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Some Properties of Extending Hypermodules and C_{11}-Hypermodules

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Abstract

In this paper, we present extending hypermodules and C_{11}-hypermodules, and then investigate some properties of these hypermodules. Especially, characterizations of extending hypermodules and characterizations of C_{11}-hypermodules are provided. Moreover, decompositions of C_{11}-hypermodules involving projection invariant subhypermodules are studied.

Keywords: hypermodules, extending hypermodules, C_{11}-hypermodules

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1. Introduction

Extending modules (also known as CS-modules) are an interesting topic in module theory which has been studied for several years. Recall that an R-module M is an *extending module* if every submodule of M is essential in a direct summand of M. Actually, there are many generalizations of extending modules, and one of generalizations of extending modules is a C_{11}-module which has been investigated by many authors (Smith & Tercan, 1993 ; Birkenmeier & Tercan, 2007 ; Tercan & Yücel, 2016). According to Birkenmeier and Tercan in 2007, an R-module M is a C_{11}*-module* if every submodule of M has a complement in M which is a direct summand of M. It can be seen that there have been many works concerning extending modules and C_{11}-modules until now (Tercan & Yücel, 2016). However, there are few works concerning these concepts in hypermodules. In this article, we extend the notions of extending modules and C_{11}-modules to hypermodules which is the main purpose of this paper. It is well-known that there are different notions of hyperrings and hypermodules (Davvaz & Leoreanu-Fotea, 2007; Siraworakun, 2012; Norouzi & Hamzekolaee, 2018). In this paper, we focus on hyperrings and hypermodules studied by Siraworakun in 2012.

For a canonical hypergroup $(H, \uplus)$, let 0 and $-a$ denote the scalar identity of H and the inverse of $a \in H$, respectively; moreover, for $\varnothing \neq A \subseteq H$ and $x_1, x_2, \ldots, x_k \in H$ with $k \in \mathbb{N}$, let $-A = \{-a : a \in A\}$, $\sum_{i=1}^{k} x_i = x_1 \uplus x_2 \uplus \cdots \uplus x_k$ and for the case $k = 1$, for each $z \in H$, "$z \in \sum_{i=1}^{k} x_i$" represents "$z = x_1$".

Definition 1. (Siraworakun, 2012) A *hyperring* is a structure $(R, \uplus, \bullet)$ where $\uplus$ and $\bullet$ are hyperoperations on R satisfying the following properties:

1. $(R, \uplus)$ is a canonical hypergroup;
2. $(R, \bullet)$ is a semihypergroup;
3. $a \bullet (b \uplus c) \subseteq (a \bullet b) \uplus (a \bullet c)$ and $(b \uplus c) \bullet a \subseteq (b \bullet a) \uplus (c \bullet a)$ for all $a, b, c \in R$; and
4. $a \bullet (-b) = (-a) \bullet b = -(a \bullet b)$ for all $a, b \in R$.

Moreover, if the equalities hold in (3), then the hyperring $(R, \uplus, \bullet)$ is said to be *strongly distributive*.

For convenience, we sometimes abbreviate a hyperring $(R, \uplus, \bullet)$ by a hyperring R and $a \bullet b$ by ab for all $a, b \in R$.

Example 2. (Siraworakun, 2012) Let $(R, +)$ be an abelian group. Define hyperoperations $\uplus$ and $\bullet$ by $a \uplus b = \{a + b\}$ and $a \bullet b = \langle a, b\rangle$, the subgroup of R generated by the set $\{a, b\}$, for all $a, b \in R$, respectively. Then $(R, \uplus, \bullet)$ is a hyperring.

Example 3. (Siraworakun, 2012) Let $a \in \mathbb{R}$ with $a \geq 1$ and $R = [a, \infty) \cup \{0\}$. Define a hyperoperation $\uplus$ on R by

$$x \uplus y = \begin{cases} \{y\}, & \text{if } x = 0, \\ \{x\}, & \text{if } y = 0, \\ [x, \infty) \cup \{0\}, & \text{if } x = y \neq 0, \\ \{\min\{x, y\}\}, & \text{if } x \neq y, x \neq 0 \text{ and } y \neq 0. \end{cases}$$

Moreover, define a hyperoperation $\bullet$ on R by $a \bullet b = \{a \cdot b\}$, where $\cdot$ is the usual multiplication on $\mathbb{R}$, for all $a, b \in R$. Then $(R, \uplus, \bullet)$ is a strongly distributive hyperring.

Example 4. (Siraworakun, 2012) Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Define a hyperoperation $\uplus$ on $\mathbb{N}_0$ by

$$m \uplus n = \begin{cases} \{m\}, & \text{if } n = 0, \\ \{n\}, & \text{if } m = 0, \\ \mathbb{N}_0, & \text{if } m = n \neq 0, \\ \{m, n\}, & \text{if } m \neq n, m \neq 0 \text{ and } n \neq 0. \end{cases}$$

Moreover, define a hyperoperation $\bullet$ on $\mathbb{N}_0$ by $m \bullet n = \{m \cdot n\}$, where $\cdot$ is the usual multiplication on $\mathbb{N}_0$, for all $m, n \in \mathbb{N}_0$. Then $(\mathbb{N}_0, \uplus, \bullet)$ is a hyperring which is not strongly distributive.

Definition 5. (Siraworakun, 2012) Let $(R, \uplus, \bullet)$ be a hyperring. An *R-hypermodule* is a structure $(M, +, \odot)$ such that $(M, +)$ is a canonical hypergroup and $\odot$ is a multivalued scalar operation, i.e., $\odot : R \times M \to \mathcal{P}^*(M)$, where $\mathcal{P}^*(M)$ is the power set of M excluding the empty set, such that for all $a, b \in R$ and $x, y \in M$,

1. $a \odot (x + y) \subseteq (a \odot x) + (a \odot y)$ and $(a \uplus b) \odot x \subseteq (a \odot x) + (b \odot x)$;
2. $(a \bullet b) \odot x = a \odot (b \odot x)$; and
3. $a \odot (-x) = (-a) \odot x = -(a \odot x)$.

If the equalities hold in (1), then the R-hypermodule M is said to be *strongly distributive.*

For convenience, we sometimes abbreviate an R-hypermodule $(M, +, \odot)$ by an R-hypermodule M and $a \odot x$ by ax for all $a \in R$ and $x \in M$. In addition, a hyperring $(R, \uplus, \bullet)$ can be viewed as an R-hypermodule by considering the hyperoperation $\bullet$ as the multivalued scalar operation. Let M be an R-hypermodule. For $\emptyset \neq A \subseteq R$, $\emptyset \neq X \subseteq M$, $r \in R$ and $y \in M$, let

$$AX = \bigcup_{a \in A, x \in X} ax, \; Ay = A\{y\} \quad \text{and} \quad rX = \{r\}X.$$

Definition 6. (Siraworakun, 2012) A nonempty subset N of an R-hypermodule M is called a *subhypermodule* of M, denoted by $N \leq M$, if N is an R-hypermodule under the same hyperoperation on M and the same multivalued scalar operation.

