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Exponential Stability for Damped Shear Beam Model and New Facts Related to the Classical Timoshenko System With a Distributed Delay Term

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Abstract

We consider in this manuscript a Timoshenko type beam model with a distributed delay term. If the distributed delay term is small enough, we prove the global existence of solutions by using the Faedo-Galerkin approximations together with some energy estimates. Under suitable assumptions, we prove exponential stability of the solution. This result is obtained by introducing a suitable Lyapounov function.

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1. Introduction

In this paper, we investigate the stability of the chear beam model with internal distributed delay term. More precisely we consider the following system :

$$\begin{cases} \rho_1\theta_{tt}(x,t) - \kappa(\theta_x(x,t) + \psi(x,t))_x + \mu_1\theta_t(x,t) + \displaystyle\int_{\tau_0}^{\tau_1} \alpha(s)\theta_t(x,t-s)ds = 0 \text{ in }]0,L[\times(0,+\infty), \\ -b\psi_{xx}(x,t) + \kappa(\theta_x(x,t) + \psi(x,t)) = 0 \text{ in }]0,L[\times(0,+\infty), \\ \theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x), \quad \psi(x,0) = \psi_0(x) \text{ in }]0,L[, \\ \psi_x(0,t) = \psi_x(L,t) = \theta(0,t) = \theta(L,t) = 0, t > 0, \\ \theta_t(x,-t) = f_0(x,-t), \ (x,t) \in]0,L[\times(0,\tau_1). \end{cases} \tag{1}$$

where the functions θ and ψ describe respectively the transverse displacement of the beam and the rotation angle of a filament of the beam.
$\rho_1, \kappa, \mu_1, b, \tau_0$ and τ_1 are element of $\mathbb{R}^+$ such that τ_0 is strictly less than τ_1.
Throughout this manuscript, we consider that $\alpha : [\tau_0;\tau_1] \to \mathbb{R}$, α is in $L^{+\infty}$ and is a bounded function satisfying

$$\int_{\tau_0}^{\tau_1} \alpha(s)ds < \mu_1. \tag{2}$$

This problem can be regarded as a problem with a memory acting only on $(t-\tau_1, t-\tau_0)$. By a change of variable, we see that

$$\int_{\tau_0}^{\tau_1} \alpha(s)\theta_t(x,t-s)ds = \int_{t-\tau_1}^{t-\tau_0} \alpha(s)\theta_t(x,t-s)ds$$

It is also well known that if $\alpha = 0$, that is in absence of delay, the corresponding following system ;

$$\begin{cases} \rho_1\theta_{tt}(x,t) - \kappa(\theta_x(x,t) + \psi(x,t))_x + \mu\theta_t(x,t) = 0 \text{ in }]0,L[\times(0,+\infty), \\ -b\psi_{xx}(x,t) + \kappa(\theta_x(x,t) + \psi(x,t)) = 0 \text{ in }]0,L[\times(0,+\infty), \\ \theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x), \quad \psi(x,0) = \psi_0(x) \text{ in }]0,L[, \\ \psi_x(0,t) = \psi_x(L,t) = \theta(0,t) = \theta(L,t) = 0, t > 0. \end{cases} \tag{3}$$

was analyzed in a recent paper by Almeida Jnior , A.J.A. Ramos and M.M. Freitas [1], where it was get the energy of the system exponentially-stable.
In the case of the wave equations, Nicaise and Pignotti [8] investigated exponential stability results with delay concentrated at τ for the system

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \text{ in } \Omega\times(0,+\infty) \\ u(x,t) = 0 \text{ on } \Gamma_D\times(0,+\infty) \\ \dfrac{\partial u}{\partial \nu}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0 \text{ on } \Gamma_N\times(0,+\infty) \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 \text{ in } \Omega \\ u(x,t-\tau) = f_0(x,t-\tau) \text{ on } \Gamma_N\times(0,\tau), \end{cases} \tag{4}$$

under the condition $\mu_2 < \mu_1$, by combining inequalities due to Carleman estimates and compactness-uniqueness arguments. Later, they also obtain in [13] the exponential stability with distributed delay of the system

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \text{ in } \Omega\times(0,+\infty) \\ u(x,t) = 0 \text{ on } \Gamma_D\times(0,+\infty) \\ \dfrac{\partial u}{\partial \nu}(x,t) + \mu_1 u_t(x,t) + \displaystyle\int_{\tau_0}^{\tau_1} \alpha(s) u_t(x,t-s)\,ds = 0 \text{ on } \Gamma_N\times(0,+\infty) \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 \text{ in } \Omega \\ u(x,-t) = f_0(x,-t) \text{ on } \Gamma_N\times(0,\tau_1), \end{cases} \tag{5}$$

under the assumption (2).

Recently in the case of the wave equation with dynamical control, Silga and Bayili [2] studied a wave equation set in a bounded domain with a dynamical control and prove that if the delay term is small enough, then the system with delay has the same (polynomial) decay rate than the one without delay.

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 \text{ in } \Omega \times (0,+\infty) \\ u(x,t) = 0 \text{ on } \Gamma_D \times (0,+\infty) \\ \dfrac{\partial u}{\partial \nu}(x,t) + \eta(x,t) = 0 \text{ on } \Gamma_N \times (0,+\infty) \\ \eta_t(x,t) - u_t(x,t) + \beta_1 \eta(x,t) + \beta_2 \eta(x,t-\tau) = 0 \text{ on } \Gamma_N \times (0,+\infty) \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 \text{ in } \Omega \\ \eta(x,0) = \eta_0 \text{ on } \Gamma_N \\ \eta(x,t-\tau) = f_0(x,t-\tau) \text{ on } \Gamma_N \times (0,\tau), \end{cases} \tag{6}$$

Later in [14] they also investigate the case of the wave equation set in a bounded domain with a distributed delay on dynamical control

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 \text{ in }]0,1[\times(0,+\infty) \\ u(0,t) = 0 \\ u_x(1,t) + \eta(t) = 0 \ \forall\, t \in (0,+\infty) \\ \eta_t(t) - u_t(1,t) + \beta_1 \eta(t) + \displaystyle\int_{\tau_0}^{\tau_1} \beta_2(s)\eta(t-s)ds = 0 \ \ \forall\, t \in (0,+\infty) \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1 \text{ in }]0,1[, \quad \eta(0) = \eta_0 \in \mathbb{R} \\ \eta(-t) = f_0(-t) \ \forall\, t \in (0,\tau_1), \end{cases} \tag{7}$$

To do this they use the assumption (2).

Motivated by the above results we will establish under the hypothesis (2) the well-posedness and the exponnential stability of the system (1).

This paper is organized as follows. In Section 2, we give the decrease for the energy of the system and the well-posedness of problems (1) using the Faedo-Galerkin method. And finally in Section 3, we will give the exponential stability of the problem using a Lyapounov fuction.

2. Well-Posedness of the Problem

In this section we will give well-posedness results for problem (1) using Faedo-Galerkin method. To this aim, we introducing the following auxiliary change of variable

$$\eta(x,\rho,t,s) = \theta_t(x,t-s\rho), \quad x \in]0,L[, \rho \in (0,1), s \in (\tau_0,\tau_1),\ t > 0. \tag{8}$$

The problem (1) is now equivalent to

$$\begin{cases} \rho_1\theta_{tt}(x,t) - \kappa(\theta_x(x,t)+\psi(x,t))_x + \mu_1\theta_t(x,t) + \displaystyle\int_{\tau_0}^{\tau_1}\alpha(s)\eta(x,1,t,s)ds = 0 \text{ in }]0,L[\times(0,+\infty), \\ -b\psi_{xx}(x,t) + \kappa(\theta_x(x,t)+\psi(x,t)) = 0 \text{ in }]0,L[\times(0,+\infty), \\ s\eta_t(\rho,t) + \eta_\rho(\rho,t) = 0 \text{ in } (0,1)\times(0,+\infty); \\ \theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x), \quad \psi(x,0) = \psi_0(x) \text{ in }]0,L[, \\ \eta(x,\rho,0,s) = f_0(x,-\rho s) \ \forall\, x \in]0,L[, \forall\, \rho \in (0,1), s\in(\tau_0,\tau_1), \\ \eta(x,0,t,s) = \theta_t(x,t) \ \forall\, x\in]0,L[, \forall\, t\in(0,+\infty), s\in(\tau_0,\tau_1), \\ \psi_x(0,t) = \psi_x(L,t) = \theta(0,t) = \theta(L,t) = 0, t>0. \end{cases} \tag{9}$$

