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European Barrier Range Accrual Option Pricing Formula Deduction and the Corresponding American Range Option Numerical Value Simulation

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Abstract

European Range Accrual Option pricing and deviation Formula has been deduced through observing the foundational asset probabilistic distribution characteristics with the help of Ito's lemma, and through relaxing the boundary assumption to infinity and zero respectively, the classical Black-Scholes option formula has been worked out. This paper subsequently articulates the numerical value simulated computation algorithm using logic program language for the corresponding demonstration. From a statistical point of view, the American range option is not definitely more valuable than the corresponding European range option and the difference between their deviations is significant.

Keywords: barrier range accrual option, pricing formula, expected mean, deviation, logic algorithm

1. Background Introduction

A traditional European option can be stricken at maturity while its American counterpart can be executed when it is in the money any time before maturity. This paper defines a kind of barrier interval option, called Barrier Range Accrual Option, which means when the foundational asset's price lies in a certain interval [L, H], the option can be executed, otherwise the payoff will be zero. Here L denotes the lower bound, and H the upper bound. We also define the strike price K which is in the interval , that is $K\in [L, H]$. Between L and K, we define the put option, and between K and H, we define the call option. The Barrier Range Accrual Option can also be classified in to American and European styles. We will subsequently discuss the two kinds of options' pricing mechanism.

2.1 European Barrier Range Accrual Option Pricing Formula Deduction

Still we describe the movement of the foundational asset satisfies

$ds_t = s_t.rdt + s_t.\sigma(t)dW_t$, the classical asset movement equation with drift term in terms of time and disturbance term which conforms to Brownian movement principle, that is $dW_t \sim(0,\mathrm{t})$, i.e., the sample mean is zero and the variance is t. The initial condition is $s_{t_0} = s_0$, and set $\mathrm{F}(s_t,t) = Ln\, s_t$, $\mathrm{k} = \mathrm{LnK}$. According to Ito's lemma,

$\mathrm{dF}(s_t,t) = \frac{\partial F}{\partial x}dx + \left[\frac{\partial F}{\partial t} + \frac{1}{2}s_t^2\sigma(t)^2\frac{\partial^2 F}{\partial^2 x}\right]d\mathrm{t}$

$=\left(r - \frac{1}{2}\sigma(t)^2\right)dt + \sigma(t)dW_t$.

Considering the initial condition $s_{t_0} = s_0$, we have

$$\mathrm{Ln}\, s_t = Ln\, s_0 + \left(r - \frac{1}{2}\sigma(t)^2\right)t + \sigma(t)\Delta W_t$$

ΔW_t of which represents the accrual variation in time span from the initial beginning to t and $\Delta W_t \sim(0,\mathrm{t})$. Hence, $\mathrm{Ln}\, s_t \sim(Ln\, s_0 + \left(r - \frac{1}{2}\sigma(t)^2\right)t,\ \sigma(t)^2 t)$, that is s_t conforms to logarithmic normal distribution, the derivation process of which is relegated to appendix 1.

From the definition of the barrier range accrual option definition above, we have the payoff formula as below, which actually is an expected mean:

$$U_{S-K} = \mathrm{E}_{t=0}(S_t - \mathrm{K})$$
$$= \mathrm{E}_{t=0}((Max(S_t - K), 0)_{1_{S_t \geq K}} + (Max(K - S_t), 0)_{1_{S_t \leq K}}$$
$$= e^{-rt}\left[\int_{Lnk}^{Lnh}(e^s - e^k)\frac{1}{\sqrt{2\pi}\sigma\sqrt{t}}e^{\frac{-\left[s-\left(Ln\, s_0+\left(r-\frac{1}{2}\sigma(t)^2\right)t\right)\right]^2}{2\sigma^2}}ds + \int_{Lnl}^{LnK}(e^k - e^s)\frac{1}{\sqrt{2\pi}\sigma\sqrt{t}}e^{\frac{-\left[s-\left(Ln\, s_0+\left(r-\frac{1}{2}\sigma(t)^2\right)t\right)\right]^2}{2\sigma^2}}ds\right].$$

From a risk neutral perspective, we should use e^{-rt} to discount the result above to the initial starting time t_0, this is why there is a e^{-rt} multiplied at the beginning of the formula.

