 30	7 / 34	7 / 36	7 / 40	7 / 46	7 / 52	7 / 54	11 / 60	11 / 60
mod 7(29)	mod 7(29)	mod 7(29)	mod 7(29)	mod 7(29)	mod 7(29)	mod 7(29)	mod 11(71)	mod 11(71)

18-pat 1	19-pat 1	20-pat 1	21-pat 1	22-pat 1	23-pat 1	c.f. 17-pat 2	c.f. 22-pat 2
13 / 70	13 / 76	29 / 80	29 / 84	19 / 90	19 / 94	17 / 66	23 / 90
11 / 68	11 / 72	Adm 23/84	Adm 23/86	17 / 90	17 / 92	Adm 13/66	Adm 19/90
mod 11 (71)	mod 11 (71)	p≤19 Adm	p≤19 Adm	mod 17 (103)	mod 17 (103)	p≤17 Adm	p≤19 Adm

Table 3. Computed data of anticipated Failure at p_{i-1} mod prime q on (term prime p) for Sub I Inadm P k-tuples that occur when k = 2, 3, 4 and 5; and 2nd V Sub II Inadm P k-tuples that occur when k ≥ 6.

As complements to the 1st V Sub II Adm P k-tuples, the Forbidden Inadmissible Prime k-tuples are equivalent to the [nonexisting] 1st V Sub II Inadm P k-tuples with their varying diameter d being much larger than the smallest possible diameter of [repeating] Sub I Adm P k-tuplets. As outlined in Appendix E, all the countably arbitrarily large number of these literally forbidden Prime k-tuples are proposed to not match any position in the sequence of prime numbers.

Remark 1.4. Reminiscent of Remark 1.1 on Gram's Law and its violations, we hereby tentatively propose *Gram's Prime k-tuple law for k ≥ 6 values* to uniquely indicate the tendency for first variety of Subtype II Inadmissible Prime k-tuples [as complement to first variety of Subtype II Admissible Prime k-tuples] to manifest failure at mod prime 5 on last terms that will alternatingly enlarge by +10 and +20; and at mod prime 7 on last terms that will alternatingly enlarge by +14 and +28. Requiring further research for answers, could similar doubling results affecting the last terms be obtained when relevant modular arithmetic is applied using other prime numbers 11, 13, 17, 19...; and could we ever encounter an arbitrarily large number of violations of this law whereby the alternatingly enlarging by +10 and +20 and/or by +14 and +20 phenomenon fails to intermittently appear?

Part II Analysis: Subtype I Admissible Prime k-tuplets + Second variety of Subtype II Admissible Prime k-tuples + Subtype I Inadmissible Prime k-tuples + Second variety of Subtype II Inadmissible Prime k-tuples. For the comparative same k value [akin to a cross-sectional study at specific k values], the countably arbitrarily large number of 2nd V Sub II Adm P k-tuples are defined as having diameter d being [slightly] larger than the corresponding diameter d allocated for Sub I Adm P k-tuplets.

In perspective, we must differentiate the nonexisting 1st V Sub II Inadm P k-tuples or Forbidden Inadmissible Prime k-tuples mentioned above in Part I Analysis from the manifestly different non-repeating and proposed-to-exist-only-once 2nd V Sub II Inadm P k-tuples mentioned here. All countably arbitrarily large number of [non-repeating] Sub I Inadm P k-tuples and [non-repeating] 2nd V Sub II Inadm P k-tuples that utilize progressively larger consecutive k values are proposed to only match one position [for each k value] in the sequence of prime numbers. The Sub I Inadm P k-tuples and 2nd V Sub II Inadm P k-tuples are selectively defined by their diameter d and first element p which are compared to the diameter d and first element p from the same k ≥ 2 value that is assigned to the eligible Sub I Adm P k-tuplet pattern. Let p_i = first element p of the eligible Sub I Adm P k-tuplet pattern for a given k ≥ 2 value. Then for the same given k ≥ 2 value, the first element p in the corresponding Sub I Inadm P k-tuple or 2nd V Sub II Inadm P k-tuple = p_{i-1} with its varying diameter d manifesting either [slightly] smaller than, equal to or larger than the smallest possible diameter d value of the eligible Sub I Adm P k-tuplet pattern. Examples of various k values arranged in increasing order are provided in Appendix F and Table 3 that contain complex intertwined Incompletely Predictable properties when we compare and contrast Sub I Adm P k-tuplets, Sub I Inadm P k-tuples, 2nd V Sub II Adm P k-tuples, and 2nd V Sub II Inadm P k-tuples. We will subsequently outline Rosser's Prime k-tuple rule for k ≥ 2 values and its violations in Remark 1.5.

In Table 3, Sub I Adm P k-tuplets as eligible pattern-1 are simultaneously listed; and p_i = 17 or 23 fail to [orderly] appear before p_i = 29 or 19 in relation to Sub I Adm P k-tuplets pattern-1 when k = 20, 21, 22 and 23. We include two randomly selected calculations using [ineligible] Prime k-tuplet pattern-2 firstly, when k = 17 with p_i = 17 and secondly, when k = 22 with p_i = 23. These calculations simply result in their respective Subtype I Prime 17-tuplet pattern-1 and Subtype I Prime 22-tuplet pattern-1.

For k = 2, 3, 4 and 5, Sub I Inadm P 2-tuple is associated with failure at mod prime 2 (term 3) when k = 2 & d = 1 [c.f. d = 2 for Prime 2-tuplet] & p_{i-1} = 2 [c.f. p_i = 3 for Prime 2-tuplet], Sub I Inadm P 3-tuple is associated with failure at mod prime 3 (term 7) when k = 3 & d = 4 [c.f. d = 6 for Prime 3-tuplet] & p_{i-1} = 3 [c.f. p_i = 5 for Prime 3-tuplet], Sub I Inadm P 4-tuple is associated with failure at mod prime 3 (term 7) when k = 4 & d = 8 [c.f. d = 8 for Prime 4-tuplet] & p_{i-1} = 3 [c.f. p_i = 5 for Prime 4-tuplet], and Sub I Inadm P 5-tuple is associated with failure at mod prime 3 (term 7) when k = 5 & d = 10 [c.f. d = 12 for Prime 5-tuplet] & p_{i-1} = 3 [c.f. p_i = 5 for Prime 5-tuplet].

For k ≥ 6, 2nd V Sub II Inadm P 6-tuple is associated with failure at mod prime 5 (term 19) when k = 6 & d = 14 [c.f. d = 16 for Prime 6-tuplet] & p_{i-1} = 5 [c.f. p_i = 7 for Prime 6-tuplet], 2nd V Sub II Inadm P 7-tuple is associated with failure at mod prime 5 (term 19) when k = 7 & d = 22 [c.f. d = 20 for Prime 7-tuplet] & p_{i-1} = 7 [c.f. p_i = 11 for Prime 7-tuplet], 2nd V Sub II Inadm P 8-tuple is associated with failure at mod prime 7 (term 29) when k = 8 & d = 24 [c.f. d = 26 for Prime 8-tuplet] & p_{i-1} = 7 [c.f. p_i = 11 for Prime 8-tuplet], etc.

The diameter d for Sub I Inadm P k-tuples is smaller than diameter d for Sub I Adm P k-tuplet when k = 2 and 3, equal to diameter d for Sub I Adm P k-tuplet when k = 4, smaller than diameter d for Sub I Adm P k-tuplet when k = 5. On the limited data presented on diameter d in Table 3 for 2nd V Sub II Inadm P k-tuples, these diameter d are seen to be *cyclical in nature*; viz, it is smaller than diameter d for Sub I Adm P k-tuplet when k = 6, it is larger than diameter d for Sub I Adm P k-tuplet when k = 7, it is smaller than diameter d for Sub I Adm P k-tuplet when k = 8, it is equal to diameter d for Sub I Adm P k-tuplet when k = 9, it is larger than diameter d for Sub I Adm P k-tuplet when k = 10, etc.

We say that violations of Rosser's Prime k-tuple rule for k ≥ 2 values in Table 3 are seen to initially occur consecutively at k = 20 with p_i = 29 and k = 21 with p_i = 29 resulting in the Missing Inadmissible Prime k-tuples. For k = 20 with p_i = 29, these are instead manifested as corresponding four 2nd V Sub II Adm P 20-tuples all with slightly larger diameter d = 84 at p_{i-1} = 23, p_{i-2} = 19, p_{i-3} = 17 and p_{i-4} = 13. For k = 21 with p_i = 29, these are instead manifested as corresponding one 2nd V Sub II Adm P 21-tuple with slightly larger diameter d = 86 at p_{i-1} = 23, two 2nd V Sub II Adm P 21-tuples with slightly larger diameter d = 88 at p_{i-2} = 19 and p_{i-4} = 13. However, k = 21 with p_{i-3} = 17 having slightly smaller diameter d = 84 corresponds instead to a 2nd V Sub II Inadm P 21-tuple due to failure at mod prime 17 (term 103).

Remark 1.5. Reminiscent of Remark 1.1 on Rosser's Rule and its violations, we hereby tentatively propose *Rosser's Prime k-tuple rule for k ≥ 2 values* to uniquely indicate each Subtype I Admissible Prime k-tuplet with its prime p_i commencing value is usually associated with its corresponding (initially) Subtype I Inadmissible Prime k-tuple [for k = 2, 3, 4 and 5] and (subsequently) second variety of Subtype II Inadmissible Prime k-tuple [for k ≥ 6] that are manifested as failure at mod prime p_{i-1} (relevant term p). However, we observe in our tabulated data in Table 3 violations of this rule that is defined as Missing Inadmissible Prime k-tuples, which refers to the complete lack of association between certain Subtype I Admissible Prime k-tuplets and their corresponding [absent] second variety of Subtype II Inadmissible Prime k-tuples. These [absent] second variety of Subtype II Inadmissible Prime k-tuples are instead replaced by the second variety of Subtype II Admissible Prime k-tuples. All of these findings are fully consistent with the iteratively computed data in Box 1 [in Appendix D]. Further research is required to determine whether these intermittent violations based on same modular arithmetic will recur arbitrarily many times for ever larger k values.

With needing to include diameter d = 2 when k = 2 [viz, s(2) = 2]; Sub I Adm P k-tuplets for k ≥ 3 can be computed recursively using the following algorithm (Forbes, 1999, p. 1740) whereby the diameter d is denoted by s(k), gcd is abbreviation for greatest common divisor, and for p prime, the notation p# is product of all primes up to and including p.

Procedure s(k):

Do S(s,3,1) for s = s(k-1)+2, s(k-1)+4,... until an admissible set B is found.

Procedure S(s,q,H):

Step 1. Set U = q#, the product of all the primes q. Set D = $\frac{U}{q}$ and h = H. *Step 2.* Set B = {i: i = 0, 2,..., s, gcd(h+i, U) =1}. *Step 3.* If B does not contain both 0 and s, go to step 8. *Step 4.* If B has less than k elements, go to step 8. *Step 5.* If B has more than k elements, do S(s,q',h), where q' is the next prime after q. Then go to step 8. *Step 6.* If B has exactly k elements and if for each prime p, q < p ≤ k, all residues modulo p are represented by B, go to step 8. *Step 7.* Indicate that B is an admissible set and report s(k) = s. *Step 8.* Add D to h. If h < H + U, go to step 2. Otherwise return.

The above algorithm is related to A008407 *Minimal difference s(n) between beginning and end of n consecutive large primes (n-tuplet) permitted by divisibility considerations.* (Forbes, n.d.) having values 0 [symbolizing the nonexisting 1-tuple], 2, 6, 8, 12, 16, 20, 26, 30, 32, 36, 42, 48, 50, 56, 60, 66, 70, 76, 80, 84, 90, 94, 100, 110, 114, 120, 126, 130, 136, 140, 146, 152, 156, 158, 162, 168, 176, 182, 186, 188, 196, 200, 210, 212, 216, 226, 236, 240, 246, 252, 254, 264, 270, 272, 278....

Remark 1.6. The first variety of Subtype II Admissible Prime k-tuples will cater more for existence of prime numbers

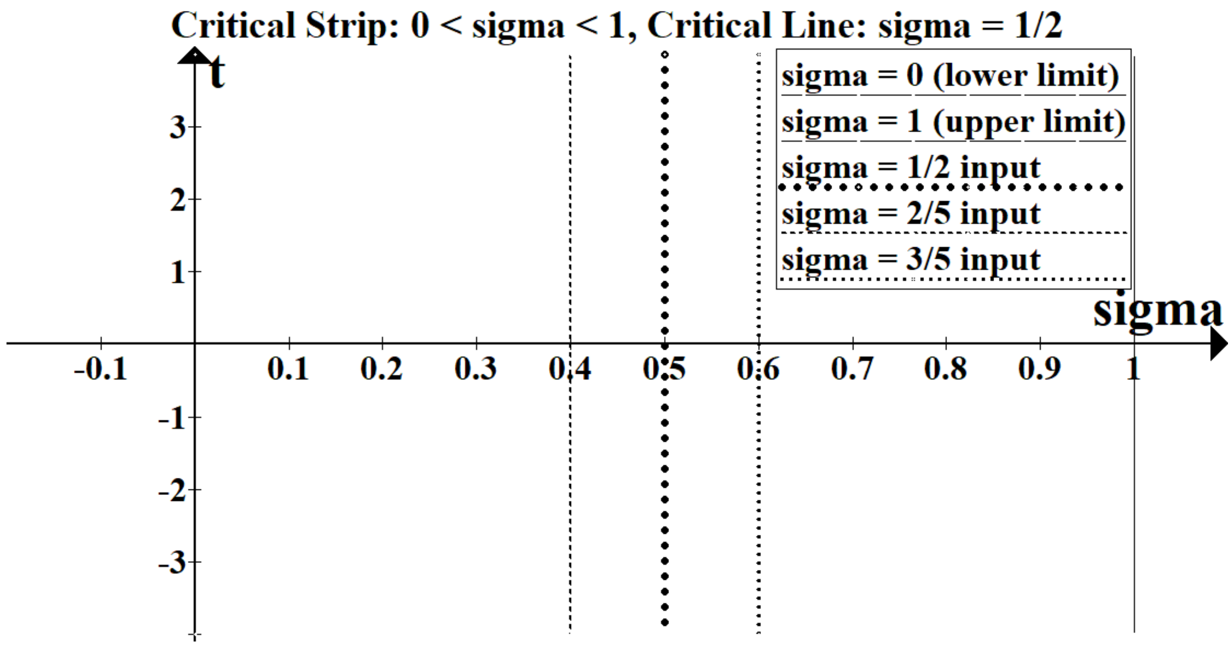

Figure 3. INPUT for $\sigma = \frac{1}{2}$, $\frac{2}{5}$, and $\frac{3}{5}$. Riemann zeta function, $\zeta(s)$, has countable infinite set of Completely Predictable trivial zeros located at s = all negative even numbers and countable infinite set of Incompletely Predictable nontrivial zeros located at $\sigma = \frac{1}{2}$ as various t values.

with large(r) prime gaps that tend to occur at large(r) range of x integer values than the second variety of Subtype II Admissible Prime k-tuples which cater more for existence of prime numbers with small(er) prime gaps that tend to occur at small(er) range of x integer values. We hypothetically deduce the second variety of Subtype II Inadmissible Prime k-tuples usually appear with the first occurring or earliest known Subtype I Admissible Prime k-tuplets pattern-1 derived with relevant k values except when violations of Rosser's Prime k-tuple rule occur. When first variety and second variety of Subtype II Admissible Prime k-tuples are combined with Subtype I Admissible Prime k-tuplets, they should, in principle, be able to represent every known odd prime numbers albeit in an overlapping manner.

We provide a short summary: As alluded to in Part I Analysis, 1st V Sub II Adm P k-tuples that are based on comparatively same k value present in [reference] Sub I Adm P k-tuplets will be associated with progressively much larger diameter d. They are simply mediated via [eligible] last prime number in involved tuples being made progressively bigger. In contrast, 2nd V Sub II Adm P k-tuples that are based on comparatively same k value present in [reference] Sub I Adm P k-tuplets will be associated with progressively slightly larger diameter d. They are simply mediated via [eligible] neighboring tuples being selectively nominated to represent them when appropriate.

Proposition 1.3. Both $f(n)$ simplified Dirichlet eta function and $F(n)$ Dirichlet Sigma-Power Law will manifest Principle of Equidistant for Multiplicative Inverse.

Proof. We use η(s) to denote $f(n)$ Dirichlet eta function containing variable n, and parameters t and σ. Here, η(s) is the proxy function for Riemann zeta function, which can be denoted by ζ(s). With also containing variable n, and parameters t and σ; the $f(n)$ simplified Dirichlet eta function, denoted by sim-η(s), is essentially obtained by applying Euler formula to η(s) and the $F(n)$ Dirichlet Sigma-Power Law, denoted by DSPL, refers to $\int$sim-η(s)dn. Let variable $\delta = \frac{1}{10}$. This will consistently generate in Figure 5 and Figure 6 the δ induced shift of [infinitely many] Varying Loops in reference to Origin; viz, the simple relationship of [more negative] left-shift given by $\zeta(\frac{1}{2} - \delta + \imath t)$ [Figure 5] < [neutral] nil-shift given by $\zeta(\frac{1}{2} + \imath t)$ [Figure 4] < [more positive] right-shift given by $\zeta(\frac{1}{2} + \delta + \imath t)$ [Figure 6]. Note: The following discussion is ahead of time as the required relevant equations for η(s), sim-η(s) and DSPL that contain transcendental functions, etc are only available for reference later on in Appendices H and I of this paper.

Given $\delta = \frac{1}{10}$, the $\sigma = \frac{1}{2} - \delta = \frac{2}{5}$-non-critical line (represented by Figure 5) and $\sigma = \frac{1}{2} + \delta = \frac{3}{5}$-non-critical line (represented by Figure 6) are **equidistant** from $\sigma = \frac{1}{2}$-critical line (represented by Figure 4). The additive inverse operation of sin(δ) + sin(-δ) = 0 indicating symmetry with respect to Origin [or cos(δ) - cos(-δ) = 0 indicating symmetry with respect to y-axis] is not applicable to our complex single sine wave [or single cosine wave] since **(2n)-complex or (2n-1)-complex**

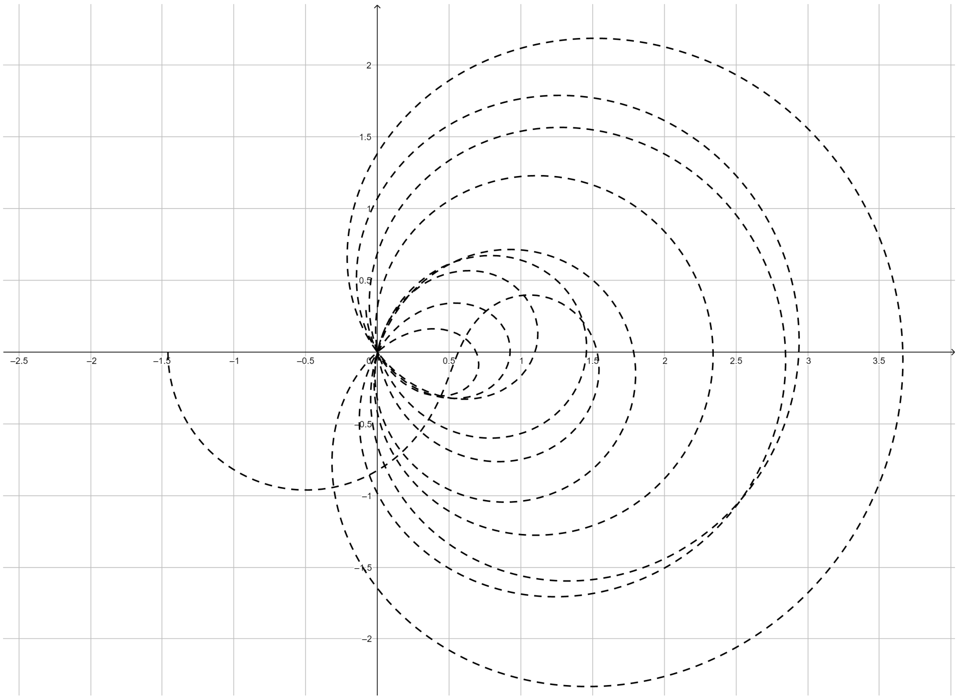

Figure 4. OUTPUT for $\sigma = \frac{1}{2}$ as Gram points. Schematically depicted polar graph of $\zeta(\frac{1}{2} + \imath t)$ plotted along critical line for real values of t running from 0 to 34, horizontal axis: $Re\{\zeta(\frac{1}{2} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{1}{2} + \imath t)\}$. Presence of Origin intercept points.

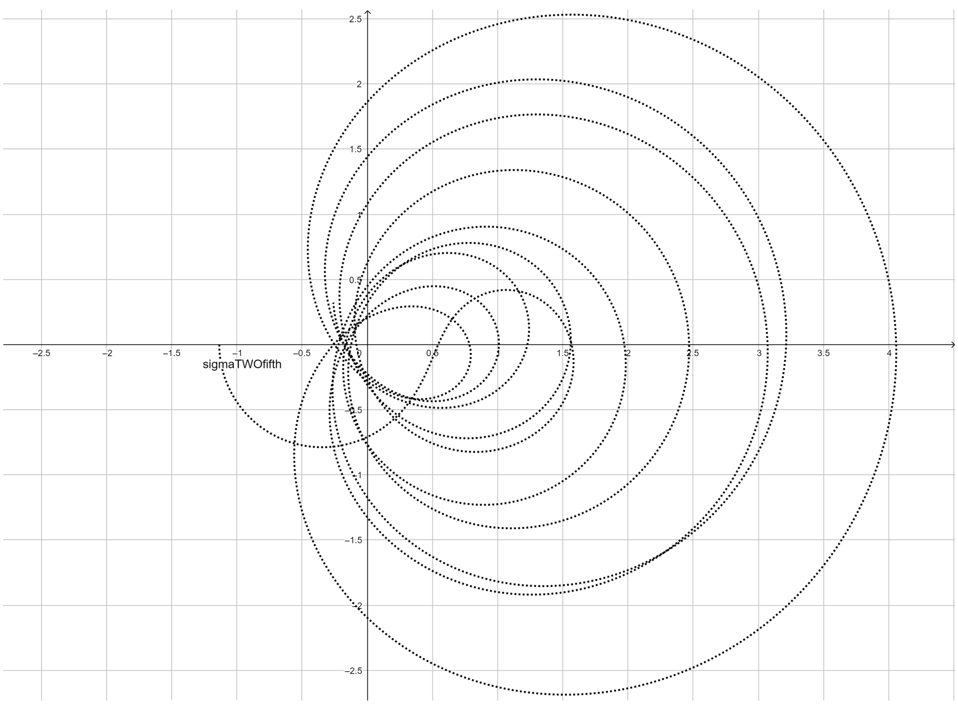

Figure 5. OUTPUT for $\sigma = \frac{2}{5}$ as virtual Gram points. Varying Loops are shifted to left of Origin with horizontal axis: $Re\{\zeta(\frac{2}{5} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{2}{5} + \imath t)\}$. Nil Origin intercept points.

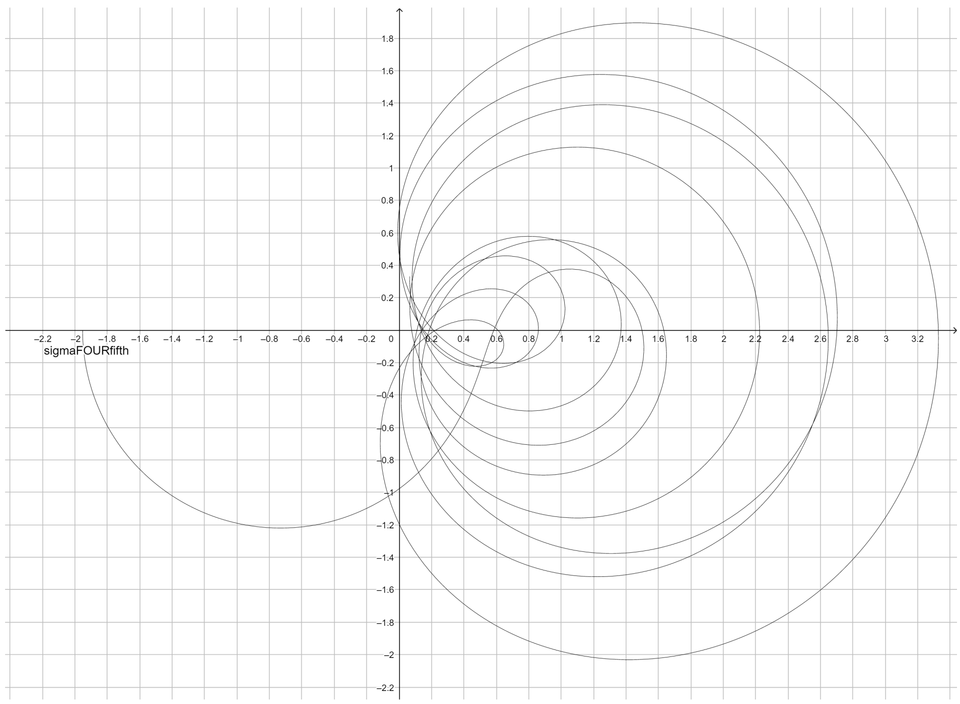

Figure 6. OUTPUT for $\sigma = \frac{3}{5}$ as virtual Gram points with horizontal axis: $Re\{\zeta(\frac{3}{5} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{3}{5} + \imath t)\}$. Varying Loops are shifted to right of Origin. Nil Origin intercept points.

term with transcendental functions consisting of sine, cosine, single sine wave, single cosine wave, natural logarithm are **independent of parameter** σ.