Proposition 7. (Siraworakun, 2012) *Let N be a nonempty subset of an R-hypermodule M. Then N is a subhypermodule of M if and only if $x - y \subseteq N$ and $rx \subseteq N$ for all $x, y \in N$ and $r \in R$.*

Example 8. (Siraworakun, 2012) Let R be the hyperring defined in Example 2 with $|R| > 1$. If $(R, \uplus, \bullet)$ is viewed as an R-hypermodule, then only R is a subhypermodule of itself.

In general, $\{0\}$ may not be a subhypermodule such as the previous example. However, we are interested in hypermodules which $\{0\}$ must be a subhypermodule. Such an R-hypermodule exists as the following examples.

Example 9. Let R be the hyperring defined in Example 3. Let $t \in \mathbb{R}$ with $0 < t \leq 1$ and $M = [0, t)$. Define a hyperoperation $+$ on M by, for any $x, y \in M$,

$$x + y = \begin{cases} \{\max\{x, y\}\}, & \text{if } x \neq y, \\ [0, x], & \text{if } x = y. \end{cases}$$

In addition, define a multivalued scalar operation $\odot$ by, for any $a \in R$ and $x \in M$,

$$a \odot x = \begin{cases} \{0\}, & \text{if } a = 0, \\ [0, \frac{x}{a}], & \text{if } a \neq 0. \end{cases}$$

It can be checked directly that $(M, +, \odot)$ is a strongly distributive R-hypermodule. Moreover, $\{0\}$ is a subhypermodule of M.

Example 10. Let $(\mathbb{N}_0, \uplus, \bullet)$ be the hyperring defined in Example 4. Consider $\mathbb{N}_0$ as an $\mathbb{N}_0$-hypermodule where its multi-valued scalar operation is the hyperoperation $\bullet$. Then $\{0\}$ and $\mathbb{N}_0$ are only subhypermodules of $\mathbb{N}_0$.

From now on, only R-hypermodules such that $\{0\}$ is a subhypermodule are considered. In general, for an R-hypermodule M and $m \in M$, $Rm = \{a \in rm : r \in R\}$ may not be a subhypermodule of M such as the previous example, $\mathbb{N}_0 2$ is not a subhypermodule of $\mathbb{N}_0$; however, Rm is always a subhypermodule of M provided that M is a strongly distributive R-hypermodule.

Proposition 11. *Let M be a strongly distributive R-hypermodule and $m \in M$. Then Rm is a subhypermodule of M.*

Proof. Note that $\varnothing \neq 0m \subseteq Rm$, so $Rm \neq \varnothing$. Let $x, y \in Rm$ and $r \in R$. Then there exist $a_1, a_2 \in R$ such that $x \in a_1 m$ and $y \in a_2 m$. Since M is strongly distributive, $a_1 m - a_2 m = (a_1 - a_2)m$. Hence,

$$x - y \subseteq a_1 m - a_2 m = (a_1 - a_2)m \subseteq Rm.$$

Moreover, $rx \subseteq r(a_1 m) = (ra_1)m \subseteq Rm$. By Proposition 7, we obtain $Rm \leq M$. □

Proposition 12. (Siraworakun, 2012) *Let K and N be subhypermodules of an R-hypermodule M. Then $K \cap N$ and $K + N$ are subhypermodules of M.*

Note that for subhypermodules K and N of an R-hypermodule M,

$$K + N = \bigcup_{x \in K, y \in N} x + y = \{z : \exists x \in K, \exists y \in N, z \in x + y\}.$$

Therefore, for any subhypermodules $N_1, N_2, \ldots, N_k$ of an R-hypermodule M with $k \in \mathbb{N}$, we can define

$$\sum_{i=1}^{k} N_i = N_1 + N_2 + \cdots + N_k = \{x : \exists n_1 \in N_1, \exists n_2 \in N_2 \ldots, \exists n_k \in N_k, x \in \sum_{i=1}^{k} n_i\}.$$

Corollary 13. *Let $N_1, N_2, \ldots, N_k$ be subhypermodules of an R-hypermodule M where $k \in \mathbb{N}$. Then $\bigcap_{i=1}^{k} N_i$ and $\sum_{i=1}^{k} N_i$ are subhypermodules of M.*

Next, we provide a proposition which can transform between the sum of subhypermodules and the intersection of subhypermodules.

Proposition 14. (Modular law) *Let H, K and L be subhypermodules of an R-hypermodule M such that $K \subseteq H$. Then $H \cap (K + L) = K + (H \cap L)$.*

Proof. Let $x \in H \cap (K + L)$. Then $x \in H$ and there exist $k \in K$ and $l \in L$ such that $x \in k + l$. Thus, $l \in x - k \subseteq H$. This means that $l \in H \cap L$, so $x \in K + (H \cap L)$. Hence, $H \cap (K + L) \subseteq K + (H \cap L)$. Since $K \subseteq H$, we get $K + (H \cap L) \subseteq H + (H \cap L) \subseteq H$. Clearly, $K + (H \cap L) \subseteq K + L$. Thus, $K + (H \cap L) \subseteq H \cap (K + L)$. Therefore, $H \cap (K + L) = K + (H \cap L)$. □

The concepts of direct sums of subhypermodules and direct summands of hypermodules are given as follows.

Definition 15. Let $N_1, N_2, \ldots, N_k$ be subhypermodules of an R-hypermodule M where $k \in \mathbb{N}$ and $k \geq 2$. We say that M is a *direct sum* of $N_1, N_2, \ldots, N_k$, denoted by $M = \bigoplus_{i=1}^{k} N_i$ or $M = N_1 \oplus N_2 \oplus \cdots \oplus N_k$, if $M = \sum_{i=1}^{k} N_i$ and for each $j \in \{1, 2, \ldots, k\}$, $N_j \cap \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i = \{0\}$.

In module theory, if a module M is a direct sum of submodules $N_1, N_2, \ldots, N_k$, then every element in M can be written uniquely as the sum of elements in $N_1, N_2, \ldots, N_k$. In hypermodules, the uniqueness involving elements in direct sums is presented as the following proposition.