Through transformation and using the definition of normal distribution function, we have the above representation formula abbreviated to

$$s_0\big(N(d1) - 2.N(d2) + N(d3)\big) - k.e^{-rt}[N(d4) - 2.N(d5) + N(d6)],$$

where d1=$\frac{Ln\frac{H}{s_0}-\left(r+\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}}$, $d2 = \frac{Ln\frac{K}{s_0}-\left(r+\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}}$, d3=$\frac{Ln\frac{L}{s_0}-\left(r+\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}}$;

d4=$\frac{Ln\frac{H}{s_0}-\left(r-\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}}$, d5=$\frac{Ln\frac{K}{s_0}-\left(r-\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}}$, d6=$\frac{Ln\frac{L}{s_0}-\left(r-\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}}$。

Note: From the beginning to here above, the signal $\sigma(t)$ and σ are used interchangeably without obfuscation, which both refer to the variation rate corresponding to the time interval [0,t]. For convenience, we set it to be constant in the numerical simulation in the following part.

The deduction for Barrier Range Accrual Option Pricing Formula has been completed and the process is demonstrated in appendix2.

2.2 The Derivation for Black-Scholes Option Formula and the Call Put Option Parity Formula

From representation formula above, we can see the first item is call option payoff function and the second is put option payoff function. If we relax the boundary assumption to infinity and zero respectively, that is, set$H \to +\infty$and $L \to 0$, we can get the Black-Scholes call option formula and the corresponding put option pricing formula, both of which will simultaneously prove S+P=C+PV(K) as the call put option parity formula, with $PV(K) = e^{-rt}K$.

3. The Corresponding Deviation Deduction

For $\mathrm{D}(S_t - \mathrm{K}) = \mathrm{E}_{t=0}(S_t - \mathrm{K})^2 - [\mathrm{E}_{t=0}(S_t - \mathrm{K})]^2$,of which the second item has already been worked out above, the deduction for the first item will be demonstrated here.

$$\mathrm{E}_{t=0}(S_t - \mathrm{K})^2 = \int_{LnL}^{LnH}[e^{-rt}(e^s - \mathrm{K})]^2\frac{1}{\sigma\sqrt{2\pi t}}\exp\left(\frac{-\left[x-\left(LNS_0+\left(r-\frac{1}{2}\sigma^2\right)t\right)\right]}{2\sigma^2 t}\right)ds$$

$$=S_0{}^2\exp(\sigma^2 t)\,[N(d7) - N(d8)] - 2Kexp(-rt)S_0[N(d9) - N(d10)] + \exp(-2rt)K^2[N(d11) - N(d12)],$$

In which,

d7=$\frac{Ln\frac{H}{S_0}-\left(r+\frac{3}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}}$,$d8 = \frac{Ln\frac{L}{S_0}-\left(r+\frac{3}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}}$, d9=$\frac{Ln\frac{H}{S_0}-\left(r+\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}}$;

d10=$\frac{Ln\frac{L}{S_0}-\left(r+\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}}$, d11=$\frac{Ln\frac{H}{S_0}-\left(r-\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}}$, d12=$\frac{Ln\frac{L}{S_0}-\left(r-\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}}$.

Hence,

$$\begin{aligned}\mathrm{D}(S_t - \mathrm{K}) = {} & S_0{}^2\exp(\sigma^2 t)\,[N(d7) - N(d8)] - 2Kexp(-rt)S_0[N(d9) - N(d10)] \\ & + \exp(-2rt)\,K^2[N(d11) - N(d12)] \\ & - \{S_0\big(N(d1) - 2.N(d2) + N(d3)\big) - K.\exp(-rt)[N(d4) - 2.N(d5) + N(d6)]\}^2\end{aligned}$$

Similarly, the same relaxation can be implemented to research the classical call option formula's deviation. The derivation process for the conclusion above can be gained via imitation of that for the payoff function above.

4. The Corresponding Statistical Test

4.1 The Expected Mean Test

For a single option, it's difficult to gain continuous data in the market. This paper utilizes stochastic simulation to acquire sample data to prove the validity of the formula. With reference to the test result, with parameter t=1 year, $\sigma(t) = 20\%$、r=5%、$S_0 = 45$, calculate $[(\overline{S-K})\text{-}U_{S-K}]/(S_{S-K}/\sqrt{n})$=0.0876<<$t_{0.10} = 1.796$, so the hypothesis that $\overline{S-K} = U_{S-K}$ can't be rejected, the process of which is specified in appendix3.

