Exponential Stability for Damped Shear Beam Model and New Facts
Related to the Classical Timoshenko System With a Distributed
Delay Term

Innocent OUEDRAOGO1, Gilbert BAYILI1

1 Département de mathématiques, Université Joseph Ki-Zerbo, Burkina Faso

Correspondence : Innocent OUEDRAOGO, Département de mathématiques, Université Joseph Ki-Zerbo, Burkina Faso.
E-mail : innocentouedraogo850@yahoo.fr ; TEL : (00226) 55112414

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Abstract
We consider in this manuscript a Timoshenko type beam model with a distributed delay term. If the distributed delay term
is small enough, we prove the global existence of solutions by using the Faedo-Galerkin approximations together with
some energy estimates. Under suitable assumptions, we prove exponential stability of the solution. This result is obtained
by introducing a suitable Lyapounov function.

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1. Introduction
In this paper, we investigate the stability of the shear beam model with internal distributed delay term. More precisely we
consider the following system:

$$
\begin{align*}
\rho_1 \theta_{tt}(x, t) - \kappa (\theta_x(x, t) + \psi(x, t))_x + \mu_1 \theta_t(x, t) + \int_{\tau_0}^{\tau_1} \alpha(s) \theta_t(x, t - s) \, ds = 0 \quad &\text{in } ]0, L[ \times (0, +\infty), \\
- b \psi_{xx}(x, t) + \kappa (\theta_x(x, t) + \psi(x, t)) = 0 \quad &\text{in } ]0, L[ \times (0, +\infty), \\
\theta(x, 0) = \theta_0(x), \quad \theta_t(x, 0) = \theta_1(x), \quad &\psi(x, 0) = \psi_0(x) \quad \text{in } ]0, L[, \\
\psi_x(0, t) = \psi_x(L, t) = \theta(0, t) = \theta(L, t) = 0, \quad &t > 0, \\
\theta_t(x, -t) = f_0(x, -t), \quad (x, t) \in ]0, L[ \times (0, \tau_1). 
\end{align*}
$$

(1)

where the functions $\theta$ and $\psi$ describe respectively the transverse displacement of the beam and the rotation angle of a
filament of the beam.

$\rho_1, \kappa, \mu_1, b, \tau_0$ and $\tau_1$ are element of $\mathbb{R}^+$ such that $\tau_0$ is strictly less than $\tau_1$.

Throughout this manuscript, we consider that $\alpha : [\tau_0; \tau_1] \to \mathbb{R}$, $\alpha$ is in $L^\infty$ and is a bounded function satisfying

$$
\int_{\tau_0}^{\tau_1} \alpha(s) \, ds < \mu_1.
$$

(2)

This problem can be regarded as a problem with a memory acting only on $(t - \tau_1, t - \tau_0)$. By a change of variable, we see that

$$
\int_{\tau_0}^{\tau_1} \alpha(s) \theta_t(x, t - s) \, ds = \int_{t - \tau_1}^{t - \tau_0} \alpha(s) \theta_t(x, t - s) \, ds
$$
It is also well known that if $\alpha = 0$, that is in absence of delay, the corresponding following system:

$$\begin{cases}
\rho_1 \theta_t(x,t) - \kappa \theta_{xx}(x,t) + \mu \theta_t(x,t) = 0 \quad \text{in } ]0, L[ \times (0, +\infty), \\
-b\psi_{xx}(x,t) + \kappa \theta_t(x,t) + \mu \psi_t(x,t) = 0 \quad \text{in } ]0, L[ \times (0, +\infty), \\
\theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x), \quad \psi(x,0) = \psi_0(x) \quad \text{in } ]0, L[,
\psi_s(0,t) = \psi_s(L,t) = \theta(0,t) = \theta(L,t) = 0, \quad t > 0.
\end{cases} \tag{3}$$

was analyzed in a recent paper by Almeida Jnior, A.J.A. Ramos and M.M. Freitas [1], where it was get the energy of the system exponentially-stable. In the case of the wave equations, Nicaise and Pignotti [8] investigated exponential stability results with delay concentrated at $\tau$ for the system

$$\begin{cases}
u_t(x,t) - \Delta u(x,t) = 0 \quad \text{in } \Omega \times (0, +\infty) \\
u(x,t) = 0 \quad \text{on } \Gamma_D \times (0, +\infty) \\
\frac{\partial u}{\partial y}(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t - \tau) = 0 \quad \text{on } \Gamma_N \times (0, +\infty) \\
u(x,0) = u_0, \quad u_t(x,0) = u_1 \quad \text{in } \Omega \\
u(x,t - \tau) = f_0(x,t - \tau) \quad \text{on } \Gamma_N \times (0, \tau),
\end{cases} \tag{4}$$

under the condition $\mu_2 < \mu_1$, by combining inequalities due to Carleman estimates and compactness-uniqueness arguments. Later, they also obtain in [13] the exponential stability with distributed delay of the system

$$\begin{cases}
u_t(x,t) - \Delta u(x,t) = 0 \quad \text{in } \Omega \times (0, +\infty) \\
u(x,t) = 0 \quad \text{on } \Gamma_D \times (0, +\infty) \\
\frac{\partial u}{\partial y}(x,t) + \mu_1 u_t(x,t) + \int_{\Gamma_N} \alpha(s) u_t(x,t - s) \, ds = 0 \quad \text{on } \Gamma_N \times (0, +\infty) \\
u(x,0) = u_0, \quad u_t(x,0) = u_1 \quad \text{in } \Omega \\
u(x,-t) = f_0(x,-t) \quad \text{on } \Gamma_N \times (0, \tau_1),
\end{cases} \tag{5}$$

under the assumption [2].

Recently in the case of the wave equation with dynamical control, Silga and Bayili [2] studied a wave equation set in a bounded domain with a dynamical control and prove that if the delay term is small enough, then the system with delay
has the same (polynomial) decay rate than the one without delay.

\[
\begin{cases}
 u_{tt}(x, t) - \Delta u(x, t) = 0 \text{ in } \Omega \times (0, +\infty) \\
 u(x, t) = 0 \text{ on } \Gamma_D \times (0, +\infty) \\
 \frac{\partial u}{\partial v}(x, t) + \eta(x, t) = 0 \text{ on } \Gamma_N \times (0, +\infty) \\
 \eta_t(x, t) - u_t(x, t) + \beta_1 \eta(x, t) + \beta_2 \eta(x, t - \tau) = 0 \text{ on } \Gamma_N \times (0, +\infty) \\
 u(0, 0) = u_0, \quad u_t(0, 0) = u_1 \text{ in } \Omega \\
 \eta(0, 0) = \eta_0 \text{ on } \Gamma_N \\
 \eta(x, t - \tau) = f_0(x, t - \tau) \text{ on } \Gamma_N \times (0, \tau),
\end{cases}
\]

(6)

Later in [14] they also investigate the case of the wave equation set in a bounded domain with a distributed delay on dynamical control

\[
\begin{cases}
 u_{tt}(x, t) - u_{xx}(x, t) = 0 \text{ in } ]0, 1[\times(0, +\infty) \\
 u(0, t) = 0 \\
 u_x(1, t) + \eta(t) = 0 \forall t \in (0, +\infty) \\
 \eta_t(t) - u_t(1, t) + \beta_1 \eta(t) + \int_{\tau_0}^{\tau_1} \beta_2(s) \eta(t - s) ds = 0 \forall t \in (0, +\infty) \\
 u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u_1 \text{ in } ]0, 1[ , \quad \eta(0) = \eta_0 \in \mathbb{R} \\
 \eta(-t) = f_0(-t) \forall t \in (0, \tau_1),
\end{cases}
\]

(7)

To do this they use the assumption (2).

Motivated by the above results we will establish under the hypothesis (2) the well-posedness and the exponential stability of the system (1).

This paper is organized as follows. In Section 2, we give the decrease for the energy of the system and the well-posedness of problems (1) using the Faedo-Galerkin method. And finally in Section 3, we will give the exponential stability of the problem using a Lyapounov function.