Proposition 16. *Let $N_1, N_2, \ldots, N_k$ be subhypermodules of an R-hypermodule M such that $M = \sum_{i=1}^{k} N_i$ where $k \in \mathbb{N}$ and $k \geq 2$. The following statements are equivalent:*

1. *for any* $j \in \{1, 2, \dots, k\}$, $N_j \cap \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i = \{0\}$;

2. *for any* $x \in M$, *there exist uniquely* $n_1 \in N_1, n_2 \in N_2, \dots, n_k \in N_k$ *such that* $x \in \sum_{i=1}^{k} n_i$;

3. *for any* $n_1 \in N_1, n_2 \in N_2, \dots, n_k \in N_k$, *if* $0 \in \sum_{i=1}^{k} n_i$, *then* $n_i = 0$ *for all* $i \in \{1, 2, \dots, k\}$.

Proof. (1)⇒(2) Assume that (1) holds. Let $x \in M$. Then there exist $n_1 \in N_1, n_2 \in N_2, \dots, n_k \in N_k$ such that $x \in \sum_{i=1}^{k} n_i$. Assume that there exist $m_1 \in N_1, m_2 \in N_2, \dots, m_k \in N_k$ such that $x \in \sum_{i=1}^{k} m_i$. Let $j \in \{1, 2, \dots, k\}$. Since $x \in \sum_{i=1}^{k} n_i$ and $x \in \sum_{i=1}^{k} m_i$, we can write $x \in n_j + y$ and $x \in m_j + z$ for some $y \in \sum_{\substack{i=1 \\ i \neq j}}^{k} n_i$ and $z \in \sum_{\substack{i=1 \\ i \neq j}}^{k} m_i$, respectively. Hence,

$$n_j \in x - y \subseteq (m_j + z) - y = m_j + (z - y) \subseteq m_j + \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i.$$

This means that $n_j \in m_j + a$ for some $a \in \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i$. Then $a \in n_j - m_j \subseteq N_j$. By assumption, $a \in N_j \cap \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i = \{0\}$, so $a = 0$. Thus, $n_j \in m_j + 0 = \{m_j\}$, i.e., $n_j = m_j$.

(2)⇒(3) Assume that (2) holds. Let $n_1 \in N_1, n_2 \in N_2, \dots, n_k \in N_k$ be such that $0 \in \sum_{i=1}^{k} n_i$. Since $0 \in 0 + \cdots + 0$ and from the assumption, $n_i = 0$ for all $i \in \{1, 2, \dots, k\}$.

(3)⇒(1) Assume that (3) holds. Let $j \in \{1, 2, \dots, k\}$ and $x \in N_j \cap \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i$. Then $x \in N_j$ and there exist $n_1 \in N_1, \dots, n_{j-1} \in N_{j-1}, n_{j+1} \in N_{j+1}, \dots, n_k \in N_k$ such that $x \in \sum_{\substack{i=1 \\ i \neq j}}^{k} n_i$. Since $N_j \leq M$, we obtain $-x \in N_j$. Thus,

$$0 \in x - x \subseteq n_1 + \cdots + n_{j-1} + (-x) + n_{j+1} + \cdots + n_k.$$

By assumption, $-x = 0$, so $x = 0$. Hence, $N_j \cap (\sum_{\substack{i=1 \\ i \neq j}}^{k} N_i) = \{0\}$. □

Proposition 17. *Let* K_1, K_2, N_1 *and* N_2 *be subhypermodules of an* R*-hypermodule* M *such that* $K_1 \leq N_1$ *and* $K_2 \leq N_2$. *If* $N_1 \cap N_2 = \{0\}$, *then* $K_1 \oplus K_2 = (K_1 \oplus N_2) \cap (N_1 \oplus K_2)$.

Proof. Assume that $N_1 \cap N_2 = \{0\}$. Then $K_1 \cap K_2 = \{0\}, K_1 \cap N_2 = \{0\}$ and $N_1 \cap K_2 = \{0\}$. Obviously, $K_1 \oplus K_2 \subseteq (K_1 \oplus N_2) \cap (N_1 \oplus K_2)$. It remains to show that $(K_1 \oplus N_2) \cap (N_1 \oplus K_2) \subseteq K_1 \oplus K_2$. Let $z \in (K_1 \oplus N_2) \cap (N_1 \oplus K_2)$. Then there exist $k_1 \in K_1, k_2 \in K_2, n_1 \in N_1$ and $n_2 \in N_2$ such that $z \in k_1 + n_2$ and $z \in n_1 + k_2$. Then $n_2 \in z - k_1 \subseteq (n_1 + k_2) - k_1 = (n_1 - k_1) + k_2$. Thus, $n_2 \in n_1' + k_2$ for some $n_1' \in n_1 - k_1 \subseteq N_1$. Then $n_1' \in n_2 - k_2 \subseteq N_2$. This means that $n_1' \in N_1 \cap N_2 = \{0\}$, i.e., $n_1' = 0$. Hence, $n_2 \in 0 + k_2 = \{k_2\} \subseteq K_2$. Then $z \in K_1 \oplus K_2$. Therefore, $K_1 \oplus K_2 = (K_1 \oplus N_2) \cap (N_1 \oplus K_2)$. □

Definition 18. Let N be a subhypermodule of an R-hypermodule M. We say that N is a *direct summand* of M, denoted by $N \leq_{\oplus} M$, if there exists $N' \leq M$ such that $M = N \oplus N'$.

Next, we introduce hypermodule homomorphisms investigated by Siraworakun in 2012, and then provide some of their properties.

Definition 19. (Siraworakun, 2012) Let M and M' be R-hypermodules. Then $f : M \to M'$ is called a *(hypermodule) homomorphism* if

$$f(x + y) = f(x) + f(y) \text{ and } f(rx) = rf(x)$$

for all $x, y \in M$ and $r \in R$.

Proposition 20. (Siraworakun, 2012) *Let* $f : M \to M'$ *be a hypermodule homomorphism. If* $f(0) = 0$, *then* $f(-x) = -f(x)$ *for all* $x \in M$.

Proposition 21. (Siraworakun, 2012) *Let* $f : M \to M'$ *be a hypermodule homomorphism such that* $f(0) = 0$. *The following statements hold.*

1. *If* $N \leq M$, *then* $f(N) \leq M'$.

2. *If* $N' \leq M'$, *then* $f^{-1}(N') \leq M$.

As the previous proposition, $\ker(f) = f^{-1}(\{0\}) \leq M$ and $f(M) \leq M'$ for any hypermodule homomorphism $f : M \to M'$ with $f(0) = 0$. In hypermodules, there is no conclusion to insist that homomorphisms send 0 to 0. However, we give a necessary and sufficient condition that makes a homomorphism sending 0 to 0 as follows.