Let us consider the Hilbert spaces

$$\mathcal{H} = H_0^1(0,L)\times L^2(0,L)\times H_*^1(0,L)\times L^2\Big((0,L)\times(0,1)\times(\tau_0,\tau_1)\Big),$$

and

$$\mathcal{H}_1 = (H^2(0,L)\cap H_0^1(0,L))^2\times H_*^2(0,L)\times H^1\Big((0,L)\times(0,1)\times(\tau_0,\tau_1)\Big).$$

where

$$L_*^2(0,L) = \left\{u\in L^2(0,L), \int_0^L u(x)dx = 0\right\}, H_*^1(0,L) = H^1(0,L)\cap L_*^2(0,L),$$

and

$$H_*^2(0,L) = H^2(0,L)\cap H_*^1(0,L).$$

We equipped $\mathcal{H}$ with the norm

$$\left\|(u,v,w,z)^{\mathsf{T}}\right\|_{\mathcal{H}}^2 = \frac{\rho_1}{2}\int_0^L |v|^2dx + \frac{b}{2}\int_0^L |w_x|^2dx + \frac{\kappa}{2}\int_0^L |u_x+w|^2dx + \frac{1}{2}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1} s\alpha(s)|\eta|^2dsd\rho dx$$

Define the energy of a solution (θ,θ_t,ψ,z) of (9) as

$$E(t) = \frac{\rho_1}{2}\int_0^L |\theta_t|^2dx + \frac{b}{2}\int_0^L|\psi_x|^2dx + \frac{\kappa}{2}\int_0^L|\theta_x+\psi|^2dx + \frac{1}{2}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1} s\alpha(s)|\eta|^2dsd\rho dx.$$

We can now prove that the energy is decreasing. More precisely, we have the following result.

Proposition 2.1. *For any regular solution of problem* (9)*, the energy is decreasing and there exists a positive constant C such that*

$$\frac{d}{dt}E(t) \le -C\Big(\int_0^L |\eta(x,0,t,s)|^2dx + \int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)+\eta(x,0,t,s)|^2dsdx\Big).$$

Démonstration. Multiplying , in (9) , the first equation by θ_t, the second equation by ψ_t and the third equation by $\alpha(s)\eta$, and then integrating the first and the second over $[0;L]$ and the third over $(0,L)\times(0,1)\times(\tau_0,\tau_1)$. By integrating by parts and using the boundary conditions, we obtain

$$\begin{aligned} &\frac{\rho_1}{2}\frac{d}{dt}\int_0^L|\theta_t(x,t)|^2dx + \kappa\int_0^L\left[\theta_x(x,t)+\psi(x,t)\right]\theta_{tx}(x,t)dx \\ &+\mu_1\int_0^L|\theta_t(x,t)|^2dx + \int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\eta(x,1,t,s)\theta_t(x,t)dsdx = 0, \end{aligned} \tag{10}$$

$$\frac{b}{2}\frac{d}{dt}\int_0^L|\psi_x(x,t)|^2dx + \kappa\int_0^L\left[\theta_x(x,t)+\psi(x,t)\right]\psi_t(x,t)dx = 0, \tag{11}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1} s\alpha(s)|\eta(x,\rho,t,s)|^2 dsd\rho dx$$
$$+\frac{1}{2}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\left[|\eta(x,1,t,s)|^2-|\eta^2(x,0,t,s)|^2\right]dsdx = 0. \tag{12}$$

Now summing relations (10), (11) and (12) we obtain

$$\begin{aligned}
&\frac{\rho_1}{2}\frac{d}{dt}\int_0^L|\theta_t(x,t)|^2dx+\kappa\int_0^L\left[\theta_x(x,t)+\psi(x,t)\right]\theta_{tx}(x,t)dx\\
&+\mu_1\int_0^L|\theta_t(x,t)|^2dx+\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\eta(x,1,t,s)\theta_t(x,t)dsdx+\frac{b}{2}\frac{d}{dt}\int_0^L|\psi_x(x,t)|^2dx\\
&+\kappa\int_0^L\left[\theta_x(x,t)+\psi(x,t)\right]\psi_t(x,t)dx+\frac{1}{2}\frac{d}{dt}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)|\eta(x,\rho,t,s)|^2dsd\rho dx\\
&+\frac{1}{2}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\left[|\eta(x,1,t,s)|^2-|\eta(x,0,t,s)|^2\right]dsdx=0,
\end{aligned}$$

which lead to

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{\rho_1}{2}\int_0^L|\theta_t(x,t)|^2dx+\frac{b}{2}\int_0^L|\psi_x(x,t)|^2dx++\frac{1}{2}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)|\eta(x,\rho,t,s)|^2dsd\rho dx\right.\\
&\left.+\frac{\kappa}{2}\int_0^L|\theta_x(x,t)+\psi(x,t)|^2dx\right\}+\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\eta(x,1,t,s)\theta_t(x,t)dsdx+\mu_1\int_0^L|\theta_t(x,t)|^2dx\\
&+\frac{1}{2}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\left[|\eta(x,1,t,s)|^2-|\eta(x,0,t,s)|^2\right]dsdx=0.
\end{aligned}$$

Then from the definition of the energy, it follows that

$$\begin{aligned}
\frac{d}{dt}E(t) &= -\mu_1\int_0^L|\theta_t(x,t)|^2dx-\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\eta(x,1,t,s)\theta_t(x,t)dsdx\\
&- \frac{1}{2}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\left[|\eta(x,1,t,s)|^2-|\eta(x,0,t,s)|^2\right]dsdx.
\end{aligned}$$

As $\theta_t(x,t)=\eta(x,0,t,s)$, we get

$$\begin{aligned}
\frac{d}{dt}E(t) &= -\mu_1\int_0^L|\eta(x,0,t,s)|^2dx-\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\eta(x,1,t,s)\eta(x,0,t,s)dsdx\\
&- \frac{1}{2}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\left[|\eta(x,1,t,s)|^2-|\eta(x,0,t,s)|^2\right]dsdx.
\end{aligned}\tag{13}$$

We also know that

$$\eta(x,1,t,s)\eta(x,0,t,s)=\frac{1}{2}|\eta(x,1,t,s)+\eta(x,0,t,s)|^2-\frac{1}{2}|\eta(x,0,t,s)|^2-\frac{1}{2}|\eta(x,1,t,s)|^2. \tag{14}$$

Then using the relations (14) in (13) , its follows that

$$\frac{d}{dt}E(t) = \left[\int_{\tau_0}^{\tau_1}\alpha(s)ds-\mu_1\right]\int_0^L|\eta(x,0,t,s)|^2dx-\frac{1}{2}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)+\eta(x,0,t,s)|^2dsdx \tag{15}$$

This proves that

$$\frac{d}{dt}E(t) \leq -C\Big(\int_0^L |\eta(x,0,t,s)|^2 dx + \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s)|\eta(x,1,t,s)+\eta(x,0,t,s)|^2 ds dx\Big). \qquad \square$$

with $C = Min\left\{-\int_{\tau_0}^{\tau_1} \alpha(s)ds + \mu_1;\ \frac{1}{2}\right\}$

We therefore deduce that the energy of the system (9) is dissipative. □

To state the main result in this section, we start by defining what we mean by a weak solution of problem (9) as following

Definition 2.1. *Given an initial data $U_0 = (\theta_0, \theta_1, \psi_0, \eta_0) \in \mathcal{H}$, a function $U = (\theta, \theta_t, \psi, \eta) \in C([0,T];\mathcal{H})$ is said to be weak solution of (9) if for almost everywhere $t \in [0,T]$,*

$$\begin{cases} \rho_1 \frac{d}{dt}\Big(\theta_t, u\Big) + \kappa\Big(\theta_x(x,t) + \psi(x,t), u_x\Big) + \mu_1(\theta_t, u) + \Big(\int_{\tau_0}^{\tau_1} \alpha(s)\eta(x,1,t,s)ds, u\Big) = 0, \\ b(\psi_x, v_x) + \kappa(\theta_x + \psi, v) = 0, \\ s(\eta_t, w) + (\eta_\rho, w) = 0, \end{cases} \tag{16}$$

for all $u \in H_0^1(0,L), v \in H_*^2(0,L), w \in L^2\Big((0,L)\times(0,1)\times(\tau_0,\tau_1)\Big)$ and $(\theta(0), \theta_t(0), \psi(0), \eta(0)) = (\theta_0, \theta_1, \psi_0, \eta_0)$.