4.2 The Deviation Test

According to the deduced deviation formula, with the aforementioned parameters, barrier range option's expected deviation is 4.6446. Based on the deviation test result $\frac{(n-1)S_{S-K}}{D(S_t-K)}$=10.302∈$(\chi^2_{0.975}(11), \chi^2_{0.025}(11))$=(3.82,21.90), the original hypothesis can't be rejected and the deduction of deviation formula is validated.

5. Logic Algorithm for American Barrier Range Accrual Option Pricing and Numerical Simulation

As widely known in academic society, there is no closed form or analytical solution formula for American Option Pricing, which is also agreed in this paper. Here we try to use logic computer algorithm to give the numerical value simulation process and the corresponding result, which will prove under the assumption at the beginning that American style Barrier Range Accrual Option is slightly not more valuable than its counter European style.

Here is the logic algorithm as follows:

$$\text{set} \quad S_{t_0} = S_0 \ , r = R, \ \sigma(t) = \sigma$$

For i=1 to n

$$S_{t_i} = S_0 + S_0.r.i + i.S_0.\sigma(t).Norm.S.Inv(Rand(\ \ ))$$

$$if\ S_{t_i} \geq L\ and\ S_{t_i} \leq H,$$

$$Payoff_i = \exp(-ri)\,.ABS(S_{t_i} - K)$$

Else $Payoff_i = 0$

Endif

AROptionValue=average($Payoff_i$)

Print "AROptionValue"

Rand() is a stochastic number occurrence function in Excel and returns stochastic number between 0 and 1. $Norm.S.Inv(\ \ )$ is the inverse standard normal distribution density function, which returns the corresponding dependent variable value according to endogenous variable generated by the computer inherent mechanism.

This paper follows the procedure above and demonstrated the process and result of the American barrier range accrual option value as in the appendix4.

6. The Comparison of Two Kinds of Opitons

6.1 Value Comparison

The author repeats 12×12 times of the above diagram computation, the sample average is 6.66. While if we use the analytical form pricing formula deducted out in the first part, we can get the expected European barrier range accrual option price is 4.98. According to statistics, $\frac{(\bar{X}-u_x)\sqrt{n}}{S_x}$ conforms to the classical t distribution with n degrees of freedom.

Through calculation, $\frac{(\bar{X}-u_x)\sqrt{n}}{S_x}$ =0.65, and with reference to the Student statistics diagram we can see P（$abs(t(12) > t_{0.10})$）$= 1.782$）, for0.62<<1.782, we can't come to the conclusion that the American range option is definitely more valuable than it's European counterpart.

From the mean comparison, according to appendix4, and through anlaysis, we can see the foundational asset's volatility expands with time lapsing on the condition that we set $\sigma(t)$ to be constant, thus the American option's payoff is reversely smaller than its European counterpart due to strike ahead of maturity, that is, the latter arbitrages relatively more profit due to strike at maturity with more volatility earnings.

6.2 Deviation Comparison

The computed result for $\frac{{S_1}^2}{{S_2}^2}$ is 4.4316, with reference to the two samples' data, while $F_{0.025}(11{,}11) \approx 3.40$, $1/F_{0.025}(11{,}11) \approx 0.29$, it can be known that $\frac{{S_1}^2}{{S_2}^2}$ lies beyond the interval(0.29,3.40), which can be inferred that the deviation difference of the American barrier option and its European counterpart is significant.

7. Remark

There can be closed or analytical form solution formula for the European barrier range accrual option pricing, and logic language algorithm can be used to compute the corresponding counter American style option. Using census and Excel stochastic number generator function, we attain twelve panels of data. From the statistical process and result, we can't not say American option is definitely more valuable than its European counterpart. From an economic point of view, this is because execution ahead of the maturity doesn't always guarantee the executor arbitrage a higher strike profit than at maturity.

The deducted pricing formula in part1 and the discrete simulated method in part5 are both very useful in option valuation, the utility of the latter will be augmented using computer programs at certain computation platforms like C++, python, etc.

What's more, we can deduce other range style option, like barrier knock-in or knock-out option, straddle, etc., with the help of analysis of the foundational asset movement characteristics. Considering the assumption at the beginning and the ensuing derivation process, we can say different foundational asset movement equation will lead to different pricing formula and the relevant statistical property conclusion.