Proposition 22. *Let $f : M \to M'$ be a hypermodule homomorphism. Then $f(0) = 0$ if and only if $0 \in f(M)$.*

Proof. $(\Rightarrow)$ If $f(0) = 0$, then $0 \in f(M)$ since $0 \in M$.

$(\Leftarrow)$ Assume that $0 \in f(M)$. Then $f(x) = 0$ for some $x \in M$. Note that $\{x\} = 0 + x$, so $f(x) \in f(0 + x) = f(0) + f(x)$. Hence, $f(0) \in f(x) - f(x) = 0 - 0 = \{0\}$, i.e., $f(0) = 0$. □

The following proposition is similar to the fact in module theory. However, for hypermodules, we require the condition that a homomorphism maps 0 to itself.

In this research, for an R-hypermodule M, let

$$\mathrm{End}_0(M) := \{f : M \to M : f \text{ is a homomorphism and } f(0) = 0\}.$$

Proposition 23. *Let M be an R-hypermodule and $f \in \mathrm{End}_0(M)$. If $f^2 = f$, then $M = f(M) \oplus \ker(f)$.*

Proof. Assume that $f^2 = f$. Claim that $M = f(M) + \ker(f)$. It suffices to show that $M \subseteq f(M) + \ker(f)$. Let $m \in M$. Then

$$\begin{aligned} f(m - f(m)) &= f(m) + f(-f(m)) \\ &= f(m) - f^2(m) \qquad \text{(by Proposition 20)} \\ &= f(m) - f(m). \end{aligned}$$

Thus, $0 \in f(m - f(m))$. Then there exists $k \in m - f(m)$ such that $f(k) = 0$, i.e., $k \in \ker(f)$. Since $k \in m - f(m)$, we get $m \in f(m) + k$. Hence, $m \in f(M) + \ker(f)$. Next, if $x \in f(M) \cap \ker(f)$, then $f(x) = 0$ and $f(y) = x$ for some $y \in M$, so $x = f(y) = f^2(y) = f(x) = 0$. This yields that $f(M) \cap \ker(f) = \{0\}$. Therefore, $M = f(M) \oplus \ker(f)$. □

Proposition 24. *Let $N_1, N_2, \ldots, N_k$ be subhypermodules of an R-hypermodule M such that $M = \bigoplus_{i=1}^{k} N_i$ where $k \in \mathbb{N}$ and $k \geq 2$. Let $j \in \{1, 2, \ldots, k\}$. Define $\pi_j : M \to N_j$ by*

$$\pi_j(x) = n_j \quad \textit{for all } x \in \textstyle\sum_{i=1}^{k} n_i.$$

Then π_j is a surjective homomorphism and $\pi_j^2 = \pi_j$.

Proof. By Proposition 16, π_j is well-defined. First, we show that π_j is a homomorphism. Let $x, y \in M$ and $r \in R$. Then there exist $n_1, n_1' \in N_1, \ldots, n_k, n_k' \in N_k$ such that $x \in \sum_{i=1}^{k} n_i$ and $y \in \sum_{i=1}^{k} n_i'$. Thus, $\pi_j(x) = n_j$ and $\pi_j(y) = n_j'$. Then

$$x + y \subseteq \textstyle\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} n_i' = \sum_{i=1}^{k} (n_i + n_i').$$

To show that $\pi_j(x + y) \subseteq \pi_j(x) + \pi_j(y)$, let $a \in x + y$. Then there exist $n_1'' \in n_1 + n_1' \subseteq N_1, \ldots, n_k'' \in n_k + n_k' \subseteq N_k$ such that $a \in \sum_{i=1}^{k} n_i''$, so $\pi_j(a) = n_j'' \in n_j + n_j' = \pi_j(x) + \pi_j(y)$. Hence, $\pi_j(x + y) \subseteq \pi_j(x) + \pi_j(y)$. Next, let $b \in \pi_j(x) + \pi_j(y)$. Since $x \in \sum_{i=1}^{k} n_i$ and $y \in \sum_{i=1}^{k} n_i'$, we can write $x \in n_j + l$ and $y \in n_j' + l'$ for some $l, l' \in \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i$. Then $n_j \in x - l$ and $n_j' \in y - l'$. Hence,

$$b \in \pi_j(x) + \pi_j(y) = n_j + n_j' \subseteq (x - l) + (y - l') = (x + y) - (l + l').$$

Then there exist $z \in x + y$ and $z' \in l + l' \subseteq \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i$ such that $b \in z - z'$. Since $z' \in \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i$, there exist $m_1 \in N_1, \ldots, m_{j-1} \in N_{j-1}, m_{j+1} \in N_{j+1}, \ldots, m_k \in N_k$ such that $z' \in \sum_{\substack{i=1 \\ i \neq j}}^{k} m_i$. Note that $b \in \pi_j(x) + \pi_j(y) \subseteq N_j$. Therefore,

$$z \in z' + b \subseteq m_1 + \cdots + m_{j-1} + b + m_{j+1} + \cdots + m_k.$$

Then $b = \pi_j(z) \in \pi_j(x + y)$. Hence, $\pi_j(x) + \pi_j(y) \subseteq \pi_j(x + y)$. This shows that $\pi_j(x + y) = \pi_j(x) + \pi_j(y)$. To show that $\pi_j(rx) \subseteq r\pi_j(x)$, let $c \in rx$. Note that $rx \subseteq r(\sum_{i=1}^{k} n_i) \subseteq \sum_{i=1}^{k} rn_i$. Then there exist $t_1 \in rn_1 \subseteq N_1, \ldots, t_k \in rn_k \subseteq N_k$ such that

$c \in \sum_{i=1}^{k} t_i$. This implies that $\pi_j(c) = t_j \in rn_j = r\pi_j(x)$. Hence, $\pi_j(rx) \subseteq r\pi_j(x)$. Next, let $d \in r\pi_j(x)$. Recall that $n_j \in x - l$ for some $l \in \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i$. Thus,

$$r\pi_j(x) = rn_j \subseteq r(x - l) \subseteq rx - rl.$$

Then there exist $p \in rx$ and $q \in rl$ such that $d \in p - q$. Since $\sum_{\substack{i=1 \\ i \neq j}}^{k} N_i \leq M$, we obtain $q \in rl \subseteq \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i$. Then there exist $q_1 \in N_1, \ldots, q_{j-1} \in N_{j-1}, q_{j+1} \in N_{j+1}, \ldots, q_k \in N_k$ such that $q \in \sum_{\substack{i=1 \\ i \neq j}}^{k} q_i$. Note that $d \in r\pi_j(x) \subseteq N_j$. Hence,

$$p \in d + q \subseteq q_1 + \cdots + q_{j-1} + d + q_{j+1} + \cdots + q_k.$$

This means that $d = \pi_j(p) \in \pi_j(rx)$. Thus, $r\pi_j(x) \subseteq \pi_j(rx)$. This shows that $r\pi_j(x) = \pi_j(rx)$. Hence, π_j is a homomorphism. Moreover, $\pi_j(x_j) = x_j$ for all $x_j \in N_j$. Therefore, π_j is surjective. Finally, let $m \in M$. Then there exist $x_1 \in N_1, \ldots, x_k \in N_k$ such that $m \in \sum_{i=1}^{k} x_i$. Thus, $\pi_j(m) = x_j$, so $\pi_j^2(m) = \pi_j(x_j) = x_j = \pi_j(m)$. Hence, $\pi_j^2 = \pi_j$. □

The map π_j as above is called the *projection map* on N_j. It is clear that projection maps send 0 to itself.