The main result of this section is the following :

Theorem 2.1.
Assume that (2) holds. Then for all $U_0 = (\theta_0, \theta_1, \psi_0, \eta_0) \in \mathcal{H}$, the problem (9) admits a unique weak solution $U = (\theta, \theta_t, \psi, \eta)$ such that :

$$\begin{cases} \theta \in L^\infty\Big(0, T, H_0^1(0,L)\Big), \\ \theta_t \in L^\infty\Big(0, T, L^2(0,L)\Big), \\ \psi \in L^\infty\Big(0, T, H_*^1(0,L)\Big), \\ \eta \in L^\infty\Big(0, T, L^2((0,L)\times(0,1)\times(\tau_0,\tau_1))\Big). \end{cases} \tag{17}$$

Moreover, if $U_0 = (\theta_0, \theta_1, \psi_0, \eta_0) \in \mathcal{H}_1$, then problem (9) admits a unique stronger weak solution $U = (\theta, \theta_t, \psi, \eta)$ which satisfies

$$\begin{cases} \theta \in L^\infty\Big(0, T, H^2(0,L) \cap H_0^1(0,L)\Big), \\ \theta_t \in L^\infty\Big(0, T, H_0^1(0,L)\Big), \\ \psi \in L^\infty\Big(0, T, H_*^2(0,L)\Big), \\ \eta \in L^\infty\Big(0, T, H^1((0,L)\times(0,1)\times(\tau_0,\tau_1))\Big). \end{cases} \tag{18}$$

In both cases, the solution $(\theta, \theta_t, \psi, \eta)$ depends continuously on the initial data in H. Particulary, the system (9) admit a unique weak solution.

Démonstration. The Faedo-Galerkin method ,see [10], [4], [12], [1], [15] or [11] will be the key to prove the existence of a global solution. The proof will be done in six steps

Step 1. ***: Faedo-Galerkin approximations***.

Let us consider initial data $(\theta_0, \theta_1, \psi_0, \eta_0) \in \mathcal{H}$.
Let $\{u^k\}, k \in \mathbb{N}^\star$ and $\{v^k\}, k \in \mathbb{N}^\star$ basics formed by eigenfunctions of $-\partial_{xx}$. This bases can be considered orthogonal in $H^2(\Omega) \cap H_0^1(\Omega)$ and $H_*^2(0,L)$ and also orthonormal in $L^2(0;L)$.

We also define the sequence $\{w^k\}, k \in \mathbb{N}^\star$ *in the following way*
$w^k(x,0,s) = u^k(x)$ *then we extend* $w^k(x,s,0)$ *by* $w^k(x,\rho,s)$ *on* $L^2\Big((0,L)\times(0,1)\times(\tau_0,\tau_1)\Big)$. *For more details see Lemma 1.1* [*11*].
We consider the finite-dimensional subspaces H_n, V_n *and* W_n *defined by*

$$H_n = span\left\{u^1,u^2,...,u^n\right\}, V_n = span\left\{v^1,v^2,...,v^n\right\} \text{ and } W_n = span\left\{w^1,w^2,...,w^n\right\}.$$

Now we will find an approximate solution in the form

$$\theta^n(t,x) = \sum_{j=1}^n a^{jn}(t)u^j(x),$$

$$\psi^n(t,x) = \sum_{j=1}^n b^{jn}(t)v^j(x)$$

and

$$\eta^n(x,\rho,t,s) = \sum_{j=1}^n c^{jn}(t)w^j(x,\rho,s)$$

of the following approximate problem

$$\begin{cases}\rho_1\displaystyle\int_0^L \theta^n_{tt}(x,t)udx - \kappa\int_0^L \left[\theta^n_x(x,t)+\psi^n(x,t)\right]_x udx + \mu_1\int_0^L \theta^n_t(x,t)udx \\ +\displaystyle\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\eta^n(x,1,t,s)udsdx = 0, \\ -b\displaystyle\int_0^L \psi^n_{xx}(x,t)vdx + \kappa\int_0^L\left[\theta^n_x(x,t)+\psi^n(x,t)\right]vdx = 0, \\ \displaystyle\int_0^L\int_0^1\int_{\tau_0}^{\tau_1} s\eta^n_t(x,\rho,t,s)wdsd\rho dx + \int_0^L\int_0^1\int_{\tau_0}^{\tau_1}\alpha(s)\eta^n_\rho(x,\rho,t,s)wdsd\rho dx = 0\end{cases} \tag{19}$$

for all $u \in H_n, v \in V_n, w \in W_n$, *with initial conditions such that*

$$\left(\theta^n(0),\theta^n_t(0),\psi^n(0),\eta^n(0)\right) = \left(\theta^n_0,\theta^n_1,\psi^n_0,\eta^n_0\right) \to (\theta_0,\theta_1,\psi_0,\eta_0) \tag{20}$$

strongly in $\mathcal{H}$ *and where* a^{jn}, b^{jn} *and* c^{jn} *are time-dependent coefficients.*
Applying standard theory of ordinary differential equations, the finite dimensional problem (19) − (20) *has a solution* $(a^{jn}, b^{jn}, c^{jn}), 1 \le j \le n$ *defined on* $[0,t_n)$ *for every* $n \in \mathbb{N}^\star$.
Then the a priori estimates that follow imply that in fact $t_n = T, \forall T > 0$.

Step 2. ***A Priori Estimate I***
By substituting u *for* θ^n_t *in* $(19)_1$, v *for* ψ^n_t *in* $(19)_2$ *and* w *by* η^n *in* $(19)_3$,*we obtain*

$$\begin{cases}\dfrac{\rho_1}{2}\dfrac{d}{dt}\displaystyle\int_0^L|\theta^n_t(x,t)|^2dx + \kappa\int_0^L\left[\theta^n_x(x,t)+\psi^n(x,t)\right]\theta^n_{tx}(x,t)dx + \mu_1\int_0^L|\theta^n_t(x,t)|^2dx \\ +\displaystyle\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\eta^n(x,1,t,s)\theta^n_t(x,t)dsdx = 0, \\ \dfrac{b}{2}\dfrac{d}{dt}\displaystyle\int_0^L|\psi^n_x(x,t)|^2dx + \kappa\int_0^L\left[\theta^n_x(x,t)+\psi^n(x,t)\right]\psi^n_t(x,t)dx = 0, \\ \dfrac{1}{2}\dfrac{d}{dt}\displaystyle\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)s|\eta^n(x,\rho,t,s)|^2dsd\rho dx + \frac{1}{2}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\left[|\eta^n(x,1,t,s)|^2-|\eta^n(x,0,t,s)|^2\right]dsdx = 0.\end{cases} \tag{21}$$

By making the same transformations as in the session of the dissipative character we obtain

$$\frac{d}{dt}E^n(t) = \left[\int_{\tau_0}^{\tau_1}\alpha(s)ds - \mu_1\right]\int_0^L|\eta^n(x,0,t,s)|^2dx - \frac{1}{2}\int_0^L\int_{\tau_0}^{\tau_1}s\alpha(s)|\eta^n(x,1,t,s)+\eta^n(x,0,t,s)|^2dsdx \tag{22}$$

where

$$E^n(t) = \frac{\rho_1}{2}\int_0^L |\theta_t^n|^2 dx + \frac{b}{2}\int_0^L |\psi_x^n|^2 dx + \frac{\kappa}{2}\int_0^L |\theta_x^n + \psi^n|^2 dx + \frac{1}{2}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1} s\alpha(s)|\eta^n|^2 ds d\rho dx.$$

By integrating (22) *from 0 to $t < t_n$, we obtain*

$$\begin{aligned} E^n(t) - E^n(0) &= \left[\int_{\tau_0}^{\tau_1}\alpha(s)ds - \mu_1\right]\int_0^t\int_0^L |\eta^n(x,0,\lambda,s)|^2 dx d\lambda \\ &- \frac{1}{2}\int_0^t\int_0^L\int_{\tau_0}^{\tau_1} s\alpha(s)|\eta^n(x,1,\lambda,s) + \eta^n(x,0,\lambda,s)|^2 ds dx d\lambda \end{aligned}$$

which means

$$\begin{aligned} E^n(t) &+ \left[\mu_1 - \int_{\tau_0}^{\tau_1}\alpha(s)ds\right]\int_0^t\int_0^L |\eta^n(x,0,\lambda,s)|^2 dx d\lambda \\ &+ \frac{1}{2}\int_0^t\int_0^L\int_{\tau_0}^{\tau_1} \alpha(s)|\eta^n(x,1,\lambda,s) + \eta^n(x,0,\lambda,s)|^2 ds dx d\lambda = E^n(0) \end{aligned}$$

where

$$\begin{aligned} E^n(0) &= \frac{\rho_1}{2}\int_0^L |\theta_t^n(x,0)|^2 dx + \frac{b}{2}\int_0^L |\psi_x^n(x,0)|^2 dx + \frac{\kappa}{2}\int_0^L |\theta_x^n(x,0) + \psi^n(x,0)|^2 dx \\ &+ \frac{1}{2}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1} s\alpha(s)|\eta^n(x,\rho,0,s)|^2 ds d\rho dx. \end{aligned}$$