However, **(2n)-complex or (2n-1)-complex term with algebraic functions** consisting of powers, fractional powers, root extraction [and scaled amplitude R on its (in)dependency on parameter t] are **dependent on parameter** σ. Let x = $(2n)$ or $\frac{1}{(2n)}$ or $(2n-1)$ or $\frac{1}{(2n-1)}$. With multiplicative inverse operation of $x^{\delta} \cdot x^{-\delta} = 1$ or $\frac{1}{x^{\delta}} \cdot \frac{1}{x^{-\delta}} = 1$ that is applicable, this imply intrinsic presence of **Multiplicative Inverse** in sim-$\eta(s)$ or DSPL for all σ values with this function or law rigidly obeying relevant trigonometric identity. We call this phenomenon **Principle of Equidistant for Multiplicative Inverse**. Finally, we note by letting $\delta = 0$, we will always generate Figure 4 representing $\sigma = \frac{1}{2}$-critical line.
The proof is now complete for Proposition 1.3□.

1.6 Infinitesimal numbers applied to Prime numbers and Nontrivial zeros

The following is considered under real-valued functions of a positive real variable: The asymptotic law of distribution of prime numbers states the limit of quotient of two functions Prime-$\pi(x)$ and $\frac{x}{\log_e x}$ as x increases without bound is 1; viz, $\lim_{x\to\infty} \frac{\text{Prime-}\pi(x)}{\left[\frac{x}{\log_e x}\right]} = 1$. Whereby the deceleratingly distributed prime numbers mathematically involves $\log_e x$, then the acceleratingly distributed composite numbers must mathematically involves e^x since these two set of numbers are [complementary] mutually exclusive entities and natural logarithm function is the inverse function of natural exponential function. Then the corresponding asymptotic law of distribution of composite numbers is $\lim_{x\to\infty} \frac{\text{Composite-}\pi(x)}{\left[\frac{x}{e^x}\right]} = 1$. A direct interpretation of Prime number theorem is average gap between all odd primes [arising from the arbitrarily large number of all even prime gaps 2, 4, 6, 8...] increases as natural logarithm of these primes, and therefore the ratio of average prime gap to all odd primes involved decreases (and is asymptotically zero). Thus Prime number theorem [which was discussed above in last paragraph of subsection 1.2 under li(x)] states the average prime gap between all odd primes increases without bound as you go out on number line.

Graphically manifesting perpetually decreasing behavior of approaching arbitrarily close to zero but never touching zero [viz, is asymptotically zero as the graphical distance gets closer to 0 or ∞]; the natural logarithm of a number is its logarithm to base of mathematical constant e – a unique irrational (transcendental) number ≈ 2.718281828459. Thus the $\log_e x$ function deceleratingly grows to $+\infty$ [conceptually a "zero"] as x increases, and deceleratingly goes to $-\infty$ [conceptually a "zero"] as x approaches 0 whereby $\log_e e = 1$ and $\log_e 1 = 0$. Let $f(x) = g(x) + h(x) + i(x)$... be a mathematically well-defined component function constituted by sum of its sub-component functions that all contain natural logarithm. Then $f(x)$ together with $g(x)$, $h(x)$, $i(x)$... must also all manifest the asymptotically zero behavior of natural logairthm. Conceptually, $f(x)$ can represent average prime gap between all odd primes [from arbitrarily large number of all even Prime gaps 2, 4, 6, 8...], $g(x)$ can represent average prime gap between all twin primes [from even Prime gap 2], $h(x)$ can represent average prime gap between all cousin primes [from even Prime gap 4], $i(x)$ can represent average prime gap between all sexy primes [from even Prime gap 6], etc whereby computations from Appendix G confirm average prime gap between all odd primes manifesting this asymptotically zero behavior of natural logarithm.

In analogy to Prime number theorem, the **first Hardy-Littlewood conjecture**, also called the Prime tuples conjecture, states that the asymptotic number of Admissible Prime k-tuplets or Prime constellations can be computed explicitly [and that every Admissible Prime k-tuplet matches an arbitrarily large number of positions in the sequence of prime numbers]. The **second Hardy-Littlewood conjecture** states that Prime-$\pi(x + y) \leq$ Prime-$\pi(x)$ + Prime-$\pi(y)$ for all $x, y \geq 2$ whereby Prime-$\pi(x)$ is the prime counting function; viz, the number of primes from $x + 1$ to $x + y$ is always less than or equal to the number of primes from 1 to y. These two Hardy-Littlewood conjectures (Hardy & Littlewood, 1923) were subsequently proven to be incompatible with each other (Hensley & Richards, 1974) with an arbitrarily large number of violations. The first such violation is expected to likely occur for very large values of x; for example, an Admissible Prime k-tuplet of 447 primes [viz, Sub I Adm P 447-tuplet with smallest possible diameter = 3158] can be found in an interval of $y = 3159$ integers, while Prime-$\pi(3159) = 446$. Although unproven, the first Hardy-Littlewood conjecture is generally considered by most people to likely be true. If that is the case, it implies that the second Hardy-Littlewood conjecture, in contrast, is false. However, we do not need to prove [or disprove] the first Hardy-Littlewood conjecture *per se* for our purpose.

For Polignac's and Twin prime conjectures to be true, none of the arbitrarily large number of countably infinite subsets of odd prime numbers as generated by corresponding even Prime gaps should ever become countably finite subsets. There are two somewhat anomalous situations.
(A) Prime numbers tend to be clustered around large or larger prime gaps occurring as multiples of 6; viz, prime gaps 6, 12, 18.... We deduce this observation do not prove or disprove Polignac's and Twin prime conjectures, and can be logically explained as follow. Excepting the first two prime gaps, all prime gaps are between numbers that are either 1 or 5 modulo 6. Under the assumption that both cases are equally likely, half the prime gaps will be between numbers in the same class, and therefore of size 0 modulo 6, and the other half will be between numbers in different classes, which split up into sizes that are 2 and 4 modulo 6. Since each of the latter cases only gets one quarter of the total, it is clear that ignoring all other factors, gaps that are 2 or 4 modulo 6 are about half as likely to occur as gaps of the same approximate magnitude that are 0 modulo 6.
(B) Here is a simple proof for two consecutive prime gaps that are equal must be of the form (6n, 6n) for n = 1, 2, 3, 4, 5...: Suppose there were two consecutive gaps between 3 consecutive prime numbers that were equal, but not divisible by 6. Then the difference is 2k where k is not divisible by 3. Therefore the (supposed) prime numbers will be p, p+2k, p+4k. But then p+4k is congruent modulo 3 to p+k. That makes the three numbers congruent modulo 3 to p, p+k, p+2k. One of those is divisible by 3 and so cannot be prime. So two consecutive gaps must be divisible by 3 and therefore (as they have to be even) by 6. However this observation do not prove or disprove Polignac's and Twin prime conjectures.

Riemann hypothesis propose all nontrivial zeros to be located on $\sigma = \frac{1}{2}$-critical line of Riemann zeta function. Previous confirmation of first 10,000,000,000,000 nontrivial zeros location on the critical line implies but does not prove Riemann hypothesis to be true. Hardy initially (Hardy, 1914), and then with Littlewood (Hardy & Littlewood, 1921), showed there are *infinitely many nontrivial zeros lying on critical line* or, equivalently, there are *infinitely many Origin intercept points lying on Origin point* by considering moments of certain functions related to Riemann zeta function. This discovery cannot constitute rigorous proof for Riemann hypothesis because they have not exclude theoretical existence of nontrivial zeros located away from the critical line when $\sigma \neq \frac{1}{2}$. Furthermore, it is literally a mathematical impossibility (*mathematical impasse*) to be able to computationally check [in a complete and successful manner] the locations of all the infinitely many nontrivial zeros to correctly be the critical line. Note: There must be infinitely many t-valued Origin intercept points lying on Origin point [and hence infinitely many t-valued nontrivial zeros] since variable t has full range of values given by $-\infty < t < +\infty$.

An infinitesimal number is a quantity that is closer to zero than any standard real number, but that is not zero. The mathematical concept of infinity can be introduced using the symbol ∞. Then the reciprocal or inverse symbol $\frac{1}{\infty}$ is the symbolic representation of the mathematical concept of infinitesimal.

Proposition 1.4. With the prevalence of various selected odd prime numbers as endpoints never becoming zero [which are conceptually defined as the *nonexisting zero* in this instance], we can apply infinitesimal numbers to rigorously show both the prevalence of total odd prime numbers having all even Prime gaps and the prevalence of subtotal odd prime numbers having corresponding even Prime gaps will never become zero.

Proof. We recall from Lemma 1.1 that all CIS-ALN-decelerating computed prime numbers are extrapolated out over a wide range of x ≥ 2 integer values; the prime counting function Prime-π(x) = number of primes ≤ x with x [conveniently] assigned to having odd number values of the form $10^n - 1$ whereby n = 1, 2, 3, 4, 5...; and the **Prevalence of prime numbers** = Prime-π(x) / x = Prime-π(x) / ($10^n - 2$) when x = 2 to $10^n - 1$. Note: The probability theory applied to n-digit primes and n-digit composites are given later on in subsection 2.3.

Apart from even Prime gaps of the form 6n with n = 1, 2, 3, 4, 5... and the [solitary] consecutive prime gaps (2, 2) present in Sub I Inadm P 3-tuple with consecutive primes (3, 5, 7), no other types of two consecutive prime gaps that are identical is possible. In reality, one could then rigorously argue from first principle alone there must be at least three even Prime gaps that will perpetually reappear over the entire sequence of prime numbers because the alternatingly appearance of just two different even Prime gaps at extremely large x integer values simply cannot occur.

We recall from Proposition 1.2 concerning a given k_0-tuple $\mathcal{H} = (h_1, \ldots, h_{k_0})$ of k_0 distinct integers for some $k_0 \geq 1$, arranged in increasing order whereby one can, in principle, find an arbitrarily large number of translates n + $\mathcal{H}$ =(n+h_1,..., n+h_{k_0}) of $\mathcal{H}$ which consists entirely of consecutive primes. The case $k_0 = 1$ is just Euclid's theorem on the infinitude of primes. From this simple theorem, we provide following mathematical arguments:
The cardinality of all prime numbers [or all odd prime numbers when we validly ignore the only even prime number 2] is given by ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating when n → ∞ in x = 2 to $10^n - 1$. The cardinality of all x = 2 to ∞ integer numbers is given by ||CIS-IM-linear|| = $\aleph_0$-linear. As n → ∞, there are an arbitrarily large number (ALN) of deceleratingly-occurring prime numbers amongst the infinitely many linearly-occurring x integer numbers; viz, x integer numbers ≫ prime numbers. Then **Prevalence of prime numbers** = Prime-π(x) / x = ALN / ∞ = an infinitesimal number symbolized by $\frac{1}{\infty}$ when x = 2 to ∞. Since Euclid's theorem holds for x = 2 to ∞, then **Prevalence of prime numbers** is constituted by an infinitesimal number but can never become zero; viz, **Prevalence of prime numbers** conceptually have a nonexisting zero.

A substantial amount of previous materials refer to proposal on subsets of odd prime numbers uniquely derived from corresponding arbitrarily large number of even Prime gaps 2i with i = 1, 2, 3, 4, 5... in that all these subsets must be arbitrarily large in number. Remark 1.2 from Proposition 1.2, in particular, support this proposal. Then there must also be full compliance with two conditions: (i) Dimensional analysis homogeneity on relevant cardinality and (ii) even Prime gaps will never terminate. All odd prime numbers having all even Prime gaps = odd prime numbers having even Prime gap 2 + odd prime numbers having even Prime gap 4 + odd prime numbers having even Prime gap 6 +... + odd prime numbers having even Prime gap 2i $\implies$ $\aleph_0$-decelerating [all odd prime numbers] = $\aleph_0$-decelerating [odd prime numbers having even Prime gap 2] + $\aleph_0$-decelerating [odd prime numbers having even Prime gap 4] + $\aleph_0$-decelerating [odd prime numbers having even Prime gap 6] +... + $\aleph_0$-decelerating [odd prime numbers having even Prime gap 2i]. Based on similar reasoning in last paragraph, we logically deduce for x = 2 to ∞, **Prevalence of various odd prime numbers as specified by their corresponding even Prime gaps 2i** can similarly all be constituted by infinitesimal numbers symbolized by $\frac{1}{\infty}$ but never become zero; viz, **Prevalence of various odd prime numbers as specified by their corresponding even Prime gaps 2i** conceptually have a nonexisting zero. *The proof is now complete for Proposition 1.4*□.

Proposition 1.5. With Origin point or $\sigma = \frac{1}{2}$-critical line of Riemann zeta function regarded as the zero endpoint [which is conceptually defined as the *existing zero* in this instance], we can apply infinitesimal numbers to rigorously show the equivalent [geometric] Origin intercept points located at the zero-dimensional Origin point and [mathematical] nontrivial zeros located at the one-dimensional $\sigma = \frac{1}{2}$-critical line will uniquely appear only when parameter $\sigma = \frac{1}{2}$.

Proof. Here, for simplicity, we use the term Riemann zeta function to also indicate Dirichlet eta function, simplified Dirichlet eta function and Dirichlet Sigma-Power Law. We recall from Lemma 1.1 that all CIS-IM-linear computed nontrivial zeros are extrapolated out over a wide range of t ≥ 0 real number values; and the Nontrivial zeros gaps, Nontrivial zeros counting function and Prevalence of nontrivial zeros can be defined. Although inevitably fluctuating, **Prevalence of nontrivial zeros** must [linearly] be a fairly constant value over t = 0 to ∞. This value is reasonably approximated by, for instance, using t = 0 to 100 range as 29/100 = 0.29 = 29% since there are precisely 29 nontrivial zeros in this range.

We recall variable δ with given value $\frac{1}{10}$ when applied to Riemann zeta function in Proposition 1.3 to confirm **Principle of Equidistant for Multiplicative Inverse** that is applicable to Figure 5 representing $\sigma = \frac{2}{5}$-non-critical line and Figure 6 representing $\sigma = \frac{3}{5}$-non-critical line. We recognize the zero-dimensional Origin point in Figure 4 is synonymous with the one-dimensional $\sigma = \frac{1}{2}$-critical line, and this particular point or line is conceptually regarded as the existing zero. Then the Varying Loop trajectory in Figure 4 will only present its CIS-linear [geometrical] Origin intercept points that is precisely equivalent to the CIS-linear [mathematical] nontrivial zeros when $\delta = 0$ since the Origin point is a zero-dimensional point that can only be touched by the trajectory when $\delta = 0$ and $\sigma = \frac{1}{2}$. We logically deduce variable δ when constituted by an infinitesimal number symbolized by $\frac{1}{\infty}$ will never become the existing zero since this equates to $\sigma \approxeq \frac{1}{2}$ [or the trajectory is extremely close to zero-dimensional Origin point] but this is categorically still not the same as $\sigma = \frac{1}{2}$ [or the trajectory touching zero-dimensional Origin point]. Thus variable δ will instead only become the existing zero when both the $\sigma = \frac{1}{2}$ and $\delta = 0$ conditions are simultaneously fully satisfied.

A worthy point of interest is the coincidental manifestation of Dimensional analysis (DA) homogeneity by parameter σ in Riemann zeta function. The *exact DA homogeneity* indicate calculated values of [exact] integer −1 and 1 as derived from $\sum$(all fractional exponents) = $2(-\sigma)$ and $2(1 - \sigma)$. Respectively, these act as surrogate markers in simplified Dirichlet eta function and Dirichlet Sigma-Power Law on the solitary unique $\sigma = \frac{1}{2}$ situation. Otherwise, for the infinitely many non-unique $\sigma \neq \frac{1}{2}$ corollary situations, calculated values of [inexact] fractional numbers ≠ integer −1 and ≠ integer 1 are derived from $\sum$(all fractional exponents) = $2(-\sigma)$ and $2(1 - \sigma)$ to indicate *inexact DA homogeneity*.
The proof is now complete for Proposition 1.5□.

2. Conjectures and Hypotheses involving Prime numbers and Nontrivial zeros

We hereby summarize using two theorems the convoluted but deceptively simple correct and complete mathematical arguments that rigorously form Algorithm-type proofs for Polignac's and Twin prime conjectures, and Equation-type proof for Riemann hypothesis.

Theorem 2.1. Modified Polignac-Twin-Prime Conjecture. The 1849 Polignac's conjecture involves studying all even Prime gaps 2, 4, 6, 8, 10... and the 1846 Twin prime conjecture involves studying [subset] even Prime gap 2. Traditionally, these two conjectures propose that the countably infinitely many elements in **Set even Prime gaps** form a countably infinite set; and their corresponding countably infinitely many elements in **Subsets odd prime numbers** form countably infinite subsets. We innovatively propose the Modified Polignac-Twin-Prime Conjecture whereby elements forming these set and subsets are instead succinctly treated as countably arbitrarily large in number.

Theorem 2.2. Riemann Hypothesis. The 1859 Riemann hypothesis proposed that the countably infinitely many nontrivial zeros of Riemann zeta function are all located on its $\sigma = \frac{1}{2}$ critical line. To generate **Set nontrivial zeros** containing countably infinitely many elements that will linearly reach an infinity value; we must instead use Dirichlet eta function, which is the proxy function for Riemann zeta function. We innovatively propose the one-dimensional [mathematical] critical line is precisely equivalent to the zero-dimensional [geometrical] Origin point.

2.1 Outline of the Proofs for Open problems in Number theory

Let us start by stating Theorem 2.1 Modified Polignac-Twin-Prime Conjecture and Theorem 2.2 Riemann Hypothesis properly. In section 1 Introduction we hinted that Theorem 2.1 will be a statement about the cardinality of Set even Prime gaps and corresponding generated Subsets odd prime numbers being sub(sets) classified as countably arbitrarily large (sub)sets; and Theorem 2.2 will be a statement about $\sigma = \frac{1}{2}$ critical line being the designated unique location for all countably infinitely large number of nontrivial zeros from Riemann zeta function. The "finitary" versions of the two theorems are as follows.

Proposition 2.3. (Modified Polignac-Twin-Prime Conjecture). To minimize complexity, we need not regurgitate the in-depth analysis previously performed on special properties in Admissible Prime k-tuplets/k-tuples and Inadmissible Prime k-tuples as this (in)action do not invalidate our on-going proposition, conjecture and theorem on the cardinality of prime numbers and prime gaps.

As the sequence of prime numbers carries on, primes with ever larger prime gaps from Set even Prime gaps will appear, but not always in sequential order e.g. prime gap 14 from prime number 113 first appear at position 30, which is earlier than prime gap 10 from prime number 139 that first appear at position 34 and prime gap 12 from prime number 199 that first appear at position 46. Let ||Set even Prime gaps|| = ||Set total odd Prime numbers|| = countably arbitrarily large number in size. Then there must be an countably arbitrarily many corresponding ||Subset odd Prime numbers|| = countably arbitrarily large number in size having the following property. If cardinality of sets and subsets are to be uniformly similar, then these intrinsically assigned cardinality must all be of countably arbitrarily large numbers in size.

Let $\mathbb{P}_i$, $\mathbb{P}_{i+1}$, $\mathbb{P}_{i+2}$ and $\mathbb{P}_{i+3}$ = four randomly selected consecutive prime numbers whereby $\mathbb{P}_{i+3} > \mathbb{P}_{i+2} > \mathbb{P}_{i+1} > \mathbb{P}_i$. If this four primes are considered in total isolation, then there are only three possible prime gaps able to be computed: Prime gap$_i$ = $\mathbb{P}_{i+1} - \mathbb{P}_i$, Prime gap$_{i+1}$ = $\mathbb{P}_{i+2} - \mathbb{P}_{i+1}$ and Prime gap$_{i+2}$ = $\mathbb{P}_{i+3} - \mathbb{P}_{i+2}$. In principle, we recognize these three prime gaps can be constituted by all possible combinations of small prime gaps 2 and 4 and/or large prime gaps ≥ 6; viz, all three prime gaps are constituted by small prime gaps, all three prime gaps are constituted by large prime gaps, and the three prime gaps are constituted by a mixture of small and large prime gaps. Intuitively, every even Prime gap 2, 4, 6, 8, 10... and its correspondingly associated odd prime numbers must exist at least once; viz, occurring only one time, occurring a finite number of times, or occurring an arbitrarily large number of times. Proving the only correct possibility of both *even Prime gaps 2, 4, 6, 8, 10... and their correspondingly associated odd prime numbers will all occur an arbitrarily large number of times* is equivalent to rigorously proving Modified Polignac-Twin-Prime Conjecture to be true. The term *prime constellation* denotes, and is synonymous with, *Admissible Prime k-tuplet* having smallest possible diameter. As explained before for same comparative k values, there are also *Admissible* and *Inadmissible Prime k-tuples* with their various subtypes and varieties. Examples of Inadmissible Prime 3-tuples occurring only once: Prime numbers (p, p+1, p+3) = (2, 3, 5) with smallest diameter = 3, whereby prime gap 1 can only occur once. Prime numbers (p, p+2, p+4) = (3, 5, 7) with smallest diameter = 4, whereby these are closest possible [non-repeatable] grouping of three prime numbers since one of every three sequential odd numbers is a multiple of three, and hence not prime (except for 3 itself).

We outline three possible trajectories of all prime gaps that are consistent with the existence of Prime-$\pi(x)$ as a stepped-mathematical function whereby we also use prime gaps 6n as common randomly chosen examples – in particular, for n = 1, 2, 3...; even Prime gaps = 6, 12, 18... [multiples of 6].

(a) **Accelerating primes:** Prime gap$_{i+2}$ – Prime gap$_{i+1}$ > Prime gap$_{i+1}$ – Prime gap$_i$ occurring an arbitrarily large number of times e.g. Admissible Prime 3-tuplet (p, p+2, p+6) with smallest possible diameter = 6, Admissible Prime 3-tuple (p+6, p+10, p+16) ≡ (p, p+4, p+10) with [not the smallest possible] diameter = 10 that is derived from Admissible Prime 18-tuplet (p, p+4, p+6, p+10, p+16, p+18, p+24, p+28, p+30, p+34, p+40, p+46, p+48, p+54, p+58, p+60, p+66, p+70) with smallest possible diameter = 70, and Admissible Prime 3-tuple (p, p+6, p+18) from ([p-24], [p-22], [p-10], p, p+6, p+18, [p+42], [p+50]) with [not the smallest possible] diameter = 18 occurring at consecutive primes (22391, 22397, 22409) with position of first p = 2506.

(b) **Decelerating primes:** Prime gap$_{i+2}$ – Prime gap$_{i+1}$ < Prime gap$_{i+1}$ – Prime gap$_i$ occurring an arbitrarily large number of times e.g. Admissible Prime 3-tuplet (p, p+4, p+6) with smallest possible diameter = 6, Admissible Prime 3-tuple (p+20, p+26, p+30) ≡ (p, p+6, p+10) with [not the smallest possible] diameter = 10 that is derived from Admissible Prime 9-tuplet (p, p+2, p+6, p+8, p+12, p+18, p+20, p+26, p+30) with smallest possible diameter = 30, and Admissible Prime 3-tuple (p, p+18, p+30) from ([p-26], [p-22], [p-12], p, p+18, p+30, [p+50], [p+54]) with [not the smallest possible] diameter = 30 occurring at consecutive primes (10193, 10211, 10223) with position of first p = 1252.

(c) **Steady primes:** Prime gap$_{i+2}$ – Prime gap$_{i+1}$ = Prime gap$_{i+1}$ – Prime gap$_i$ that should occur an arbitrarily large number of times [albeit on extremely rare occasions] and can only involve prime gaps 6n. For instance, the Admissible Prime 3-tuple (p, p+6, p+12) from ([p-2], p, p+6, p+12, [p+18], [p+28], [p+36]) with [not the smallest possible] diameter = 12 occurring at consecutive primes (63691, 63697, 63703) with position of first p = 6386; and Admissible Prime 3-tuple (p, p+18, p+36) from ([p-2], p, p+18, p+36, [p+54], [p+60]) with [not the smallest possible] diameter = 36 occurring at consecutive primes (76543, 76561, 76579) with position of first p = 7531. An exception is the solitary Inadmissible Prime 3-tuple (p, p+2, p+4) with smallest diameter = 4 occurring at consecutive primes (3, 5, 7) ≡ cummulative prime gaps (0, 2, 4). We can explain using either (3, 5, 7) tuple or (0, 2, 4) tuple why this particular Prime 3-tuple is inadmissible, and we choose the former tuple. k = 3, prime q ≤ k $\implies$ prime q = 2 and 3 which are required for modular q. For modular 2: 3 ≡ 1 (mod 2), 5 ≡ 1 (mod 2), 7 ≡ 1 (mod 2) $\implies$ these three primes did not take on all two residue values 0 and 1 [considered as success]. However, for modular 3: 3 ≡ 0 (mod 3), 5 ≡ 2 (mod 3), 7 ≡ 1 (mod 3) $\implies$ these three primes did take on all three residue values 0, 1 and 2 [considered as failure]. By definition, this failure occurrence $\implies$ the three primes are inadmissible since they would always include a multiple of 3 and therefore could not all be prime unless one of the numbers is 3 itself with finite one prime placement.