Proposition 25. *Let H, L and N be subhypermodules of an R-hypermodule M such that $M = H \oplus L$ and let $\pi_L : M \to L$ be the projection map on L. If $H \cap N = \{0\}$, then $H \oplus N = H \oplus \pi_L(N)$.*

Proof. Since $H \cap \pi_L(N) \leq H \cap L = \{0\}$, we obtain $H \cap \pi_L(N) = \{0\}$. Let $x \in H \oplus N$. Then $x \in h_1 + n_1$ for some $h_1 \in H$ and $n_1 \in N$. Since $M = H \oplus L$, there exist $h_2 \in H$ and $l \in L$ such that $n_1 \in h_2 + l$. Then $\pi_L(n_1) = l$. Hence,

$$x \in h_1 + n_1 \subseteq h_1 + (h_2 + l) = (h_1 + h_2) + \pi_L(n_1) \subseteq H \oplus \pi_L(N).$$

This shows that $H \oplus N \subseteq H \oplus \pi_L(N)$. Next, let $y \in H \oplus \pi_L(N)$. Then there exist $h' \in H$ and $n' \in N$ such that $y \in h' + \pi_L(n')$. Since $M = H \oplus L$, there exist $h'' \in H$ and $l' \in L$ such that $n' \in h'' + l'$. Then $\pi_L(n') = l' \in n' - h''$. Thus,

$$y \in h' + \pi_L(n') \subseteq h' + (n' - h'') = (h' - h'') + n' \subseteq H \oplus N.$$

This means that $H \oplus \pi_L(N) \subseteq H \oplus N$. Therefore, $H \oplus N = H \oplus \pi_L(N)$. □

Definition 26. Let P be a subhypermodule of an R-hypermodule M. We say that P is a *projection invariant subhypermodule* of M, denoted by $P \leq_p M$, if $f(P) \subseteq P$ for all $f^2 = f \in \mathrm{End}_0(M)$.

Proposition 27. *Let K, N and P be subhypermodules of an R-hypermodule M such that $M = K \oplus N$. If $P \leq_p M$, then $P = (P \cap K) \oplus (P \cap N)$.*

Proof. Assume that $P \leq_p M$. Clearly, $(P \cap K) \cap (P \cap N) = \{0\}$. Moreover, it suffices to show that $P \subseteq (P \cap K) \oplus (P \cap N)$. Let $p \in P$. Then $p \in k + n$ for some $k \in K$ and $n \in N$. Let π_K and π_N be the projection maps on K and N, respectively. Then $\pi_K^2 = \pi_K, \pi_N^2 = \pi_N \in \mathrm{End}_0(M)$. Since $P \leq_p M$, we obtain $\pi_K(P) \subseteq P$. Hence, $k = \pi_K(p) \in P$. Similarly, $n \in P$. This shows that $p \in (P \cap K) \oplus (P \cap N)$. □

2. Essential Subhypermodules, Complements and Closed Subhypermodules

In this section, we give notions of some special subhypermodules consisting of essential subhypermodules, complements and closed subhypermodules. These subhypermodules are important in order to define extending hypermodules and C_{11}-hypermodules in the next section.

Definition 28. Let N be a subhypermodule of an R-hypermodule M. We say that N is an *essential subhypermodule* of M (or *essential* in M), denoted by $N \leq_e M$, if $N \cap L \neq \{0\}$ for all $\{0\} \neq L \leq M$.

Remark 29. For each $N \leq M$, we see that $N \leq_e M$ if and only if $L = \{0\}$ for any $L \leq M$ with $N \cap L = \{0\}$.

Proposition 30. *Let M and M' be R-hypermodules. The following statements hold.*

1. *Let $K \leq N \leq M$. Then $K \leq_e M$ if and only if $K \leq_e N$ and $N \leq_e M$.*

2. *Let $f : M \to M'$ be a homomorphism such that $f(0) = 0$ and $N' \leq M'$. If $N' \leq_e M'$, then $f^{-1}(N') \leq_e M$.*

3. *Let $K_1, K_2, N_1, N_2 \leq M$. If $K_1 \leq_e N_1$ and $K_2 \leq_e N_2$, then $K_1 \cap K_2 \leq_e N_1 \cap N_2$.*

4. *Let* $K_1,\ K_2,\ N_1,\ N_2 \le M$ *be such that* $K_1 \cap K_2 = \{0\}$. *If* $K_1 \le_e N_1$ *and* $K_2 \le_e N_2$, *then* $K_1 \oplus K_2 \le_e N_1 \oplus N_2$.

Proof. (1) Assume that $K \le_e M$. We immediately obtain that $K \le_e N$. Moreover, if $\{0\} \ne L \le M$, then $\{0\} \ne K \cap L \le N \cap L$ since $K \le_e M$. Hence, $N \le_e M$.
Conversely, assume that $K \le_e N$ and $N \le_e M$. Let $L \le M$ be such that $K \cap L = \{0\}$. Then $K \cap (N \cap L) = \{0\}$. Now, $N \cap L = \{0\}$ because $K \le_e N$. Since $N \le_e M$, we obtain $L = \{0\}$. This shows that $K \le_e M$.

(2) Assume that $N' \le_e M'$. Then $f^{-1}(N') \le M$ by Proposition 21(2). Next, let $\{0\} \ne L \le M$. If $f(L) = \{0\}$, then $L \le \ker(f) \le f^{-1}(N')$, so $\{0\} \ne L = f^{-1}(N') \cap L$. Suppose that $f(L) \ne \{0\}$. By Proposition 21(1), $\{0\} \ne f(L) \le M'$. Then $N' \cap f(L) \ne \{0\}$ since $N' \le_e M'$. Thus, there exists $0 \ne l \in L$ such that $0 \ne f(l) \in N'$. This means that $0 \ne l \in f^{-1}(N') \cap L$. Hence, $f^{-1}(N') \le_e M$.