As $(\theta^n(0), \theta_t^n(0), \psi^n(0), \eta^n(0)) = \left(\theta_0^n, \theta_1^n, \psi_0^n, \eta_0^n\right) \to (\theta_0, \theta_1, \psi_0, \eta_0)$ *strongly in* $\mathcal{H}$ *then there exists a positive constant* C_1 *such that* $E^n(0) \le C_1$. *Hence*

$$\begin{aligned} &E^n(t) + \left[\mu_1 - \int_{\tau_0}^{\tau_1}\alpha(s)ds\right]\int_0^t\int_0^L |\eta^n(x,0,\lambda,s)|^2 dx d\lambda \\ &+\frac{1}{2}\int_0^t\int_0^L\int_{\tau_0}^{\tau_1} \alpha(s)|\eta^n(x,1,\lambda,s) + \eta^n(x,0,\lambda,s)|^2 ds dx d\lambda \le C_1 \end{aligned}$$

which means that

$$\begin{aligned} &\frac{\rho_1}{2}\int_0^L |\theta_t^n|^2 dx + \frac{b}{2}\int_0^L |\psi_x^n|^2 dx + \frac{\kappa}{2}\int_0^L |\theta_x^n + \psi^n|^2 dx + \frac{1}{2}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1} s\alpha(s)|\eta^n|^2 ds d\rho dx \\ &+\frac{1}{2}\int_0^t\int_0^L\int_{\tau_0}^{\tau_1} \alpha(s)|\eta^n(x,1,\lambda,s) + \eta^n(x,0,\lambda,s)|^2 ds dx d\lambda \\ &+\left[\mu_1 - \int_{\tau_0}^{\tau_1}\alpha(s)ds\right]\int_0^t\int_0^L |\eta^n(x,0,\lambda,s)|^2 dx d\lambda \le C_1 \end{aligned} \tag{23}$$

As the constant C_1 does not depend on n we can therefore take $t_n = T$. for all $T > 0$

Step 3. ***A Priori Estimate II***
Let us derive the equation $(19)_1$ *with respect to t and then replacing u by θ_{tt}^n. We obtain*

$$\begin{aligned} &\rho_1\int_0^L \theta_{ttt}^n(x,t)\theta_{tt}^n(x,t)dx - \kappa\int_0^L \left[\theta_{xt}^n(x,t) + \psi_t^n(x,t)\right]_x \theta_{tt}^n(x,t)dx \\ &+\mu_1\int_0^L \theta_{tt}^n(x,t)\theta_{tt}^n(x,t)dx + \int_0^L\int_{\tau_0}^{\tau_1} \alpha(s)\eta_t^n(x,1,t,s)\theta_{tt}^n(x,t)ds dx = 0. \end{aligned}$$

By integrating by parts, we obtain

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^L |\theta^n_{tt}(x,t)|^2dx - \kappa\left[(\theta^n_{xt}+\psi^n_t)\theta^n_{tt}(x,t)\right]_0^L + \kappa\int_0^L \left[\theta^n_{xt}(x,t)+\psi^n_t(x,t)\right]\theta^n_{xtt}(x,t)dx$$
$$+\mu_1\int_0^L |\theta^n_{tt}(x,t)|^2dx + \int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\eta^n_t(x,1,t,s)\theta^n_{tt}(x,t)dsdx = 0.$$

As θ is null at 0 and at L we have

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^L |\theta^n_{tt}(x,t)|^2dx + \kappa\int_0^L \left[\theta^n_{xt}(x,t)+\psi^n_t(x,t)\right]\theta^n_{xtt}(x,t)dx$$
$$+\mu_1\int_0^L |\theta^n_{tt}(x,t)|^2dx + \int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\eta^n_t(x,1,t,s)\theta^n_{tt}(x,t)dsdx = 0. \tag{24}$$

Let us derive the equation $(19)_2$ *with respect to t and then replacing v by ψ^n_{tt}. We obtain*

$$-b\int_0^L \psi^n_{xxt}(x,t)\psi^n_{tt}(x,t)dx + \kappa\int_0^L \left[\theta^n_{xt}(x,t)+\psi^n_t(x,t)\right]\psi^n_{tt}(x,t)dx = 0.$$

Integrating by parts, we obtain

$$-b\left[\psi^n_{xt}\psi^n_{tt}(x,t)\right]_0^L + b\int_0^L \psi^n_{xt}(x,t)\psi^n_{xtt}(x,t)dx + \kappa\int_0^L \left[\theta^n_{xt}(x,t)+\psi^n_t(x,t)\right]\psi^n_{tt}(x,t)dx = 0.$$

Since ψ_x is zero at the edge we have

$$b\int_0^L \psi^n_{xt}(x,t)\psi^n_{xtt}(x,t)dx + \kappa\int_0^L \left[\theta^n_{xt}(x,t)+\psi^n_t(x,t)\right]\psi^n_{tt}(x,t)dx = 0.$$

what is written

$$\frac{b}{2}\frac{d}{dt}\int_0^L |\psi^n_{xt}(x,t)|^2dx + \kappa\int_0^L \left[\theta^n_{xt}(x,t)+\psi^n_t(x,t)\right]\psi^n_{tt}(x,t)dx = 0. \tag{25}$$

Summing (24) *and* (25) *we obtain*

$$\frac{1}{2}\frac{d}{dt}\left\{\rho_1\int_0^L |\theta^n_{tt}(x,t)|^2dx + k\int_0^L |\theta^n_{xt}(x,t)+\psi^n_t(x,t)|^2dx + b\int_0^L |\psi^n_{xt}(x,t)|^2dx\right\}$$
$$+\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\eta^n_t(x,1,t,s)\theta^n_{tt}(x,t)dsdx + \mu_1\int_0^L |\theta^n_{tt}(x,t)|^2dx = 0$$

which means that

$$\frac{d}{dt}G^n(t) + \int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\eta^n_t(x,1,t,s)\theta^n_{tt}(x,t)dsdx + \mu_1\int_0^L |\theta^n_{tt}(x,t)|^2dx = 0 \tag{26}$$

with

$$G^n(t) = \frac{\rho_1}{2}\int_0^L |\theta^n_{tt}(x,t)|^2dx + \frac{k}{2}\int_0^L |\theta^n_{xt}(x,t)+\psi^n_t(x,t)|^2dx + \frac{b}{2}\int_0^L |\psi^n_{xt}(x,t)|^2dx.$$

By integrating (26) *from 0 to t we have*

$$G^n(t) + \int_0^t\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\eta^n_\lambda(x,1,\lambda,s)\theta^n_{\lambda\lambda}(x,\lambda)dsdxd\lambda + \mu_1\int_0^t\int_0^L |\theta^n_{\lambda\lambda}(x,\lambda)|^2dxd\lambda = G(0)$$

where

$$G^n(0) = \frac{\rho_1}{2}\int_0^L |\theta^n_{tt}(x,0)|^2dx + \frac{k}{2}\int_0^L |\theta^n_{xt}(x,0)+\psi^n_t(x,0)|^2dx + \frac{b}{2}\int_0^L |\psi^n_{xt}(x,0)|^2dx.$$

As $(\theta^n(0),\theta^n_t(0),\psi^n(0),\eta^n(0)) = \left(\theta^n_0,\theta^n_1,\psi^n_0,\eta^n_0\right) \to (\theta_0,\theta_1,\psi_0,\eta_0)$ *strongly in $\mathcal{H}$ then there exists a positive constant C_2 such that $G^n(0)\le C_2$. Hence*

$$G^n(t) + \int_0^t\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\eta^n_\lambda(x,1,\lambda,s)\theta^n_{\lambda\lambda}(x,\lambda)dsd\lambda dx + \mu_1\int_0^L |\theta^n_{tt}(x,t)|^2dx \le C_2 \tag{27}$$

Step 4. ***Passage to Limit***

From (23) *and* (27) *we have*
$\{\theta^n\}$ *is bounded in* $L^\infty\left(0,T,H_0^1(0,L)\right)$,
$\{\theta_t^n\}$ *is bounded in* $L^\infty\left(0,T,L^2(0,L)\right)$,
$\{\theta_{tt}^n\}$ *is bounded in* $L^\infty\left(0,T,L^2(0,L)\right)$,
$\{\psi^n\}$ *is bounded in* $L^\infty\left(0,T,H_*^1(0,L)\right)$,
$\{\psi_t^n\}$ *is bounded in* $L^\infty\left(0,T,L^2(0,L)\right)$,
$\{\eta^n\}$ *is bounded in* $L^\infty\left(0,T,L^2((0,L)\times(0,1)\times(\tau_0,\tau_1))\right)$.