2. Well-Posedness of the Problem

In this section we will give well-posedness results for problem (1) using Faedo-Galerkin method. To this aim, we introducing the following auxiliary change of variable

\[
\eta(x, \rho, t, s) = \theta_s(x, t - s \rho), \quad x \in [0, L], \rho \in (0, 1), s \in (\tau_0, \tau_1), \quad t > 0.
\]

(8)

The problem (1) is now equivalent to
\[
\begin{aligned}
&\rho_1 (\theta_\alpha (x, t) - \kappa (\theta_\alpha (x, t) + \psi (x, t)))_x + \mu_1 \theta_\alpha (x, t) + \int_{\tau_0}^{t_1} \alpha(s) \eta(x, 1, t, s)ds = 0 \text{ in } [0, L] \times (0, +\infty), \\
&-b \psi_{xx} (x, t) + \kappa (\theta_\alpha (x, t) + \psi (x, t)) = 0 \text{ in } [0, L] \times (0, +\infty), \\
&\eta (\rho, t) + \eta_\rho (\rho, t) = 0 \text{ in } (0, 1) \times (0, +\infty), \\
&\theta(x, 0) = \theta_0 (x), \quad \theta_\alpha (x, 0) = \theta_1 (x), \quad \psi (x, 0) = \psi_0 (x) \text{ in } [0, L], \\
&\eta(x, \rho, 0, s) = f_0 (x - \rho s), \forall x \in [0, L], \forall \rho \in (0, 1), s \in (\tau_0, \tau_1), \\
&\eta(x, 0, t, s) = \theta_1 (x, t) \forall x \in [0, L], \forall t \in (0, +\infty), s \in (\tau_0, \tau_1), \\
&\psi (x, 0, t) = \psi (L, t) = \theta (0, t) = 0(L, t) = 0, t > 0.
\end{aligned}
\]

Let us consider the Hilbert spaces
\[
\mathcal{H} = H_0^1 (0, L) \times L^2 (0, L) \times H_0^1 (0, L) \times L^2 \left( (0, L) \times (0, 1) \times (\tau_0, \tau_1) \right),
\]
and
\[
\mathcal{H}_1 = (H^2 (0, L) \cap H_0^1 (0, L))^2 \times H_0^2 (0, L) \times H^1 \left( (0, L) \times (0, 1) \times (\tau_0, \tau_1) \right).
\]

where
\[
L^2 (0, L) = \left\{ u \in L^2 (0, L), \int_0^L u(x) dx = 0 \right\}, H_0^1 (0, L) = H^1 (0, L) \cap L^2 (0, L),
\]
and
\[
H_0^2 (0, L) = H^2 (0, L) \cap H_0^1 (0, L).
\]

We equipped \( \mathcal{H} \) with the norm
\[
\| (u, v, w, z)^T \|_{\mathcal{H}}^2 = \frac{\rho_1}{2} \int_0^L |v_x|^2 dx + \frac{b}{2} \int_0^L |w|^2 dx + \frac{\kappa}{2} \int_0^L |u_x + w|^2 dx + \frac{1}{2} \int_0^L \int_{\tau_0}^{t_1} \alpha(s) |\eta|^2 ds dx.
\]

Define the energy of a solution \((\theta, \theta_\alpha, \psi, z)\) of (9) as
\[
E(t) = \frac{\rho_1}{2} \int_0^L |\theta|^2 dx + \frac{b}{2} \int_0^L |\psi_x|^2 dx + \frac{\kappa}{2} \int_0^L |\theta_\alpha + \psi|^2 dx + \frac{1}{2} \int_0^L \int_{\tau_0}^{t_1} \alpha(s) |\eta|^2 ds dx.
\]

We can now prove that the energy is decreasing. More precisely, we have the following result.

**Proposition 2.1.** For any regular solution of problem (9), the energy is decreasing and there exists a positive constant \( C \) such that
\[
\frac{d}{dt} E(t) \leq -C \left( \int_0^L |\eta(x, 0, t, s)|^2 dx + \int_0^L \int_{\tau_0}^{t_1} \alpha(s) |\eta(x, 1, t, s) + \eta(x, 0, t, s)|^2 dx ds \right).
\]

**Démonstration.** Multiplying \((9)\), the first equation by \( \theta_\alpha \), the second equation by \( \psi_x \) and the third equation by \( \alpha(s) \eta \), and then integrating the first and the second over \([0; L]\) and the third over \((0, L) \times (0, 1) \times (\tau_0, \tau_1)\). By integrating by parts and using the boundary conditions, we obtain

\[
\begin{aligned}
\frac{\rho_1}{2} \frac{d}{dt} \int_0^L |\theta_\alpha (x, t)|^2 dx + \kappa \int_0^L \left[ \theta_\alpha (x, t) + \psi (x, t) \right] \theta_\alpha (x, t) dx \\
+ \mu_1 \int_0^L |\theta_\alpha (x, t)|^2 dx + \int_{\tau_0}^{t_1} \alpha(s) \eta(x, 1, t, s) \theta_\alpha (x, t) ds dx = 0,
\end{aligned}
\]

and

\[
\begin{aligned}
\frac{b}{2} \frac{d}{dt} \int_0^L |\psi_x (x, t)|^2 dx + \kappa \int_0^L \left[ \theta_\alpha (x, t) + \psi (x, t) \right] \psi_x (x, t) dx = 0,
\end{aligned}
\]

(10) (11)
and

\[ \frac{1}{2} \frac{d}{dt} \int_0^L \int_0^1 \int_{\tau_0}^{\tau_1} \alpha(s) \eta(x, \rho, t, s)^2 ds dx + \frac{1}{2} \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s) \left[ \eta(x, 1, t, s)^2 - |\eta(x, 0, t, s)|^2 \right] ds dx = 0. \] (12)

Now summing relations (10), (11) and (12) we obtain

\[ \frac{\rho_1}{2} \frac{d}{dt} \int_0^L |\theta_1(x,t)|^2 dx + \frac{b}{2} \int_0^L |\theta_2(x,t)|^2 dx + \mu_1 \int_0^L |\theta_4(x,t)|^2 dx + \frac{1}{2} \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s) \eta(x, 1, t, s) \theta_4(x,t) ds dx + \mu_1 \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s) \eta(x, 1, t, s) \theta_4(x,t) ds dx \]

which lead to

\[ \frac{d}{dt} \left( \frac{\rho_1}{2} \int_0^L |\theta_1(x,t)|^2 dx + \frac{b}{2} \int_0^L |\theta_2(x,t)|^2 dx + \frac{1}{2} \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s) \eta(x, 1, t, s) \theta_4(x,t) ds dx \right) \]

Then from the definition of the energy, it follows that

\[ \frac{d}{dt} E(t) = -\mu_1 \int_0^L |\theta_1(x,t)|^2 dx - \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s) \eta(x, 1, t, s) \theta_4(x,t) ds dx - \frac{1}{2} \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s) \left[ \eta(x, 1, t, s)^2 - |\eta(x, 0, t, s)|^2 \right] ds dx. \]

As \( \theta_4(x,t) = \eta(x, 0, t, s) \), we get

\[ \frac{d}{dt} E(t) = -\mu_1 \int_0^L |\eta(x, 0, t, s)|^2 dx - \frac{1}{2} \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s) \left[ \eta(x, 1, t, s)^2 - |\eta(x, 0, t, s)|^2 \right] ds dx. \] (13)

We also know that

\[ \eta(x, 1, t, s) \eta(x, 0, t, s) = \frac{1}{2} |\eta(x, 1, t, s) + \eta(x, 0, t, s)|^2 - \frac{1}{2} |\eta(x, 0, t, s)|^2 - \frac{1}{2} |\eta(x, 1, t, s)|^2. \] (14)

Then using the relations (14) in (13), it follows that

\[ \frac{d}{dt} E(t) = \left[ \int_{\tau_0}^{\tau_1} \alpha(s) ds - \mu_1 \right] \int_0^L |\eta(x, 0, t, s)|^2 dx - \frac{1}{2} \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s) |\eta(x, 1, t, s) + \eta(x, 0, t, s)|^2 ds dx \] (15)

This proves that
Given an initial data \( U \)

To state the main result in this section, we start by defining what we mean by a weak solution of problem (9) as following

We therefore deduce that the energy of the system (9) is dissipative. \( \square \)

To state the main result in this section, we start by defining what we mean by a weak solution of problem (9) as following

**Definition 2.1.** Given an initial data \( U_0 = (\theta_0, \theta_1, \psi_0, \eta_0) \in \mathcal{H} \) a function \( U = (\theta, \theta_1, \psi, \eta) \in C([0, T]; \mathcal{H}) \) is said to be weak solution of (9) if for almost everywhere \( t \in [0, T] \),

\[
\begin{align*}
\left\{ \begin{array}{l}
\rho_1 \frac{d}{dt}(\theta_1, u) + \kappa(\theta_1(x, t) + \psi(x, t), u_x) + \mu_1(\theta_1, u) + \left( \int_{\tau_0}^{\tau_1} \alpha(s)\eta(x, 1, t, s)ds, u \right) = 0, \\
b(\psi_1, v) + \kappa(\theta_1 + \psi, v) = 0, \\
s(\eta_1, w) + (\eta_0, w) = 0,
\end{array} \right.
\end{align*}
\]

for all \( u \in H_0^1(0, L), v \in H^2(0, L), w \in L^2((0, L) \times (0, 1) \times (\tau_0, \tau_1)) \) and \( (\theta(0), \theta_1(0), \psi(0), \eta(0)) = (\theta_0, \theta_1, \psi_0, \eta_0) \).