(3) Assume that $K_1 \le_e N_1$ and $K_2 \le_e N_2$. Let $\{0\} \ne L \le N_1 \cap N_2$. Then $K_2 \cap L \ne \{0\}$ since $K_2 \le_e N_2$. Because $K_1 \le_e N_1$, it follows that $(K_1 \cap K_2) \cap L \ne \{0\}$. Therefore, $K_1 \cap K_2 \le_e N_1 \cap N_2$.

(4) Assume that $K_1 \le_e N_1$ and $K_2 \le_e N_2$. Then $\{0\} = K_1 \cap K_2 \le_e N_1 \cap N_2$. This forces that $N_1 \cap N_2 = \{0\}$. For each $i \in \{1, 2\}$, let $\pi_i : N_1 \oplus N_2 \to N_i$ be the projection map on N_i. By (2), $\pi_1^{-1}(K_1) \le_e N_1 \oplus N_2$ and $\pi_2^{-1}(K_2) \le_e N_1 \oplus N_2$. Moreover, by (3), we get $\pi_1^{-1}(K_1) \cap \pi_2^{-1}(K_2) \le_e N_1 \oplus N_2$. Claim that $\pi_1^{-1}(K_1) = K_1 \oplus N_2$ and $\pi_2^{-1}(K_2) = N_1 \oplus K_2$. To prove the claim, let $x \in \pi_1^{-1}(K_1)$. Then there exist $n_1 \in N_1$ and $n_2 \in N_2$ such that $x \in n_1 + n_2$. Therefore, $n_1 = \pi_1(x) \in K_1$. This means that $x \in K_1 \oplus N_2$. Hence, $\pi_1^{-1}(K_1) \subseteq K_1 \oplus N_2$. Next, let $y \in K_1 \oplus N_2$. Then there exist $k_1 \in K_1 \le N_1$ and $n_2 \in N_2$ such that $y \in k_1 + n_2$. Thus, $\pi_1(y) = k_1 \in K_1$, i.e., $y \in \pi_1^{-1}(K_1)$. This shows that $K_1 \oplus N_2 \subseteq \pi_1^{-1}(K_1)$. Therefore, $\pi_1^{-1}(K_1) = K_1 \oplus N_2$. Similarly, $\pi_2^{-1}(K_2) = N_1 \oplus K_2$. By Proposition 17,

$$K_1 \oplus K_2 = (K_1 \oplus N_2) \cap (N_1 \oplus K_2) = \pi_1^{-1}(K_1) \cap \pi_2^{-1}(K_2) \le_e N_1 \oplus N_2.$$

This completes the proof. □

Next, we provide a characterization of essential subhypermodules of an R-hypermodule under certain conditions.

Proposition 31. *Let M be a strongly distributive R-hypermodule such that $m \in Rm$ for all $m \in M$, and let $N \le M$. Then $N \le_e M$ if and only if $N \cap Rx \ne \{0\}$ for all $0 \ne x \in M$.*

Proof. Assume that $N \le_e M$. Let $0 \ne x \in M$. The essentiality of N in M yields $N \cap Rx \ne \{0\}$.
Conversely, assume that $N \cap Rx \ne \{0\}$ for all $0 \ne x \in M$. Let $\{0\} \ne L \le M$. Then there exists $0 \ne y \in L$. By assumption, $N \cap Ry \ne \{0\}$. Note that $Ry \subseteq L$ because $L \le M$. Then $\{0\} \ne N \cap Ry \subseteq N \cap L$. Therefore, $N \le_e M$. □

In general, R-hypermodules M might not satisfy the condition that $m \in Rm$ for all $m \in M$ such as the R'-hypermodule M' in Example 9 where $M' = [0, \frac{1}{2}]$ and $R' = \{0\} \cup [2, \infty)$ because $R'\frac{1}{2} = [0, \frac{1}{4}]$, so $\frac{1}{2} \notin R'\frac{1}{2}$. However, if we let $R'' = \{0\} \cup [1, \infty)$, then the R''-hypermodule M' defined in Example 9 satisfies the condition that $m \in R''m$ for all $m \in M'$.

Definition 32. Let N be a subhypermodule of an R-hypermodule M. A subhypermodule K of M is called a *complement* of N in M if it is maximal under inclusion in the set $\{L \le M : L \cap N = \{0\}\}$, i.e., $K \cap N = \{0\}$ and $K = K'$ for any $K \le K' \le M$ with $K' \cap N = \{0\}$.

One can check that every subhypermodule of an R-hypermodule always has a complement. Moreover, for any subhypermodules H and N of an R-hypermodule M such that $H \cap N = \{0\}$, there exists a complement K of N in M such that $H \le K$.

Remark 33. Let K and K' be subhypermodules of an R-hypermodule M such that $M = K \oplus K'$. Then K and K' are complements of each other in M.

Proposition 34. *Let K and N be subhypermodules of an R-hypermodule M. If K is a complement of N in M, then $N \oplus K \le_e M$.*

Proof. Assume that K is a complement of N in M. Let $L \le M$ be such that $(N \oplus K) \cap L = \{0\}$. Claim that $N \cap (K + L) = \{0\}$. Let $x \in N \cap (K + L)$. Then $x \in N$ and $x \in k + l$ for some $k \in K$ and $l \in L$. Thus $l \in x - k \subseteq N \oplus K$. This means that $l \in (N \oplus K) \cap L = \{0\}$, i.e., $l = 0$. Then $x \in k + 0 = \{k\} \subseteq K$, so $x \in N \cap K = \{0\}$, i.e., $x = 0$. Hence, $N \cap (K + L) = \{0\}$ as claimed. Since K is a complement of N in M, we obtain $K = K + L$. Then $L \le K + L = K \le N \oplus K$ leading to $L = (N \oplus K) \cap L = \{0\}$. Therefore, $N \oplus K \le_e M$. □

The converse of the previous proposition does not hold in general because hypermodules generalize modules, and there is an example in modules insisting this conclusion (consider $\langle\overline{15}\rangle \oplus \langle\overline{6}\rangle$ in the $\mathbb{Z}$-module $\mathbb{Z}_{30}$). However, we give a sufficient condition such that the converse of Proposition 34 holds as follows.