From above commentaries on possible trajectories of all even Prime gaps, we reach an important logical deduction. Even if the first Hardy-Littlewood conjecture (k-tuple conjecture) is theoretically proven to be false in the sense that every, or some, Admissible Prime k-tuplets having smallest possible diameter do not match an arbitrarily large number of positions in the sequence of prime numbers; this finding will not, in principle, exclude Polignac's and Twin prime conjectures to be true. This is because Admissible Prime k-tuples having ever larger diameter [which is always greater than the smallest possible diameter of Admissible Prime k-tuplets] can still be [conveniently] created that will match an arbitrarily large number of positions in the sequence of prime numbers. In particular, apart from the Admissible Prime 2-tuplet of twin primes with diameter [prime gap] = 2 that can or will repeat itself an arbitrarily large number of times; the most *extreme* but simpliest Admissible Prime 2-tuples having ever larger diameter [prime gap] = 4, 6, 8, 10, 12,... can still be created whereby each Admissible Prime 2-tuple can or will repeat itself an arbitrarily large number of times. In effect, we do not need to involve the first Hardy-Littlewood conjecture to prove Polignac's and Twin prime conjectures because apart from the only countably finite even prime number 2, all the countably arbitrarily large number of odd prime numbers 3, 5, 7, 11, 13... can be fully represented by the solitary Admissible Prime 2-tuplet system that represent even Prime gap 2 and the arbitrarily large number of Admissible Prime 2-tuples system that represent even Prime gaps 4, 6, 8, 10, 12....

It is a breakthrough insight to correctly express the ultimate significance arising from observing the cardinality of (sub)sets of even Prime gaps and odd prime numbers reaching countably arbitrarily large number in size *not* in a infinite-scale smooth manner but instead in a infinite-scale stepwise manner. We deduce this infinite-scale stepwise phenomenon must reflect the above-mentioned repeated groupings of small and/or large prime gaps on an eternal basis resulting in inevitable presence of frequent repeating accelerating and decelerating primes [and also infrequent repeating steady primes]. Finding the correct dependence of ||Set even Prime gaps|| = ||Set total odd Prime numbers|| = ||Subsets odd Prime numbers|| = countably arbitrarily large number in size on this infinite-scale stepwise phenomenon is the famous open problem of Modified Polignac-Twin-Prime Conjecture.

Proposition 2.4. (Polignac's and Twin prime conjectures on cardinality of odd prime numbers and even Prime gaps to be countably arbitrarily large in numbers). We recall from Proposition 1.4 the all-important mathematical argument based on first principle: There must be at least three even Prime gaps that will perpetually reappear over entire sequence of prime numbers because the alternating appearance of just two different even Prime gaps at extremely large x integer values is simply not viable. There is categorically no known *discriminatory* property present in any particular even Prime gaps that will mathematically or probabilistically prohibit them from perpetually reappearing over the entire sequence of prime numbers.

The countably arbitrarily large number of small and large prime gaps that form all even Prime gaps are synonymous with, and generate, all [corresponding] known countably arbitrarily large number of odd prime numbers [with all these entities deceleratingly reaching arbitrarily large number values]. Just as both small and large prime gaps must appear less frequently at ever larger range of x integer values due to prime numbers overall becoming progressively rarer [conceptually as progressive diminishing non-zero probability that never become zero probability] at ever larger range of x integer values, so must repeated groupings of small and/or large prime gaps that manifest infinite-scale stepwise phenomenon overall becoming progressively rarer [conceptually as progressive diminishing non-zero probability that never become zero probability] at ever larger range of x integer values.

Since this imply there is zero probability that any particular small and/or large prime gap(s) present in the countably arbitrarily large number of repeated groupings derived from these prime gaps will abruptly terminate or disappear, we consequently deduce both even Prime gaps and corresponding odd prime numbers must, by default, also recur on an eternal basis to form countably arbitrarily large numbers of both elements [with all these entities deceleratingly reaching arbitrarily large number values].
*This **encoded** property would imply Theorem 2.1 Modified Polignac-Twin-Prime Conjecture*□.

Proposition 2.5. (Riemann Hypothesis). To minimize complexity, we need not consider here the in-depth analysis on special properties of Gram's Law and Rosser's Rule as this (in)action do not invalidate our proposition, conjecture and theorem on location of nontrivial zeros.

Let there be three mutually exclusive but dependent **Set nontrivial zeros**, **Set Gram[y=0] points** and **Set Gram[x=0] points** generated by Dirichlet eta function when $\sigma = \frac{1}{2}$. Let there be two mutually exclusive but dependent **Set virtual Gram[y=0] points** and **Set virtual Gram[x=0] points** generated by Dirichlet eta function when $\sigma \neq \frac{1}{2}$. Let ||Set nontrivial zeros|| = ||Set Gram[y=0] points|| = ||Set Gram[x=0] points|| = countably infinite large number in size that will linearly reach an infinity value. Let ||Set virtual Gram[y=0] points|| = ||Set virtual Gram[x=0] points|| = countably infinite large number in size that will linearly reach an infinity value.

As schematically depicted by Figure 3, the $\sigma = \frac{1}{2}$ critical line is located in, and bisect, the $0 < \sigma < 1$ critical strip into two equal regions $0 < \sigma < \frac{1}{2}$ and $\frac{1}{2} < \sigma < 1$. It is a breakthrough insight to correctly determine the ultimate significance arising from observing solitary [mathematical] $\sigma = \frac{1}{2}$ critical line [as 1-dimensional line] will precisely represent solitary [geometric] Origin point [as 0-dimensional point], while the infinitely many [mathematical] $\sigma \neq \frac{1}{2}$ non-critical lines that are located in $0 < \sigma < \frac{1}{2}$ or $\frac{1}{2} < \sigma < 1$ region will never represent the solitary [geometric] Origin point.

Proposition 2.6. Conjecture for Riemann hypothesis on location of all nontrivial zeros to be the critical line. The countably infinitely large number of nontirvial zeros that will linearly reach an infinity value are generated only when parameter $\sigma = \frac{1}{2}$ in Dirichlet eta function, and not when parameter $\sigma \neq \frac{1}{2}$ in Dirichlet eta function.

Since this then imply there is zero probability that any particular parameter $\sigma \neq \frac{1}{2}$ values that do occur in Dirichlet eta function will mathematically represent the $\sigma = \frac{1}{2}$ critical line [or geometrically represent the analogous $\sigma = \frac{1}{2}$ Origin point], we consequently deduce all countably infinitely large number of nontrivial zeros that linearly reach an infinity value as generated from Dirichlet eta function when parameter $\sigma = \frac{1}{2}$ will, by default, also have to be located on the $\sigma = \frac{1}{2}$ critical line. *This **encoded** property would imply Theorem 2.2 Riemann Hypothesis*□.

2.2 General notations

The following is a list of abbreviations used by this paper.
CP entities: Completely Predictable entities which will manifest CP *independent* properties.
IP entities: Incompletely Predictable entities which will manifest IP *dependent* properties.
ζ**(s)**: $f(n)$ Riemann zeta function containing variable n, and parameters t and σ will generate [via its *proxy* Dirichlet eta function] Zeroes when $\sigma = \frac{1}{2}$ and virtual Zeroes when $\sigma \neq \frac{1}{2}$.
η**(s)**: $f(n)$ Dirichlet eta function, which represents the analytic continuation of ζ(s), containing variable n, and parameters t and σ will generate Zeroes when $\sigma = \frac{1}{2}$ and virtual Zeroes when $\sigma \neq \frac{1}{2}$.
sim-η**(s)**: $f(n)$ simplified Dirichlet eta function, which is essentially derived by applying Euler formula to η(s), containing variable n, and parameters t and σ will generate Zeroes when $\sigma = \frac{1}{2}$ and virtual Zeroes when $\sigma \neq \frac{1}{2}$.
DSPL: $F(n)$ Dirichlet Sigma-Power Law = $\int sim - \eta(s)dn$ containing variable n, and parameters t and σ will generate Pseudo-zeroes when $\sigma = \frac{1}{2}$ and virtual Pseudo-zeroes when $\sigma \neq \frac{1}{2}$.
NTZ: nontrivial zeros or **G[x=0,y=0]P**: Gram[x=0,y=0] points = Origin intercept points located at the solitary-positioned zero-dimensional Origin point are generated by equation G[x=0,y=0]P-η(s) [containing exponent = $\frac{1}{2}$] when $\sigma = \frac{1}{2}$.
GP or G[y=0]P: 'usual' (or 'traditional') Gram points = Gram[y=0] points = x-axis intercept points located at the multiple-positioned one-dimensional x-axis line are generated by equation G[y=0]P-η(s) when $\sigma = \frac{1}{2}$. We deduce that Riemann hypothesis can also be usefully stated as none of the [additional] virtual G[x=0]P generated by equation G[x=0]P-η(s) when $\sigma \neq \frac{1}{2}$ – as demonstrated by Figure 10 for $\sigma = \frac{1}{3}$ – can be constituted by t values of transcendental numbers that [incorrectly] coincide with t values of transcendental numbers for NTZ when $\sigma = \frac{1}{2}$.
G[x=0]P: Gram[x=0] points = y-axis intercept points located at the multiple-positioned one-dimentional y-axis line are generated by equation G[x=0]P-η(s) when $\sigma = \frac{1}{2}$.
virtual NTZ: virtual nontrivial zeros or **virtual G[x=0,y=0]P**: virtual Gram[x=0,y=0] points. These are virtual Origin intercept points located at the multiple-positioned virtual Origin points which are generated by equation G[x=0,y=0]P-η(s) containing exponent values $\neq \frac{1}{2}$ when $\sigma \neq \frac{1}{2}$. We note that each virtual NTZ when $\sigma < \frac{1}{2}$ in Figure 5 equates to an [additional] negative virtual G[y=0]P located at IP varying positions on horizontal axis, and each virtual NTZ when $\sigma > \frac{1}{2}$ in Figure 6 equates to an [additional] positive virtual G[y=0]P located at IP varying positions on horizontal axis. We observe overall less virtual G[x=0]P when $\sigma > \frac{1}{2}$, and overall more virtual G[x=0]P when $\sigma < \frac{1}{2}$.

2.3 Probability theory applied to n-digit Primes and Composites

Probability = 100% X Proportion. Probability and Proportion are literally equivalent to each other for our analysis on prime and composite numbers (and nontrivial zeros). If the probability [range between 0 or 0% and 1 or 100%] of an event occurring is Y, then the probability [range between 0 or 0% and 1 or 100%] of the event not occurring is 1–Y. The odds of an event represent the ratio *Probability that the event will occur* : *Probability that the event will not occur*. This can be succinctly expressed as Odds of event = $\dfrac{Y}{1-Y}$.

Based on cardinality of (sub)sets of prime and composite numbers used in Prime-Composite quotient as outlined in subsection 1.3, we interpret their Probability or Proportion will satisfy the following statement:
$$\text{P(odd primes)} \approx \frac{1}{2 \cdot \text{P(Gap 1-even composites)} + 2 \cdot \text{P(Gap 1-odd composites)}}.$$

P(any number is divisible by a prime p, or in fact any integer) = $1/p$. Let there be k randomly chosen integers. When k = 2, P(two numbers are both divisible by p) = $1/p^2$, and P(at least one of the two numbers is not divisible by p) = $1 - 1/p^2$. Any finite collection of divisibility events associated to distinct primes is mutually independent. For example, in the case of two events, a number is divisible by primes p and q *iff* it is divisible by pq; the latter event has probability 1/pq. We make the heuristic assumption that such reasoning can be extended to infinitely many divisibility events. Then,
$$\text{P(two numbers are coprime)} = \prod_{\text{prime } p}\left(1 - \frac{1}{p^2}\right) = \left(\prod_{\text{prime } p}\frac{1}{1-p^{-2}}\right)^{-1} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \approx 0.607927102 \approx 61\%$$
– a product over all primes. More generally, P(k randomly chosen integers being coprime) = $\dfrac{1}{\zeta(k)}$.

The fundamental theorem of arithmetic asserts that every nonzero integer can be written as a product of primes in a unique way, up to ordering and multiplication by units. Prime numbers are defined as *All integers apart from 0 and 1 that are evenly divisible by itself and by 1.* Composite numbers are defined as *All integers apart from 0 and 1 that are evenly divisible by numbers other than itself and 1.* The integer numbers ($\mathbb{Z}$) = {0, 1, 2, 3, 4...}, prime numbers ($\mathbb{P}$) = {2, 3, 5, 7, 11...} and composite numbers ($\mathbb{C}$) = {4, 6, 8, 9, 10...} can all be analyzed in terms of their corresponding unique n-digit numbers.

n	0	1	2	3	4	5	6	7	8	9	10	.....
A006879$_n$	0	4	21	143	1061	8363	68906	586081	5096876	45086079	404204977	.....
A006880$_n$	0	4	25	168	1229	9592	78498	664579	5761455	50847534	455052511	.....

A006879 Number of primes with n digits. Number of primes between $10^{(n-1)}$ and 10^n (Sloane & Plouffe1, 1995 i). Using our unique n-digit $\mathbb{P}$ grouping, this statement is mathematically equivalent to Number of primes between $10^{(n-1)}$ and $10^n - 1$ since the integer 10^n itself can never be prime.

A006880 Number of primes < 10^n. Number of primes with at most n digits or Prime counting function $\mathbb{P}$-$\pi(< 10^n)$ defined as $\|\mathbb{P} < 10^n\|$ (Sloane & Plouffe1, 1995 ii). Using our unique n-digit $\mathbb{P}$ and n-digit $\mathbb{C}$ groupings, Prime counting function $\mathbb{P}$-$\pi(\leq 10^n - 1)$ is defined as $\|\mathbb{P} \leq 10^n - 1\|$; and Composite counting function $\mathbb{C}$-$\pi(\leq 10^n - 1)$ as $\|\mathbb{C} \leq 10^n - 1\|$.

The above two integer sequences A006879 and A006880 are directly related to our unique n-digit $\mathbb{P}$ and n-digit $\mathbb{C}$ groupings whereby n = 0, 1, 2, 3, 4... [to an arbitrarily large number]. A006880 forms the partial sums of A006879. Using n-digit $\mathbb{P}$ grouping, A006879 can be alternatively defined as *The number of primes between* $10^{(n-1)}$ *and* $10^n - 1$ *which supply precisely the original and identical A006879$_n$ as n-digit prime number values.* Only by employing similar crucial step of using n-digit $\mathbb{C}$ grouping *The number of composites between* $10^{(n-1)}$ *and* $10^n - 1$*, will we obtain the complementary-A006879$_n$ as n-digit composite number values.* There are precisely $10^n - 1$ minus $10^{(n-1)}$ plus $1 = 10^n - 10^{(n-1)}$ integer numbers between $10^{(n-1)}$ and $10^n - 1$. The important implication is that we are now always dealing with the same n-digit integer, prime and composite numbers whereby the relationship n-digit $\mathbb{Z}$ = n-digit $\mathbb{P}$ + n-digit $\mathbb{C}$ will always hold [except for when n = 1 because 0 and 1 are neither prime nor composite]. We note from A006879 and A006880 the number of primes that are still constituted by very large number values will proportionately decline in an accelerating manner with two progressively larger n values in $10^{(n-1)}$ and $10^n - 1$. Despite this phenomenon, one could aesthetically speculate there will always be many allocated primes to theoretically represent all even Prime gaps in the sequence of prime numbers.

For i = 1, 2, 3, 4, 5..., Set of $\mathbb{Z}_i$ {0, 1, 2, 3, 4...} as **CIS-IM-linear** = Set of neither $\mathbb{P}$ nor $\mathbb{C}$ {0, 1} as **CFS** + Set of $\mathbb{P}_i$ {2, 3, 5, 7, 11...} as **CIS-ALN-decelerating** + Set of $\mathbb{C}_i$ {4, 6, 8, 9, 10...} as **CIS-IM-accelerating**. All $\mathbb{P}$ are odd except for the first and only even $\mathbb{P}$ 2. There is only one solitary even $\mathbb{P}$ 2 and one solitary odd $\mathbb{P}$ 5 that are not $\mathbb{C}$. Otherwise, all $\mathbb{Z}$ with their last digit ending as even numbers 0, 2, 4, 6 or 8, or odd number 5 must always be $\mathbb{C}$. Apart from $\mathbb{P}$ 2 and $\mathbb{P}$ 5, all $\mathbb{P}$ have their last digit ending as odd numbers 1, 3, 7 or 9. But not all $\mathbb{Z}$ with their last digit ending as odd numbers 1, 3, 7 or 9 are $\mathbb{P}$ – in fact, these numbers are more likely to be $\mathbb{C}$ than $\mathbb{P}$. We deduce that for ≥2-digit numbers, (i) $\mathbb{C}$ can have their last digit ending in 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9 but (ii) $\mathbb{P}$ can only have their last digit ending in 1, 3, 7 or 9; and thus (iii) all $\mathbb{Z}$ with their last digit ending in 0, 2, 4, 5, 6 or 8 must be $\mathbb{C}$.

For n = 1, 2, 3, 4, 5...[to an arbitrarily large number]; we apply probability theory to the generated subsets of n-digit $\mathbb{P}$ as **CIS-ALN-decelerating** and n-digit $\mathbb{C}$ as **CIS-IM-accelerating**. With probability 1, all randomly selected $\mathbb{Z}$ that has its last digit ending with 0, 2, 4, 5, 6 or 8 must be *almost surely* $\mathbb{C}$. This is equivalently stated as: With probability 0, all randomly selected $\mathbb{Z}$ that has its last digit ending with 0, 2, 4, 5, 6 or 8 must be *almost never* $\mathbb{P}$. Thus, P(randomly selected $\mathbb{Z}$ is $\mathbb{C}$ with 100% certainty) = 0.6 [except for the isolated 1-digit $\mathbb{Z}$ 0 and 1-digit $\mathbb{P}$ 2 and 5 which are not $\mathbb{C}$]. The terms *almost surely* and *almost never* can now be replaced with *surely* and *never* when we disregard the 1-digit $\mathbb{P}$ and 1-digit $\mathbb{C}$. Since the condition "randomly selected $\mathbb{Z}$ that has its last digit ending with 0, 2, 4, 5, 6 or 8 must be *surely* $\mathbb{C}$" will always apply to any chosen subsets of ≥2-digit $\mathbb{Z}$, the consequently derived equivalent condition "60% of all $\mathbb{Z}$ being $\mathbb{C}$ with 100% certainty" will also always apply to these same subsets. Then, with 60% of all $\mathbb{Z}$ being $\mathbb{C}$ with 100% certainty, there are always more $\mathbb{C}$ than $\mathbb{P}$ for any chosen corresponding subsets of ≥2-digit $\mathbb{P}$ and ≥2-digit $\mathbb{C}$.

Constraints on Prime numbers and Prime gaps: We define Prime gap$_i$ = $\mathbb{P}_{i+1} - \mathbb{P}_i$. We ignore $\mathbb{P}_1 = 2$ and $\mathbb{P}_3 = 5$. We convey the paired list of (last digit for $\mathbb{P}_i$, last digit for $\mathbb{P}_{i+1}$) as full range of choices permissible for corresponding specified groupings of prime gaps:

CIS-ALN-decelerating $\mathbb{P}_i$ selected from Prime gap$_i$ = 2, 12, 22, 32...[to an arbitrarily large number as **CIS-ALN-decelerating**] → (1, 3), (7, 9), (9, 1). The last digit of $\mathbb{P}_i$ with prime gap having last digit ending in 2 cannot end in 3 or 5 but can end in 1, 7 or 9.

CIS-ALN-decelerating $\mathbb{P}_i$ selected from Prime gap$_i$ = 4, 14, 24, 34...[to an arbitrarily large number as **CIS-ALN-decelerating**] → (3, 7), (7, 1), (9, 3). The last digit of $\mathbb{P}_i$ with prime gap having last digit ending in 4 cannot end in 1 or 5 but can end in 3, 7 or 9.

CIS-ALN-decelerating $\mathbb{P}_i$ selected from Prime gap$_i$ = 6, 16, 26, 36...[to an arbitrarily large number as **CIS-ALN-decelerating**] → (1, 7), (3, 9), (7, 3). The last digit of $\mathbb{P}_i$ with prime gap having last digit ending in 6 cannot end in 5 or 9 but can end in 1, 3 or 7.

CIS-ALN-decelerating $\mathbb{P}_i$ selected from Prime gap$_i$ = 8, 18, 28, 38...[to an arbitrarily large number as **CIS-ALN-decelerating**] → (1, 9), (3, 1), (9, 7). The last digit of $\mathbb{P}_i$ with prime gap having last digit ending in 8 cannot end in 5 or 7 but can end in 1, 3 or 9.

CIS-ALN-decelerating $\mathbb{P}_i$ selected from Prime gap$_i$ = 10, 20, 30, 40...[to an arbitrarily large number as **CIS-ALN-decelerating**] → (1, 1), (3, 3), (7, 7), (9, 9). The last digit of $\mathbb{P}_i$ with prime gap having last digit ending in 0 cannot end in 5 but can end in 1, 3, 7 or 9.

The last digit of $\mathbb{P}_i$ ending in 1 is associated with prime gap having last digit ending in 0, 2, 6 or 8.

The last digit of $\mathbb{P}_i$ ending in 3 is associated with prime gap having last digit ending in 0, 4, 6 or 8.

The last digit of $\mathbb{P}_i$ ending in 7 is associated with prime gap having last digit ending in 0, 2, 4 or 6.

The last digit of $\mathbb{P}_i$ ending in 9 is associated with prime gap having last digit ending in 0, 2, 4 or 8.

We note **CIS-ALN-decelerating** $\mathbb{P}_i$ having Prime gap$_i$ [given as multiples of 10] with last digit ending in 0 is associated with four choices that are available arbitrarily often. Otherwise, **CIS-ALN-decelerating** $\mathbb{P}_i$ having Prime gap$_i$ with last digit ending in 2, 4, 6 or 8 is each associated with three choices that are available arbitrarily often. Statistically, the last digit of $\mathbb{Z}_i$ ending in 1, 3, 7 or 9 are more likely to be just $\mathbb{O}_i$ than [odd] $\mathbb{P}_i$. At ever larger range of numbers for the paired list of (last digit for $\mathbb{P}_i$, last digit for $\mathbb{P}_{i+1}$), we can intuitively surmise that $\mathbb{P}$ associated with progressively larger prime gaps moving from left to right and from top to bottom should occur relatively more often than $\mathbb{P}$ associated with comparatively smaller prime gaps. However, both $\mathbb{P}$ associated with progressively larger prime gaps and $\mathbb{P}$ associated with comparatively smaller prime gaps should generally occur less often at ever larger range of numbers. Thus, although prime gap having last digit ending in 0 can be associated with last digit of $\mathbb{P}_i$ ending in 1, 3, 7 or 9 as four choices [instead of just three choices]; these prime gaps as a unique group will still always constitute larger prime gaps that will overall intrinsically occur less often at ever larger range of numbers.

The crucial inference here is all known last digit of $\mathbb{P}_i$ ending in 1, 3, 7 or 9 that will represent all existing even Prime gaps must do so on the eternal basis thus confirming Polignac's and Twin prime conjectures to be true. Our modified Polignac's conjecture [involving all even Prime gaps 2, 4, 6, 8, 10...] together with its subset Twin prime conjecture [involving only even Prime gap 2] can now be categorically stated as: **||Set all even Prime gaps|| = ||Subset odd prime numbers associated with each even Prime gap|| = CIS-ALN-decelerating**. These two conjectures are traditionally stated together in a less informative manner as **||Set all even Prime gaps|| = ||Subset odd prime numbers associated with each even Prime gap|| = CIS**.

Constraints on Composite numbers and Composite gaps: We define Composite gap$_i$ = $\mathbb{C}_{i+1} - \mathbb{C}_i$. For 1-digit $\mathbb{P}$ and 1-digit $\mathbb{C}$ that are members of 1-digit $\mathbb{Z}$, there are always more $\mathbb{P}$ than $\mathbb{C}$ except at $\mathbb{Z}$ = 9 which is $\mathbb{C}$ and whereby now $||\mathbb{P}|| = ||\mathbb{C}||$ = 4. There will always be more $\mathbb{C}$ as **CIS-IM-accelerating** than $\mathbb{P}$ as **CIS-ALN-decelerating** when $\mathbb{Z} \geq 10$ with $||\mathbb{C}|| = 2\ ||\mathbb{P}||$ at $\mathbb{Z}$ = 14 and $||\mathbb{C}|| > 2\ ||\mathbb{P}||$ at $\mathbb{Z} > 14$.

Let P(certain $\mathbb{C}$) denote P(randomly selected $\mathbb{Z}$ is $\mathbb{C}$ with 100% certainty). Then, P(certain even $\mathbb{C}$) = 0.5 for all subsets ≥2-digit $\mathbb{C}$ having elements with their last digit ending in 0, 2, 4, 6 or 8 [as **CIS-IM-linear**] and P(certain odd $\mathbb{C}$) = 0.1 for subset ≥2-digit $\mathbb{C}$ having elements with their last digit ending in 5 [as **CIS-IM-linear**].