The main result of this section is the following :

**Theorem 2.1.** Assume that [2] holds. Then for all \( U_0 = (\theta_0, \theta_1, \psi_0, \eta_0) \in \mathcal{H} \), the problem (9) admits a unique weak solution \( U = (\theta, \theta_1, \psi, \eta) \) such that :

\[
\begin{align*}
\theta & \in L^\infty\left(0, T, H_0^3(0, L)\right), \\
\theta_1 & \in L^\infty\left(0, T, L^2(0, L)\right), \\
\psi & \in L^\infty\left(0, T, H_0^1(0, L)\right), \\
\eta & \in L^\infty\left(0, T, L^2((0, L) \times (0, 1) \times (\tau_0, \tau_1))\right).
\end{align*}
\]

Moreover, if \( U_0 = (\theta_0, \theta_1, \psi_0, \eta_0) \in \mathcal{H}_1 \), then problem (9) admits a unique stronger weak solution \( U = (\theta, \theta_1, \psi, \eta) \) which satisfies

\[
\begin{align*}
\theta & \in L^\infty\left(0, T, H^2(0, L) \cap H_0^3(0, L)\right), \\
\theta_1 & \in L^\infty\left(0, T, H_0^3(0, L)\right), \\
\psi & \in L^\infty\left(0, T, H_0^1(0, L)\right), \\
\eta & \in L^\infty\left(0, T, H^1((0, L) \times (0, 1) \times (\tau_0, \tau_1))\right).
\end{align*}
\]

In both cases, the solution \( (\theta, \theta_1, \psi, \eta) \) depends continuously on the initial data in \( \mathcal{H} \). Particular, the system (9) admit a unique weak solution.

**Démonstration.** The Faedo-Galerkin method, see [10, 4, 12, 11, 15] or [11] will be the key to prove the existence of a global solution. The proof will be done in six steps

**Step 1. : Faedo-Galerkin approximations.**

Let us consider initial data \( (\theta_0, \theta_1, \psi_0, \eta_0) \in \mathcal{H} \).

Let \( \{u^k\}, k \in \mathbb{N}^* \) and \( \{v^k\}, k \in \mathbb{N}^* \) basics formed by eigenfunctions of \( -\partial_{xx} \). This bases can be considered orthogonal in \( H^2(\Omega) \cap H_0^3(\Omega) \) and \( H_0^2(0, L) \) and also orthonormal in \( L^2(0; L) \).
We also define the sequence \( \{w^k\}, k \in \mathbb{N}^* \) in the following way
\[ w^k(x, 0, s) = u^k(x) \] then we extend \( w^k(x, \rho, s) \) by \( w^k(x, \rho, s) \) on \( L^2((0, L) \times (0, 1) \times (\tau_0, \tau_1)) \). For more details see Lemma 1.1 [17].
We consider the finite-dimensional subspaces \( H_n, V_n \) and \( W_n \) defined by
\[ H_n = \text{span} \{ u^1, u^2, ..., u^n \}, \quad V_n = \text{span} \{ v^1, v^2, ..., v^n \} \quad \text{and} \quad W_n = \text{span} \{ w^1, w^2, ..., w^n \}. \]
Now we will find an approximate solution in the form
\[ \theta^n(t, x) = \sum_{j=1}^{n} a^{jn}(t)u^j(x), \]
\[ \psi^n(t, x) = \sum_{j=1}^{n} b^{jn}(t)v^j(x) \]
and
\[ \eta^n(x, \rho, t, s) = \sum_{j=1}^{n} c^{jn}(t)w^j(x, \rho, s) \]
of the following approximate problem
\begin{align}
\left\{ \begin{array}{l}
\rho_1 \int_{0}^{L} \theta^n_{, t}(x, t)udx - \kappa \int_{0}^{L} \left[ \theta^n(x, t) + \psi^n(x, t) \right] \theta^n(x, t) dx + \mu_1 \int_{0}^{L} \theta^n_{, t}(x, t) dx \\
+ \int_{0}^{\tau_1} \int_{\tau_0}^{\tau_1} \alpha(s) \eta^n(x, 1, t, s)udx ds = 0, \\
-b \int_{0}^{L} \psi^n_{, t}(x, t)dx + \kappa \int_{0}^{L} \left[ \theta^n(x, t) + \psi^n(x, t) \right] \psi^n(x, t) dx = 0,
\end{array} \right.
\end{align}
(19)
for all \( u \in H_n, v \in V_n, w \in W_n \), with initial conditions such that
\begin{align}
(\theta^n(0), \theta^n_1(0), \psi^n(0), \eta^n(0)) = (\theta^n_0, \theta^n_1, \psi^n_0, \eta^n_0) \to (\theta_0, \theta_1, \psi_0, \eta_0)
\end{align}
(20)
strongly in \( \mathcal{H} \) and where \( a^{jn}, b^{jn} \) and \( c^{jn} \) are time-dependent coefficients.
Applying standard theory of ordinary differential equations, the finite dimensional problem (19) – (20) has a solution \((a^{jn}, b^{jn}, c^{jn}), 1 \leq j \leq n\) defined on \([0, t_n]\) for every \( n \in \mathbb{N}^* \).
Then the a priori estimates that follow imply that in fact \( t_n = T, \forall T > 0 \).

**Step 2. A Priori Estimate I**

By substituting \( u \) for \( \theta^n \) in (19), \( v \) for \( \psi^n \) in (19), and \( w \) by \( \eta^n \) in (19), we obtain
\begin{align}
\left\{ \begin{array}{l}
\frac{\rho_1}{2} \frac{d}{dt} \int_{0}^{L} \left| \theta^n_{, t}(x, t) \right|^2 dx + \kappa \int_{0}^{L} \left[ \theta^n(x, t) + \psi^n(x, t) \right] \theta^n_{, t}(x, t) dx + \mu_1 \int_{0}^{L} \left| \theta^n_{, t}(x, t) \right|^2 dx \\
+ \int_{0}^{\tau_1} \int_{\tau_0}^{\tau_1} \alpha(s) \eta^n(x, 1, t, s)\theta^n_{, t}(x, t)dx ds = 0, \\
-b \frac{d}{dt} \int_{0}^{L} \left| \psi^n_{, t}(x, t) \right|^2 dx + \kappa \int_{0}^{L} \left[ \theta^n(x, t) + \psi^n(x, t) \right] \psi^n_{, t}(x, t) dx = 0,
\end{array} \right.
\end{align}
(21)
By making the same transformations as in the section of the dissipative character we obtain
\begin{align}
\frac{d}{dt} E^n(t) &= \int_{\tau_0}^{\tau_1} \alpha(s)ds - \mu_1 \int_{0}^{L} \left| \eta^n(x, 0, t, s) \right|^2 dx - \frac{1}{2} \int_{0}^{L} \int_{\tau_0}^{\tau_1} \alpha(s) \left[ \left| \eta^n(x, 1, t, s) \right|^2 - \left| \eta^n(x, 0, t, s) \right|^2 \right] dx ds = 0.
\end{align}
(22)
where
\[ E^n(t) = \frac{\rho_1}{2} \int_0^L |\theta^n_t|^2 dx + \frac{b}{2} \int_0^L |\psi^n_x|^2 dx + \frac{\kappa}{2} \int_0^L |\theta^n_x + \psi^n|^2 dx + \frac{1}{2} \int_0^L \int_{\tau_0}^{t_1} \sigma(s) |\eta^n|^2 d\tau ds dx. \]