Proposition 35. *Let K and N be subhypermodules of an R-hypermodule M. If $K \leq_\oplus M$ and $N \oplus K \leq_e M$, then K is a complement of N in M.*

Proof. Assume that $K \leq_\oplus M$ and $N \oplus K \leq_e M$. Then $M = K \oplus K'$ for some $K' \leq M$. Let $L \leq M$ be such that $K \subseteq L$ and $L \cap N = \{0\}$. We show that $K = L$. By the modular law,

$$K \oplus (L \cap K') = L \cap (K \oplus K') = L \cap M = L.$$

Claim that $L \cap K' = \{0\}$. By the essentiality of $N \oplus K$ in M, it suffices to show that $(N \oplus K) \cap (L \cap K') = \{0\}$. Let $x \in (N \oplus K) \cap (L \cap K')$. Then $x \in n + k$ for some $n \in N$ and $k \in K$. Thus, $n \in x - k \subseteq L$, so $n \in L \cap N = \{0\}$, i.e., $n = 0$. Then $x \in 0 + k = \{k\} \subseteq K$. Hence, $x \in K \cap K' = \{0\}$, i.e., $x = 0$. This shows that $(N \oplus K) \cap (L \cap K') = \{0\}$. Therefore, K is a complement of N in M. □

Proposition 36. *Let H, L and N be subhypermodules of an R-hypermodule M such that $H \leq_e L$. Then N is a complement of L in M if and only if N is a complement of H in M.*

Proof. First, assume that N is a complement of L in M. Clearly, $N \cap H = \{0\}$. Let $N' \leq M$ be such that $N \subseteq N'$ and $N' \cap H = \{0\}$. Then $H \cap (L \cap N') = \{0\}$. Thus, $L \cap N' = \{0\}$ since $H \leq_e L$. It follows that $N' = N$ because N is a complement of L in M. Hence, N is a complement of H in M.
Conversely, assume that N is a complement of H in M. Then $H \cap (N \cap L) = \{0\}$. Since $H \leq_e L$, we obtain $N \cap L = \{0\}$. Moreover, if $N \leq N'' \leq M$ with $N'' \cap L = \{0\}$, then $N'' \cap H \leq N'' \cap L = \{0\}$, so $N = N''$ since N is a complement of H in M. Therefore, N is a complement of L in M. □

Proposition 37. *Let K, L and N be subhypermodules of an R-hypermodule M such that $K \leq L$. If N is a complement of K in M and $N \cap L = \{0\}$, then $K \leq_e L$.*

Proof. Assume that N is a complement of K in M and $N \cap L = \{0\}$. Let $H \leq L$ be such that $K \cap H = \{0\}$. Note that $N \cap H \leq N \cap L = \{0\}$. Claim that $K \cap (H \oplus N) = \{0\}$. Let $x \in K \cap (H \oplus N)$. Then $x \in K$ and $x \in h + n$ for some $h \in H$ and $n \in N$. Thus, $n \in x - h \subseteq L$, so $n \in N \cap L = \{0\}$, i.e., $n = 0$. It follows that $x \in h + 0 = \{h\} \subseteq H$ which implies $x \in K \cap H = \{0\}$, i.e., $x = 0$. Hence, $K \cap (H \oplus N) = \{0\}$. Since N is a complement of K in M, we obtain $H \oplus N = N$. Then $H \leq H \oplus N = N$, so $H = N \cap H = \{0\}$. Therefore, $K \leq_e L$. □

We end this section by proposing and investigating closed subhypermodules.

Definition 38. Let C be a subhypermodule of an R-hypermodule M. We say that C is a *closed subhypermodule* of M (or *closed* in M), denoted by $C \leq_c M$, if there exists $C' \leq M$ such that C is a complement of C' in M.

By Remark 33, every direct summand of an R-hypermodule M is a closed subhypermodule of M.

Next, we give an equivalent condition of closed subhypermodules concerning the essentiality of subhypermodules of an R-hypermodule.

Proposition 39. *Let C be a subhypermodule of an R-hypermodule M. Then $C \leq_c M$ if and only if $C = L$ for any $L \leq M$ with $C \leq_e L$.*

Proof. Assume that $C \leq_c M$. Then there exists $C' \leq M$ such that C is a complement of C' in M. Let $L \leq M$ be such that $C \leq_e L$. Note that $C \cap (L \cap C') \leq C \cap C' = \{0\}$, but then $C \leq_e L$, so $L \cap C' = \{0\}$. Since C is a complement of C' in M, we obtain $C = L$.
Conversely, assume that $C = L$ for any $L \leq M$ with $C \leq_e L$. Let N be a complement of C in M. Claim that C is a complement of N in M. Let $H \leq M$ be such that $C \subseteq H$ and $N \cap H = \{0\}$. Thus, $C \leq_e H$ by Proposition 37 and then $C = H$. Therefore, C is a complement of N in M. We conclude that $C \leq_c M$. □

Proposition 40. *Let K and N be subhypermodules of an R-hypermodule M with $K \leq N$. If $K \leq_c N$ and $N \leq_c M$, then $K \leq_c M$.*

Proof. Assume that $K \leq_c N$ and $N \leq_c M$. Then there exist $K' \leq N$ and $N' \leq M$ such that K is a complement of K' in N and N is a complement of N' in M, respectively. To show that $K \leq_c M$, let $L \leq M$ be such that $K \leq_e L$. Claim that $K = L$. We divide the details of the proof into three steps as follows.

Step (i) We show that $L \cap (K' + N') = \{0\}$.
Claim that $K \cap (K' + N') = \{0\}$. Let $x \in K \cap (K' + N')$. Then $x \in K$ and $x \in k' + n'$ for some $k' \in K'$ and $n' \in N'$. Thus $n' \in x - k' \subseteq N$, so $n' \in N \cap N' = \{0\}$, i.e., $n' = 0$. Then $x \in k' + 0 = \{k'\} \subseteq K'$. This means that $x \in K \cap K' = \{0\}$, i.e., $x = 0$. Hence, $K \cap (K' + N') = \{0\}$. Then $K \cap [L \cap (K' + N')] \leq K \cap (K' + N') = \{0\}$. However, since $K \leq_e L$, this concludes that $L \cap (K' + N') = \{0\}$.

Step (ii) We show that $K = N \cap (L + N')$.
We first show that $K' \cap (L + N') = \{0\}$. Let $y \in K' \cap (L + N')$. Then $y \in K'$ and there exist $l \in L$ and $n' \in N'$ such that $y \in l + n'$. Thus, $l \in y - n' \subseteq K' + N'$. Hence, $l = 0$ from Step (i). Then $y \in 0 + n' = \{n'\} \subseteq N'$. This means that $y \in K' \cap N' \leq N \cap N' = \{0\}$, i.e., $y = 0$. Therefore, $K' \cap (L + N') = \{0\}$. Then $K' \cap [N \cap (L + N')] = K' \cap (L + N') = \{0\}$. Note that $K \leq N \cap (L + N')$. Since K is a complement of K' in N, we get $K = N \cap (L + N')$.