One can more closely analyze $\mathbb{P}$-$\mathbb{C}$ **identifier grouping** which was defined in section 1 Introduction. Event 1: P(uncertain even $\mathbb{C}$ with last digit ending in 0, 2, 6 or 8) [as **CIS-ALN-decelerating**] can [partially and orderly] represent Gap-2-E-$\mathbb{C}_1$ that always occur before every O-$\mathbb{P}_i$ with last digit ending in 1, 3, 7 or 9. Event 2: P(uncertain even $\mathbb{C}$ with last digit ending in 0, 2, 4 or 8) [as **CIS-IM-accelerating**] can [partially and orderly] represent Gap-1-E-$\mathbb{C}_2$ that always occur after every O-$\mathbb{P}_i$ with last digit ending in 1, 3, 7 or 9. Event 3: P(uncertain odd $\mathbb{C}$ with last digit ending in 1, 3, 7 or 9) [as **CIS-IM-accelerating**] can [partially and orderly] represent, and P(certain odd $\mathbb{C}$ with last digit ending in 5) [as **CIS-IM-linear**] can [totally and orderly] represent, Gap-1-O-$\mathbb{C}_3$ that always occur after Gap-1-E-$\mathbb{C}_2$. Event 4: P(uncertain even $\mathbb{C}$ with last digit ending in 0, 2, 4, 6 or 8) [as **CIS-IM-accelerating**] can [partially and orderly] represent Gap-1-E-$\mathbb{C}_4$ that always occur after Gap-1-O-$\mathbb{C}_3$... until P(uncertain even $\mathbb{C}$ with last digit ending in 0, 2, 6 or 8) [as **CIS-ALN-decelerating**] can [partially and orderly] represent Gap-2-E-$\mathbb{C}_n$ that always occur before every O-$\mathbb{P}_{i+1}$ with last digit ending in 1, 3, 7 or 9. Only Event 1 and Event 2 can occur for twin primes. In Event 2, there are four choices for Gap-1-E-$\mathbb{C}$ as opposed to Event 4 whereby there are, instead, five choices for Gap-1-E-$\mathbb{C}$.

The conjecture on all Subsets of Prime numbers derived from Set of even Prime gaps manifesting cardinality **CIS-ALN-decelerating** can be algorithmatically proven by deriving above correct and complete mathematical arguments that led to **$\mathbb{P}$-$\mathbb{C}$ identifier grouping** which must also obey *Constraints on Prime numbers and Prime gaps* and *Constraints on Composite numbers and Composite gaps*. Computed data on n-digit prime numbers including their average prime gaps are supplied in Appendix G.

Based on last digit ending in even or odd numbers, the condition P(uncertain even and odd $\mathbb{C}$ with odd composite gap 1) + P(uncertain even $\mathbb{C}$ with even composite gap 2) + P(uncertain odd $\mathbb{P}$) = 0.4 <=> P(uncertain even and odd $\mathbb{C}$ with odd com-

posite gap 1) + 2 P(uncertain odd ℙ) = 0.4. Now 2·P(uncertain odd ℙ) = $\frac{1}{\text{P(uncertain even and odd } \mathbb{C} \text{ with odd composite gap 1)}}$.
Based on imaginary unit i, the following are derived abstract statements confirming P(uncertain even and odd ℂ) = 2·P(uncertain odd ℙ) that are interesting but likely have no known mathematical correlation to the real physical world:
(i) P(uncertain even and odd ℂ with odd composite gap 1) + $\frac{1}{\text{P(uncertain even and odd } \mathbb{C} \text{ with odd composite gap 1)}}$ = 0.4. Let x = P(uncertain even and odd ℂ with odd composite gap 1) $\implies$ $10x^2 - 4x + 10 = 0$. Solving this quadratic equation: x = $\frac{1 \pm \sqrt{24}i}{5}$ whereby $i = \sqrt{-1}$
(ii) $\frac{1}{2 \cdot \text{P(uncertain odd } \mathbb{P})}$ + 2 P(uncertain odd ℙ) = 0.4. Let y = P(uncertain odd ℙ) $\implies$ $40y^2 - 8y + 10 = 0$. Solving this quadratic equation: y = $\frac{1 \pm \sqrt{24}i}{10}$ whereby $i = \sqrt{-1}$

P(uncertain even and odd ℂ with odd composite gap 1) ≥ 1 P(uncertain even ℂ with even composite gap 2) can only occur when P(uncertain even and odd ℂ with odd composite gap 1) ≥ 2/15 and P(uncertain odd ℙ) ≤ 2/15, (ii) P(uncertain even and odd ℂ with odd composite gap 1) ≥ 2 P(uncertain even ℂ with even composite gap 2) can only occur when P(uncertain even and odd ℂ with odd composite gap 1) ≥ 4/15 and P(uncertain odd ℙ) ≤ 1/15, (iii) P(uncertain even and odd ℂ with odd composite gap 1) ≥ 3 P(uncertain even ℂ with even composite gap 2) can only occur when P(uncertain even and odd ℂ with odd composite gap 1) ≥ 14/45 and P(uncertain odd ℙ) ≤ 2/45, etc. In general, for n = 1, 2, 3, 4, 5...; P(uncertain even and odd ℂ with odd composite gap 1) ≥ n P(uncertain even ℂ with even composite gap 2) can ***practically*** and ***dynamically*** occur only when P(uncertain even and odd ℂ with odd composite gap 1) ≥ 2/5 – 4/(15n) and P(uncertain odd ℙ) ≤ 2/(15n).

Some of the properties of co-prime numbers are as follows. 1 is coprime with every number. Any two prime numbers are coprime to each other: As every prime number has only two factors 1 and the number itself, the only common factor of two prime numbers will be 1. For example, 2 and 3 are two prime numbers. Factors of 2 are 1, 2, and factors of 3 are 1, 3. The only common factor is 1 and hence they are coprime. Any two successive numbers / integers are always coprime: Take any consecutive numbers such as 2, 3, or 3, 4 or 5, 6, and so on; they have 1 as their highest common factor (HCF). The sum of any two coprime numbers are always coprime with their product: 2 and 3 are coprime and have 5 as their sum (2+3) and 6 as the product (2X3). Hence, 5 and 6 are coprime to each other. Two even numbers can never form a coprime pair as all the even numbers have a common factor as 2. If two numbers have their unit digits as 0 and 5, then they are not coprime to each other. For example 10 and 15 are not coprime since their HCF is 5 (or divisible by 5).

We know that coprime numbers are numbers whose HCF is 1; i.e., two numbers whose common factor is 1 only are called coprime numbers. On the other hand, twin primes are prime numbers whose difference is always equal to 2. For example, the difference between 3 and 5 is 2, and hence 3 and 5 are twin primes. The major contrasting features between twin primes and coprime numbers are as follows. The difference between two twin primes is always equal to 2, whereas the difference between two coprime numbers can be any number. Twin primes are always prime numbers, whereas coprime numbers can also be composite numbers. It is a mathematical impossibility that $P_{i+1} - P_i$ be constituted by a random integer with last digit ending in 3 and 7. Thus, twin prime (Gap 2) can only arise with last digit combination (1, 3), (7, 9), (9, 1). In general [as previously shown above]:
Gap 2 can end with last digit combinations as (1, 3), (7, 9), (9, 1)
Gap 4 can end with last digit combinations as (3, 7), (7, 1), (9, 3)
Gap 6 can end with last digit combinations as (1, 7), (3, 9), (7, 3)
Gap 8 can end with last digit combinations as (1, 9), (3, 1), (9, 7)
Gap 10 can end with last digit combinations as (1, 1), (3, 3), (7, 7), (9, 9)
Gap 12 can end with last digit combinations as (1, 3), (7, 9), (9, 1)
...repeating cycles...

We outline the following interesting concepts from formal language theory (FLT). A string a is a subsequence of another string b, if a can be obtained from b by deleting zero or more of the characters in b. Example, 517 is a substring of 251667. The empty string is a subsequence of every string. Two strings a and b are comparable if either a is a substring of b, or b is a substring of a. A string a in a set of strings S is minimal if whenever b (an element of S) is a substring of a, we have $b = a$. Then, every set of pairwise incomparable strings is finite (Lothaire, 1983). Consequently, from any set of strings we can find its minimal elements.

Prime numbers are defined as *All Natural numbers apart from 1 that are evenly divisible by itself and by 1.* Every prime number, when written in base ten, has one of the following [finite] 26 primes as a substring: 2, 3, 5, 7, 11, 19, 41, 61, 89,

x	P_i/C_i Gaps	ΣPC_x-Gaps	Dim	x	P_i/C_i Gaps	ΣPC_x-Gaps	Dim
1	N/A	0	2x-2	33	C21, 1	58	2x-8
2	P1, 1	0	2x-4	34	C22, 1	59	2x-9
3	P2, 2	1	2x-5	35	C23, 1	60	2x-10
4	C1, 2	1	Y	36	C24, 2	61	2x-11
5	P3, 2	3	Y	37	P12, 4	67	Y
6	C2, 2	5	Y	38	C25, 1	69	Y
7	P4, 4	7	Y	39	C26, 1	70	2x-8
8	C3, 1	9	Y	40	C27, 1	71	2x-9
9	C4, 1	10	2x-8	41	P13, 2	75	Y
10	C5, 2	11	2x-9	42	C28, 2	77	Y
11	P5, 2	15	Y	43	P14, 4	79	Y
12	C6, 2	17	Y	44	C29, 1	81	Y
13	P6, 4	19	Y	45	C30, 1	82	2x-8
14	C7, 1	21	Y	46	C31, 2	83	2x-9
15	C8, 1	22	2x-8	47	P15, 6	87	Y
16	C9, 1	23	2x-9	48	C32, 1	89	Y
17	P7, 2	27	Y	49	C33, 1	90	2x-8
18	C10, 2	29	Y	50	C34, 1	91	2x-9
19	P8, 4	31	Y	51	C35, 1	92	2x-10
20	C11, 1	33	Y	52	C36, 1	93	2x-11
21	C12, 1	34	2x-8	53	P16, 6	99	Y
22	C13, 2	35	2x-9	54	C37, 1	101	Y
23	P9, 6	39	Y	55	C38, 1	102	2x-8
24	C14, 1	41	Y	56	C39, 1	103	2x-9
25	C15, 1	42	2x-8	57	C40, 1	104	2x-10
26	C16, 1	43	2x-9	58	C41, 1	105	2x-11
27	C17, 1	44	2x-10	59	P17, 2	111	Y
28	C18, 2	45	2x-11	60	C42, 2	113	Y
29	P10, 2	51	Y	61	P18, 6	115	Y
30	C19, 2	53	Y	62	C43, 1	117	Y
31	P11, 6	55	Y	63	C44, 1	118	2x-8
32	C20, 1	57	Y	64	C45, 1	119	2x-9

Legend: C = composite, **P** = prime, Dim = Dimension, Y = 2x - 7, N/A = Not Applicable.

Table 4. Prime-Composite finite scale mathematical (tabulated) landscape for x = 2 to 64.

409, 449, 499, 881, 991, 6469, 6949, 9001, 9049, 9649, 9949, 60649, 666649, 946669, 60000049, 66000049, 66600049. Algorithmically defined using above concepts from FLT (Shallit, 1999-2000), we named these 26 primes the *Minimal set of prime-strings in base 10* or *Minimal primes*.

Composite numbers are defined as *All Natural numbers apart from 1 that are evenly divisible by numbers other than itself and 1*. Every composite number, when written in base ten, has one of the following [finite] 32 composites as a substring: 4, 6, 8, 9, 10, 12, 15, 20, 21, 22, 25, 27, 30, 32, 33, 35, 50, 51, 52, 55, 57, 70, 72, 75, 77, 111, 117, 171, 371, 711, 713, 731. Algorithmically defined using above concepts from FLT, we named these 32 composites the *Minimal set of composite-strings in base 10* or *Minimal composites*.

3. Anatomy of Prime-Composite Varying Loop

Figure 7 geometrically depict Incompletely Predictable Prime-Composite Varying Loops is provided. This allows visual representation of two algorithms in action; viz, Sieve-of-Eratosthenes algorithm that generate all primes and Complement-Sieve-of-Eratosthenes algorithm that generate all composites. The tabulated and graphed Prime-Composite finite scale mathematical landscape, and its derivation are also provided.

Let **N** = natural numbers, **P** = prime numbers, and **C** = composite numbers. Based on our innovative Dimension (2x - N) system with N = 2x - ΣPC_x-Gap and x = all integers commencing from 1; Dimension (2x - N) when expanded is numerically just equal to ΣPC_x-Gap since Dimension (2x - N) = 2x - 2x + ΣPC_x-Gap = ΣPC_x-Gap. Definition for this system is explained using position x = 31 and 32. For i and x ∈ **N** [as per data in Table 4]; ΣPC_x-Gap = ΣPC_{x-1}-Gap

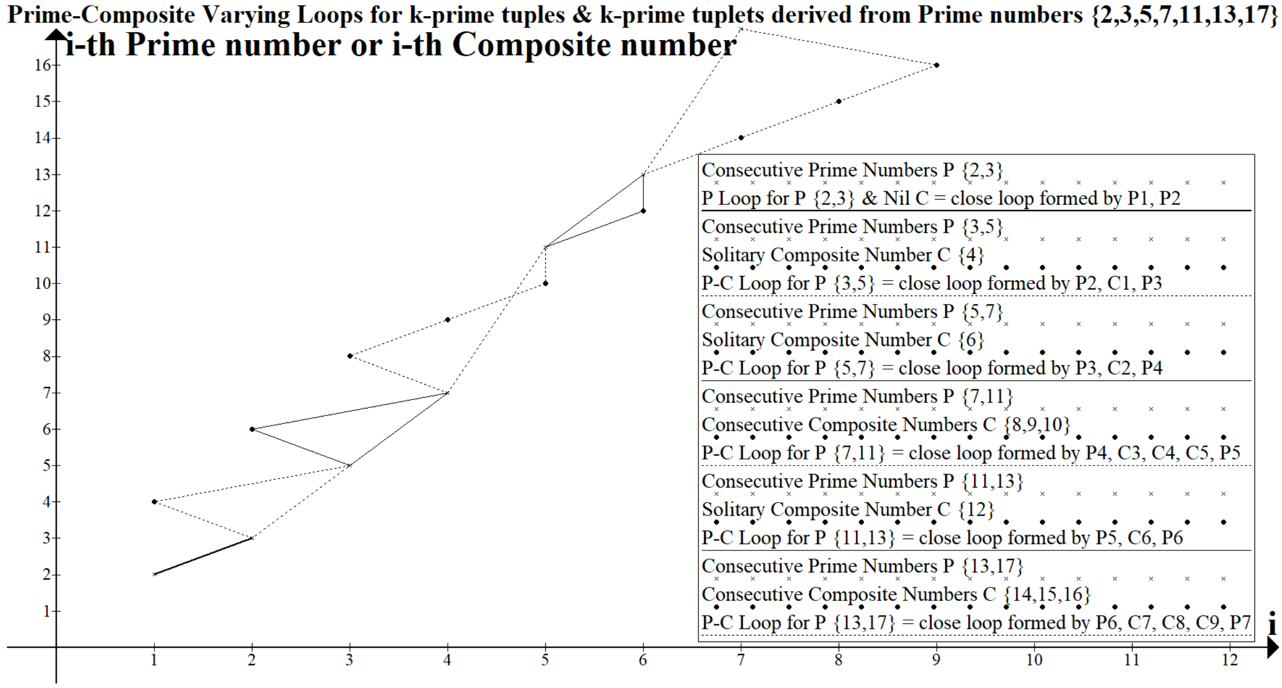

Figure 7. Prime-Composite Varying Loops. This figure is a geometric representation of prime and composite numbers computed for prime numbers {2, 3, 5, 7, 11, 13 and 17}.

+ Gap value at $\mathbf{P}_{i-1}$ or Gap value at $\mathbf{C}_{i-1}$ whereby (i) $\mathbf{P}_i$ or $\mathbf{C}_i$ at position x is determined by whether relevant x value belongs to a **P** or **C**, and (ii) both ΣPC_1-Gap and ΣPC_2-Gap = 0. Example, for position x = 31: 31 is **P** (**P11**). Desired Gap value at **P10** = 2. ΣPC_{31}-Gap (55) = ΣPC_{30}-Gap (53) + Gap value at **P10** (2). Example, for position x = 32: 32 is **C** (**C20**). Desired Gap value at **C19** = 2. ΣPC_{32}-Gap (57) = ΣPC_{31}-Gap (55) + Gap value at **C20** (2).

Plus-Minus Gap 2 Composite Number Alternating Law refers to *rhythmic patterns of alternating presence and absence* for relevant Gap 2 Composite Numbers. Mathematically, it has built-in intrinsic mechanism to automatically generate all prime numbers from prime gaps ≥ 4 appearances in a consistent *ad infinitum* manner. Plus Gap 2 Composite Number Continuous Law refers to *(non-)rhythmic patterns with continual presence* for relevant Gap 2 Composite Numbers. Mathematically, it has built-in intrinsic mechanism to automatically generate all prime numbers from prime gap = 2 appearances in a consistent *ad infinitum* manner. These two deduced Laws ***that crucially involve both prime and composite numbers being dependently and algorithmically tabulated together with subsequent analysis on their [combined] corresponding gaps*** will qualitatively confirm Polignac's and Twin prime conjectures to be true.

4. Anatomy of Nontrivial Zeros-Gram Points Varying Loop

Let Origin intercept point = nontrivial zero (or NTZ) = Gram[x=0,y=0] point (or G[x=0,y=0]P); x-axis intercept point = Gram[y=0] point (or G[y=0]P aka the *'usual'* / *'traditional'* Gram point); and y-axis intercept point = Gram[x=0] point (or G[x=0]P). We follow the peculiar choice of the index n used for Gram points and NTZ [depicted in order of their initial appearances for $\sigma = \frac{1}{2}$ and positive t values]: n = –3 for 1^{st} –ve G[y=0]P, n = –1 for 1^{st} –ve G[x=0]P, n = –2 for 2^{nd} +ve G[y=0]P, n = –1 for 3^{rd} +ve G[y=0]P, n = 1 for 1^{st} NTZ, n = 0 for 2^{nd} +ve G[x=0]P, n = 0 for 4^{th} +ve G[y=0]P, n = 1 for 3^{rd} –ve G[x=0]P, n = 2 for 2^{nd} NTZ, n = 1 for 5^{th} +ve G[y=0]P, n = 3 for 3^{rd} NTZ, n = 2 for 4^{th} +ve G[x=0]P, n = 2 for 6^{th} +ve G[y=0]P, n = 3 for 5^{th} –ve G[x=0]P, and so on. Thus, we observe different varieties of Nontrivial Zeros-Gram Points Varying Loops commencing from 1^{st} NTZ: (A) NTZ, +ve G[x=0]P, +ve G[y=0]P, –ve G[x=0]P, NTZ; (B) NTZ, +ve G[y=0]P, NTZ; (C) NTZ, +ve G[x=0]P, +ve G[y=0]P, –ve G[x=0]P, NTZ; (D) NTZ, +ve G[y=0]P, NTZ; (E) NTZ, +ve G[x=0]P, +ve G[y=0]P, –ve G[x=0]P, NTZ; (F) NTZ, +ve G[y=0]P, –ve G[x=0]P, NTZ; (G) NTZ, +ve G[y=0]P, (H) NTZ; (I) NTZ, +ve G[x=0]P, +ve G[y=0]P, –ve G[x=0]P, NTZ; etc.

We geometrically depict $\sigma = \frac{1}{2}$ as Gram points in Figure 9, Close-up view of virtual Origin points when $\sigma = \frac{1}{3}$ in Figure 10, and Simulated dynamic trajectories showing Origin intercept points when $\sigma = \frac{1}{2}$ and virtual Origin intercept points when $\sigma = \frac{2}{5}$ and $\sigma = \frac{4}{5}$ in Figure 11. As demonstrated in Figure 11, two different trajectories as specified by two different σ values will always form two colinear lines (co-lines) [**which is conveniently defined as two parallel curved lines that will never cross over**]. In particular, we crucially note the unique trajectory formed by solitary $\sigma = \frac{1}{2}$ value will also

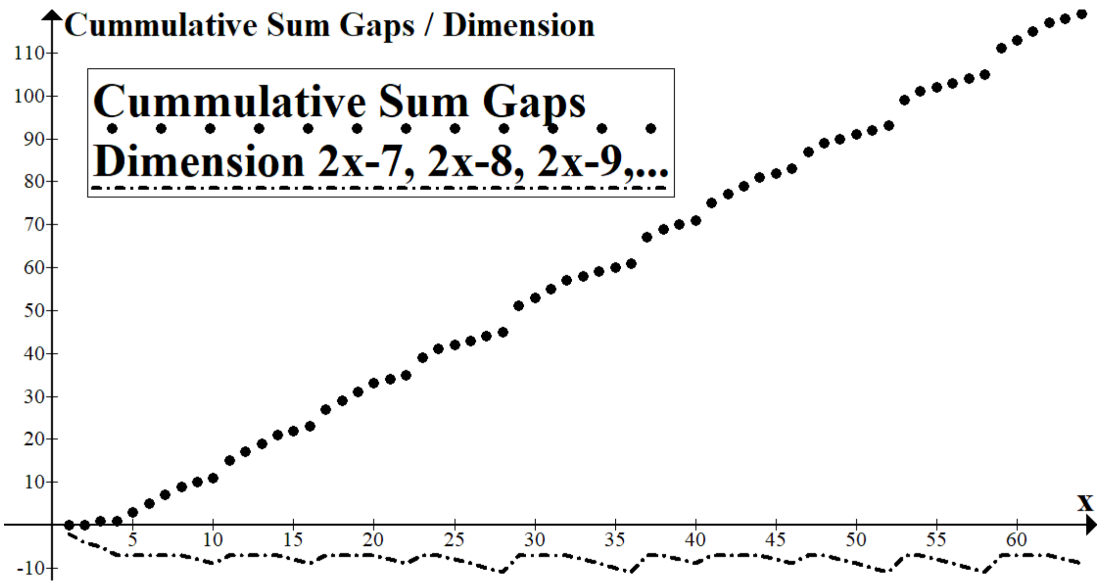

Figure 8. Prime-Composite finite scale mathematical (graphed) landscape for x = 2 to 64.

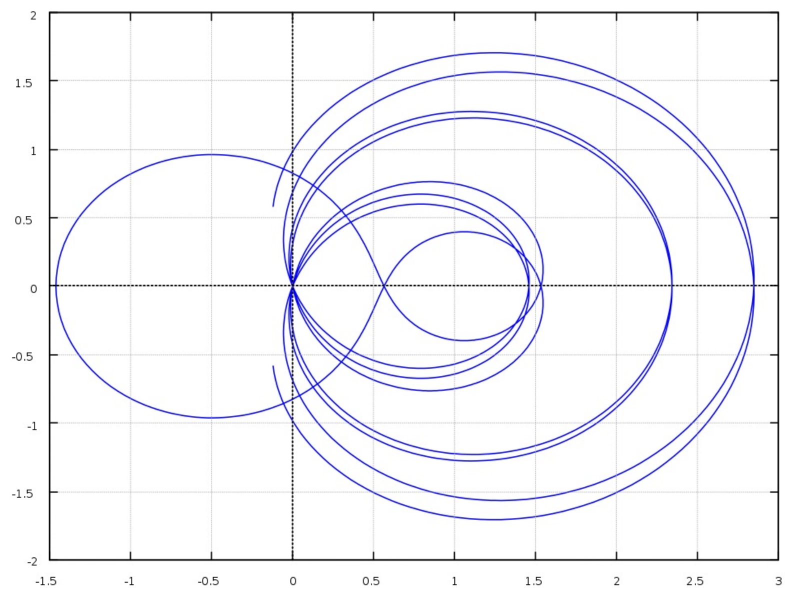

Figure 9. OUTPUT for $\sigma = \frac{1}{2}$ as Gram points. Polar graph of $\zeta(\frac{1}{2} + \imath t)$ plotted along critical line for real values of t running between -30 and 30, horizontal axis: $Re\{\zeta(\frac{1}{2} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{1}{2} + \imath t)\}$. Total presence of all Origin intercept points at the Origin. Perfect symmetry about the horizontal axis. At a Gram[y=0] point g_n, $\zeta\left(\frac{1}{2} + ig_n\right) = \cos(\theta(g_n))Z(g_n) = (-1)^n Z(g_n)$, and if this is positive at two successive Gram[y=0] points, $Z(t)$ must have a Origin intercept point (nontrivial zero) in the interval. We note the θ-function oscillates for absolute-small real arguments and therefore is not uniquely invertible in the interval [–24,24]. Thus the odd θ-function has its symmetric negative Gram[y=0] point with value 0 at index –3.

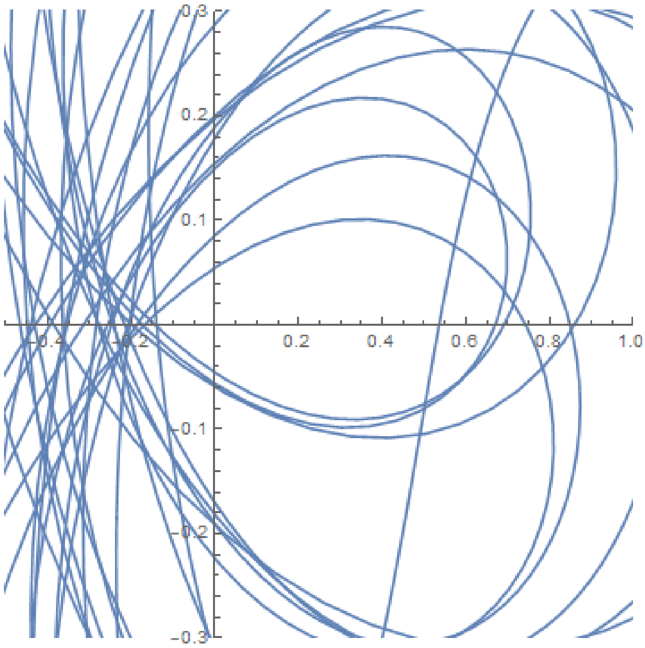

Figure 10. Close-up view of virtual Origin points when $\sigma = \frac{1}{3}$. OUTPUT for $\sigma = \frac{1}{3}$ [$\sigma < \frac{1}{2}$ situation] as virtual Gram points. Polar graph of $\zeta(\frac{1}{3} + \imath t)$ plotted along non-critical line for real values of t running between 0 and 100, horizontal axis: $Re\{\zeta(\frac{1}{2} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{1}{2} + \imath t)\}$. Total absence of all Origin intercept points at "static" Origin point. Total presence of all virtual Origin intercept points (as additional negative virtual Gram[y=0] points on x-axis) at "varying" [infinitely many] virtual Origin points.