By integrating (22) from 0 to \( t < t_n \), we obtain
\[
E^n(t) - E^n(0) = \left[ \int_{\tau_0}^{t_1} \alpha(s) ds - \mu_1 \right] \int_0^L |\eta^n(x, 0, \alpha, s)|^2 d\alpha d\lambda \\
\quad - \frac{1}{2} \int_0^\alpha \int_{\tau_0}^{t_1} \alpha(s) |\eta^n(x, 1, \lambda, s) + \eta^n(x, 0, \lambda, s)|^2 d\alpha d\lambda.
\]

which means
\[
E^n(t) = \frac{\rho_1}{2} \int_0^L |\theta^n_t|^2 dx + \frac{b}{2} \int_0^L |\psi^n_x|^2 dx + \frac{\kappa}{2} \int_0^L |\theta^n_x + \psi^n|^2 dx \\
\quad + \frac{1}{2} \int_0^L \int_{\tau_0}^{t_1} \alpha(s) |\eta^n(x, 0, 0, s)|^2 d\alpha d\lambda.
\]

As \((\theta^n(0), \theta^n_t(0), \psi^n(0), \eta^n(0)) = (\theta^0, \theta^1, \psi^0, \eta^0)\) strongly in \( H \) then there exists a positive constant \( C_1 \) such that \( E^n(0) \leq C_1 \). Hence
\[
E^n(t) + \left[ \mu_1 - \int_{\tau_0}^{t_1} \alpha(s) ds \right] \int_0^L |\eta^n(x, 0, \alpha, s)|^2 d\alpha d\lambda \\
\quad + \frac{1}{2} \int_0^\alpha \int_{\tau_0}^{t_1} \alpha(s) |\eta^n(x, 1, \lambda, s) + \eta^n(x, 0, \lambda, s)|^2 d\alpha d\lambda \leq C_1
\]

which means that
\[
\frac{\rho_1}{2} \int_0^L |\theta^n_t|^2 dx + \frac{b}{2} \int_0^L |\psi^n_x|^2 dx + \frac{\kappa}{2} \int_0^L |\theta^n_x + \psi^n|^2 dx + \frac{1}{2} \int_0^L \int_{\tau_0}^{t_1} \alpha(s) |\eta^n|^2 d\tau ds dx
\]
\[
\quad + \frac{1}{2} \int_0^L \int_{\tau_0}^{t_1} \alpha(s) |\eta^n(x, 1, \lambda, s) + \eta^n(x, 0, \lambda, s)|^2 d\alpha d\lambda \\
\quad + \left[ \mu_1 - \int_{\tau_0}^{t_1} \alpha(s) ds \right] \int_0^L |\eta^n(x, 0, 0, s)|^2 d\alpha d\lambda \leq C_1
\]

(23)

As the constant \( C_1 \) does not depend on \( n \) we can therefore take \( t_n = T \) for all \( T > 0 \)

**Step 3. A Priori Estimate II**
Let us derive the equation (19) with respect to \( t \) and then replacing \( u \) by \( \theta^n \). We obtain
\[
\rho_1 \int_0^L \theta^n_m(x, t) \theta^n_0(x, t) dx - \kappa \int_0^L \left[ \theta^n_m(x, t) + \psi^n(x, t) \right] \theta^n_0(x, t) dx
\]
\[
+ \mu_1 \int_0^L \theta^n_m(x, t) \theta^n_0(x, t) dx + \int_{\tau_0}^{t_1} \alpha(s) |\eta^n(x, 1, t, s) \theta^n_0(x, t)| ds dx = 0.
\]

By integrating by parts, we obtain
Integrating by parts, we obtain

\[ \psi \int_0^t \int_L \frac{d}{dr} \left[ \theta_{nt} (x, t) + \psi^n (x, t) \right] \theta_{nt} (x, t) dx \]

As \( \theta \) is null at 0 and at \( L \) we have

\[ \frac{\rho_1}{2} \frac{d}{dt} \int_0^L \left| \theta_{nt} (x, t) \right|^2 dx + \kappa \int_0^L \left[ \theta_{nt} (x, t) + \psi^n (x, t) \right] \theta_{nt} (x, t) dx \]

+ \mu_1 \int_0^L \left| \theta_{nt} (x, t) \right|^2 dx + \int_0^t \int_0^L \alpha(s) \eta^n (x, 1, t, s) \theta_{nt} (x, t) ds dx = 0. \tag{24} \]

Let us derive the equation \( (19) \) with respect to \( t \) and then replacing \( \psi \) by \( \psi^n \). We obtain

\[ -b \int_0^L p_{nt} (x, t) \psi^n (x, t) dx + \kappa \int_0^L \left[ \theta_{nt} (x, t) + \psi^n (x, t) \right] \psi^n (x, t) dx = 0. \]

Integrating by parts, we obtain

\[ -b \int_0^L \psi^n (x, t) \psi_{nt} (x, t) dx + \kappa \int_0^L \left[ \theta_{nt} (x, t) + \psi^n (x, t) \right] \psi^n (x, t) dx = 0. \]

Since \( \psi \) is zero at the edge we have

\[ b \int_0^L \psi^n (x, t) \psi_{nt} (x, t) dx + \kappa \int_0^L \left[ \theta_{nt} (x, t) + \psi^n (x, t) \right] \psi^n (x, t) dx = 0. \]

what is written

\[ b \frac{d}{dt} \int_0^L \left| \psi_{nt} (x, t) \right|^2 dx + \kappa \int_0^L \left[ \theta_{nt} (x, t) + \psi^n (x, t) \right] \psi^n (x, t) dx = 0. \tag{25} \]

Summing \( (24) \) and \( (25) \) we obtain

\[ \frac{1}{2} \frac{d}{dt} \left( \frac{\rho_1}{2} \int_0^L \left| \theta_{nt} (x, t) \right|^2 dx + \kappa \int_0^L \left[ \theta_{nt} (x, t) + \psi^n (x, t) \right] \theta_{nt} (x, t) dx \right) \]

+ \int_0^t \int_0^L \alpha(s) \eta^n (x, 1, t, s) \theta_{nt} (x, t) ds dx + \mu_1 \int_0^L \left| \theta_{nt} (x, t) \right|^2 dx = 0 \]

which means that

\[ \frac{d}{dt} G^n (t) + \int_0^t \int_0^L \alpha(s) \eta^n (x, 1, t, s) \theta_{nt} (x, t) ds dx + \mu_1 \int_0^L \left| \theta_{nt} (x, t) \right|^2 dx = 0 \tag{26} \]

with

\[ G^n (t) = \frac{\rho_1}{2} \int_0^L \left| \theta_{nt} (x, t) \right|^2 dx + \kappa \int_0^L \left[ \theta_{nt} (x, t) + \psi^n (x, t) \right] \theta_{nt} (x, t) dx + \frac{b}{2} \int_0^L \left| \psi_{nt} (x, t) \right|^2 dx. \]

By integrating \( (26) \) from 0 to \( t \) we have

\[ G^n (t) + \int_0^t \int_0^L \alpha(s) \eta^n (x, 1, \lambda, s) \theta_{nt} (x, \lambda) ds dx + \mu_1 \int_0^L \left| \theta_{nt} (x, \lambda) \right|^2 dx = G^n (0) \]

where

\[ G^n (0) = \frac{\rho_1}{2} \int_0^L \left| \theta_{nt} (x, 0) \right|^2 dx + \kappa \int_0^L \left[ \theta_{nt} (x, 0) + \psi^n (x, 0) \right] \theta_{nt} (x, 0) dx + \frac{b}{2} \int_0^L \left| \psi_{nt} (x, 0) \right|^2 dx. \]

As \( (\theta^n (0), \theta^n (0), \psi^n (0), \eta^n (0)) \) \( \rightarrow (\theta_0, \theta_1, \psi_0, \eta_0) \) strongly in \( \mathcal{H} \) then there exists a positive constant \( C_2 \) such that \( G^n (0) \leq C_2 \). Hence

\[ G^n (t) + \int_0^t \int_0^L \alpha(s) \eta^n (x, 1, \lambda, s) \theta_{nt} (x, \lambda) ds dx + \mu_1 \int_0^L \left| \theta_{nt} (x, \lambda) \right|^2 dx \leq C_2 \tag{27} \]
Step 4. Passage to Limit

From (23) and (27) we have
\( \{ \theta^n \} \) is bounded in \( L^\infty(0, T, H^1_0(0, L)) \),
\( \{ \theta_t^n \} \) is bounded in \( L^\infty(0, T, L^2(0, L)) \),
\( \{ \theta_{tt}^n \} \) is bounded in \( L^\infty(0, T, L^2(0, L)) \),
\( \{ \psi^n \} \) is bounded in \( L^\infty(0, T, H^1_0(0, L)) \),
\( \{ \psi_t^n \} \) is bounded in \( L^\infty(0, T, L^2(0, L)) \),
\( \{ \eta^n \} \) is bounded in \( L^\infty(0, T, L^2((0, L) \times (0, 1) \times (\tau_0, \tau_1))) \).