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Appendix

1. $\mathrm{dF}(s_t, t) = \frac{\partial F}{\partial x}dx + \left[\frac{\partial F}{\partial t} + \frac{1}{2}s_t^2\sigma(t)^2\frac{\partial^2 F}{\partial^2 x}\right]dt$

$=\frac{1}{s_t}(s_t.rdt + s_t.\sigma(t)dW_t) + [0 - \frac{1}{2}s_t^2\sigma(t)^2\frac{1}{s_t^2}]dt$

$=\left(r - \frac{1}{2}\sigma(t)^2\right)dt + \sigma(t)dW_t$

2. $e^{-rt}[\int_{Lnk}^{Lnh}(e^s - e^k)\frac{1}{\sqrt{2\pi}\sigma\sqrt{t}}e^{\frac{-\left[s-\left(Ln\,s_0+\left(r-\frac{1}{2}\sigma(t)^2\right)t\right)\right]^2}{2\sigma^2}}ds + \int_{Lnl}^{LnK}(e^k - e^s)\frac{1}{\sqrt{2\pi}\sigma\sqrt{t}}e^{\frac{-\left[s-\left(Ln\,s_0+\left(r-\frac{1}{2}\sigma(t)^2\right)t\right)\right]^2}{2\sigma^2}}ds]$

$=e^{-rt}[\int_{Lnk}^{Lnh}(e^s - K)\frac{1}{\sqrt{2\pi}\sigma\sqrt{t}}e^{\frac{-\left[s-\left(Ln\,s_0+\left(r-\frac{1}{2}\sigma(t)^2\right)t\right)\right]^2}{2\sigma^2}}ds + \int_{Lnl}^{LnK}(\mathrm{K} - e^s)\frac{1}{\sqrt{2\pi}\sigma\sqrt{t}}e^{\frac{-\left[s-\left(Ln\,s_0+\left(r-\frac{1}{2}\sigma(t)^2\right)t\right)\right]^2}{2\sigma^2}}ds].$

$=s_0[N(\frac{Ln\frac{H}{s_0}-\left(r+\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}})\text{-}\ N(\frac{Ln\frac{K}{s_0}-\left(r+\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}})]\text{- K.}\,e^{-rt}[N\left(\frac{Ln\frac{H}{s_0}-\left(r-\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}}\right) - N(\frac{Ln\frac{K}{s_0}-\left(r-\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}})]+\{\ \mathrm{K}.\,e^{-rt}[N\left(\frac{Ln\frac{K}{s_0}-\left(r-\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}}\right) - (\frac{Ln\frac{L}{s_0}-\left(r-\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}})] - s_0[N(\frac{Ln\frac{K}{s_0}-\left(r+\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}})\text{-}\ N(\frac{Ln\frac{L}{s_0}-\left(r+\frac{1}{2}\sigma^2\right)t}{\sigma(t)\sqrt{t}})]\}$

$=s_0\left(N(d1) - 2.N(d2) + N(d3)\right) - k.e^{-rt}[N(d4) - 2.N(d5) + N(d6)]$

3. European range accrual option strike at maturity

Sample	12.18	4.71	6.84	6.45	1.69	-	0.14	3.22	3.35	7.14	4.39	10.84
Mean	5.08											
Deviation	14.69											
Statitical variant	0.088											

4.

Sequence	S_0	Annual risk free rate	σ	InverseNd.Rand()	S_i	L	H	K	Payoff	Discounted Payoff
1	45	0.05	0.2	0.31	45.42	30	45	60	0.42	0.4187

Note: InverseNd.Rand()=NORM.S.INV(RAND())*Sequence/12

Logic juncture / Sequence	Discounted payoff											
	1	2	3	4	5	6	7	8	9	10	11	12
1	0.42	3.10	4.43	6.34	10.69			13.78	9.10			
2	1.18	1.64	6.75	9.01							7.85	
3	0.64	0.04	4.94	5.86		12.84			12.35	11.12		
4	1.10	3.75	5.94	0.21		13.86			3.67			10.63
5	1.18	1.77	2.64	1.77	6.59				10.84			
6	0.52	4.13	6.62			5.29	6.42					
7	0.49	6.02	3.55	4.57	10.13	13.07		4.27				
8	0.82	1.82	2.58	0.49	3.57	6.78						
9	0.19	0.59	3.84									
10	0.73	7.79	2.94	9.84	11.57					0.30		
11	0.90	2.74	0.72	11.57		9.27		7.02				
12	0.75	2.50	4.48	4.43								
Sample mean	0.74	2.99	4.12	5.41	8.51	10.19	6.42	8.36	8.99	5.71	7.85	10.63
Statistical mean	6.66											
Root of deviation	8.99											