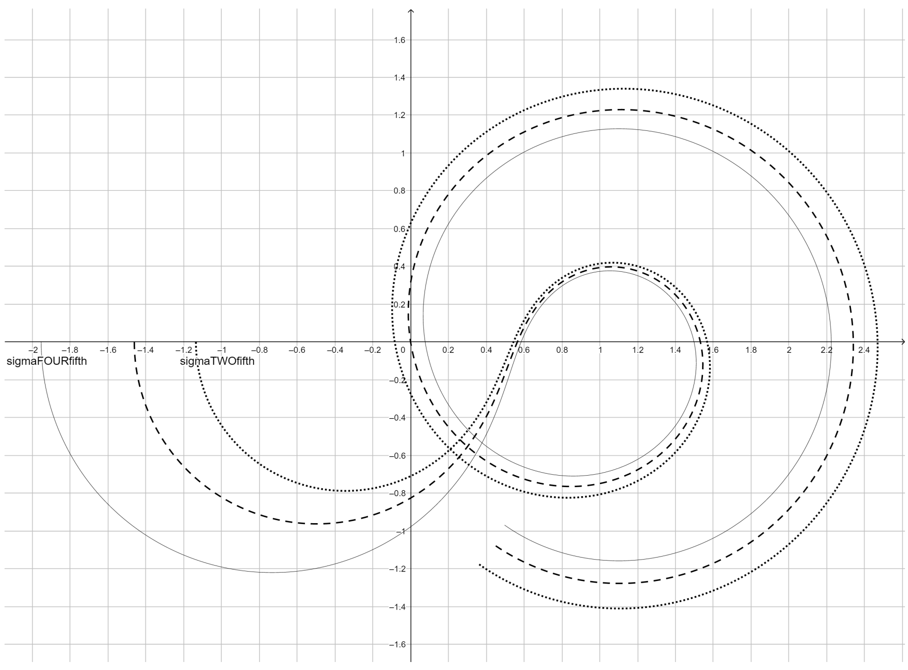

Figure 11. Simulated dynamic trajectories showing Origin intercept points when $\sigma = \frac{1}{2}$ and virtual Origin intercept points when $\sigma = \frac{2}{5}$ and $\sigma = \frac{4}{5}$. Horizontal axis: $Re\{\zeta(\sigma + \imath t)\}$, and vertical axis: $Im\{\zeta(\sigma + \imath t)\}$. Total absence of all Origin intercept points at the [static] Origin point. Total presence of all virtual Origin intercept points (as additional negative virtual Gram[y=0] points on the x-axis) at the [infinitely many varying] virtual Origin points; viz, these negative virtual Gram[y=0] points on the x-axis cannot exist at the solitary Origin point since the two trajectories form two co-lines.

always form co-lines with other trajectories formed by any of arbitrarily chosen $\sigma \neq \frac{1}{2}$ values. Since only the trajectory formed by $\sigma = \frac{1}{2}$ value will intersect with Origin point thus giving rise to Origin intercept points [nontrivial zeros], all other trajectories formed by $\sigma \neq \frac{1}{2}$ values will never intersect with Origin point.

In Figure 10 for the $\sigma = \frac{1}{3}$ [$\sigma < \frac{1}{2}$ situation], there are relatively more virtual Gram[x=0] points existing as y-axis intercept points. On the contrary $\sigma > \frac{1}{2}$ situation e.g. when $\sigma = \frac{2}{3}$, there will instead be virtual Origin intercept points (as additional positive virtual Gram[y=0] points on x-axis) at the "varying" [infinitely many] virtual Origin points with relatively less virtual Gram[x=0] points existing as y-axis intercept points. Then proof for Riemann hypothesis can be stated as fulfilling two conditions: The position of Origin point when $\sigma = \frac{1}{2}$ is uniquely a solitary point, and the positions of virtual Origin points for any σ values when $\sigma \neq \frac{1}{2}$ are non-uniquely infinitely many points but these cannot include the position of Origin point.

The Incompletely Predictable Nontrivial zeros-Gram points Varying Loops (NTZ-GP VL), indicating NTZ gaps as geometrically depicted in Figure 4, are dynamically defined by the line tracing joining n^{th} NTZ to $(n+1)^{th}$ NTZ with the [solitary] Origin point acting as the unique $\sigma = \frac{1}{2}$-Attractor. The four boundaries in a usual NTZ-GP VL on the short range scale will typically consist of the two sequential patterns n^{th} NTZ, then a [alternatingly] positive and negative G[x=0]P (or *vice versa*), then a positive G[y=0]P, and finally $(n+1)^{th}$ NTZ. The area enclosed by each NTZ-GP VL can be obtained by integrating the relevant equation for each Varying Loop in interval from 0π to 2π.

5. Conclusions

Treated as *Incompletely Predictable problems*, we provide a comparatively elementary algorithm-type proof for Polignac's and Twin prime conjectures. This statement can be conceptually stated as Plus-Minus Gap 2 Composite Number Alternating Law and Plus Gap 2 Composite Number Continuous Law that are applicable on the finite (small) scale, are also applicable on the infinite (large) scale. There is zero probability that any particular prime gaps from eternal repeated groupings of small and/or large prime gaps that faithfully generate all the countably arbitrarily large number of odd primes (colloquially in one Complex Container) will abruptly terminate or disappear.

The *strong* principle argument is [full] presence of DA homogeneity in cardinality of Odd Primes as CIS-ALN-decelerating for all even Prime gaps equates to complete subsets of Odd Primes whereas [partial] presence of DA non-homogeneity in one or more cardinality of Odd Primes being CFS will not equate to complete subsets of Odd Primes. One could also advocate for a *weak* principle argument supporting DA homogeneity for cardinality of Odd Primes in that nature should not *(dis)favor* any particular cardinality to be CFS; and therefore DA non-homogeneity simply cannot exist for these cardinality. The *Law of Continuity* is a heuristic principle *whatever succeed for the [small] finite, also succeed for the [large] infinite*. This principle is categorically applicable to p_1 that represents all the arbitrarily large number of primes 2, 3, 5, 7, 11, 13... since we importantly recognize under Proposition 1.2 and Box 1 in Appendix D that, for example, the multiplicative group of integers modulo p_1 can always be used to uniquely generate relevant admissible k-tuples and inadmissible (k+1)-tuples for each and every small or large p_1 values.

Treated as *Incompletely Predictable problems*, we provide a comparatively elementary equation-type proof on Riemann hypothesis while explaining the existence of mutually exclusive three types of Gram points (colloquially in three separate Complex Containers) and two types of virtual Gram points (colloquially in two separate Complex Containers). We conduct appropriate analysis to obtain complex properties present in (i) Dirichlet Sigma-Power Law, Gram[y=0] points-Dirichlet Sigma-Power Law and Gram[x=0] points-Dirichlet Sigma-Power Law that give rise to relevant Pseudo-Gram points; and in (ii) virtual Gram[y=0] points-Dirichlet Sigma-Power Law and virtual Gram[x=0] points-Dirichlet Sigma-Power Law that give rise to relevant virtual Pseudo-Gram points. There is zero probability that any of the countably infinitely many nontrivial zeros can be located away from [geometric] Origin point, which is equivalent to [mathematical] critical line.

With methods historically based on blends of computational, analytic, algebraic and geometric number theory; the geometrical-mathematical unified approach used in our proofs is analogically similar to the algebra-geometry unified approach of geometric Langlands program formalized by Professor Peter Scholze and Professor Laurent Fargues (2021). Our Algebra and Number Theory achievements represent solving *colloquially, from outside the Complex Containers*, overall complex (meta-)properties on Incompletely Predictable problems on prime numbers and nontrivial zeros.

Dedicated research is required to determine many interesting [unsolved] complex properties *colloquially, from inside the Complex Containers* at finer level e.g., on our proposed *Gram's Prime k-tuple law for k ≥ 6 values* (in Remark 1.4) and *Rosser's Prime k-tuple rule for k ≥ 2 values* (in Remark 1.5). Finally, the p-adic number system for any prime number p extends ordinary arithmetic of rational numbers in a different way from extension of the rational number system to real and complex number systems. The p-adic Riemann zeta function $\zeta_p(s)$ is a function analogous to Riemann zeta function $\zeta(s)$, or more general L-functions, but whose domain and target are p-adic. Its values at negative odd integers are those of Riemann zeta function at negative odd integers (up to an explicit correction factor).

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Appendix

A. Gram's Law and Rosser's Rule

Named after Danish mathematician Jørgen Pedersen Gram (June 27, 1850 – April 29, 1916), ['traditional' / 'usual'] Gram points or (mathematical) Gram[y=0] points or (geometrical) x-axis intercept points are other conjugate pairs values in Riemann zeta function $\zeta(s)$ on $\sigma = \frac{1}{2}$ critical line. Then s = $\frac{1}{2} + \imath t$ gives rise to $\zeta(\frac{1}{2} + \imath t)$ on critical line; and Gram points when defined in terms of $\zeta(s)$ is given by $\sum ReIm\{\zeta(s)\} = Re\{\zeta(s)\} + 0$, or simply $\text{Im}\{\zeta(s)\} = 0$. Alternatively defined using expression denoting $\zeta(s)$ on critical line $\zeta(\frac{1}{2} + \imath t) = Z(t)e^{-\imath\theta(t)}$ whereby Hardy's function, Z, is real for real t, and θ is Riemann–Siegel theta function given in terms of gamma function as $\theta(t) = \arg\left(\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right) - \frac{\ln\pi}{2}t$ for real values of t; we note that $\zeta(s)$ is real when $\sin(\theta(\text{t})) = 0$. This implies that θ(t) is an integer multiple of π which allows for location of Gram points to be calculated easily by inverting the formula for θ. Gram points are historically [crudely] numbered as g_n for n = 0, 1, 2, 3,..., whereby g_n is the unique solution of $\theta(\text{t}) = \text{n}\pi$. Here, n = 0 is the [first] g_0 value of 17.8455995405... which is larger than the smallest [first] positive nontrivial zeros (NTZ) value of 14.13472515.... Thus, n = -3 correspond to $\text{g}_{-3} = 0$, n = -2 correspond to $\text{g}_{-2} = 3.4362182261...$, and n = -1 correspond to $\text{g}_{-1} = 9.6669080561....$

Paired [infinite-length] integer sequences with prestigious connections:
A100967+0, which is A100967 (Noe, 2004), is precisely defined as "Least k such that binomial(2k+1, k-n-1) ≥ binomial(2k, k) viz. (2k+1)!k!k! ≥ (2k)!(k-n-1)!(k+n+2)!". The terms commencing from Position 0, 1, 2, 3,... of A100967+0 are listed below: 3, 9, 18, 29, 44, 61, 81, 104, 130, 159, 191, 225, 263, 303, 347, 393, 442, 494, 549, 606, 667, 730, 797, 866, 938, 1013, 1091, 1172, 1255, 1342, 1431, 1524, 1619, 1717, 1818, 1922, 2029, 2138, 2251, 2366, 2485, 2606, 2730, 2857, 2987, 3119, 3255, 3394, 3535,....

A100967+1 is precisely defined as "Add 1 to each and every terms from A100967+0". The terms commencing from Position 0, 1, 2, 3,... of A100967+1 are listed below: 4, 10, 19, 30, 45, 62, 82, 105, 131, 160, 192, 226, 264, 304, 348, 394, 443, 495, 550, 607, 668, 731, 798, 867, 939, 1014, 1092, 1173, 1256, 1343, 1432, 1525, 1620, 1718, 1819, 1923, 2030, 2139, 2252, 2367, 2486, 2607, 2731, 2858, 2988, 3120, 3256, 3395, 3536,....

A228186 (Ting, 2013) is defined as "Greatest natural number k > n such that calculated peak values for ratio R = $\frac{\text{Combinations With Repetition}}{\text{Combinations Without Repetition}} = \frac{(k+n-1)!(n-k)!}{n!(n-1)!}$ belong to maximal rational numbers < 2". It is also defined as "Smallest natural number k > n such that (k+n+1)!(k-n-2)! < 2k!(k-1)!". The terms commencing from Position 0, 1, 2, 3,... of A228186 are listed below: 4, 9, 18, 29, 44, 61, 81, 104, 130, 159, 191, 226, 263, 304, 347, 393, 442, 494, 549, 607, 667, 731, 797, 866, 938, 1013, 1091, 1172, 1256, 1342, 1432, 1524, 1619, 1717, 1818, 1922, 2029, 2139, 2251, 2367, 2485, 2606, 2730, 2857, 2987, 3120, 3255, 3394, 3535,....

Unexpected connection [and unrelated to NTZ and Gram points]: A228186 can be considered an innovative [infinite-length] *"Hybrid integer sequence"* identical to *"non-Hybrid integer sequence"* A100967+0 except for the interspersed [finite] 21 'exceptional' terms located at Position 0, 11, 13, 19, 21, 28, 30, 37, 39, 45, 50, 51, 52, 55, 57, 62, 66, 70, 73, 77, and 81 with their corresponding 21 values exactly specified by [infinite-length] *"non-Hybrid integer sequence"* A100967+1.

A114856-"bad"-Gram-points, which is A114856 (Weisstein, 2006), is precisely defined as "Indices n of Gram points g_n for which $(-1)^n\text{Z}(\text{g}_n) < 0$ with Z(t) being Riemann-Siegel Z-function [and full given range of values n = 0, 1, 2, 3,...]". The terms of A114856-"bad"-Gram-points are: 126, 134, 195, 211, 232, 254, 288, 367, 377, 379, 397, 400, 461, 507, 518, 529, 567, 578, 595, 618, 626, 637, 654, 668, 692, 694, 703, 715, 728, 766, 777, 793, 795, 807, 819, 848, 857, 869, 887, 964, 992, 995, 1016, 1028, 1034, 1043, 1046, 1071, 1086,....

A114856-"good"-Gram-points, given by "total"-Gram points minus A114856-"bad"-Gram-points, is precisely defined as "Indices n of Gram points g_n for which $(-1)^n Z(g_n) > 0$ with Z(t) being Riemann-Siegel Z-function [and full given range of values n = 0, 1, 2, 3,...]". The derived terms of A114856-"good"-Gram-points: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50,....

A216700 (Greathouse, 2012) is precisely defined as "Violations of Rosser's rule: numbers n such that the Gram block [g_n, g_{n+k}] contains fewer than k points t such that Z(t) = 0 with Z(t) being Riemann-Siegel Z-function [and full given range of values n = 0, 1, 2, 3,...]". The terms of A216700 are 13999525, 30783329, 30930927, 37592215, 40870156, 43628107, 46082042, 46875667, 49624541, 50799238, 55221454, 56948780, 60515663, 61331766, 69784844, 75052114, 79545241, 79652248, 83088043, 83689523, 85348958, 86513820, 87947597,....

Expected connection: All NTZ (as conjectured by Riemann hypothesis) and Gram points (by definition) are located on the same critical line of Riemann zeta function. Counting NTZ can be validly reduced to counting all Gram points where Gram's Law is satisfied and adding count of NTZ inside each Gram block. With this process, we need not locate NTZ but just have to accurately compute Z(t) to show that it changes sign.

Gram's Law is the observation that there is [usually] exactly one NTZ (Gram[x=0,y=0] points or Origin intercept points) between any two "good" Gram points. Examples of closely related statements equivalent to Gram's law are: $(-1)^n Z(g_n)$ is [usually] positive or Z(t) [usually] has opposite sign at consecutive Gram points. Thus, a t-valued Gram point is called a "good" Gram point if $\zeta(s)$ is positive at $\frac{1}{2} + \imath t$ with $(-1)^n Z(g_n) > 0$ and a "bad" Gram point if $\zeta(s)$ is negative at $\frac{1}{2} + \imath t$ with $(-1)^n Z(g_n) < 0$. The indices of "bad" Gram points where Z has the 'wrong' sign are given by A114856 in OEIS. A Gram block [g_n, g_{n+k}] is a half-open interval bounded by two "good" Gram points g_n and g_{n+k} such that all Gram points g_{n+1},..., g_{n+k-1} between them are "bad" Gram points. A refinement of Gram's Law is known as Rosser's Rule (Rosser, Yohe & Schoenfeld, 1969) which stated that Gram blocks [usually] have the expected number of NTZ in them (identical to number of Gram intervals), even though some of the individual Gram intervals in the block may not have exactly one NTZ in them. Example, the interval bounded by g_{125} and g_{127} is a Gram block containing a unique "bad" Gram point g_{126} and expected number 2 of NTZ although neither of its two Gram intervals contains a unique NTZ.

Gram's Law and Rosser's Rule both imply that in some sense NTZ do not stray too far from their expected positions, and that they hold most of the time but are violated infinitely often (in an Incompletely Predictable manner) (Trudgian, 2011 & 2014). Professor Timothy Trudgian in 2011 explicitly showed that both Gram's Law and Rosser's Rule fail in a positive proportion of cases. In particular, it is expected that in about 73% [$\approx\frac{3}{4}$] one NTZ is enclosed by two successive Gram points [and thus Gram's Law fails for about 27% $\approx\frac{1}{4}$ of all Gram intervals to contain exactly one NTZ], but in about 14% no NTZ and in about 13% two NTZ are in such a Gram interval on the long run.

B. Freebasic programme to elucidate all patterns of Prime k-tuplets for k = 2 to 50

Patterns for first 50 Prime k-tuples including data on p_1 congruent to p (modulo q) can be obtained from website **Patterns of prime k-tuplets & the Hardy-Littlewood constants** with URL https://pzktupel.de/ktpatt_hl.php maintained by Norman Luhn (Email address: pzktupel@pzktupel.de). Luhn's programme to provide all possible patterns of Subtype I Admissible Prime k-tuplets for k = 2 to 50 is reproduced with permission.

```
#INCLUDE "windows.bi"
#INCLUDE "vbcompat.bi"
DIM AS UINTEGER G,i,j,k,mini,l,m,n,o,p
DIM AS STRING S1,S2
START:
INPUT "k= "; k
mini=500
G=2*3*5*7*11*13*17*19*23
REDIM F(G\7) AS UBYTE
REDIM Z(100000000) AS UINTEGER
REDIM S(2000000) AS STRING
REDIM R(2000000) AS UINTEGER
i=3
WHILE i<24
FOR j=i*i TO G STEP i*2
F(j SHR 3)=BITSET(F(j SHR 3),j MOD 8)
```

```
NEXT j
i+=2
WEND
j=0
REM Collect all free Cells with no factor <29 and count in Array Z()
FOR i=3 TO G\8
IF BIT(F(I),1)=0 THEN j+=1:Z(j)=i*8+1
IF BIT(F(I),3)=0 THEN j+=1:Z(j)=i*8+3
IF BIT(F(I),5)=0 THEN j+=1:Z(j)=i*8+5
IF BIT(F(I),7)=0 THEN j+=1:Z(j)=i*8+7
NEXT i
REM start with 29, calculate the diameter Z(i+ k-condition)-Z(i) < mini ? Yes, set mini=Z(i+k)-Z(i)
FOR i=29 TO j STEP 2
IF Z(i+k-1)-Z(i)¡mini THEN mini=Z(i+k-1)-Z(i)
NEXT i
PRINT ”Diameter found for k: ”;mini
REM Musterermittlung
FOR i=1 TO j
IF Z(i+k-1)-Z(i)<>mini THEN GOTO NXTI
S1=” ”
FOR l=i TO i+k-1
S1=S1+STR(Z(l)-Z(i))+” ”
NEXT l
m+=1:S(m)=S1:R(m)=Z(i) MOD 30
IF m>9999 THEN m=9999
NXTI:NEXT i
PRINT ”Write pattern in file pattern.txt”
OPEN ”pattern.txt” FOR OUTPUT as #1
PRINT ”k=”;k
FOR n=1 TO m
FOR o=1 TO n-1
IF S(o)=S(n) THEN GOTO NXTN
PRINT ”Member of: N=”;R(n);”+30k NEXT o,
Pattern d=”;S(n)
PRINT #1,”Member of: N=”;R(n);”+30k, Pattern d=”;S(nNXTN:NEXT n
PRINT #1,”Member of: N=”;R(n);”+30k, Pattern d=”;S(n)
CLOSE #1
PRINT ”done, ENTER!”
SLEEP
```

C. The 18 patterns of Subtype I Admissible Prime 25-tuplets

The 18 patterns for the [randomly-selected] Prime 25-tuplets are depicted as cummulative prime gaps and progressive prime gaps. Frequency of patterns [that progressively decreasing by 8, 4 and 2 as related by 2^{-1}] containing prime gap 8 = 16/18, prime gap 10 = 8/18, prime gap 12 = 4/18, and prime gap 14 = 2/18. Frequency of patterns [that are progressively increasing by 8, 4 and 2 as related by 2^{-1}] NOT containing prime gap 8 = 2/18, prime gap 10 = 10/18, prime gap 12 = 14/18, and prime gap 14 = 16/18.

Pat-1 (0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50, 56, 62, 68, 72, 78, 86, 90, 96, 98, 102, 110) ≡ (0, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6, 6, 6, 4, 6, 8, 4, 6, 2, 4, 8); #Gap 8 = 2, #Gap 10 = 0, #Gap 12 = 0, #Gap 14 = 0

Pat-2 (0, 2, 6, 8, 12, 20, 26, 30, 36, 38, 42, 48, 56, 66, 68, 72, 78, 80, 86, 90, 92, 96, 98, 108, 110) ≡ (0, 2, 4, 2, 4, 8, 6, 4, 6, 2, 4, 6, 8, 10, 2, 4, 6, 2, 6, 4, 2, 4, 2, 10, 2); #Gap 8 = 2, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0

Pat-3 (0, 2, 6, 8, 12, 20, 26, 30, 36, 38, 42, 48, 50, 56, 66, 68, 72, 78, 80, 86, 90, 92, 98, 108, 110) ≡ (0, 2, 4, 2, 4, 8, 6, 4, 6, 2, 4, 6, 2, 6, 10, 2, 4, 6, 2, 6, 4, 2, 6, 10, 2); #Gap 8 = 1, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0

Pat-4 (0, 2, 6, 8, 12, 20, 26, 30, 36, 38, 42, 50, 56, 66, 68, 72, 78, 80, 86, 90, 92, 96, 98, 108, 110) ≡ (0, 2, 4, 2, 4, 8, 6, 4, 6, 2, 4, 8, 6, 10, 2, 4, 6, 2, 6, 4, 2, 4, 2, 10, 2); #Gap 8 = 2, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0

Pat-5 (0, 2, 6, 8, 12, 20, 26, 30, 36, 38, 42, 50, 56, 62, 66, 68, 72, 78, 80, 86, 90, 92, 96, 108, 110) ≡ (0, 2, 4, 2, 4, 8, 6, 4, 6, 2, 4, 8, 6, 6, 4, 2, 4, 6, 2, 6, 4, 2, 4, 12, 2); #Gap 8 = 2, #Gap 10 = 0, #Gap 12 = 1, #Gap 14 = 0

Pat-6 (0, 2, 6, 8, 12, 20, 26, 30, 36, 38, 42, 48, 56, 62, 66, 68, 72, 78, 80, 86, 90, 92, 96, 108, 110) ≡ (0, 2, 4, 2, 4, 8, 6, 4,