So we can extract subsequences \( \{ \theta^n \}, \{ \psi^n \} \) and \( \{ \eta^n \} \) such as

\( \theta^n \rightharpoonup \theta \) in \( L^\infty(0, T, H^1_0(0, L)) \),
\( \theta_t^n \rightharpoonup \theta_t \) in \( L^\infty(0, T, L^2(0, L)) \),
\( \theta_{tt}^n \rightharpoonup \theta_{tt} \) in \( L^\infty(0, T, L^2(0, L)) \),
\( \psi^n \rightharpoonup \psi \) in \( L^\infty(0, T, H^1_0(0, L)) \),
\( \psi_t^n \rightharpoonup \psi_t \) in \( L^\infty(0, T, L^2(0, L)) \),
\( \eta^n \rightharpoonup \eta \) in \( L^\infty(0, T, L^2((0, L) \times (0, 1) \times (\tau_0, \tau_1))) \).

Moreover, from (23) we have
\( \{ \theta^n \} \) is bounded in \( L^2(0, T, H^1_0(0, L)) \),
\( \{ \theta_t^n \} \) is bounded in \( L^2(0, T, L^2(0, L)) \).

Since the embedding of \( H^1_0(0, L) \) in \( L^2(0, L) \) is compact, we have by Aubin-Lions theorem (27) that
\( \{ \theta^n \} \rightharpoonup \theta \) strongly in \( L^\infty(0, T, L^2(0, L)) \),

We also show that
\( \{ \theta_t^n \} \rightharpoonup \theta_t \) strongly in \( L^\infty(0, T, L^2(0, L)) \),
\( \{ \psi^n \} \rightharpoonup \psi \) strongly in \( L^\infty(0, T, H^1_0(0, L)) \).

Then we can pass to limit the approximate problem (19) – (20) in order to get a weak solution of problem (9).
And we use density arguments to get problems (9) admit a global weak solution satisfaisant

\[
\begin{align*}
\theta & \in L^\infty(0, T, H^1_0(0, L)), \\
\theta_t & \in L^\infty(0, T, L^2(0, L)), \\
\psi & \in L^\infty(0, T, H^1_0(0, L)), \\
\eta & \in L^\infty(0, T, L^2((0, L) \times (0, 1) \times (\tau_0, \tau_1))).
\end{align*}
\] (28)

Step 5. A Priori Estimate III

Suppose that the problem (19) satisfies \( (\theta_0, \theta_1, \psi_0, \eta_0) \in \mathcal{H}_1 \) and
\( (\theta^n_0, \theta^n_1, \psi^n_0, \eta^n_0) \rightharpoonup (\theta_0, \theta_1, \psi_0, \eta_0) \) strongly in \( \mathcal{H}_1 \).

Replacing \( u \) by \( -\theta^n_{\text{ext}} \) in (19), \( v \) by \( -\psi^n_{\text{ext}} \) in (19), and \( w \) by \( \eta^n_{\text{ext}} \) in (19) we arrive at

\[
\frac{d}{dt} Q(t) = \left[ \int_{\tau_0}^{\tau_1} \alpha(s) ds - \mu_1 \right] \int_0^L |\theta^n_{\text{ext}}(x, 0, t, s)|^2 dx - \frac{1}{2} \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s)|\theta^n_{\text{ext}}(x, 1, t, s) + \eta^n_{\text{ext}}(x, 0, t, s)|^2 ds dx
\] (30)
where
\[ Q^n(t) = \frac{\rho_1}{2} \int_0^t \left[ \theta_{t,x}^2 \right] dx + \frac{b}{2} \int_0^t \left[ \theta_{t,x}^2 \right] dx + \frac{\kappa}{2} \int_0^t \left[ \theta_{t,x}^2 + \psi_t^2 \right] dx + \frac{1}{2} \int_0^t \int_0^{t_1} \sigma(s) \eta_t^2 dsdxdt \]

Integrating (30) from 0 to \( t < T \), we get,
\[ Q^n(t) - Q^n(0) = \left[ \int_{\tau_0}^t \alpha(s) ds - \mu_1 \right] \left[ \int_0^t \left[ \eta_t(x, 0, \lambda, s) \right]^2 ds \right] dx \]
\[ - \frac{1}{2} \int_{\tau_0}^t \int_0^L \int_{\tau_0}^{t_1} \sigma(s) \eta_t^2(x, 1, \lambda, s) + \eta_t^2(x, 0, \lambda, s) dsdxdt \]

which means
\[ Q^n(t) + \left[ \mu_1 - \int_{\tau_0}^t \alpha(s) ds \right] \left[ \int_0^t \left[ \eta_t(x, 0, \lambda, s) \right]^2 ds \right] dx \]
\[ + \frac{1}{2} \int_{\tau_0}^t \int_0^L \int_{\tau_0}^{t_1} \alpha(s) \eta_t^2(x, 1, \lambda, s) + \eta_t^2(x, 0, \lambda, s) dsdxdt = Q^n(0) \]

where
\[ Q^n(0) = \frac{\rho_1}{2} \int_0^L \left[ \theta_{t,x}^2 \right] dx + \frac{b}{2} \int_0^L \left[ \theta_{t,x}^2 \right] dx + \frac{\kappa}{2} \int_0^L \left[ \theta_{t,x}^2 + \psi_t^2 \right] dx + \frac{1}{2} \int_0^L \int_{\tau_0}^{t_1} \sigma(s) \eta_t^2 dsdxdt. \]

As \((\theta^n(0), \theta^n_t(0), \psi^n(0), \eta^n(0)) = (\theta_0, \theta_t, \psi_0, \eta_0) \) strongly in \( H_1 \), we can deduce that there exists a positive constant \( C_3 \) such that \( Q^n(0) \leq C_3 \). Hence
\[ Q^n(t) + \left[ \mu_1 - \int_{\tau_0}^t \alpha(s) ds \right] \left[ \int_0^t \left[ \eta_t(x, 0, \lambda, s) \right]^2 ds \right] dx \]
\[ + \frac{1}{2} \int_{\tau_0}^t \int_0^L \int_{\tau_0}^{t_1} \alpha(s) \eta_t^2(x, 1, \lambda, s) + \eta_t^2(x, 0, \lambda, s) dsdxdt \leq C_3 \]

which means that
\[ \frac{\rho_1}{2} \int_0^L \left[ \theta_{t,x}^2 \right] dx + \frac{b}{2} \int_0^L \left[ \theta_{t,x}^2 \right] dx + \frac{\kappa}{2} \int_0^L \left[ \theta_{t,x}^2 + \psi_t^2 \right] dx + \frac{1}{2} \int_0^L \int_{\tau_0}^{t_1} \sigma(s) \eta_t^2 dsdxdt \]
\[ + \frac{1}{2} \int_{\tau_0}^t \int_0^L \int_{\tau_0}^{t_1} \alpha(s) \eta_t^2(x, 1, \lambda, s) + \eta_t^2(x, 0, \lambda, s) dsdxdt \]
\[ + \left[ \mu_1 - \int_{\tau_0}^t \alpha(s) ds \right] \left[ \int_0^t \left[ \eta_t(x, 0, \lambda, s) \right]^2 ds \right] dx \leq C_3 \]

where \( C_3 \geq 0 \) also independent of \( t \) and \( n \) but depending on initial data. Then we can conclude that
\[ (\theta^n) \text{ is bounded in } L^\infty \left( 0, T, H^2(0, L) \cap H^1_0(0, L) \right), \]
\[ (\theta^n_t) \text{ is bounded in } L^\infty \left( 0, T, H^1_0(0, L) \right), \]
\[ (\psi^n) \text{ is bounded in } L^\infty \left( 0, T, H^2_0(0, L) \right), \]
\[ (\eta^n) \text{ is bounded in } L^\infty \left( 0, T, H^1((0, L) \times (0, 1) \times (\tau_0, \tau_1)) \right). \]