So we can extract subsequences $\{\theta^n\}$,$\{\psi^n\}$ *and* $\{\eta^n\}$ *such as*

$\{\theta^n\} \rightharpoonup \star\ \theta$ *in* $L^\infty\left(0,T,H_0^1(0,L)\right)$,
$\{\theta_t^n\} \rightharpoonup \star\ \theta_t$ *in* $L^\infty\left(0,T,L^2(0,L)\right)$,
$\{\theta_{tt}^n\} \rightharpoonup \star\ \theta_{tt}$ *in* $L^\infty\left(0,T,L^2(0,L)\right)$,
$\{\psi^n\} \rightharpoonup \star\ \psi$ *in* $L^\infty\left(0,T,H_*^1(0,L)\right)$,
$\{\psi_t^n\} \rightharpoonup \star\ \psi_t$ *in* $L^\infty\left(0,T,L^2(0,L)\right)$,
$\{\eta^n\} \rightharpoonup \star\ z$ *in* $L^\infty\left(0,T,L^2((0,L)\times(0,1)\times(\tau_0,\tau_1))\right)$.

Moreover, from (23) *we have*
$\{\theta^n\}$ *is bounded in* $L^2\left(0,T,H_0^1(0,L)\right)$,
$\{\theta_t^n\}$ *is bounded in* $L^2\left(0,T,L^2(0,L)\right)$.

Since the embedding of $H_0^1(0,L)$ *in* $L^2(0,L)$ *is compact, we have by Aubin-Lions theorem* [7] *that*

$\{\theta^n\} \to \theta$ *strongly in* $L^\infty\left(0,T,L^2(0,L)\right)$,

We also show that

$\{\theta_t^n\} \to \theta_t$ *strongly in* $L^\infty\left(0,T,L^2(0,L)\right)$,
$\{\psi^n\} \to \psi$ *strongly in* $L^\infty\left(0,T,H_0^1(0,L)\right)$,

Then we can pass to limit the approximate problem (19) − (20) *in order to get a weak solution of problem* (9).
And we use density arguments to get problems (9) *admit a global weak solution satisfaisant*

$$\begin{cases} \theta \in L^\infty\left(0,T,H_0^1(0,L)\right), \\ \theta_t \in L^\infty\left(0,T,L^2(0,L)\right), \\ \psi \in L^\infty\left(0,T,H_\star^1(0,L)\right), \\ \eta \in L^\infty\left(0,T,L^2((0,L)\times(0,1)\times(\tau_0,\tau_1))\right). \end{cases} \tag{28}$$

Step 5. ***A Priori Estimate III***

Suppose that the problem (19) *satisfies* $(\theta_0,\theta_1,\psi_0,\eta_0) \in \mathcal{H}_1$ *and*

$$\left(\theta_0^n,\theta_1^n,\psi_0^n,\eta_0^n\right) \to (\theta_0,\theta_1,\psi_0,\eta_0) \ \textit{strongly in } \mathcal{H}_1. \tag{29}$$

Replacing u by $-\theta_{xxt}^n$ *in* $(19)_1$, *v by* $-\psi_{xxt}^n$ *in* $(19)_2$ *and w by* η_{xx}^n *in* $(19)_3$ *we arrive at*

$$\frac{d}{dt}Q^n(t) = \left[\int_{\tau_0}^{\tau_1}\alpha(s)ds-\mu_1\right]\int_0^L|\eta_x^n(x,0,t,s)|^2dx-\frac{1}{2}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta_x^n(x,1,t,s)+\eta_x^n(x,0,t,s)|^2dsdx \tag{30}$$

where

$$Q^n(t) = \frac{\rho_1}{2}\int_0^L |\theta^n_{tx}|^2 dx + \frac{b}{2}\int_0^L |\psi^n_{xx}|^2 dx + \frac{\kappa}{2}\int_0^L |\theta^n_{xx} + \psi^n_x|^2 dx + \frac{1}{2}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1} s\alpha(s)|\eta^n_x|^2 ds d\rho dx$$

.

Integrating (30) *from 0 to* $t < T$*, we get,*

$$\begin{aligned} Q^n(t) - Q^n(0) &= \left[\int_{\tau_0}^{\tau_1} \alpha(s)ds - \mu_1\right]\int_0^t\int_0^L |\eta^n_x(x,0,\lambda,s)|^2 dx d\lambda \\ &- \frac{1}{2}\int_0^t\int_0^L\int_{\tau_0}^{\tau_1} s\alpha(s)|\eta^n_x(x,1,\lambda,s) + \eta^n_x(x,0,\lambda,s)|^2 ds dx d\lambda \end{aligned}$$

which means

$$\begin{aligned} & Q^n(t) + \left[\mu_1 - \int_{\tau_0}^{\tau_1}\alpha(s)ds\right]\int_0^t\int_0^L |\eta^n_x(x,0,\lambda,s)|^2 dx d\lambda \\ & +\frac{1}{2}\int_0^t\int_0^L\int_{\tau_0}^{\tau_1} \alpha(s)|\eta^n_x(x,1,\lambda,s) + \eta^n_x(x,0,\lambda,s)|^2 ds dx d\lambda = Q^n(0) \end{aligned}$$

where

$$\begin{aligned} Q^n(0) &= \frac{\rho_1}{2}\int_0^L |\theta^n_{tx}(x,0)|^2 dx + \frac{b}{2}\int_0^L |\psi^n_{xx}(x,0)|^2 dx + \frac{\kappa}{2}\int_0^L |\theta^n_{xx}(x,0) + \psi^n_x(x,0)|^2 dx \\ &+ \frac{1}{2}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1} s\alpha(s)|\eta^n_x(x,\rho,0,s)|^2 ds d\rho dx. \end{aligned}$$

As $(\theta^n(0), \theta^n_t(0), \psi^n(0), \eta^n(0)) = \left(\theta^n_0, \theta^n_1, \psi^n_0, \eta^n_0\right) \to (\theta_0, \theta_1, \psi_0, \eta_0)$ *strongly in* $\mathcal{H}_1$*. We can deduce that there exists a positive constant* C_3 *such that* $Q^n(0) \le C_3$*. Hence*

$$\begin{aligned} & Q^n(t) + \left[\mu_1 - \int_{\tau_0}^{\tau_1}\alpha(s)ds\right]\int_0^t\int_0^L |\eta^n_x(x,0,\lambda,s)|^2 dx d\lambda \\ & +\frac{1}{2}\int_0^t\int_0^L\int_{\tau_0}^{\tau_1} \alpha(s)|\eta^n_x(x,1,\lambda,s) + \eta^n_x(x,0,\lambda,s)|^2 ds dx d\lambda \le C_3 \end{aligned}$$

which means that

$$\begin{aligned} & \frac{\rho_1}{2}\int_0^L |\theta_{tx}|^2 dx + \frac{b}{2}\int_0^L |\psi_{xx}|^2 dx + \frac{\kappa}{2}\int_0^L |\theta_{xx} + \psi_x|^2 dx + \frac{1}{2}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1} s\alpha(s)|\eta_x|^2 ds d\rho dx \\ & +\frac{1}{2}\int_0^t\int_0^L\int_{\tau_0}^{\tau_1} \alpha(s)|\eta^n_x(x,1,\lambda,s) + \eta^n_x(x,0,\lambda,s)|^2 ds dx d\lambda \\ & +\left[\mu_1 - \int_{\tau_0}^{\tau_1}\alpha(s)ds\right]\int_0^t\int_0^L |\eta^n_x(x,0,\lambda,s)|^2 dx d\lambda \le C_3 \end{aligned} \tag{31}$$

where $C_3 \ge 0$ *also independent of t and n but depending on initial data. Then we can conclude that*

$\{\theta^n\}$ *is bounded in* $L^\infty\left(0, T, H^2(0,L) \cap H^1_0(0,L)\right)$,
$\{\theta^n_t\}$ *is bounded in* $L^\infty\left(0, T, H^1_0(0,L)\right)$,
$\{\psi^n\}$ *is bounded in* $L^\infty\left(0, T, H^2_\star(0,L)\right)$,
$\{\eta^n\}$ *is bounded in* $L^\infty(0, T, H^1((0,L)\times(0,1)\times(\tau_0,\tau_1)))$.