6, 2, 4, 6, 8, 6, 4, 2, 4, 6, 2, 6, 4, 2, 4, 12, 2); #Gap 8 = 2, #Gap 10 = 0, #Gap 12 = 1, #Gap 14 = 0
Pat-7 (0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 50, 56, 62, 68, 72, 78, 86, 90, 92, 96, 98, 102, 110) ≡ (0, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 8, 6, 6, 6, 4, 6, 8, 4, 2, 4, 2, 4, 8); #Gap 8 = 3, #Gap 10 = 0, #Gap 12 = 0, #Gap 14 = 0
Pat-8 (0, 2, 6, 12, 14, 20, 24, 26, 30, 32, 42, 44, 54, 56, 60, 66, 72, 74, 80, 86, 90, 96, 102, 104, 110) ≡ (0, 2, 4, 6, 2, 6, 4, 2, 4, 2, 10, 2, 10, 2, 4, 6, 6, 2, 6, 6, 4, 6, 6, 2, 6); #Gap 8 = 0, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0
Pat-9 (0, 6, 8, 14, 20, 24, 30, 36, 38, 44, 50, 54, 56, 66, 68, 78, 80, 84, 86, 90, 96, 98, 104, 108, 110) ≡ (0, 6, 2, 6, 6, 4, 6, 6, 2, 6, 6, 4, 2, 10, 2, 10, 2, 4, 2, 4, 6, 2, 6, 4, 2); #Gap 8 = 0, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0
Pat-10 (0, 2, 8, 12, 14, 18, 24, 30, 32, 38, 42, 44, 50, 54, 60, 68, 72, 74, 78, 80, 84, 98, 102, 108, 110) ≡ (0, 2, 6, 4, 2, 4, 6, 6, 2, 6, 4, 2, 6, 4, 6, 8, 4, 2, 4, 2, 4, 14, 4, 6, 2); #Gap 8 = 1, #Gap 10 = 0, #Gap 12 = 0, #Gap 14 = 1
Pat-11 (0, 2, 8, 12, 26, 30, 32, 36, 38, 42, 50, 56, 60, 66, 68, 72, 78, 80, 86, 92, 96, 98, 102, 108, 110) ≡ (0, 2, 6, 4, 14, 4, 2, 4, 2, 4, 8, 6, 4, 6, 2, 4, 6, 2, 6, 6, 4, 2, 4, 6, 2); #Gap 8 = 1, #Gap 10 = 0, #Gap 12 = 0, #Gap 14 = 1
Pat-12 (0, 8, 12, 14, 18, 20, 24, 32, 38, 42, 48, 54, 60, 68, 74, 78, 80, 84, 90, 92, 98, 102, 104, 108, 110) ≡ (0, 8, 4, 2, 4, 2, 4, 8, 6, 4, 6, 6, 6, 8, 6, 4, 2, 4, 6, 2, 6, 4, 2, 4, 2); #Gap 8 = 3, #Gap 10 = 0, #Gap 12 = 0, #Gap 14 = 0
Pat-13 (0, 8, 12, 14, 20, 24, 32, 38, 42, 48, 54, 60, 62, 68, 74, 78, 80, 84, 90, 92, 98, 102, 104, 108, 110) ≡ (0, 8, 4, 2, 6, 4, 8, 6, 4, 6, 6, 6, 2, 6, 6, 4, 2, 4, 6, 2, 6, 4, 2, 4, 2); #Gap 8 = 2, #Gap 10 = 0, #Gap 12 = 0, #Gap 14 = 0
Pat-14 (0, 2, 12, 14, 18, 20, 24, 30, 32, 38, 42, 44, 54, 60, 68, 72, 74, 80, 84, 90, 98, 102, 104, 108, 110) ≡ (0, 2, 10, 2, 4, 2, 4, 6, 2, 6, 4, 2, 10, 6, 8, 4, 2, 6, 4, 6, 8, 4, 2, 4, 2); #Gap 8 = 2, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0
Pat-15 (0, 2, 12, 18, 20, 24, 30, 32, 38, 42, 44, 54, 60, 62, 68, 72, 74, 80, 84, 90, 98, 102, 104, 108, 110) ≡ (0, 2, 10, 6, 2, 4, 6, 2, 6, 4, 2, 10, 6, 2, 6, 4, 2, 6, 4, 6, 8, 4, 2, 4, 2); #Gap 8 = 1, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0
Pat-16 (0, 2, 12, 14, 18, 20, 24, 30, 32, 38, 42, 44, 54, 62, 68, 72, 74, 80, 84, 90, 98, 102, 104, 108, 110) ≡ (0, 2, 10, 2, 4, 2, 4, 6, 2, 6, 4, 2, 10, 8, 6, 4, 2, 6, 4, 6, 8, 4, 2, 4, 2); #Gap 8 = 2, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0
Pat-17 (0, 2, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48, 54, 62, 68, 72, 74, 80, 84, 90, 98, 102, 104, 108, 110) ≡ (0, 2, 12, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 8, 6, 4, 2, 6, 4, 6, 8, 4, 2, 6, 2); #Gap 8 = 2, #Gap 10 = 0, #Gap 12 = 1, #Gap 14 = 0
Pat-18 (0, 2, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48, 54, 60, 68, 72, 74, 80, 84, 90, 98, 102, 104, 108, 110) ≡ (0, 2, 12, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 8, 4, 2, 6, 4, 6, 8, 4, 2, 4, 2); #Gap 8 = 2, #Gap 10 = 0, #Gap 12 = 1, #Gap 14 = 0

D. Admissible Prime k-tuplets/k-tuples and Inadmissible Prime k-tuples for nominated p_1 commencing values

Box 1. Admissible Prime k-tuplets/k-tuples and Inadmissible Prime k-tuples for initial 15 p_1 commencing values [out of an arbitrarily large number of other commencing values].

p_1 commencing value = 2. Set Admissible Prime k-tuples as k-value = 0 [empty set] with its cardinality = ||CFS|| = 0. Set Inadmissible Prime k-tuples as k-value = 2 [having nadir diameter d = 1 and nadir average gap = 1/2 = 0.5], 3, 4, 5, 6... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 2, failure at mod 2 (term 3) first occur at k = 2 with minimum diameter d = 1.

p_1 commencing value = 3. Set Admissible Prime k-tuplets as k-value = 2 [having zenith diameter d = 2 and zenith average gap = 2/1 = 2] with its cardinality = ||CFS|| = 1. Set Inadmissible Prime k-tuples as k-value = 3 [having nadir diameter d = 4 and nadir average gap = 4/3 = 1.33], 4, 5, 6, 7... with its cardinality = ||CIS-ALN-decelerating|| = symbolically $\aleph_0$. At p_1 = 3, failure at mod 3 (term 7) first occur at k = 3 with minimum diameter d = 4.

p_1 commencing value = 5. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5 [having zenith diameter d = 12 and zenith average gap = 12/5 = 2.4] with its cardinality = ||CFS|| = 4. Set Inadmissible Prime k-tuples as k-value = 6 [having nadir diameter d = 14 and nadir average gap = 14/6 = 2.33], 7, 8, 9, 10... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 5, failure at mod 5 (term 19) first occur at k = 6 with minimum diameter d = 14.

p_1 commencing value = 7. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5, 6 [having zenith diameter d = 16 and zenith average gap = 16/6 = 2.67] with its cardinality = ||CFS|| = 5. Set Inadmissible Prime k-tuples as k-value = 7 [having nadir diameter d = 22 and nadir average gap = 22/7 = 3.14], 8, 9, 10, 11... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 7, failure at mod 7 (term 29) first occur at k = 7 with minimum diameter d = 22.

p_1 commencing value = 11. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 15 [having zenith diameter d = 56 and zenith average gap = 56/15 = 3.73] with its cardinality = ||CFS|| = 14. Set Inadmissible Prime k-tuples as k-value = 16 [having nadir diameter d = 60 and nadir average gap = 60/16 = 3.75], 17, 18, 19, 20... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 11, failure at mod 11 (term 71) first occur at k = 16 with minimum diameter d = 60.

p_1 commencing value = 13. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 21 [having zenith diameter d = 88 and zenith average gap = 88/21 = 4.19] with its cardinality = ||CFS|| = 20. Set Inadmissible Prime k-tuples as k-value = 22 [having nadir diameter d = 90 and nadir average gap = 90/22 = 4.09], 23, 24, 25, 26... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 13, failure at mod 13 (term 103) first occur at k = 22 with minimum diameter d = 90.

p_1 commencing value = 17. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 20 [having zenith diameter d = 84 and zenith average gap = 84/20 = 4.2] with its cardinality = ||CFS|| = 19. Set Inadmissible Prime k-tuples as k-value = 21 [having nadir diameter d = 86 and nadir average gap = 86/21 = 4.10], 22, 23, 24, 25... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 17, failure at mod 17 (term 103) first occur at k = 21 with minimum diameter d = 86.

p_1 commencing value = 19. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 35 [having zenith diameter d = 162 and zenith average gap = 162/35 = 4.63] with its cardinality = ||CFS|| = 34. Set Inadmissible Prime k-tuples as k-value = 36 [having nadir diameter d = 172 and nadir average gap = 172/36 = 4.78], 37, 38, 39, 40... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 19, failure at mod 19 (term 191) first occur at k = 36 with minimum diameter d = 172.

p_1 commencing value = 23. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 39 [having zenith diameter d = 188 and zenith average gap = 188/39 = 4.82] with its cardinality = ||CFS|| = 38. Set Inadmissible Prime k-tuples as k-value = 40 [having nadir diameter d = 200 and nadir average gap = 200/40 = 5], 41, 42, 43, 44... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 23, failure at mod 23 (term 223) first occur at k = 40 with minimum diameter d = 200.

p_1 commencing value = 29. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 59 [having zenith diameter d = 308 and zenith average gap = 308/59 = 5.22] with its cardinality = ||CFS|| = 58. Set Inadmissible Prime k-tuples as k-value = 60 [having nadir diameter d = 318 and nadir average gap = 318/60 = 5.3], 61, 62, 63, 64... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 29, failure at mod 29 (term 347) first occur at k = 60 with minimum diameter d = 318.

p_1 commencing value = 31. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 95 [having zenith diameter d = 540 and zenith average gap = 540/95 = 5.68] with its cardinality = ||CFS|| = 94. Set Inadmissible Prime k-tuples as k-value = 96 [having nadir diameter d = 546 and nadir average gap = 546/96 = 5.69], 97, 98, 99, 100... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 31, failure at mod 31 (term 577) first occur at k = 96 with minimum diameter d = 546.

p_1 commencing value = 37. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 73 [having zenith diameter d = 396 and zenith average gap = 396/73 = 5.42] with its cardinality = ||CFS|| = 72. Set Inadmissible Prime k-tuples as k-value = 74 [having nadir diameter d = 402 and nadir average gap = 402/74 = 5.43], 75, 76, 77, 78... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 37, failure at mod 37 (term 439) first occur at k = 74 with minimum diameter d = 402.

p_1 commencing value = 41. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 94 [having zenith diameter d = 536 and zenith average gap = 536/94 = 5.70] with its cardinality = ||CFS|| = 93. Set Inadmissible Prime k-tuples as k-value = 95 [having nadir diameter d = 546 and nadir average gap = 546/95 = 5.75], 96, 97, 98, 99... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 41, failure at mod 41 (term 587) first occur at k = 95 with minimum diameter d = 546.

p_1 commencing value = 43. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 77 [having zenith diameter d = 420 and zenith average gap = 420/77 = 5.45] with its cardinality = ||CFS|| = 76. Set Inadmissible Prime k-tuples as k-value = 78 [having nadir diameter d = 424 and nadir average gap = 424/78 = 5.44], 79, 80, 81, 82... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 43, failure at mod 43 (term 467) first occur at k = 78 with minimum diameter d = 424.

p_1 commencing value = 47. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 78 [having zenith diameter d = 432 and zenith average gap = 432/78 = 5.54] with its cardinality = ||CFS|| = 77. Set Inadmissible Prime k-tuples as k-value = 79 [having nadir diameter d = 440 and nadir average gap = 440/79 = 5.57], 80, 81, 82, 83... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 47, failure at mod 47 (term 487) first occur at k = 79 with minimum diameter d = 440.

E. The forbidden Prime k-tuples

Using initial examples [out of an arbitrarily large number of examples], we show the countably arbitrarily large number of forbidden Prime k-tuples do not match any position in sequence of prime numbers.

For k = 2, p = 2 pattern Sub I Inadm P 2-tuple [smallest possible diameter d = 1] starts at p = 2 pattern with first and only occurrence as (p+0, p+1) = (2, 3). There is no 1st V Sub II Adm P 2-tuple with [consecutive] larger diameter (prime gap) = 3 as (p+0, p+3), 5 as (p+0, p+5), 7 as (p+0, p+7), 9 as (p+0, p+9), 11 as (p+0, p+11)....

For k = 2, p = 3 pattern Sub I Adm P 2-tuplet [smallest possible diameter d = 2] starts at p = 3 pattern with first occurrence as (p+0, p+2) = (3, 5). There are arbitrarily large number of 1st V Sub II Adm P 2-tuples with [consecutive] larger diameter (prime gap) = 4 as (p+0, p+4), 6 as (p+0, p+6), 8 as (p+0, p+8), 10 as (p+0, p+10), 12 as (p+0, p+12)... $\Longrightarrow$ there is no existing Forbidden Inadmissible Prime 2-tuple.

For k = 3, p = 5 pattern-1 Sub I Adm P 3-tuplet [smallest possible diameter d = 6] starts at p = 5 pattern-1 with first occurrence as (p+0, p+2, p+6) = (5, 7, 11). There are arbitrarily large number of 1st V Sub II Adm P 3-tuples with [non-consecutive] larger diameter = 8 as (p+0, p+2, p+8), 12 as (p+0, p+2, p+12), 14 as (p+0, p+2, p+14), 18 as (p+0, p+2, p+18), 20 as (p+0, p+2, p+20), 24 as (p+0, p+2, p+24)... $\Longrightarrow$ the non-existing [non-consecutive] larger diameter d = 10 as (p+0, p+2, p+10), 16 as (p+0, p+2, p+16), 22 as (p+0, p+2, p+22), 28 as (p+0, p+2, p+28)... are Forbidden Inadmissible Prime 3-tuples all with failure at mod prime 3 (last term = 10, 16, 22, 28...). For n = 0, 1, 2, 3... in Prime 3-tuples of the format (p+0, p+2, p+4+6n), we notice that these are actually non-existing 1st V Sub II Inadm P 3-tuples apart from the solitary Sub I Inadm P 3-tuple (p+0, p+2, p+4) located at consecutive prime numbers (3, 5, 7) when n = 0.

For k = 3, p = 7 pattern-2 Sub I Adm P 3-tuplet [smallest possible diameter d = 6] starts at p = 7 pattern-2 with first occurrence as (p+0, p+4, p+6) = (7, 11, 13). There are arbitrarily large number of 1st V Sub II Adm P 3-tuples with [non-consecutive] larger diameter = 10 as (p+0, p+4, p+10), 12 as (p+0, p+4, p+12), 16 as (p+0, p+4, p+16), 18 as (p+0, p+4, p+18), 22 as (p+0, p+4, p+22), 24 as (p+0, p+4, p+24), 28 as (p+0, p+4, p+28)... $\Longrightarrow$ the non-existing [non-consecutive] larger diameter d = 8 as (p+0, p+4, p+8), 14 as (p+0, p+4, p+14), 20 as (p+0, p+4, p+20)... are Forbidden Inadmissible Prime 3-tuples all with failure at mod prime 3 (last term = 8, 14, 20, 26...). For n = 0, 1, 2, 3... in Prime 3-tuples of the one format (p+0, p+4, p+8+6n), we notice that these are actually non-existing 1st V Sub II Inadm P 3-tuples.

For k = 4, p = 5 pattern Sub I Adm P 4-tuplet [smallest possible diameter d = 8] starts at p = 5 pattern with first occurrence as (p+0, p+2, p+6, p+8) = (5, 7, 11, 13). There are arbitrarily large number of 1st V Sub II Adm P 4-tuples with [non-consecutive] larger diameter = 12 as (p+0, p+2, p+6, p+12), 14 as (p+0, p+2, p+6, p+14), 18 as (p+0, p+2, p+6, p+18), 20 as (p+0, p+2, p+6, p+20), 24 as (p+0, p+2, p+6, p+24), 26 as (p+0, p+2, p+6, p+26)... $\Longrightarrow$ the non-existing [non-consecutive] larger diameter d = 10 as (p+0, p+2, p+6, p+10), 16 as (p+0, p+2, p+6, p+16), 22 as (p+0, p+2, p+6, p+22), 28 as (p+0, p+2, p+6, p+28)... are Forbidden Inadmissible Prime 4-tuples all with failure at mod prime 3 (last term = 10, 16, 22, 28...). For n = 0, 1, 2, 3... in Prime 4-tuples of the one format (p+0, p+2, p+6, p+10+6n), we notice that these are actually non-existing 1st V Sub II Inadm P 4-tuples.

For k = 5, p = 5 pattern-1 Sub I Adm P 5-tuplet [smallest possible diameter d = 12] starts at p = 5 pattern-1 with first occurrence as (p+0, p+2, p+6, p+8, p+12) = (5, 7, 11, 13, 17). There are arbitrarily large number of 1st V Sub II Adm P 5-tuples with [non-consecutive] larger diameter = 18 as (p+0, p+2, p+6, p+8, p+18), 20 as (p+0, p+2, p+6, p+8, p+20), 26 as (p+0, p+2, p+6, p+8, p+26), 30 as (p+0, p+2, p+6, p+8, p+30), 32 as (p+0, p+2, p+6, p+8, p+32), 36 as (p+0, p+2, p+6, p+8, p+36)... $\Longrightarrow$ the non-existing [non-consecutive] larger diameter d = 14 as (p+0, p+2, p+6, p+8, p+14), 16 as (p+0, p+2, p+6, p+8, p+16), 22 as (p+0, p+2, p+6, p+8, p+22), 24 as (p+0, p+2, p+6, p+8, p+24), 28 as (p+0, p+2, p+6, p+8, p+28)... are Forbidden Inadmissible Prime 5-tuples all with failure at mod prime 3 (last term = 16, 22, 28, 34, 40, 46, 52...) or mod prime 5 (last term = 14, 24, 44, 54...). For n = 0, 1, 2, 3... in Prime 5-tuples of the two formats (p+0, p+2, p+6, p+8, p+16+6n) and (p+0, p+2, p+6, p+8, p+14+10n/20n), we notice that these are actually non-existing 1st V Sub II Inadm P 5-tuples.

For k = 5, p = 7 pattern-2 Sub I Adm P 5-tuplet [smallest possible diameter d = 12] starts at p = 7 pattern-2 with first occurrence as (p+0, p+4, p+6, p+10, p+12) = (7, 11, 13, 17, 19). There are arbitrarily large number of 1st V Sub II Adm P 5-tuples with [non-consecutive] larger diameter = 16 as (p+0, p+4, p+6, p+10, p+16), 18 as (p+0, p+4, p+6, p+10, p+18), 22 as (p+0, p+4, p+6, p+10, p+22), 24 as (p+0, p+4, p+6, p+10, p+24), 28 as (p+0, p+4, p+6, p+10, p+28), 30 as (p+0, p+4, p+6, p+10, p+30)... $\Longrightarrow$ the non-existing [non-consecutive] larger diameter d = 14 as (p+0, p+4, p+6, p+10, p+14), 20 as (p+0, p+4, p+6, p+10, p+20), 26 as (p+0, p+4, p+6, p+10, p+26), 32 as (p+0, p+4, p+6, p+10, p+32), 38 as (p+0, p+4, p+6, p+10, p+38)... are Forbidden Inadmissible Prime 5-tuples all with failure at mod prime 3 (last term = 14, 20, 26, 32, 38, 44...). For n = 0, 1, 2, 3... in Prime 5-tuples of the one format (p+0, p+4, p+6, p+10, p+14+6n), we notice that these are actually non-existing 1st V Sub II Inadm P 5-tuples.

For k = 6, p = 7 pattern Sub I Adm P 6-tuplet [smallest possible diameter d = 16] starts at p = 7 pattern with first occurrence as (p+0, p+4, p+6, p+10, p+12, p+16) = (7, 11, 13, 17, 19, 23). There are arbitrarily large number of 1st V Sub II Adm P 6-tuples with [non-consecutive] larger diameter = 22 as (p+0, p+4, p+6, p+10, p+12, p+22), 24 as (p+0, p+4, p+6, p+10, p+12, p+24), 30 as (p+0, p+4, p+6, p+10, p+12, p+30), 34 as (p+0, p+4, p+6, p+10, p+12, p+34), 36 as (p+0, p+4, p+6, p+10, p+12, p+36), 40 as (p+0, p+4, p+6, p+10, p+12, p+40)... $\Longrightarrow$ the non-existing [non-consecutive] larger diameter d = 18 as (p+0, p+4, p+6, p+10, p+12, p+18), 20 as (p+0, p+4, p+6, p+10, p+12, p+20), 26 as (p+0, p+4, p+6, p+10, p+12, p+26), 28 as (p+0, p+4, p+6, p+10, p+12, p+28), 32 as (p+0, p+4, p+6, p+10, p+12, p+32)... are Forbidden Inadmissible Prime 6-tuples all with failure at mod prime 3 (last term = 20, 26, 32, 38, 44, 50, 56, 62, 68...) or mod prime 5 (last term = 18, 28, 48, 58, 78, 88...). For n = 0, 1, 2, 3... in Prime 6-tuples of the two formats (p+0, p+4, p+6, p+10, p+12, p+18+10n/20n) and (p+0, p+4, p+6, p+10, p+12, p+20+6n), we notice that these are actually non-existing 1st V Sub II Inadm P 6-tuples.

For k = 7, p = 11 pattern-1 Sub I Adm Prime 7-tuplet [smallest possible diameter d = 20] starts at p = 11 pattern-1 with first occurrence as (p+0, p+2, p+6, p+8, p+12, p+18, p+20) = (11, 13, 17, 19, 23, 29, 31). There are arbitrarily large number of 1st V Sub II Adm P 7-tuples with [non-consecutive] larger diameter = 26 as (p+0, p+2, p+6, p+8, p+12, p+18, p+26), 30 as (p+0, p+2, p+6, p+8, p+12, p+18, p+30), 32 as (p+0, p+2, p+6, p+8, p+12, p+18, p+32), 36 as (p+0, p+2, p+6, p+8, p+12, p+18, p+36), 42 as (p+0, p+2, p+6, p+8, p+12, p+18, p+42), 48 as (p+0, p+2, p+6, p+8, p+12, p+18, p+48)... $\Longrightarrow$ the non-existing [non-consecutive] larger diameter d = 22 as (p+0, p+2, p+6, p+8, p+12, p+18, p+22), 24 as (p+0, p+2, p+6, p+8, p+12, p+18, p+24), 28 as (p+0, p+2, p+6, p+8, p+12, p+18, p+28), 34 as (p+0, p+2, p+6, p+8, p+12, p+18, p+34), 38 as (p+0, p+2, p+6, p+8, p+12, p+18, p+38)... are Forbidden Inadmissible Prime 7-tuples all with failure at mod prime 3 (last term = 22, 28, 34, 40, 46, 52, 58...) or mod prime 5 (last term = 24, 44, 54, 74, 84, 104...) or mod prime 7 (last term = 38, 66, 80, 108, 122, 150...). For n = 0, 1, 2, 3... in Prime 7-tuples of the three formats (p+0, p+2, p+6, p+8, p+12, p+18, p+22+6n), (p+0, p+2, p+6, p+8, p+12, p+18, p+24+20n/10n) and (p+0, p+2, p+6, p+8, p+12, p+18, p+38+28n/14n), we notice that these are actually non-existing 1st V Sub II Inadm P 7-tuples.

For k = 7, p = 11 pattern-2 Sub I Adm P 7-tuplet [smallest possible diameter d = 20] starts at p = 5639 pattern-2 with first occurrence as (p+0, p+2, p+8, p+12, p+14, p+18, p+20) = (5639, 5641, 5647, 5651, 5653, 5657, 5659). There are arbitrarily large number of 1st V Sub II Adm P 7-tuples with [non-consecutive] larger diameter = 24 as (p+0, p+2, p+8, p+12, p+14, p+18, p+24), 30 as (p+0, p+2, p+8, p+12, p+14, p+18, p+30), 32 as (p+0, p+2, p+8, p+12, p+14, p+18, p+32), 38 as (p+0, p+2, p+8, p+12, p+14, p+18, p+38), 42 as (p+0, p+2, p+8, p+12, p+14, p+18, p+42), 44 as (p+0, p+2, p+8, p+12, p+14, p+18, p+44)... $\Longrightarrow$ the non-existing [non-consecutive] larger diameter d = 22 as (p+0, p+2, p+8, p+12, p+14, p+18, p+22), 26 as (p+0, p+2, p+8, p+12, p+14, p+18, p+26), 28 as (p+0, p+2, p+8, p+12, p+14, p+18, p+28), 34 as (p+0, p+2, p+8, p+12, p+14, p+18, p+34), 36 as (p+0, p+2, p+8, p+12, p+14, p+18, p+36)... are Forbidden Inadmissible Prime 7-tuples all with failure at mod prime 3 (last term = 22, 28, 34, 40, 46, 52, 58, 64, 70, 76...) or mod prime 5 (last term = 26, 36, 56, 66, 86, 96, 116, 126, 146...). For n = 0, 1, 2, 3... in Prime 7-tuples of the two formats (p+0, p+2, p+6, p+8, p+12, p+18, p+22+6n) and (p+0, p+2, p+6, p+8, p+12, p+18, p+26+10n/20n), we notice that these are actually non-existing 1st V Sub II Inadm P 7-tuples.

F. Failures at p_{i-1} mod prime q on term prime p for Subtype I Inadmissible Prime k-tuples

Using the following initial examples [out of an arbitrarily large number of examples], we show failures occurring at p_{i-1} mod prime q on term prime p for various Sub I Inadm P k-tuples.

For k = 2, p_i = 3 pattern and p_{i-1} = 2 Sub I Adm P 2-tuplet (3, 5): k = 2, p_i = 3, d = 2. Sub I Inadm P 2-tuple (2, 3) with failure at mod prime 2 (term 3): k = 2, p_{i-1} = 2, d = 1. By incorporating ever larger consecutive k values of 3, 4, 5, 6..., there is a countably arbitrarily large number of Sub I Inadm P k-tuples when first prime p = 2.

For k = 3, p_i = 5 pattern-1 and p_{i-1} = 3 Sub I Adm P 3-tuplet (5, 7, 11): k = 3, p_i = 5, d = 6. Sub I Inadm P 3-tuple (3, 5, 7) with failure at mod prime 3 (term 7): k = 3, p_{i-1} = 3, d = 4. By incorporating ever larger consecutive k values of 4, 5, 6, 7..., there is a countably arbitrarily large number of Sub I Inadm P k-tuples when first prime p = 3.

For k = 3, p_i = 7 pattern-2 and p_{i-1} = 5 Sub I Adm P 3-tuplet (7, 11, 13): k = 3, p_i = 7, d = 6. There is no Sub I Inadm P 3-tuple for k = 3 when p_{i-1} = 5 since Prime 3-tuple (5, 7, 11) with d = 6 is just the previous Sub I Adm P 3-tuplet. However, there is a Sub I Inadm P 3-tuple (2, 3, 5) with failure at mod prime 2 (term 3): k = 3, p_{i-2} = 2 [as special exception], d = 3. By incorporating ever larger consecutive k values of 4, 5, 6, 7..., there is a countably arbitrarily large number of Sub I Inadm P k-tuples when first prime p = 2.