This implies that
\[ (\theta^n) \to \theta \text{ in } L^\infty \left( 0, T, H^2(0, L) \cap H^1_0(0, L) \right), \]
\[ (\theta^n_t) \to \theta_t \text{ in } L^\infty \left( 0, T, H^1_0(0, L) \right), \]
\[ (\psi^n) \to \psi \text{ in } L^\infty \left( 0, T, H^2_0(0, L) \right), \]

This completes the proof.
Step 6. Continuous dependence

Let $U_1(t) = (\theta, \theta_0, \psi, \eta)$ and $U_2(t) = (\theta', \theta'_0, \psi', \eta')$ two solutions of the problem [9] with initial datas $U_1(0) = (\theta_0, \theta_1, \psi_0, \eta_0)$, $U_2(0) = (\theta'_0, \theta'_1, \psi', \eta'_0) \in \mathcal{H}_1$. Then $(\Phi, \Phi_0, \Psi, \Gamma) = U_1(t) - U_2(t)$ is solution of the system

\[
\begin{aligned}
\rho_1 \Phi_t(x, t) - \kappa(\Phi_t(x, t) + \Psi(x, t))_x + \mu_1 \Phi_t(x, t) + \int_{t_0}^{t} \alpha(s) \Gamma(x, 1, t, s) ds &= 0 \quad \text{in} \quad [0, L] \times (0, +\infty), \\
-b \Phi_t(x, t) + \kappa(\Phi_t(x, t) + \Psi(x, t)) &= 0 \quad \text{in} \quad [0, L] \times (0, +\infty), \\
\sigma(t, x, t, s) + \Gamma_p(x, \rho, t, s) &= 0 \quad \text{for} \quad (x, \rho, t, s) \in (0, L) \times (0, 1) \times (0, +\infty) \times (\tau_0, \tau_1).
\end{aligned}
\]

with initial data $(\Phi(0), \Phi_t(0), \Psi(0), \Gamma(0)) = U_1(0) - U_2(0)$. Multiply [33] by $\Phi_t$ and [33] by $\Psi_t$ and integrate. we obtain

\[
\frac{d}{dt} \mathcal{P}(t) + \int_0^L \theta_t(x, t) \int_{t_0}^{t_1} \alpha(s) \Gamma(x, 1, t, s) ds dx + \mu_1 \int_0^L |\Phi_t(x, t)|^2 dx = 0
\]

avec

\[
\mathcal{P}(t) = \frac{\rho_1}{2} \int_0^L |\Phi_t(x, t)|^2 dx + k \int_0^L |\Phi_t(x, t) + \Psi(x, t)|^2 dx + b \int_0^L |\Psi_t(x, t)|^2 dx
\]

Applying Young’s inequality to $\int_0^L \theta_t(x, t) \int_{t_0}^{t_1} \alpha(s) \Gamma(x, 1, t, s) ds dx$ and the fact that $\mu_2$ is bounded, we obtain the existence of a constant $M_1$ such that

\[
\frac{d}{dt} \mathcal{P}(t) \leq M_1 \int_0^L |\Phi_t(x, t)|^2 dx 
\]

\[
\leq M_1 \left[ \int_0^L |\Phi_t(x, t)|^2 dx + \int_0^L |\Phi_t(x, t) + \Psi(x, t)|^2 dx + \int_0^L |\Psi_t(x, t)|^2 dx \right] 
\]

(34)

And by integrating (34) from 0 to $t$ we get

\[
\mathcal{P}(t) \leq \mathcal{P}(0) + M_1 \int_0^t \left[ \int_0^L |\Phi_t(x, \lambda)|^2 dx + \int_0^L |\Phi_t(x, \lambda) + \Psi(x, \lambda)|^2 dx + \int_0^L |\Psi_t(x, \lambda)|^2 dx \right] d\lambda 
\]

(35)

On the other hand we know that for $M_2 = \min \left\{ \frac{\rho_1}{2}, \frac{k}{2}, \frac{b}{2} \right\}$, we have

\[
\mathcal{P}(\lambda) \geq M_2 \left[ \int_0^L |\Phi_t(x, \lambda)|^2 dx + \int_0^L |\Phi_t(x, \lambda) + \Psi(x, \lambda)|^2 dx + \int_0^L |\Psi_t(x, \lambda)|^2 dx \right] 
\]

(36)

From (35) and (36) we have
Let the assumption

Theorem 3.1.

result.

In this section, our aim is to study the exponential stability of the system (1). More precisely we prove the following

3. Exponential Stability

In this section, our aim is to study the exponential stability of the system (1). More precisely we prove the following result.

Theorem 3.1. Let the assumption (2) be satisfied. Then, there exist positive constants M and K such that, for any solution of (9)

\[ E(t) \leq ME(0)e^{-Kt}, \forall t \geq 0. \]

Démostración. Let

\[ Q(t) = 2c\mu_1 \int_0^L \theta \psi \, dx + \mu_1 \int_0^L |\theta|^2 \, dx \]

where \( c \) is a constant whose conditions we will specify later.

Multiply (9) by \( 2c\theta \) and integrate from 0 to L. We obtain

\[
2c\mu_1 \int_0^L \theta \psi \, dx - 2cK \int_0^L \frac{d}{dt} \left( \theta \psi \right) \, dx + 2cK \int_0^L \frac{d}{dx} \left( \theta \psi \right) \, dx + \mu_1 \int_0^L \frac{d}{dx} \left( |\theta|^2 \right) \, dx + 2c \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s) \eta(x, 1, t, s) \theta(x, t) \, ds \, dx = 0.
\]

As \( \frac{d}{dt} \left( \theta \psi \right) - |\theta|^2 \) we have

\[
2c\mu_1 \int_0^L \frac{d}{dt} \left( \theta \psi \right) \, dx - 2cK \int_0^L \frac{d}{dx} \left( \theta \psi \right) \, dx + 2cK \int_0^L \frac{d}{dx} \left( \theta \psi \right) \, dx + \mu_1 \int_0^L \frac{d}{dx} \left( |\theta|^2 \right) \, dx + 2c \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s) \eta(x, 1, t, s) \theta(x, t) \, ds \, dx = 0.
\]

which means that

\[
\frac{d}{dt} \int_0^L \left[ 2c\mu_1 \theta \psi \, dx + c\mu_1 |\theta|^2 \right] \, dx - 2cK \int_0^L \frac{d}{dx} \left( |\theta|^2 \right) \, dx + 2cK \int_0^L \frac{d}{dx} \left( \theta \psi \right) \, dx + 2c \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s) \eta(x, 1, t, s) \theta(x, t) \, ds \, dx = 0.
\]

Multiply (9) by \( 2c\psi \) and integrate over 0 to L. we obtain

\[
-2cb \int_0^L \psi \psi_x \, dx + 2cK \int_0^L \theta \psi \, dx + 2cK \int_0^L \frac{d}{dx} \left( \theta \psi \right) \, dx = 0.
\]

By integrating by parts, we have
\[-2cb \left[ \psi, \psi \right]_0^L + 2cb \int_0^L |\psi|^2_{0} \psi + 2c \int_0^L \left[ \theta(x, t) + \psi(x, t) \right] \psi \, dx = 0.\]

Since \( \psi \) is zero at 0 and L, we have

\[ 2cb \int_0^L |\psi|^2_{0} \psi + 2c \int_0^L \left[ \theta(x, t) + \psi(x, t) \right] \psi \, dx = 0. \]  

(40)

By summing (39) – (40) we obtain

\[
\frac{d}{dt} Q(t) = 2c \rho_1 \int_0^L |\theta|^2_{0} \theta - 2c \int_0^L |\theta(x, t) + \psi(x, t)|^2 \, dx - 2c \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) \eta(x, 1, t, s) \theta(x, t) \, ds \, dx \\
- 2cb \int_0^L |\psi|^2_{0} \psi.
\]