This implique that

$\{\theta^n\} \rightharpoonup \star\ \theta$ *in* $L^\infty\left(0, T, H^2(0,L) \cap H^1_0(0,L)\right)$,
$\{\theta^n_t\} \rightharpoonup \star\ \theta_t$ *in* $L^\infty\left(0, T, H^1_1(0,L)\right)$,
$\{\psi^n\} \rightharpoonup \star\ \psi$ *in* $L^\infty\left(0, T, H^2_\star(0,L)\right)$,

$\{\eta^n\} \rightharpoonup \star\ \eta$ *in* $L^\infty\left(0, T, H^1((0,L)\times(0,1)\times(\tau_0,\tau_1))\right)$.

We conclude that $(\theta, \theta_t, \psi, \eta)$ *is a strong weak solution (as in* [15]*) such that*

$$\begin{cases} \theta \in L^\infty\left(0, T, H^2(0,L)\cap H_0^1(0,L)\right), \\ \theta_t \in L^\infty\left(0, T, H_0^1(0,L)\right), \\ \psi \in L^\infty\left(0, T, H_\star^2(0,L)\right), \\ \eta \in L^\infty\left(0, T, H^1((0,L)\times(0,1)\times(\tau_0,\tau_1))\right). \end{cases} \tag{32}$$

Step 6. ***Continuous dependence***

Let $U_1(t) = (\theta, \theta_t, \psi, \eta)$ *and* $U_2(t) = (\theta', \theta'_t, \psi', \eta')$ *two solutions of the problem* (9) *with initial datas* $U_1(0) = (\theta_0, \theta_1, \psi_0, \eta_0)$, $U_2(0) = \left(\theta'_0, \theta'_1, \psi'_0, \eta'_0\right) \in \mathcal{H}_1$. *Then* $(\Phi, \Phi_t, \Psi, \Gamma) = U_1(t) - U_2(t)$ *is solution of the system*

$$\begin{cases} \rho_1\Phi_{tt}(x,t) - \kappa(\Phi_x(x,t) + \Psi(x,t))_x + \mu_1\Phi_t(x,t) + \displaystyle\int_{\tau_0}^{\tau_1} \alpha(s)\Gamma(x,1,t,s)ds = 0 \ in \ \ ]0,L[\times(0,+\infty), \\ -b\Phi_{xx}(x,t) + \kappa(\Phi_x(x,t) + \Psi(x,t)) = 0 \ in \ \ ]0,L[\times(0,+\infty), \\ s\varsigma_t(x,\rho,t,s) + \Gamma_\rho(x,\rho,t,s) = 0 \ \ for \ \ (x,\rho,t,s) \in (0,L)\times(0,1)\times(0,+\infty)\times(\tau_0,\tau_1). \end{cases} \tag{33}$$

with initial data $\left(\Phi(0), \Phi_t(0), \Psi(0), \Gamma(0)\right) = U_1(0) - U_2(0)$.
Multiply $(33)_1$ *by* Φ_t *and* $(33)_2$ *by* Ψ_t *and integrate. we obtain*

$$\frac{d}{dt}\mathcal{P}(t) + \int_0^L \theta_t(x,t)\int_{\tau_0}^{\tau_1}\alpha(s)\Gamma(x,1,t,s)dsdx + \mu_1\int_0^L |\Phi_t(x,t)|^2dx = 0$$

avec

$$\mathcal{P}(t) = \frac{\rho_1}{2}\int_0^L |\Phi_t(x,t)|^2dx + \frac{k}{2}\int_0^L |\Phi_x(x,t) + \Psi(x,t)|^2dx + \frac{b}{2}\int_0^L |\Psi_x(x,t)|^2dx$$

Applying Young's inequality to $\int_0^L \theta_t(x,t)\int_{\tau_0}^{\tau_1}\alpha(s)\Gamma(x,1,t,s)dsdx$ *and the fact that* μ_2 *is bounded, we obtain the existence of a constant* M_1 *such that*

$$\begin{aligned} \frac{d}{dt}\mathcal{P}(t) &\leq M_1\int_0^L |\Phi_t(x,t)|^2dx \\ &\leq M_1\Big[\int_0^L |\Phi_t(x,t)|^2dx + \int_0^L |\Phi_x(x,t) + \Psi(x,t)|^2dx + \int_0^L |\Psi_x(x,t)|^2dx\Big] \end{aligned} \tag{34}$$

And by integrating (34) *from 0 to t we get*

$$\mathcal{P}(t) \leq \mathcal{P}(0) + M_1\int_0^t \Big[\int_0^L |\Phi_\lambda(x,\lambda)|^2dx + \int_0^L |\Phi_x(x,\lambda) + \Psi(x,\lambda)|^2dx + \int_0^L |\Psi_x(x,\lambda)|^2dx\Big]d\lambda \tag{35}$$

On the other hand we know that for $M_2 = min\left\{\frac{\rho_1}{2}, \frac{k}{2}, \frac{b}{2}\right\}$, *we have*

$$\mathcal{P}(\lambda) \geq M_2\Big[\int_0^L |\Phi_\lambda(x,\lambda)|^2dx + \int_0^L |\Phi_x(x,\lambda) + \Psi(x,\lambda)|^2dx + \int_0^L |\Psi_x(x,\lambda)|^2dx\Big] \tag{36}$$

From (35) *and* (36) *we have*

$$\mathcal{P}(t) \leq \mathcal{P}(0) + \frac{M_1}{M_2}\int_0^t d\lambda \tag{37}$$

Applying Gronwall's inequality we have

$$\mathcal{P}(t) \leq \mathcal{P}(0) e^{\frac{M_1 . t}{M_2}} \tag{38}$$

Hence the continuous dependence and the uniqueness of the solution. □

□

3. Exponential Stability

In this section, our aim is to study the exponential stability of the systelm (1) . More precisely we prove the following result.

Theorem 3.1. *Let the assumption* (2) *be satisfied. Then, there exist positive constants M and K such that, for any solution of* (9)

$$E(t) \leq ME(0)e^{-Kt}, \forall t \geq 0.$$

Démonstration. Let

$$Q(t) = 2c\rho_1 \int_0^L \theta_t \theta dx + \mu_1 c \int_0^L |\theta|^2 dx$$

where c is a constant whose conditions we will specify later.

Multiply $(9)_1$ by $2c\theta$ and integrate from 0 to L. We obtain

$$2c\rho_1 \int_0^L \theta_{tt}(x,t)\theta(x,t)dx - 2c\kappa \int_0^L \left[\theta_x(x,t) + \psi(x,t)\right]_x \theta(x,t)dx$$
$$+2c\mu_1 \int_0^L \theta_t(x,t)\theta(x,t)dx + 2c\int_0^L \int_{\tau_0}^{\tau_1} \alpha(s)\eta(x,1,t,s)\theta(x,t)dsdx = 0.$$

As $\theta_{tt}\theta = \dfrac{\partial}{\partial t}(\theta_t\theta) - |\theta_t|^2$ we have

$$2c\rho_1 \int_0^L \frac{\partial}{\partial t}[\theta_t(x,t)\theta(x,t)]dx - 2c\rho_1 \int_0^L |\theta_t|^2 dx + 2c\kappa \int_0^L \left[\theta_x(x,t) + \psi(x,t)\right]\theta_x(x,t)dx +$$
$$c\mu_1 \int_0^L \frac{\partial}{\partial t}|\theta(x,t)|^2 dx + 2c\int_0^L \int_{\tau_0}^{\tau_1} \alpha(s)\eta(x,1,t,s)\theta(x,t)dsdx = 0.$$

which means that

$$\frac{\partial}{\partial t}\int_0^L \left[2c\rho_1\theta_t(x,t)\theta(x,t) + c\mu_1|\theta(x,t)|^2\right]dx - 2c\rho_1\int_0^L |\theta_t|^2 dx$$
$$+2c\kappa \int_0^L \left[\theta_x(x,t) + \psi(x,t)\right]\theta_x(x,t)dx + 2c\int_0^L \int_{\tau_0}^{\tau_1} \alpha(s)\eta(x,1,t,s)\theta(x,t)dsdx = 0. \tag{39}$$

Multiply $(9)_2$ by $2c\psi$ and integrate over 0 to L. we obtain

$$-2cb\int_0^L \psi_{xx}(x,t)\psi dx + 2c\kappa \int_0^L \left[\theta_x(x,t) + \psi(x,t)\right]\psi dx = 0.$$

By integrating by parts, we have

$-2cb\left[\psi_x\psi\right]_0^L + 2cb\int_0^L |\psi_x|^2dx + 2c\kappa\int_0^L \left[\theta_x(x,t)+\psi(x,t)\right]\psi dx = 0.$
Since ψ_x is zero at 0 and L, we have

$$2cb\int_0^L |\psi_x|^2dx + 2c\kappa\int_0^L \left[\theta_x(x,t)+\psi(x,t)\right]\psi dx = 0. \tag{40}$$