For k = 4, p_i = 5 pattern and p_{i-1} = 3 Sub I Adm P 4-tuplet (5, 7, 11, 13): k = 4, p_i = 5, d = 8. Sub I Inadm P 4-tuple (3, 5, 7, 11) with failure at mod prime 3 (term 7): k = 4, p_{i-1} = 3, d = 8. By incorporating ever larger consecutive k values of 5, 6, 7, 8..., there is a countably arbitrarily large number of Sub I Inadm P k-tuples when first prime p = 3.

For k = 5, p_i = 5 pattern-1 and p_{i-1} = 3 Sub I Adm P 5-tuplet (5, 7, 11, 13, 17): k = 5, p_i = 5, d = 12. Sub I Inadm P 5-tuple (3, 5, 7, 11, 13) with failure at mod prime 3 (term 7): k = 5, p_{i-1} = 3, d = 10. By incorporating ever larger consecutive k values of 6, 7, 8, 9..., there is a countably arbitrarily large number of Sub I Inadm P k-tuples when first prime p = 3.

For k = 5, p_i = 7 pattern-2 and p_{i-1} = 5 Sub I Adm P 5-tuplet (7, 11, 13, 17, 19): k = 4, p_i = 7, d = 12. There is no Sub I Inadm P 5-tuple for k = 5 when p_{i-1} = 5 since Prime 5-tuple (5, 7, 11, 13, 17) with d = 12 is a Sub I Adm P 5-tuplet. However, there is Sub I Inadm P 5-tuple (3, 5, 7, 11, 13) with failure at mod prime 3 (term 7): k = 5, p_{i-2} = 3 [as special exception], d = 10. By incorporating ever larger consecutive k values of 6, 7, 8, 9..., there is a countably arbitrarily large number of Sub I Inadm P k-tuples when first prime p = 3.

For k = 6, p_i = 7 pattern and p_{i-1} = 5 Sub I Adm P 6-tuplet (7, 11, 13, 17, 19, 23): k = 6, p_i = 7, d = 16. 2nd V Sub II Inadm P 6-tuple (5, 7, 11, 13, 17, 19) with failure at mod prime 5 (term 19): k = 6, p_{i-1} = 5, d = 14. By incorporating ever larger consecutive k values of 7, 8, 9, 10..., there is a countably arbitrarily large number of 2nd V Sub II Inadm P k-tuples when first prime p = 5.

For k = 7, p_i = 11 pattern-1 and p_{i-1} = 7 Sub I Adm P 7-tuplet (11, 13, 17, 19, 23, 29, 31): k = 7, p_i = 11, d = 20. 2nd V Sub II Inadm P 7-tuple (7, 11, 13, 17, 19, 23, 29) with failure at mod prime 7 (term 29): k = 7, p_{i-1} = 7, d = 22. By incorporating ever larger consecutive k values of 8, 9, 10, 11..., there is a countably arbitrarily large number of 2nd V Sub II Inadm P k-tuples when first prime element p = 7.

For k = 7, p_i = 5639 pattern-2 and p_{i-1} = 5623 Sub I Adm P 7-tuplet (5639, 5641, 5647, 5651, 5653, 5657, 5659): k = 7, p_i = 5639, d = 20. There is no 2nd V Sub II Inadm P 7-tuple for k = 7 when p_{i-1} = 5623 with d = 34 or p_{i-2} = 5591 with d = 62 since these two Prime 7-tuples are 2nd V Sub II Adm P 7-tuples.

For k = 8, p_i = 11 pattern-1 and p_{i-1} = 7 Sub I Adm P 8-tuplet (11, 13, 17, 19, 23, 29, 31, 37): k = 8, p_i = 11, d = 26. 2nd V Sub II Inadm P 8-tuple (7, 11, 13, 17, 19, 23, 29, 31) with failure at mod prime 7 (term 29): k = 8, p_{i-1} = 7, d = 24. By incorporating ever larger consecutive k values of 9, 10, 11, 12..., there is a countably arbitrarily large number of 2nd V Sub II Inadm P k-tuples when first prime p = 7.

For k = 8, p_i = 17 pattern-2 and p_{i-1} = 13 Sub I Adm P 8-tuplet (17, 19, 23, 29, 31, 37, 41, 43): k = 8, p_i = 17, d = 26. There is no 2nd V Sub II Inadm P 8-tuple for k = 8 when p_{i-1} = 13 with d = 28 or p_{i-2} = 11 with d = 26 since these two Prime 8-tuples are 2nd V Sub II Adm P 8-tuples.

For k = 8, p_i = 88793 pattern-3 and p_{i-1} = 88789 Sub I Adm P 8-tuplet (88793, 88799, 88801, 88807, 88811, 88813, 88817, 88819): k = 8, p_i = 88793, d = 26. There is no 2nd V Sub II Inadm P 8-tuple for k = 8 when p_{i-1} = 88789 with d = 28 or p_{i-2} = 88771 with d = 42 since these two Prime 8-tuples are 2nd V Sub II Adm P 8-tuples.

G. Computed data on n-digit primes with average prime gaps

The following are examples of computed data on n-digit prime numbers that include their average prime gaps [whereby n can be arbitrarily large in magnitude].

Corresponding subsets 1-digit $\mathbb{P}$ {2, 3, 5 and 7} and $\mathbb{C}$ {4, 6, 8 and 9} that are dervied from subset 1-digit $\mathbb{Z}$ {0, 1, 2, 3, 4, 5, 6, 7, 8 and 9} with cardinality of 10 have both equal cardinality of 4. First 1-digit $\mathbb{P}_i$ occurs at i = 1 (odd position) and last 1-digit $\mathbb{P}_i$ ends at i = 4 (even position). Average $\mathbb{P}$ gap for 1-digit $\mathbb{P}$ = 10/4 = 2.5.

Corresponding subsets 2-digit $\mathbb{P}$ {11, 13, 17, 19, 23...} with cardinality of 21 and $\mathbb{C}$ {10, 12, 14, 15, 16...} with cardinality of 69 together form subset 2-digit $\mathbb{Z}$ {10, 11, 12, 13, 14,..., 99} with cardinality of 90. There are 60% of 90 $\mathbb{Z}$ = 54 $\mathbb{Z}$ being $\mathbb{C}$ with 100% certainty. Consequently, there are 21 $\mathbb{P}$ and 69 - 54 = 15 $\mathbb{C}$ that together constitute the P(uncertain $\mathbb{P}$ + uncertain $\mathbb{C}$) = 0.4 whereby we note that there are more uncertain $\mathbb{P}$ [21/36 = 58.3%] than uncertain $\mathbb{C}$ [15/36 = 41.7%]. First 2-digit $\mathbb{P}_i$ starts at i = 5 (odd position) and last 2-digit $\mathbb{P}_i$ ends at i = 25 (odd position). Average $\mathbb{P}$ gap for 2-digit $\mathbb{P}$ = 90/21 = 4.29.

Corresponding subsets 3-digit $\mathbb{P}$ {101, 103, 107, 109, 113...} with cardinality of 143 and $\mathbb{C}$ {100, 102, 104, 105, 106...} with cardinality of 757 together form subset 3-digit $\mathbb{Z}$ {100, 101, 102, 103, 104..., 999} with cardinality of 900. There are 60% of 900 $\mathbb{Z}$ = 540 $\mathbb{Z}$ being $\mathbb{C}$ with 100% certainty. Consequently, there are 143 $\mathbb{P}$ and 757 - 540 = 217 $\mathbb{C}$ that together constitute the P(uncertain $\mathbb{P}$ + uncertain $\mathbb{C}$) = 0.4 whereby we note that there are less uncertain $\mathbb{P}$ [143/360 = 39.7%] than uncertain $\mathbb{C}$ [217/360 = 60.3%]. First 3-digit $\mathbb{P}_i$ starts at i = 26 (even position) and last 3-digit $\mathbb{P}_i$ ends at i = 168 (even position). Average $\mathbb{P}$ gap for 3-digit $\mathbb{P}$ = 900/143 = 6.29.

Corresponding subsets 4-digit $\mathbb{P}$ {1009, 1013, 1019, 1021, 1031...} with cardinality of 1061 and $\mathbb{C}$ {1000, 1001, 1002, 1003, 1004...} with cardinality of 7939 together form subset 4-digit $\mathbb{Z}$ {1000, 1001, 1002, 1003, 1004..., 9999} with cardinality of 9000. There are 60% of 9000 $\mathbb{Z}$ = 5400 $\mathbb{Z}$ being $\mathbb{C}$ with 100% certainty. Consequently, there are 1061 $\mathbb{P}$ and 7939 - 5400 = 2539 $\mathbb{C}$ that together constitute the P(uncertain $\mathbb{P}$ + uncertain $\mathbb{C}$) = 0.4 whereby we note that there are less uncertain $\mathbb{P}$ [1061/3600 = 29.5%] than uncertain $\mathbb{C}$ [2539/3600 = 70.5%]. First 4-digit $\mathbb{P}_i$ starts at i = 169 (odd position) and last 4-digit $\mathbb{P}_i$ ends at i = 1229 (odd position). Average $\mathbb{P}$ gap for 4-digit $\mathbb{P}$ = 9000/1061 = 8.48.

Corresponding subsets 5-digit $\mathbb{P}$ {10007, 10009, 10037, 10039, 10061...} with cardinality of 8363 and $\mathbb{C}$ {10000, 10001, 10002, 10003, 10004...} with cardinality of 81637 together form subset 5-digit $\mathbb{Z}$ {10000, 10001, 10002, 10003, 10004..., 99999} with cardinality of 90000. There are 60% of 90000 $\mathbb{Z}$ = 54000 $\mathbb{Z}$ being $\mathbb{C}$ with 100% certainty. Consequently, there are 8363 $\mathbb{P}$ and 81637 - 54000 = 27637 $\mathbb{C}$ that together constitute the P(uncertain $\mathbb{P}$ + uncertain $\mathbb{C}$) = 0.4 whereby we note that there are less uncertain $\mathbb{P}$ [8363/36000= 23.2%] than uncertain $\mathbb{C}$ [27637/36000 = 76.8%]. First 5-digit $\mathbb{P}_i$ starts at i = 1230 (even position) and last 5-digit $\mathbb{P}_i$ ends at i = 9592 (even position). Average $\mathbb{P}$ gap for 5-digit $\mathbb{P}$ = 90000/8363 = 10.76.

Corresponding subsets 6-digit $\mathbb{P}$ {100003, 100019, 100043, 100049, 100057...} with cardinality of 68906 and $\mathbb{C}$ {100000, 100001, 100002, 100004, 100005...} with cardinality of 831094 together form subset 6-digit $\mathbb{Z}$ {100000, 100001, 100002, 100003, 100004..., 999999} with cardinality of 900000. There are 60% of 900000 $\mathbb{Z}$ = 540000 $\mathbb{Z}$ being $\mathbb{C}$ with 100% certainty. Consequently, there are 68906 $\mathbb{P}$ and 831094 - 540000 = 291094 $\mathbb{C}$ that together constitute the P(uncertain $\mathbb{P}$ + uncertain $\mathbb{C}$) = 0.4 whereby we note that there are less uncertain $\mathbb{P}$ [68906/360000 = 19.1%] than uncertain $\mathbb{C}$ [291094/360000 = 80.9%]. First 6-digit $\mathbb{P}_i$ starts at i = 9593 (0dd position) and last 6-digit $\mathbb{P}_i$ ends at i = 78498 (even position). Average $\mathbb{P}$ gap for 6-digit $\mathbb{P}$ = 900000/68906 = 13.06.

H. Equations from Riemann zeta function and related functions

We follow the abbreviations listed under General notations in subsection 2.2.

Derived f(n) = 0 and F(n) = 0 equations *– representative examples comply with exact DA homogeneity at $\sigma = \frac{1}{2}$ critical line and inexact DA homogeneity at $\sigma \neq \frac{1}{2}$ non-critical lines.* NTZ are synonymous with Gram[x=0,y=0] points which is one type of Gram points. Whenever applicable, all modified equations below are expressed using trigonometric identities. Together with Gram[y=0] points and Gram[x=0] points as remaining two types of Gram points, these three types of Gram points are fully **located** in their complex equations (akin to *Complex Containers*) as IP entities whereby their overall location [but not actual positions] are **intrinsically incorporated** in these complex equations. Eqs. (1), (3), (5), (6), (7) and (8) that comply with exact DA homogeneity at $\sigma = \frac{1}{2}$ all have fractional exponents $\frac{1}{2}$. Eqs. (2) and (4) that comply with inexact DA homogeneity at $\sigma = \frac{2}{5}$ have fractional exponents $\frac{2}{5}$ in the former and $\frac{3}{5}$ in the later that are mixed with fractional exponents $\frac{1}{2}$.

$$\sum_{n=1}^{\infty}(2n)^{-\frac{1}{2}}2^{\frac{1}{2}}\cos(t\ln(2n)+\frac{1}{4}\pi)-\sum_{n=1}^{\infty}(2n-1)^{-\frac{1}{2}}2^{\frac{1}{2}}\cos(t\ln(2n-1)+\frac{1}{4}\pi)=0 \tag{1}$$

With exact DA homogeneity, Eq. (1) is f(n) sim-η(s) at $\sigma = \frac{1}{2}$ that will incorporate all NTZ [as Zeroes]. There is total absence of (non-existent) virtual NTZ [as virtual Zeroes].

$$\sum_{n=1}^{\infty}(2n)^{-\frac{2}{5}}2^{\frac{1}{2}}\cos(t\ln(2n)+\frac{1}{4}\pi)-\sum_{n=1}^{\infty}(2n-1)^{-\frac{2}{5}}2^{\frac{1}{2}}\cos(t\ln(2n-1)+\frac{1}{4}\pi)=0 \tag{2}$$

With inexact DA homogeneity, Eq. (2) is f(n) sim-η(s) at $\sigma = \frac{2}{5}$ that will incorporate all (non-existent) virtual NTZ [as virtual Zeroes]. There is total absence of NTZ [as Zeroes].

$$\frac{1}{2^{\frac{1}{2}}}\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}\left[(2n)^{\frac{1}{2}}\cos(t\ln(2n)-\frac{1}{4}\pi)-(2n-1)^{\frac{1}{2}}\cos(t\ln(2n-1)-\frac{1}{4}\pi)+C\right]_1^{\infty}=0 \tag{3}$$

With exact DA homogeneity, Eq. (3) is F(n) DSPL at $\sigma = \frac{1}{2}$ that will incorporate all NTZ [as Pseudo-zeroes to Zeroes conversion]. There is total absence of (non-existent) virtual NTZ [as virtual Pseudo-zeroes to virtual Zeroes conversion].

$$\frac{1}{2^{\frac{1}{2}}}\left(t^2+\frac{9}{25}\right)^{\frac{1}{2}}\left[(2n)^{\frac{3}{5}}\cos(t\ln(2n)-\frac{1}{4}\pi)-(2n-1)^{\frac{3}{5}}\cos(t\ln(2n-1)-\frac{1}{4}\pi)+C\right]_1^{\infty}=0 \tag{4}$$

With inexact DA homogeneity, Eq. (4) is F(n) DSPL at $\sigma = \frac{2}{5}$ that will incorporate all (non-existent) virtual NTZ [as virtual Pseudo-zeroes to virtual Zeroes conversion]. There is total absence of NTZ [as Pseudo-zeroes to Zeroes conversion].

$$\sum_{n=1}^{\infty}(2n)^{-\frac{1}{2}}\sin(t\ln(2n))-\sum_{n=1}^{\infty}(2n-1)^{-\frac{1}{2}}\sin(t\ln(2n-1))=0 \tag{5}$$

Eq. (5) can also be equivalently written as
$\sum_{n=1}^{\infty}(2n)^{-\frac{1}{2}}\cos(t\ln(2n)-\frac{1}{2}\pi)-\sum_{n=1}^{\infty}(2n-1)^{-\frac{1}{2}}\cos(t\ln(2n-1)-\frac{1}{2}\pi)=0.$

With exact DA homogeneity, Eq. (5) is f(n) Gram[y=0] points-sim-η(s) at $\sigma = \frac{1}{2}$ that will incorporate all Gram[y=0] points [as Zeroes]. There is total absence of virtual Gram[y=0] points [as virtual Zeroes].

$$-\frac{1}{2(t^2+\frac{1}{4})^{\frac{1}{2}}}\cdot\left[(2n)^{\frac{1}{2}}(\cos(t\ln(2n)-\frac{1}{4}\pi)-\cos(t\ln(2n-1)-\frac{1}{4}\pi))+C\right]_1^{\infty}=0 \tag{6}$$

Eq. (6) can also be equivalently written as
$\frac{1}{2(t^2+\frac{1}{4})^{\frac{1}{2}}}\cdot\left[(2n)^{\frac{1}{2}}(\cos(t\ln(2n)+\frac{3}{4}\pi)-\cos(t\ln(2n-1)+\frac{3}{4}\pi))+C\right]_1^{\infty}=0.$

With exact DA homogeneity, Eq. (6) is F(n) Gram[y=0] points-DSPL at $\sigma = \frac{1}{2}$ that will incorporate all Gram[y=0] points [as Pseudo-zeroes to Zeroes conversion]. There is total absence of virtual Gram[y=0] points [as virtual Pseudo-zeroes to virtual Zeroes conversion].

$$\sum_{n=1}^{\infty}(2n)^{-\frac{1}{2}}\cos(t\ln(2n))-\sum_{n=1}^{\infty}(2n-1)^{-\frac{1}{2}}\cos(t\ln(2n-1))=0 \tag{7}$$

With exact DA homogeneity, Eq. (7) is f(n) Gram[x=0] points-sim-η(s) at $\sigma = \frac{1}{2}$ that will incorporate all Gram[x=0] points [as Zeroes]. There is total absence of virtual Gram[x=0] points [as virtual Zeroes].

$$\frac{1}{2(t^2+\frac{1}{4})^{\frac{1}{2}}} \cdot \left[(2n)^{\frac{1}{2}}(\cos(t\ln(2n)-\frac{3}{4}\pi)-\cos(t\ln(2n-1)-\frac{3}{4}\pi))+C\right]_1^\infty = 0 \tag{8}$$

With exact DA homogeneity, Eq. (8) is F(n) Gram[x=0] points-DSPL at $\sigma = \frac{1}{2}$ that will incorporate all Gram[x=0] points [as Pseudo-zeroes to Zeroes conversion]. There is total absence of virtual Gram[x=0] points [as virtual Pseudo-zeroes to virtual Zeroes conversion].

We outline sim-η(s) as Eq. (2) and DSPL as Eq. (4) that comply with inexact DA homogeneity at $\sigma = \frac{2}{5}$ non-critical line (depicted by Figure 5) whereby $\sigma = \frac{2}{5}$ [instead of $\sigma = \frac{1}{2}$] is substituted into these two equations. Using [selective] trigonometric identity for linear combination of sine and cosine function whenever applicable to relevant f(n) = 0 and F(n) = 0 equations, we outline exact DA homogeneity at $\sigma = \frac{1}{2}$ critical line (depicted by Figure 4) for Gram[x=0,y=0] points (NTZ) as Eq. (1), Gram[y=0] points as Eq. (5) and Gram[x=0] points as Eq. (7). However, f(n) = 0 equations for Gram[y=0] points as Eq. (5) and Gram[x=0] points as Eq. (7) with exact DA homogeneity at $\sigma = \frac{1}{2}$ critical line are not amendable to treatments using trigonometric identity with implication that their corollary situation endowed with inexact DA homogeneity at $\sigma \neq \frac{1}{2}$ non-critical lines (depicted by Figures 5 and 6) will only manifest solitary [unmixed] $\neq \frac{1}{2}$ fractional exponents. We provide [self-explanatory] corresponding f(n) = 0 equations below for Gram[y=0] points and Gram[x=0] points corollary situation when $\sigma = \frac{2}{5}$.

$$\sum_{n=1}^{\infty}(2n)^{-\frac{2}{5}}\sin(t\ln(2n)) - \sum_{n=1}^{\infty}(2n-1)^{-\frac{2}{5}}\sin(t\ln(2n-1)) = 0$$

$$\sum_{n=1}^{\infty}(2n)^{-\frac{2}{5}}\cos(t\ln(2n)) - \sum_{n=1}^{\infty}(2n-1)^{-\frac{2}{5}}\cos(t\ln(2n-1)) = 0$$

We arbitrarily chose single cosine wave with format R$cos(n \pm \alpha)$ to use where R is scaled amplitude and α is phase shift. In equations for NTZ, Gram[y=0] points and Gram[x=0] points; all their approximate Areas of Varying Loops $\propto$ precise Areas of Varying Loops with R validly treated as a proportionality factor. We analyze f(n) = 0 and F(n) = 0 equations at $\sigma = \frac{1}{2}$ critical line for NTZ where R = $2^{\frac{1}{2}}(2n)^{-\frac{1}{2}}$ or $2^{\frac{1}{2}}(2n-1)^{-\frac{1}{2}}$ in f(n)'s Eq. (1) and R = $\frac{1}{2^{\frac{1}{2}}\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n)^{\frac{1}{2}}$ or $\frac{1}{2^{\frac{1}{2}}\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n-1)^{\frac{1}{2}}$ in F(n)'s Eq. (3).

Remark H.1. Whereas for NTZ F(n) Eq. (3) that exactly represent precise Areas of Varying Loops and f(n) Eq. (1) [when interpreted as Riemann sum] that exactly represent approximate Areas of Varying Loops in a proportionate manner; so must the associated scaled amplitude R from Eq. (3) **which is dependent on parameter t** and Eq. (1) **which is independent of parameter t** represent [in a surrogate manner] corresponding precise and approximate Areas of Varying Loops in a proportionate manner.

We analyze f(n) = 0 equations [relevant to approximate Areas of Varying Loops] at $\sigma = \frac{1}{2}$ critical line for Gram[y=0] points as Eq. (5) and Gram[x=0] points as Eq. (7) whereby we validly designate R = $(2n)^{-\frac{1}{2}}$ or $(2n-1)^{-\frac{1}{2}}$ as the assigned scaled amplitude and [unwritten] $\alpha = 0$ as the assigned phase shift.

Relevant to precise Areas of Varying Loops at $\sigma = \frac{1}{2}$ critical line for Gram[y=0] points F(n) Eq. (6) with R = $-\frac{1}{2\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n)^{\frac{1}{2}}$ or $-\frac{1}{2\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n-1)^{\frac{1}{2}}$ and Gram[x=0] points F(n) Eq. (8) with R = $\frac{1}{2\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n)^{\frac{1}{2}}$ or $\frac{1}{2\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n-1)^{\frac{1}{2}}$, we observe the former R to be the negative of the later R. However, this observation is context-sensitive because when Eq. (6) is written in its equivalent format above, the former R is identical to the later R. Both R are now just given by $\frac{1}{2\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n)^{\frac{1}{2}}$ or $\frac{1}{2\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n-1)^{\frac{1}{2}}$.

Remark H.2. Whereas for Gram[y=0] points F(n) Eq. (6) that exactly represent precise Areas of Varying Loops and f(n) Eq. (5) [when interpreted as Riemann sum] that exactly represent approximate Areas of Varying Loops in a proportionate manner; so must the associated scaled amplitude R in Eq. (6) **which is dependent on parameter t** and Eq. (5) **which is**

independent of parameter t represent [in a surrogate manner] corresponding precise and approximate Areas of Varying Loops in a proportionate manner.

Remark H.3. Whereas for Gram[x=0] points F(n) Eq. (8) that exactly represent precise Areas of Varying Loops and f(n) Eq. (7) [when interpreted as Riemann sum] that exactly represent approximate Areas of Varying Loops in a proportionate manner; so must the associated scaled amplitude R in Eq. (8) **which is dependent on parameter t** and Eq. (7) **which is independent of parameter t** represent [in a surrogate manner] corresponding precise and approximate Areas of Varying Loops in a proportionate manner.

Finally, we analyze f(n) = 0 and F(n) = 0 equations at $\sigma = \frac{1}{2}$ critical line for NTZ where phase shift $\alpha = \frac{1}{4}\pi$ in NTZ f(n) Eq. (1) and $-\frac{1}{4}\pi$ in NTZ F(n) Eq. (3); and F(n) = 0 equations at $\sigma = \frac{1}{2}$ critical line for Gram[y=0] points and Gram[x=0] points where phase shift $\alpha = -\frac{1}{4}\pi$ (or $\frac{3}{4}\pi$ when written in its equivalent format above) in Gram[y=0] points F(n) Eq. (6) and $-\frac{3}{4}\pi$ in Gram[x=0] points F(n) Eq. (8). Always being $\frac{1}{2}\pi$ out-of-phase with each other, trigonometric functions sine and cosine are cofunctions with sin n = cos ($\frac{\pi}{2}$ - n) or cos (n - $\frac{\pi}{2}$), cos n = sin ($\frac{\pi}{2}$ - n) or sin (n + $\frac{\pi}{2}$), $\frac{d(\sin n)}{dn} = \cos n$, $\frac{d(\cos n)}{dn} = -\sin n$, $\int \sin n \cdot dn = -\cos n + C$ [= sin (n - $\frac{\pi}{2}$) + C] and $\int \cos n \cdot dn = \sin n + C$ [= cos (n - $\frac{\pi}{2}$) + C]. Last two integrals explain relation between f(n)'s Zeroes and F(n)'s Pseudo-zeroes when they involve simple sine and/or cosine terms viz, f(n)'s CP Zeroes = F(n)'s CP Pseudo-zeroes – $\frac{1}{2}\pi$ with CP Zeroes and CP Pseudo-zeroes being $\frac{1}{2}\pi$ out-of-phase with each other.