Applying Young’s inequality at \(-2c \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) \eta(x, 1, t, s) \theta(x, t) \, ds \, dx\) we have

\[
\frac{d}{dt} Q(t) \leq 2c \rho_1 \int_0^L |\theta|^2_{0} \theta - 2c \int_0^L |\theta(x, t) + \psi(x, t)|^2 \, dx + \frac{(2c)^2}{4c} \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) \eta(x, 1, t, s) \theta(x, t) \, ds \, dx \\
+ \epsilon \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) \theta(x, t) |\theta(x, t)|^2 \, ds \, dx - 2cb \int_0^L |\psi|^2_{0} \psi, \forall \epsilon > 0.
\]

By applying the inequality of Poincare at \(\epsilon \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) |\theta(x, t)|^2 \, ds \, dx\) we obtain

\[
\frac{d}{dt} Q(t) \leq 2c \rho_1 \int_0^L |\theta|^2_{0} \theta - 2c \int_0^L |\theta(x, t) + \psi(x, t)|^2 \, dx + 2L^2 \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) |\theta(x, t)|^2 \, ds \, dx \\
+ \frac{c^2}{\epsilon} \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) \eta(x, 1, t, s) |\theta(x, t)|^2 \, ds \, dx - 2cb \int_0^L |\psi|^2_{0} \psi, \forall \epsilon > 0. \]  

(41)

We know that

\[ \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) |\theta(x, t)|^2 \, ds \, dx \leq 2 \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) |\theta(x, t)| \, ds \, dx + 2 \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) |\psi(x, t)|^2 \, ds \, dx. \]

By Poincare’s inequality we have

\[ \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) |\theta(x, t)|^2 \, ds \, dx \leq 2 \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) |\theta(x, t)| + \psi(x, t)|^2 \, ds \, dx + 2L^2 \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) |\psi(x, t)|^2 \, ds \, dx. \]  

(42)

Multiplying (42) by \(\epsilon L^2\) we have

\[
\epsilon L^2 \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) |\theta(x, t)|^2 \, ds \, dx \leq 2\epsilon L^2 \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) |\theta(x, t)| + \psi(x, t)|^2 \, ds \, dx \\
+ 2\epsilon L^2 \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) |\psi(x, t)|^2 \, ds \, dx \\
\leq \frac{2\epsilon L^2 \mu_1}{k} \int_0^L |\theta(x, t)| + \psi(x, t)|^2 \, dx + \frac{2\epsilon L^2 \mu_1}{b} \int_0^L |\psi(x, t)|^2 \, dx.
\]

Choosing \(c = \max \left\{ \frac{2\epsilon L^2 \mu_1}{k}, \frac{2\epsilon L^2 \mu_1}{b} \right\}\) we obtain

\[ \epsilon L^2 \int_0^L \int_{\tau_1}^{\tau_2} \alpha(s) |\theta(x, t)|^2 \, ds \, dx \leq c \int_0^L \theta(x, t) + \psi(x, t)|^2 \, dx + cb \int_0^L |\psi(x, t)|^2 \, dx. \]  

(43)
From (41) and (43) we can write

\[
\frac{d}{dt} Q(t) \leq 2c_1 \int_0^L \theta_1^2 dx - 2c_0 \int_0^L \theta_1(x, t) + \psi(x, t)]^2 dx + c_0 \int_0^L \theta_1(x, t) + \psi(x, t)]^2 dx \\
+ \frac{c_0}{c} \int_0^L |\theta_1(x, t)|^2 dx + \frac{c_0^2}{c} \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s) \eta(x, 1, t, s)^2 ds dx - 2cb \int_0^L |\psi_1^2 dx, \forall \epsilon > 0.
\]

which means that

\[
\frac{d}{dt} Q(t) \leq 2c_1 \int_0^L \theta_1^2 dx - 2c_0 \int_0^L \theta_1(x, t) + \psi(x, t)]^2 dx + c_0 \int_0^L \theta_1(x, t) + \psi(x, t)]^2 dx \\
- \frac{c_0}{c} \int_0^L |\theta_1(x, t)|^2 dx, \forall \epsilon > 0. \quad (44)
\]

Let’s put

\[
I(t) = e^{2\tau_1} \int_0^L \int_0^{\tau_1} \int_{\tau_0} \alpha(s) e^{-2\psi} |\eta|^2 dx ds dx.
\]

we have

\[
\frac{d}{dt} I(t) = 2e^{2\tau_1} \int_0^L \int_0^{\tau_1} \int_{\tau_0} \alpha(s) e^{-2\psi} \eta \eta dx ds dx.
\]

From (9), we have \( s \eta = -\eta \), so

\[
\frac{d}{dt} I(t) = -2e^{2\tau_1} \int_0^L \int_0^{\tau_1} \int_{\tau_0} \alpha(s)e^{-2\psi} \eta \eta dx ds dx \\
= -2e^{2\tau_1} \int_0^L \int_0^{\tau_1} \int_{\tau_0} \alpha(s)e^{-2\psi} |\eta|^2 dx ds dx - e^{2\tau_1} \int_0^L \int_0^{\tau_1} \int_{\tau_0} \alpha(s) \frac{d}{dp} (e^{-2\psi} |\eta|^2) dx ds dp \\
= -2e^{2\tau_1} \int_0^L \int_0^{\tau_1} \int_{\tau_0} \alpha(s)e^{-2\psi} |\eta|^2 dx ds dx - e^{2\tau_1} \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s)e^{-2\psi} |\eta(x, 1, t, s)|^2 dx ds dx \\
+ e^{2\tau_1} \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s)e^{-2\psi} \eta(x, 0, t, s)^2 ds dx \\
\leq -2 \int_0^L \int_0^{\tau_1} \int_{\tau_0} \alpha(s)|\eta|^2 ds dx - \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s)|\eta(x, 1, t, s)|^2 dx ds dx \\
+ e^{2\tau_1} \left( \int_{\tau_0}^{\tau_1} \alpha(s) ds \right) \int_0^L \eta(x, 0, t, s)^2 dx \\
\leq -\int_0^L \int_0^{\tau_1} \int_{\tau_0} \alpha(s)|\eta|^2 ds dx - \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s)|\eta(x, 1, t, s)|^2 dx ds dx \\
+ e^{2\tau_1} \mu_1 \int_0^L \eta(x, 0, t, s)^2 dx
\]

however

\[
\frac{d}{dt} I(t) \leq -\int_0^L \int_0^{\tau_1} \int_{\tau_0} \alpha(s)|\eta|^2 ds dx - \int_0^L \int_{\tau_0}^{\tau_1} \alpha(s)|\eta(x, 1, t, s)|^2 dx ds dx \\
+ e^{2\tau_1} \mu_1 \int_0^L \eta(x, 0, t, s)^2 dx \quad (45)
\]
Summing (44) and (45) we have

\[
\frac{d}{dt}(I(t) + Q(t)) \leq 2cp_1 \int_0^L |\theta_s(x, t) + \psi(x, t)|^2 dx - cb \int_0^L |\theta_s(x, t) + \psi(x, t)|^2 dx + \frac{c^2}{E} \int_0^L \int_{t_0}^{T_1} \alpha(s)|\eta(x, 1, t, s)|^2 ds dx 
- \frac{c^2}{E} \int_0^L \int_{t_0}^{T_1} \alpha(s)|\eta(x, 1, t, s)|^2 ds dx + \frac{c^2}{E} \int_0^L \int_{t_0}^{T_1} \alpha(s)|\eta(x, 1, t, s)|^2 ds dx + 3cp_1 \int_0^L |\theta_s(x, t)|^2 dx 
+ e^{2\tau_1} \mu_1 \int_0^L |\eta(x, 0, t, s)|^2 dx, \forall \epsilon > 0 
\]