By summing (39) – (40) we obtain

$$\begin{aligned}\frac{d}{dt}Q(t) \quad &= \quad 2c\rho_1\int_0^L |\theta_t|^2dx - 2c\kappa\int_0^L |\theta_x(x,t)+\psi(x,t)|^2dx - 2c\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\eta(x,1,t,s)\theta(x,t)dsdx \\ &-2cb\int_0^L |\psi_x|^2dx.\end{aligned}$$

Applying Young's inegality at $-2c\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)\eta(x,1,t,s)\theta(x,t)dsdx$ we have

$$\begin{aligned}\frac{d}{dt}Q(t) &\le 2c\rho_1\int_0^L |\theta_t|^2dx - 2c\kappa\int_0^L |\theta_x(x,t)+\psi(x,t)|^2dx + \frac{(2c)^2}{4\varepsilon}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx \\ &+ \varepsilon\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\theta(x,t)|^2dsdx - 2cb\int_0^L|\psi_x|^2dx, \forall\varepsilon>0 \\ &\le 2c\rho_1\int_0^L |\theta_t|^2dx - 2c\kappa\int_0^L |\theta_x(x,t)+\psi(x,t)|^2dx + \varepsilon\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\theta(x,t)|^2dsdx \\ &+ \frac{c^2}{\varepsilon}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx - 2cb\int_0^L|\psi_x|^2dx, \forall\varepsilon>0.\end{aligned}$$

By applying the inequality of Poincare at $\varepsilon\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\theta(x,t)|^2dsdx$ we obtain

$$\begin{aligned}\frac{d}{dt}Q(t) &\le 2c\rho_1\int_0^L |\theta_t|^2dx - 2c\kappa\int_0^L |\theta_x(x,t)+\psi(x,t)|^2dx + \varepsilon L^2\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\theta_x(x,t)|^2dsdx \\ &+ \frac{c^2}{\varepsilon}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx - 2cb\int_0^L|\psi_x|^2dx, \forall\varepsilon>0.\end{aligned} \tag{41}$$

We know that

$$\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\theta_x(x,t)|^2dsdx \le 2\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\theta_x(x,t)+\psi(x,t)|^2dsdx + 2\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\psi(x,t)|^2dsdx.$$

By Poincare's inequality we have

$$\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\theta_x(x,t)|^2dsdx \le 2\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\theta_x(x,t)+\psi(x,t)|^2dsdx + 2L^2\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\psi_x(x,t)|^2dsdx. \tag{42}$$

Multiplying (42) by εL^2 we have

$$\begin{aligned}\varepsilon L^2\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\theta_x(x,t)|^2dsdx &\le 2\varepsilon L^2\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\theta_x(x,t)+\psi(x,t)|^2dsdx \\ &+ 2\varepsilon L^4\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\psi_x(x,t)|^2dsdx \\ &\le \frac{2\varepsilon L^2\mu_1}{k}.k\int_0^L|\theta_x(x,t)+\psi(x,t)|^2dx + \frac{2\varepsilon L^4\mu_1}{b}.b\int_0^L|\psi_x(x,t)|^2dx.\end{aligned}$$

Choosing $c = max\left\{\frac{2\varepsilon L^2\mu_1}{k}, \frac{2\varepsilon L^4\mu_1}{b}\right\}$ we obtain

$$\varepsilon L^2\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\theta_x(x,t)|^2dsdx \le ck\int_0^L|\theta_x(x,t)+\psi(x,t)|^2dx + cb\int_0^L|\psi_x(x,t)|^2dx. \tag{43}$$

From (41) and (43) we can write

$$\begin{aligned}\frac{d}{dt}Q(t) &\leq 2c\rho_1\int_0^L|\theta_t|^2dx - 2c\kappa\int_0^L|\theta_x(x,t)+\psi(x,t)|^2dx + ck\int_0^L|\theta_x(x,t)+\psi(x,t)|^2dx \\ &+ cb\int_0^L|\psi_x(x,t)|^2dx + \frac{c^2}{\varepsilon}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx - 2cb\int_0^L|\psi_x|^2dx, \forall\varepsilon>0.\end{aligned}$$

which means that

$$\begin{aligned}\frac{d}{dt}Q(t) &\leq 2c\rho_1\int_0^L|\theta_t|^2dx - c\kappa\int_0^L|\theta_x(x,t)+\psi(x,t)|^2dx + \frac{c^2}{\varepsilon}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx \\ &- cb\int_0^L|\psi_x|^2dx, \forall\varepsilon>0. \end{aligned} \tag{44}$$

Let's put

$$I(t) = e^{2\tau_1}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)e^{-2s\rho}|\eta|^2dsd\rho dx.$$

we have

$$\frac{d}{dt}I(t) = 2e^{2\tau_1}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)e^{-2s\rho}\eta\eta_t dsd\rho dx.$$

From $(9)_3$ we have $s\eta_t = -\eta_\rho$ so

$$\begin{aligned}\frac{d}{dt}I(t) &= -2e^{2\tau_1}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}\alpha(s)e^{-2s\rho}\eta\eta_\rho dsd\rho dx \\ &= -2e^{2\tau_1}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)e^{-2s\rho}|\eta|^2dsd\rho dx - e^{2\tau_1}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}\alpha(s)\frac{d}{d\rho}\Big(e^{-2s\rho}|\eta|^2\Big)dsdxd\rho \\ &= -2e^{2\tau_1}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)e^{-2s\rho}|\eta|^2dsd\rho dx - e^{2\tau_1}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)e^{-2s}|\eta(x,1,t,s)|^2dsdx \\ &+ e^{2\tau_1}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)e^0|\eta(x,0,t,s)|^2dsdx \\ &\leq -2e^{2\tau_1}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)e^{-2\tau_1}|\eta|^2dsd\rho dx - e^{2\tau_1}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)e^{-2\tau_1}|\eta(x,1,t,s)|^2dsdx \\ &+ e^{2\tau_1}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,0,t,s)|^2dsdx \\ &\leq -2\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)|\eta|^2dsd\rho dx - \int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx \\ &+ e^{2\tau_1}\Big(\int_{\tau_0}^{\tau_1}\alpha(s)ds\Big)\int_0^L|\eta(x,0,t,s)|^2dx \\ &\leq -\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)|\eta|^2dsd\rho dx - \int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx \\ &+ e^{2\tau_1}\mu_1\int_0^L|\eta(x,0,t,s)|^2dx\end{aligned}$$

however

$$\begin{aligned}\frac{d}{dt}I(t) &\leq -\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)|\eta|^2dsd\rho dx - \int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx \\ &+ e^{2\tau_1}\mu_1\int_0^L|\eta(x,0,t,s)|^2dx \end{aligned} \tag{45}$$

Summing (44) and (45) we have

$$\begin{aligned}
\frac{d}{dt}\Big(I(t)+Q(t)\Big) &\leq 2c\rho_1\int_0^L |\theta_t|^2dx - c\kappa\int_0^L |\theta_x(x,t)+\psi(x,t)|^2dx + \frac{c^2}{\varepsilon}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx \\
&- cb\int_0^L |\psi_x|^2dx - \int_0^L\int_0^1\int_{\tau_0}^{\tau_1} s\alpha(s)|\eta|^2dsd\rho dx - \int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx \\
&+ e^{2\tau_1}\mu_1\int_0^L |\eta(x,0,t,s)|^2dx., \forall\varepsilon>0 \\
&\leq -c\rho_1\int_0^L |\theta_t|^2dx - c\kappa\int_0^L |\theta_x(x,t)+\psi(x,t)|^2dx - cb\int_0^L |\psi_x|^2dx \\
&- \int_0^L\int_0^1\int_{\tau_0}^{\tau_1} s\alpha(s)|\eta|^2dsd\rho dx + \frac{c^2}{\varepsilon}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx + 3c\rho_1\int_0^L |\theta_t|^2dx \\
&- \int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx + e^{2\tau_1}\mu_1\int_0^L |\eta(x,0,t,s)|^2dx
\end{aligned}$$

Choosing $c_1 = min\{1; c\}$, we obtain

$$\begin{aligned}
\frac{d}{dt}\Big(I(t)+Q(t)\Big) &\leq -c_1\rho_1\int_0^L |\theta_t|^2dx - c_1\kappa\int_0^L |\theta_x(x,t)+\psi(x,t)|^2dx - c_1 b\int_0^L |\psi_x|^2dx \\
&- c_1\int_0^L\int_0^1\int_{\tau_0}^{\tau_1} s\alpha(s)|\eta|^2dsd\rho dx + \frac{c^2}{\varepsilon}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx \\
&+ 3c\rho_1\int_0^L |\theta_t|^2dx - \int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx + e^{2\tau_1}\mu_1\int_0^L |\eta(x,0,t,s)|^2dx \\
&\leq -2c_1E(t) + \Big(\frac{c^2}{\varepsilon}-1\Big)\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx \\
&+ \Big(e^{2\tau_1}\mu_1 + 3c\rho_1\Big)\int_0^L |\eta(x,0,t,s)|^2dx.
\end{aligned}$$