We show that NTZ obtained directly from IP Zeroes and indirectly from IP Pseudo-zeroes behave in accordance with complex sine and/or cosine terms present in their equations that are $\frac{1}{2}\pi$ out-of-phase with each other. Involving trigonometric functions as complex sine and/or cosine terms: f(n)'s IP NTZ or [non-existent] f(n)'s IP virtual NTZ (in t values) = F(n)'s IP Pseudo-NTZ or [non-existent] F(n)'s IP virtual Pseudo-NTZ (in t values) – $\frac{1}{2}\pi$; f(n)'s IP Gram[y=0] points or f(n)'s IP virtual Gram[y=0] points (in t values) = F(n)'s IP Pseudo-Gram[y=0] points or F(n)'s IP virtual Pseudo-Gram[y=0] points (in t values) – $\frac{3}{4}\pi$; and f(n)'s IP Gram[x=0] points or f(n)'s IP virtual Gram[x=0] points (in t values) = F(n)'s IP Pseudo-Gram[x=0] points or F(n)'s IP virtual Pseudo-Gram[x=0] points (in t values) – $\frac{3}{4}\pi$.

$\int f(n)dn = F(n) + C$ where $F'(n) = f(n)$. f(n) and F(n) are literally [connected] **bijective (both injective and surjective or a one-to-one correspondence) functions**. Underlying f(n) as equation and F(n) as law (equation) that generate their CIS of IP Zeroes, IP virtual Zeroes, IP Pseudo-zeroes and IP virtual Pseudo-zeroes are precisely related as $\frac{1}{2}\pi$ (for NTZ case) or $\frac{3}{4}\pi$ (for Gram[y=0] points and Gram[x=0] points cases) out-of-phase with each other. Peculiar to IP NTZ as Origin intercept points, we crucially note only they will uniquely behave in accordance with complex sine and/or cosine terms present in their equations that generate corresponding IP Zeroes and IP Pseudo-zeroes which are $\frac{1}{2}\pi$ [but not $\frac{3}{4}\pi$] out-of-phase with each other.

We show that corresponding paired IP two types of Gram points [as Zeroes] situation, paired IP two types of virtual Gram points [as virtual Zeroes] situation, paired IP two types of Pseudo-Gram points [as Pseudo-zeroes] situation, and paired IP two types of virtual Pseudo-Gram points [as virtual Pseudo-zeroes] situation are always $\frac{1}{2}\pi$ out-of-phase with each other in every one of these situations. The x-axis and y-axis are orthogonal to each other with angle between them = $\frac{1}{2}\pi$ radian. Involving trigonometric functions as complex sine and/or cosine terms: f(n)'s IP Gram[y=0] points or f(n)'s IP virtual Gram[y=0] points (in t values) = f(n)'s IP Gram[x=0] points or f(n)'s IP virtual Gram[x=0] points (in t values) + $\frac{1}{2}\pi$; and F(n)'s IP Pseudo-Gram[y=0] points or F(n)'s IP virtual Pseudo-Gram[y=0] points (in t values) = F(n)'s IP Pseudo-Gram[x=0] points or F(n)'s IP virtual Pseudo-Gram[x=0] points (in t values) + $\frac{1}{2}\pi$.

These observations imply underlying f(n) as equation and F(n) as law (equation) that generate corresponding paired IP two types of Gram points [as Zeroes] situation, paired IP two types of virtual Gram points [as virtual Zeroes] situation, paired IP two types of Pseudo-Gram points [as Pseudo-zeroes] situation, and paired IP two types of virtual Pseudo-Gram points [as virtual Pseudo-zeroes] situation are always $\frac{1}{2}\pi$ out-of-phase with each other in every one of these mentioned situations.

The $\sigma = \frac{1}{2}$ NTZ computed from Eq. (1) – $\sigma \neq \frac{1}{2}$ (non-existent) virtual NTZ computed from Eq. (2) Pairing. For i = 1, 2, 3,..., ∞; let mutually exclusive ith NTZ = NTZ$_i$ and ith virtual NTZ = vNTZ$_i$, and ith NTZ gaps = NTZ-Gap$_i$ and ith virtual NTZ gaps = vNTZ-Gap$_i$. Eq. (1) and Eq. (2) are dependently identical except for associated σ values. They are used to precisely, tediously and dependently calculate all NTZ$_i$ and vNTZ$_i$ with their ith positions being IP.

I. NTZ or Gram [x=0,y=0] points as geometrical Origin intercept points are mathematically defined by $\sum ReIm\{\eta(s)\} = Re\{\eta(s)\} + Im\{\eta(s)\} = 0$. General equation for f(n)'s sim-η(s) as Zeroes is given by

$$\sum_{n=1}^{\infty} -(2n)^{-\sigma}(\sin(t\ln(2n)) - \cos(t\ln(2n)))$$
$$-\sum_{n=1}^{\infty} -(2n-1)^{-\sigma}(\sin(t\ln(2n-1)) - \cos(t\ln(2n-1))) = 0 \tag{9}$$

General equation for F(n)'s DSPL with ability for Pseudo-zeroes to Zeroes conversion is given by

$$\begin{aligned}\frac{1}{2\left(t^2+(\sigma-1)^2\right)} \cdot \Big[&(2n)^{1-\sigma}((t+\sigma-1)\sin(t\ln(2n)) + (t-\sigma+1)\\ &\cdot\cos(t\ln(2n))) - (2n-1)^{1-\sigma}((t+\sigma-1)\\ &\cdot\sin(t\ln(2n-1)) + (t-\sigma+1)\cos(t\ln(2n-1))) + C\Big]_1^{\infty} = 0\end{aligned} \tag{10}$$

II. Gram[y=0] points as geometrical x-axis intercept points are mathematically defined by $\sum \mathrm{ReIm}\{\eta(s)\} = \mathrm{Re}\{\eta(s)\} + 0$, or simply Im $\{\eta(s)\} = 0$. General equation for f(n)'s Gram[y=0] points-sim-η(s) as Zeroes is given by

$$\sum_{n=1}^{\infty}(2n)^{-\sigma}\sin(t\ln(2n)) - \sum_{n=1}^{\infty}(2n-1)^{-\sigma}\sin(t\ln(2n-1)) = 0 \tag{11}$$

General equation for F(n)'s Gram[y=0] points-DSPL with ability for Pseudo-zeroes to Zeroes conversion is given by

$$-\frac{1}{2\left(t^2+(\sigma-1)^2\right)} \cdot \Big[(2n)^{1-\sigma}((\sigma-1)\sin(t\ln(2n)) + t\cos(t\ln(2n)))-$$
$$(2n-1)^{1-\sigma}((\sigma-1)\sin(t\ln(2n-1)) + t\cos(t\ln(2n-1))) + C\Big]_1^{\infty} = 0 \tag{12}$$

III. Gram[x=0] points as geometrical y-axis intercept points are mathematically defined by $\sum \mathrm{ReIm}\{\eta(s)\} = 0 + \mathrm{Im}\{\eta(s)\}$, or simply Re$\{\eta(s)\} = 0$. General equation for f(n)'s Gram[x=0] points-sim-η(s) as Zeroes is given by

$$\sum_{n=1}^{\infty}(2n)^{-\sigma}\cos(t\ln(2n)) - \sum_{n=1}^{\infty}(2n-1)^{-\sigma}\cos(t\ln(2n-1)) = 0 \tag{13}$$

General equation for F(n)'s Gram[x=0] points-DSPL with ability for Pseudo-zeroes to Zeroes conversion is given by

$$\frac{1}{2\left(t^2+(\sigma-1)^2\right)} \cdot \Big[(2n)^{1-\sigma}(t\sin(t\ln(2n)) - (\sigma-1)\cos(t\ln(2n)))-$$
$$(2n-1)^{1-\sigma}(t\sin(t\ln(2n-1)) - (\sigma-1)\cos(t\ln(2n-1))) + C\Big]_1^{\infty} = 0 \tag{14}$$

I. Riemann zeta function and related functions

We follow the abbreviations listed under General notations in subsection 2.2. $\zeta(s)$ is a function of complex variable s (= $\sigma \pm \imath$t) that analytically continues sum of infinite series $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$. The common convention is to write s as $\sigma + \imath$t with $\imath = \sqrt{-1}$, and with σ and t real. Valid for $\sigma > 0$, we write $\zeta(s)$ as $Re\{\zeta(s)\}+\imath Im\{\zeta(s)\}$ and note that $\zeta(\sigma + \imath t)$ when $0 < t < +\infty$ is the complex conjugate of $\zeta(\sigma - \imath t)$ when $-\infty < t < 0$.

Also known as alternating zeta function, $\eta(s)$ must act as *proxy* for $\zeta(s)$ in critical strip (viz, $0 < \sigma < 1$) containing critical line (viz, $\sigma = \frac{1}{2}$) because $\zeta(s)$ only converges when $\sigma > 1$. This implies $\zeta(s)$ is undefined to left of this $\sigma > 1$ region [in the critical strip] which then requires $\eta(s)$ representation instead. They are related to each other as $\zeta(s) = \gamma \cdot \eta(s)$ with proportionality factor $\gamma = \dfrac{1}{(1-2^{1-s})}$ and $\eta(s) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^s} = \dfrac{1}{1^s} - \dfrac{1}{2^s} + \dfrac{1}{3^s} - \cdots$.

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \\ &= \Pi_{p\ prime} \frac{1}{(1-p^{-s})} \\ &= \frac{1}{(1-2^{-s})}.\frac{1}{(1-3^{-s})}.\frac{1}{(1-5^{-s})}.\frac{1}{(1-7^{-s})}.\frac{1}{(1-11^{-s})}\cdots\frac{1}{(1-p^{-s})}\cdots\end{aligned} \tag{15}$$

Eq. (15) is defined for only $1 < \sigma < \infty$ region where $\zeta(s)$ is absolutely convergent with no zeros located here. In Eq. (15), equivalent Euler product formula with product over prime numbers [instead of summation over natural numbers] also represents $\zeta(s) \implies$ all prime and, by default, composite numbers are intrinsically encoded in $\zeta(s)$.

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \cdot \Gamma(1-s) \cdot \zeta(1-s) \tag{16}$$

With $\sigma = \frac{1}{2}$ as symmetry line of reflection, Eq. (16) is Riemann's functional equation valid for $-\infty < \sigma < \infty$. It can be used to find all trivial zeros on horizontal line at $\imath$t = 0 occurring when σ = -2, -4, -6, -8, -10,..., ∞ whereby $\zeta(s) = 0$ because factor $\sin(\dfrac{\pi s}{2})$ vanishes. Γ is gamma function, an extension of factorial function [a product function denoted by ! notation whereby n! = $n(n-1)(n-2)\ldots(n-(n-1))$] with its argument shifted down by 1, to real and complex numbers. That is, if n is a positive integer, $\Gamma(n) = (n-1)!$

$$\begin{aligned}\zeta(s) &= \frac{1}{(1-2^{1-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \\ &= \frac{1}{(1-2^{1-s})}\left(\frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots\right)\end{aligned} \tag{17}$$

Eq. (17) is defined for all $\sigma > 0$ values except for simple pole at $\sigma = 1$. As alluded to above, ζ(s) without $\dfrac{1}{(1-2^{1-s})}$ viz. $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^s}$ is η(s). It is a holomorphic function of s defined by analytic continuation and is also defined at $\sigma = 1$ whereby analogous trivial zeros with presence for η(s) [but not for ζ(s)] on vertical straight line $\sigma = 1$ are found at $s = 1 \pm \imath \dfrac{2\pi k}{\ln(2)}$ where k = 1, 2, 3, 4,..., ∞.

Euler formula can be stated as $e^{\imath n} = \cos n + \imath \cdot \sin n$. Euler identity (where $n = \pi$) is $e^{\imath\pi} = \cos\pi + \imath \cdot \sin\pi = -1 + 0$ [or stated as $e^{\imath\pi} + 1 = 0$]. The n^s of $\zeta(s)$ is expanded to $n^s = n^{(\sigma+\imath t)} = n^{\sigma} e^{t\ln(n)\cdot\imath}$ since $n^t = e^{t\ln(n)}$. Apply Euler formula to n^s result in $n^s = n^{\sigma}(\cos(t\ln(n)) + \imath \cdot \sin(t\ln(n))$. This is written in trigonometric form [designated by short-hand notation $n^s(Euler)$] whereby n^{σ} is modulus and $t\ln(n)$ is polar angle (argument).

We apply $n^s(Euler)$ to Eq. (17) to obtain f(n) general sim-$\eta(s)$ for determining $\sigma = \frac{1}{2}$ NTZ versus (non-existent) $\sigma \neq \frac{1}{2}$ virtual NTZ (Ting, 2020, p. 24 - 28). At $\sigma = \frac{1}{2}$, this is given as Eq. (9) and with the trigonometric identity application as Eq. (1). Integrate f(n) general sim-$\eta(s)$ to obtain F(n) general DSPL for determining $\sigma = \frac{1}{2}$ Pseudo-zeroes versus (non-existent) $\sigma \neq \frac{1}{2}$ virtual Pseudo-zeroes. Pseudo-zeroes and (non-existent) virtual Pseudo-zeroes can be converted to Zeroes (NTZ) and (non-existent) virtual Zeroes (virtual NTZ). At $\sigma = \frac{1}{2}$, this is given as Eq. (10) and with the trigonometric identity application as Eq. (3).

We provide f(n) general Gram[y=0] points-sim-$\eta(s)$ for determining $\sigma = \frac{1}{2}$ Gram[y=0] points versus $\sigma \neq \frac{1}{2}$ virtual Gram[y=0] points (Ting, 2020, p. 28 - 30). At $\sigma = \frac{1}{2}$, this is given as Eq. (11) but we are unable to apply the trigonometric identity. Integrate f(n) general Gram[y=0] points-sim-$\eta(s)$ to obtain F(n) general Gram[y=0] points-DSPL for determining $\sigma = \frac{1}{2}$ Pseudo-zeroes versus $\sigma \neq \frac{1}{2}$ virtual Pseudo-zeroes. Pseudo-zeroes and virtual Pseudo-zeroes can be converted to Zeroes (Gram[y=0] points) and virtual Zeroes (virtual Gram[y=0] points). At $\sigma = \frac{1}{2}$, this is given as Eq. (12) and with the trigonometric identity application as Eq. (6).

We provide f(n) general Gram[x=0] points-sim-$\eta(s)$ for determining $\sigma = \frac{1}{2}$ Gram[x=0] points versus $\sigma \neq \frac{1}{2}$ virtual Gram[x=0] points (Ting, 2020, p. 28 - 30). At $\sigma = \frac{1}{2}$, this is given as Eq. (13) but we are unable to apply the trigonometric identity. Integrate f(n) general Gram[x=0] points-sim-$\eta(s)$ to obtain F(n) general Gram[x=0] points-DSPL for determining $\sigma = \frac{1}{2}$ Pseudo-zeroes versus $\sigma \neq \frac{1}{2}$ virtual Pseudo-zeroes. Pseudo-zeroes and virtual Pseudo-zeroes can be converted to Zeroes (Gram[x=0] points) and virtual Zeroes (virtual Gram[x=0] points). At $\sigma = \frac{1}{2}$, this is given as Eq. (14) and with the trigonometric identity application as Eq. (8).

J. Coprimality and Basic arithmetic operations

We ignore the negative integers. For i = 1, 2, 3, 4, 5..., Set of integer numbers ($\mathbb{Z}_i$) = {0, 1, 2, 3, 4...} as **CIS-IM-linear**. For i = 0, 1, 2, 3, 4..., Set of even numbers ($\mathbb{E}_i$) = {0, 2, 4, 6, 8...} as **CIS-IM-linear** complies with congruence **n** ≡ 0 (mod 2) viz, "Any integer that can be divided exactly by 2 with last digit always being 0, 2, 4, 6 or 8" and with [extra] zeroth even number $\mathbb{E}_0 = 0$. For i = 1, 2, 3, 4, 5..., Set of odd numbers ($\mathbb{O}_i$) = {1, 3, 5, 7, 9...} as **CIS-IM-linear** complies with congruence **n** ≡ 1 (mod 2) viz, "Any integer that cannot be divided exactly by 2 with last digit always being 1, 3, 5, 7 or 9". Then, Set $\mathbb{Z}$ = Set $\mathbb{E}$ + Set $\mathbb{O}$. The four basic arithmetic operations are addition, subtraction, multiplication, and division. We apply these operations to $\mathbb{E}$ and $\mathbb{O}$:
$\mathbb{E}+\mathbb{E} = \mathbb{E}$; $\mathbb{O}+\mathbb{O} = \mathbb{E}$; $\mathbb{E}+\mathbb{O} = \mathbb{O}$; $\mathbb{O}+\mathbb{O}+\mathbb{O} = \mathbb{O}$, $\mathbb{O}+\mathbb{O}+\mathbb{O}+\mathbb{O} = \mathbb{E}$, etc.
We can ignore substraction since it is simply equivalent to addition of negative numbers. Addition of multiple $\mathbb{O}$ with even number of elements will always give rise to an $\mathbb{E}$, and odd number of elements will always give rise to an $\mathbb{O}$. Thus, addition of various combinations of multiple $\mathbb{E}$ and multiple $\mathbb{O}$ that will result in either $\mathbb{E}$ or $\mathbb{O}$ will only depend on whether the number of elements for $\mathbb{O}$ is even or odd.
$\mathbb{E}$ X $\mathbb{E}$ = $\mathbb{E}$; $\mathbb{O}$ X $\mathbb{O}$ = $\mathbb{O}$; $\mathbb{E}$ X $\mathbb{O}$ = $\mathbb{E}$; $\mathbb{O}$ X $\mathbb{O}$ X $\mathbb{E}$ = $\mathbb{E}$, $\mathbb{O}$ X $\mathbb{E}$ X $\mathbb{E}$ = $\mathbb{E}$, etc.
Multiplication of various combinations of multiple $\mathbb{E}$ and multiple $\mathbb{O}$ will only depend on the presence of $\mathbb{E}$, and will always result in $\mathbb{E}$ irrespective of the number of elements for $\mathbb{O}$.
$\mathbb{E}\div\mathbb{E} = \mathbb{E}$ [for exact divisions]; $\mathbb{O}\div\mathbb{O} = \mathbb{O}$ [for exact divisions]; $\mathbb{E}\div\mathbb{O} = \mathbb{E}$ [for exact divisions]; $\mathbb{O}\div\mathbb{E}$ = rational numbers containing terminating decimal numbers or rational numbers containing non-terminating decimal numbers with infinitely repeating patterns. Only inexact divisions are possible for $\mathbb{O}\div\mathbb{E}$.
Division is defined as dividend÷divisor = quotient whereby the dividend and divisor do not have common factors other than 1; and is undefined when the divisor = 0. Division is also identical to multiplying the dividend by the inverse of the divisor. The product of a fraction and its reciprocal is 1, hence the reciprocal is the multiplicative inverse of a fraction. Fraction = Numerator [or Dividend] / Denominator [or Divisor] which in its simpliest form can have either terminating or non-terminating decimal representations. All terminating decimals can be expressed as $\frac{a}{10^n}$. Fractions whose denominator does not include 5 and 2 as factors are non-terminating decimals. Fractions whose denominator is a power of 2 (e.g. 1/4 and 1/8) are terminating decimals. Fractions whose denominator is a power of 5 (e.g. 1/5, 1/25, 1/125) are terminating decimals. Fractions whose denominator includes 2 as a factor and includes a factor not equal to 5 are non-terminating decimals (e.g. 1/6 and 1/14). Fractions whose denominator includes 5 as a factor and includes a factor not equal to 2 (e.g. 1/15) are non-terminating decimals. From these observations, we conjecture a fraction is a terminating decimal if its factors are 2 or its powers, 5 or its powers, or both. If a fractions is in the form $\frac{xy}{(x)(5^c)(2^d)}$, it is terminating.

Algorithm to decide whether a fraction will result in a terminating decimal include the following steps:
1. Reduce the fraction to its simpliest form in which the numerator x and denominator y are integers that have no other common divisors than 1 (and −1, when negative numbers are considered). Thus, $\frac{x}{y}$ is irreducible *iff* x and y are coprime $\implies$ no prime number divides both x and y; viz, x and y have their greatest common divisor (GCD) = 1.
2. If denominator y ends in 0 or 5, divide x by 5. Repeat until y does not end in 0 or 5.
3. If denominator y ends in 0 or 2 or 4 or 6 or 8, divide y by 2. Repeat until y is an odd number.
4. If denominator y = 1, the fraction will result in a terminating decimal. Otherwise, the fraction will not result in a terminating decimal.

A set of integers S = $\{a_1, a_2, a_3, ..., a_n\}$ is setwise coprime if GCD of all its elements = 1. If every pair in this set is coprime, then the set is pairwise coprime. Pairwise coprimality is a stronger condition than setwise coprimality. Every pairwise coprime finite set is also setwise coprime but the reverse is not true. It is possible for an infinite set of integers to be [completely] pairwise coprime with notable examples being set of all prime numbers, set of elements in Sylvester's sequence, and set of all Fermat numbers. Then it is also possible for an infinite set of integers to not be [completely] pairwise coprime with simplest example being set of all composite numbers. In perspective, prime numbers are (smaller deceleratingly reaching) countably arbitrarily large in number whereas [complementary] composite number are (larger acceleratingly reaching) countably infinitely many in number.

The Chinese remainder theorem says we can uniquely solve every pair of, or larger congregation of, congruences having relatively prime (coprime) moduli. Let m and n be relatively prime (coprime) positive integers. For all integers a and b, the pair of congruences x ≡ a mod m, x ≡ b mod n has a solution, and this solution is uniquely determined modulo mn. More generally, for r ≥ 2, let m_1, m_2,..., m_r be nonzero integers that are pairwise relatively prime (coprime): (m_i, m_j) = 1 for i = j. Then, for all integers a_1, a_2,..., a_r, the system of congruences x ≡ a_1 mod m_1, a_2 mod m_2,..., a_r mod m_r, has a solution, and this solution is uniquely determined modulo $m_1m_2...m_r$. With implications for previous paragraph, then employing Chinese remainder theorem meant that we can uniquely solve every pair of, or larger congregation of, congruences having pairwise, and not just setwise, relatively prime (coprime) moduli.

K. Rational and irrational numbers

The decimal system, or base-10 system with 10 as its base, will be used throughout this paper. Other base systems include binary (base-2), ternary (base-3), quaternary (base-4), quinary (base-5), senary (base-6), etc. Since the base is just a representation of the physical amount; integers such as prime and composite numbers, and transcendental numbers such as nontrivial zeros and closely related two types of Gram points are intuitively expected to generally have the same mathematical properties across any nominated base system. Based on the digits after decimal point, decimal numbers are divided into three types: (i) terminating, (ii) non-terminating but recurring, and (iii) non-terminating and non-recurring [with (i) and (ii) constituting rational numbers, and (iii) constituting algebraic and transcendental irrational numbers]. Note also that (i), (ii) and (iii) as numbers all represent zero-dimensional points whereby (i) and (ii) are able to do so with 100% accuracy but (iii) is only able to do so with accuracy that progressively increases with higher number of non-terminating and non-recurring digits after decimal point being supplied.

An algebraic irrational number [e.g. golden ratio ϕ, $\sqrt{2}$, etc which together form a countably infinite set] is a number that is a root of a non-zero polynomial in one variable with integer (or, equivalently, rational) coefficients. A transcendental irrational number [e.g. nontrivial zeros, Liouville numbers, π, e, etc which together form an uncountably infinite set] is a number that is not an algebraic irrational number; viz, it is not the root of a non-zero polynomial of finite degree with rational coefficients. Thus, irrational numbers in totality will constitute an uncountably infinite set with both types of irrational numbers containing non-terminating and non-recurring digits. An important property of algebraic irrational numbers essentially states that these numbers cannot be well approximated by rational numbers. Since transcendental irrational numbers do not possess this propoerty, they cannot be algebraic and must be transcendental. Irrational numbers can be further classifed in a useful manner under Isolated countably finitely-sized group or Connected countably infinitely-sized group. For instance, the [isolated] countably finite π, e and ϕ belong to the former group, and the [connected] countably infinite nontrivial zeros and closely related two types of Gram points, square roots, cube roots and Liouville numbers belong to the later group.

A Liouville number is a transcendental number having very close rational number approximations. An irrational number x is called a Liouville number if, for each positive integer n, there exist a pair of integers (p, q) with p > 0 and q > 1 such that $0 <| x - \frac{p}{q} |< \frac{1}{q^n}$. For any integer b ≥ 2 [viz, base ≥ 2] and any sequence of integers (a_1, a_2, a_3..., a_k) such that $a_k \in$ {0, 1, 2, 3,... b-1} for all k and $a_k \neq 0$ for infinitely many k, we define the constructed Liouville number as $x = \sum_{k=1}^{\infty} \frac{a_k}{b^{k!}}$. In the special case when b = 10, and $a_k = 1$ for all k, the resulting number x is called Liouville's constant – a decimal fraction with a 1 in each decimal place corresponding to a factorial n!, and zeros everywhere else. Liouville numbers are precisely those transcendental irrational numbers that can be more closely approximated by rational numbers than any algebraic irrational number.

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