Choosing \( c_1 = \min \{1; c\} \), we obtain

\[
\frac{d}{dt}(I(t) + Q(t)) \leq -c_1p_1 \int_0^L |\theta_s(x, t) - c_1k \int_0^L \theta_s(x, t) + \psi(x, t)|^2 dx - c_1b \int_0^L |\psi_s(x, t)|^2 dx 
- c_1 \int_0^L \int_0^1 \int_{t_0}^{T_1} \alpha(s)|\eta(x, 1, t, s)|^2 ds dx + \frac{c^2}{E} \int_0^L \int_{t_0}^{T_1} \alpha(s)|\eta(x, 1, t, s)|^2 ds dx 
+ 3cp_1 \int_0^L |\theta_s(x, t)|^2 dx - \int_0^L \int_{t_0}^{T_1} \alpha(s)|\eta(x, 1, t, s)|^2 ds dx + e^{2\tau_1} \mu_1 \int_0^L |\eta(x, 0, t, s)|^2 dx 
\]

Thereby

\[
\frac{d}{dt}(I(t) + Q(t)) \leq -2c_1E(t) + \left(\frac{c^2}{E} - 1\right) \int_0^L \int_{t_0}^{T_1} \alpha(s)|\eta(x, 1, t, s)|^2 ds dx 
+ \left(e^{2\tau_1} \mu_1 + 3cp_1\right) \int_0^L |\eta(x, 0, t, s)|^2 dx. 
\] (46)

Let us introduce the Lyapunov functional

\[ M(t) = NE(t) + I(t) + Q(t), \forall t > 0 \]

where \( N \) is a suitable positive constant that will precised later on. We obtain from (15) and (46)

\[
\frac{d}{dt} M(t) \leq N \int_{t_0}^{T_1} \alpha(s) ds - \mu_1 \left(\int_0^L |\eta(x, 0, t, s)|^2 dx - N \frac{1}{2} \int_{t_0}^{T_1} \alpha(s)|\eta(x, 1, t, s) + \eta(x, 0, t, s)|^2 ds dx 
- 2c_1E(t) + \left(\frac{c^2}{E} - 1\right) \int_0^L \int_{t_0}^{T_1} \alpha(s)|\eta(x, 1, t, s)|^2 ds dx + \left(e^{2\tau_1} \mu_1 + 3cp_1\right) \int_0^L |\eta(x, 0, t, s)|^2 dx \right) \] (47)
Take $c_2 = \max \left\{ \frac{2L^2 \mu_1}{k}, \frac{2L^4 \mu_1}{b} \right\}$, we have $c = ec_2$. In (47) we get

$$\frac{d}{dt} M(t) \leq N \left[ \int_{\tau_0}^{\tau_1} \alpha(s)ds - \mu_1 \right] \int_{0}^{L} |\eta(x, 0, t, s)|^2 dx - N \frac{1}{2} \int_{0}^{L} |\eta(x, 1, t, s) + \eta(x, 0, t, s)|^2 ds dx$$

$$- 2c_1 E(t) + \left( ec_2^2 - 1 \right) \int_{0}^{L} \int_{\tau_0}^{\tau_1} \alpha(s) |\eta(x, 1, t, s)|^2 ds dx + \left( e^{2\tau_1} \mu_1 + 3c_1 \right) \int_{0}^{L} |\eta(x, 0, t, s)|^2 dx$$

$$\leq \left[ N \left( \int_{\tau_0}^{\tau_1} \alpha(s)ds - \mu_1 \right) + e^{2\tau_1} \mu_1 + 3c_1 \right] \int_{0}^{L} |\eta(x, 0, t, s)|^2 dx$$

$$- N \frac{1}{2} \int_{0}^{L} |\eta(x, 1, t, s) + \eta(x, 0, t, s)|^2 ds dx - 2c_1 E(t) + \left( ec_2^2 - 1 \right) \int_{0}^{L} \int_{\tau_0}^{\tau_1} \alpha(s) |\eta(x, 1, t, s)|^2 ds dx$$

For

$$N > \frac{-e^{2\tau_1} \mu_1 - 3c_1}{\int_{\tau_0}^{\tau_1} \alpha(s)ds - \mu_1} \text{ et } e < \frac{1}{c_2^2}$$

we have

$$\frac{d}{dt} M(t) \leq -2c_1 E(t). \quad (48)$$

Before continuing our proof let us prove the following lemma

**Lemma 3.1.** There are two positive constants $\gamma$ and $\beta$ such that

$$\gamma E(t) \leq M(t) \leq \beta E(t), \forall t \geq 0. \quad (49)$$

**Démonstration.** Let’s put

$$\varepsilon(t) = M(t) - NE(t)$$

We have

$$|\varepsilon(t)| = |M(t) - NE(t)|$$

$$|\varepsilon(t)| = 2c_1 \int_{0}^{L} \theta \theta dx + \mu_1 \int_{0}^{L} |\theta|^2 dx + e^{2\tau_1} \int_{0}^{L} \int_{\tau_0}^{\tau_1} \alpha(s) e^{-2\theta} |\eta|^2 ds dx$$

Applying Youg’s and Poincare’s inequalities we obtain

$$|\varepsilon(t)| \leq c_1 \int_{0}^{L} |\theta|^2 dx + (c_1 + \mu_1) L^2 \int_{0}^{L} |\theta|^2 dx + e^{2\tau_1} \int_{0}^{L} \int_{\tau_0}^{\tau_1} \alpha(s) |\eta|^2 ds dx$$

$$\leq c_1 \int_{0}^{L} |\theta|^2 dx + (c_1 + \mu_1) L^2 \int_{0}^{L} |\theta| + \psi |^2 dx + e^{2\tau_1} \int_{0}^{L} \int_{\tau_0}^{\tau_1} \alpha(s) |\eta|^2 ds dx$$

$$\leq c_1 \int_{0}^{L} |\theta|^2 dx + (c_1 + \mu_1) L^2 \int_{0}^{L} |\theta| + |\psi|^2 dx + (c_1 + \mu_1) L^2 \int_{0}^{L} |\psi|^2 dx$$

$$+ e^{2\tau_1} \int_{0}^{L} \int_{\tau_0}^{\tau_1} \alpha(s) |\eta|^2 ds dx$$
Applying Poincaré inequality to \((c\rho_1 + \mu_1 c)L^2 \int_0^L |\psi|^2 \, dx\) we can say

\[
|\varepsilon(t)| \leq c_\rho \int_0^L |\theta_\varepsilon|^2 \, dx + (c\rho_1 + \mu_1 c)L^2 \int_0^L |\theta_x + \psi|^2 \, dx + (c\rho_1 + \mu_1 c)L^4 \int_0^L |\psi_x|^2 \, dx
\]

we can say

\[
|\varepsilon(t)| \leq c_\rho \int_0^L |\theta_\varepsilon|^2 \, dx + (c\rho_1 + \mu_1 c)L^2 \int_0^L |\theta_x + \psi|^2 \, dx + (c\rho_1 + \mu_1 c)L^4 \int_0^L |\psi_x|^2 \, dx
\]

Finally

\[
|\varepsilon(t)| \leq \delta E(t)
\]

with \(\delta = \max\left\{2c, \frac{2(c\rho_1 + \mu_1 c)L^2}{k}, \frac{2(c\rho_1 + \mu_1 c)L^4}{k}, 2e^{2\tau_1}\right\}\)

So when we put \(\gamma = N - \delta\) and \(\beta = N + \delta\) we have (49).

We return to the proof of theorem 3.1

By (48) and (49) we obtain

\[
\frac{d}{dt} \mathcal{M}(t) \leq -2c_1 E(t) \leq -2c_1 E(t) \frac{\beta E(t)}{\beta E(t)}.
\]

which means that

\[
\frac{d}{dt} \mathcal{M}(t) \leq -2c_1 \beta.
\]

Integrating (50) from 0 to \(t\) we have

\[
\ln \mathcal{M}(t) - \ln \mathcal{M}(0) \leq \frac{-2c_1}{\beta} t.
\]

which means that

\[
\mathcal{M}(t) \leq \mathcal{M}(0)e^{\frac{-2c_1}{\beta} t}.
\]

By (49) we obtain

\[
\gamma E(t) \leq \mathcal{M}(t),
\]

and

\[
\mathcal{M}(0) \leq \beta E(0).
\]

However (51) can be written

\[
E(t) \leq \frac{\beta E(0)}{\gamma} e^{\frac{-2c_1}{\gamma} t}.
\]

We obtain by taking \(M = \frac{\beta}{\gamma}\) and \(K = \frac{2c_1}{\beta}\)

\[
E(t) \leq ME(0)e^{-Kt}, \forall t \geq 0.
\]

This completes the proof.
Références


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