Thereby

$$\begin{aligned}
\frac{d}{dt}\Big(I(t)+Q(t)\Big) &\leq -2c_1E(t) + \Big(\frac{c^2}{\varepsilon}-1\Big)\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx \\
&+ \Big(e^{2\tau_1}\mu_1 + 3c\rho_1\Big)\int_0^L |\eta(x,0,t,s)|^2dx.
\end{aligned} \tag{46}$$

Let us introduce the Lyapunov functional

$$\mathcal{M}(t) = NE(t) + I(t) + Q(t), \forall t>0$$

where N is a suitable positive constant that will precised later on. We obtain from (15) and (46)

$$\begin{aligned}
\frac{d}{dt}\mathcal{M}(t) &\leq N\left[\int_{\tau_0}^{\tau_1}\alpha(s)ds - \mu_1\right]\int_0^L |\eta(x,0,t,s)|^2dx - N\frac{1}{2}\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)+\eta(x,0,t,s)|^2dsdx \\
&- 2c_1E(t) + \Big(\frac{c^2}{\varepsilon}-1\Big)\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx + \Big(e^{2\tau_1}\mu_1 + 3c\rho_1\Big)\int_0^L |\eta(x,0,t,s)|^2dx
\end{aligned} \tag{47}$$

Take $c_2 = max\left\{\frac{2L^2\mu_1}{k}, \frac{2L^4\mu_1}{b}\right\}$, we have $c = \varepsilon c_2$. In (47) we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{M}(t) &\leq N\left[\int_{\tau_0}^{\tau_1}\alpha(s)ds - \mu_1\right]\int_0^L |\eta(x,0,t,s)|^2dx - N\frac{1}{2}\int_0^L |\eta(x,1,t,s)+\eta(x,0,t,s)|^2dsdx \\
&- 2c_1E(t) + \left(\varepsilon c_2^2 - 1\right)\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx + \left(e^{2\tau_1}\mu_1 + 3c\rho_1\right)\int_0^L|\eta(x,0,t,s)|^2dx \\
&\leq \left[N\Big(\int_{\tau_0}^{\tau_1}\alpha(s)ds - \mu_1\Big) + e^{2\tau_1}\mu_1 + 3c\rho_1\right]\int_0^L|\eta(x,0,t)|^2dx \\
&- N\frac{1}{2}\int_0^L|\eta(x,1,t,s)+\eta(x,0,t,s)|^2dsdx - 2c_1E(t) + \left(\varepsilon c_2^2-1\right)\int_0^L\int_{\tau_0}^{\tau_1}\alpha(s)|\eta(x,1,t,s)|^2dsdx
\end{aligned}$$

For

$$N > \frac{-e^{2\tau_1}\mu_1 - 3c\rho_1}{\displaystyle\int_{\tau_0}^{\tau_1}\alpha(s)ds - \mu_1}\ et\ \varepsilon < \frac{1}{c_2^2}$$

we have

$$\frac{d}{dt}\mathcal{M}(t) \leq -2c_1E(t). \tag{48}$$

Before continuing our proof let us prove the following lemma

Lemma 3.1. *There are two positive constants γ and β such that*

$$\gamma E(t) \leq \mathcal{M}(t) \leq \beta E(t), \forall t \geq 0. \tag{49}$$

Démonstration. Let's put

$$\varepsilon(t) = \mathcal{M}(t) - NE(t)$$

We have

$$\begin{aligned}
|\varepsilon(t)| &= |\mathcal{M}(t) - NE(t)| \\
|\varepsilon(t)| &= \left|2c\rho_1\int_0^L\theta_t\theta dx + \mu_1 c\int_0^L|\theta|^2dx + e^{2\tau_1}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)e^{-2s\rho}|\eta|^2dsd\rho dx\right| \\
&\leq \left|2c\rho_1\int_0^L\theta_t\theta dx\right| + \left|\mu_1 c\int_0^L|\theta|^2dx\right| + \left|e^{2\tau_1}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)e^{-2s\rho}|\eta|^2dsd\rho dx\right|
\end{aligned}$$

Appliying Youg's and Poincare's inegalities we obtaint

$$\begin{aligned}
|\varepsilon(t)| &\leq c\rho_1\int_0^L|\theta_t|^2dx + (c\rho_1+\mu_1 c)L^2\int_0^L|\theta_x|^2dx + +e^{2\tau_1}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)|\eta|^2dsd\rho dx \\
&\leq c\rho_1\int_0^L|\theta_t|^2dx + (c\rho_1+\mu_1 c)L^2\int_0^L|\theta_x+\psi-\psi|^2dx + e^{2\tau_1}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)|\eta|^2dsd\rho dx \\
&\leq c\rho_1\int_0^L|\theta_t|^2dx + (c\rho_1+\mu_1 c)L^2\int_0^L|\theta_x+\psi|^2dx + (c\rho_1+\mu_1 c)L^2\int_0^L|\psi|^2dx \\
&+ e^{2\tau_1}\int_0^L\int_0^1\int_{\tau_0}^{\tau_1}s\alpha(s)|\eta|^2dsd\rho dx
\end{aligned}$$

Appliying Poincare inegality to $(c\rho_1 + \mu_1 c)L^2 \int_0^L |\psi|^2 dx$ we can say

$$\begin{aligned} |\varepsilon(t)| &\leq c\rho_1 \int_0^L |\theta_t|^2 dx + (c\rho_1 + \mu_1 c)L^2 \int_0^L |\theta_x + \psi|^2 dx + (c\rho_1 + \mu_1 c)L^4 \int_0^L |\psi_x|^2 dx \\ &+ e^{2\tau_1} \int_0^L \int_0^1 \int_{\tau_0}^{\tau_1} s\alpha(s)|\eta|^2 ds d\rho dx \\ &\leq 2c.\frac{\rho_1}{2} \int_0^L |\theta_t|^2 dx + \frac{2(c\rho_1 + \mu_1 c)L^2}{k}.\frac{k}{2} \int_0^L |\theta_x + \psi|^2 dx + \frac{2(c\rho_1 + \mu_1 c)L^4}{k}.\frac{b}{2} \int_0^L |\psi_x|^2 dx \\ &+ 2e^{2\tau_1} \frac{1}{2} \int_0^L \int_0^1 \int_{\tau_0}^{\tau_1} s\alpha(s)|\eta|^2 ds d\rho dx \end{aligned}$$

Finally

$$|\varepsilon(t)| \leq \delta E(t)$$

with $\delta = max\left\{2c, \frac{2(c\rho_1 + \mu_1 c)L^2}{k}, \frac{2(c\rho_1 + \mu_1 c)L^4}{k}, 2e^{2\tau_1}\right\}$
So when we put $\gamma = N - \delta$ and $\beta = N + \delta$ we have (49). □

□

We return to the proof of theorem 3.1.
By (48) and (49) we obtain

$$\frac{\frac{d}{dt}\mathcal{M}(t)}{\mathcal{M}(t)} \leq \frac{-2c_1 E(t)}{\mathcal{M}(t)} \leq \frac{-2c_1 E(t)}{\beta E(t)}.$$

which means that

$$\frac{\frac{d}{dt}\mathcal{M}(t)}{\mathcal{M}(t)} \leq \frac{-2c_1}{\beta}. \tag{50}$$

Integrating (50) from 0 to t we have

$$ln\mathcal{M}(t) - ln\mathcal{M}(0) \leq \frac{-2c_1}{\beta} t.$$

which means that

$$\mathcal{M}(t) \leq \mathcal{M}(0)e^{\frac{-2c_1}{\beta}t}. \tag{51}$$

By (49) we obtain

$$\gamma E(t) \leq \mathcal{M}(t),$$

and

$$\mathcal{M}(0) \leq \beta E(0).$$

However (51) can be written

$$E(t) \leq \frac{\beta E(0)}{\gamma} e^{\frac{-2c_1 t}{\beta}}.$$

We obtain by taking $M = \frac{\beta}{\gamma}$ and $K = \frac{2c_1}{\beta}$

$$E(t) \leq ME(0)e^{-Kt}, \forall t \geq 0.$$

This completes the proof. □ □

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