Step (iii) We show that $L \leq N$.
Claim that $N' \cap (N + L) \leq L$. Let $z \in N' \cap (N + L)$. Then $z \in N'$ and there exist $n \in N$ and $l \in L$ such that $z \in n + l$. Thus, $n \in z - l \subseteq l + N'$, so $n \in N \cap (L + N') = K \leq L$ by Step (ii). Then $z \in n + l \subseteq L$. Therefore, $N' \cap (N + L) \leq L$. We observe that $K \cap [N' \cap (N + L)] \leq N \cap N' = \{0\}$, but $K \leq_e L$, this implies that $N' \cap (N + L) = \{0\}$. It follows that $N = N + L$ since N is a complement of N' in M. Hence, $L \leq N + L = N$.

Now, $L \leq N$ from Step (iii) and recall that $K \leq_c N$ and $K \leq_e L$. Therefore, $K = L$ from Proposition 39. Again, by Proposition 39, $K \leq_c M$. □

Proposition 41. *Let K be a subhypermodule of an R-hypermodule M. Then there exists $L \leq M$ such that $K \leq_e L$ and $L \leq_c M$.*

Proof. Let N be a complement of K in M. Then $N \cap K = \{0\}$. By Zorn's lemma, there exists $L \leq M$ such that L is a complement of N in M and $K \leq L$. Then $L \leq_c M$. By Proposition 37, $K \leq_e L$. □

3. Extending Hypermodules and C_{11}-Hypermodules

In this section, we provide the concepts of extending hypermodules and C_{11}-hypermodules which generalize extending modules and C_{11}-modules, respectively. The main purpose of this section is to provide characterizations of extending hypermodules and C_{11}-hypermodules. Moreover, decompositions of C_{11}-hypermodules are investigated. Although most properties of extending hypermodules and C_{11}-hypermodules are similar to those of extending modules and C_{11}-modules in module theory, the important point in this paper is to develop tools in hypermodules needed to prove those properties.

Definition 42. An R-hypermodule M is called an *extending hypermodule* if for each $N \leq M$, there exists $D \leq_\oplus M$ such that $N \leq_e D$.

First, characterizations of extending hypermodules regarding closed subhypermodules and the essentiality of direct sums of two subhypermodules are given. In addition, we characterize a strongly distributive extending hypermodule M satisfying the condition that $m \in Rm$ for all $m \in M$ by using the lifting of homomorphisms from some subhypermodules of M into M itself.

Theorem 43. *Let M be an R-hypermodule. The following statements are equivalent:*

1. *M is an extending hypermodule;*
2. *every closed subhypermodule of M is a direct summand of M;*
3. *for any $K, L \leq M$ with $K \cap L = \{0\}$, there exists a direct summand D of M such that $L \leq D$ and $D \oplus K \leq_e M$.*

Proof. (1)⇒(2) Assume that M is an extending hypermodule. Let $C \leq_c M$. Then there exists $C' \leq_\oplus M$ such that $C \leq_e C'$. Hence, $C = C'$ by Proposition 39.

(2)⇒(3) Assume (2) holds. Let $K, L \leq M$ be such that $K \cap L = \{0\}$. Then there exists a complement D of K in M such that $L \leq D$. Thus, $D \leq_c M$. By assumption and Proposition 34, we obtain $D \leq_\oplus M$ and $D \oplus K \leq_e M$, respectively.

(3)⇒(1) Assume (3) holds. Let $L \leq M$ and let K be a complement of L in M. Then $K \cap L = \{0\}$. By assumption, there exists $D \leq_\oplus M$ such that $L \leq D$ and $D \oplus K \leq_e M$. Thus, $L \leq_e D$ by Proposition 37. Therefore, M is an extending hypermodule. □

Recall that every direct summand is a closed subhypermodule, but the converse does not hold in general. However, we can summarize from Theorem 43 that direct sumands and closed subhypermodules of an R-hypermodule are identical provided that the R-hypermodule is an extending hypermodule.

Proposition 44. *Let M be a strongly distributive R-hypermodule such that $m \in Rm$ for all $m \in M$. Then M is an extending hypermodule if and only if for every closed subhypermodule K of M there exists a complement L of K in M such that every homomorphism $f : K \oplus L \to M$ with $f(0) = 0$ can be extended to a homomorphism $\bar{f} : M \to M$.*

Proof. Assume that M is an extending hypermodule. Let $K \leq_c M$. Then $K \leq_\oplus M$ by Theorem 43, so $M = K \oplus L$ for some $L \leq M$. As a result, L is a complement of K in M and the result regarding homomorphisms is clear.
Conversely, let $C \leq_c M$. By hypothesis, there exists a complement D of C in M such that every homomorphism $f : C \oplus D \to M$ with $f(0) = 0$ can be lifted to a homomorphism $\bar{f} : M \to M$. We show that $C \leq_\oplus M$. Let $\pi_C : C \oplus D \to M$ be the projection map on C. Note that π_C is a homomorphism with $\pi_C(0) = 0$. Then there exists a homomorphism $\bar{\pi}_C : M \to M$ such that $\bar{\pi}_C(a) = \pi_C(a)$ for all $a \in C \oplus D$. Especially, $\bar{\pi}_C(c) = c$ for all $c \in C$ and $\bar{\pi}_C(d) = 0$ for all $d \in D$. Note that $C \leq \bar{\pi}_C(M)$ and $D \leq \ker(\bar{\pi}_C)$. To show that $C \leq_e \bar{\pi}_C(M)$, let $0 \neq z \in \bar{\pi}_C(M)$. Then $\bar{\pi}_C(y) = z$ for some $y \in M$. We observe that $y \notin \ker(\bar{\pi}_C)$ since $z \neq 0$. Thus, $y \notin D$. By assumption, $y \in Ry \leq D + Ry$. This implies that $D \subsetneq D + Ry$. Since D is a complement of C in M, we obtain $C \cap (D + Ry) \neq \{0\}$. Let $0 \neq z_0 \in C \cap (D + Ry)$. Then $z_0 \in C$ and there exist $d \in D$ and $y_0 \in Ry$ such that $z_0 \in d + y_0$. Since $y_0 \in Ry$, there exists $r \in R$ such that $y_0 \in ry$. Therefore,

$$z_0 = \bar{\pi}_C(z_0) \in \bar{\pi}_C(d + y_0) = \bar{\pi}_C(d) + \bar{\pi}_C(y_0) \subseteq 0 + \bar{\pi}_C(ry) = \bar{\pi}_C(ry) = r\bar{\pi}_C(y) = rz.$$

This means that $z_0 \in Rz$. Hence, $0 \neq z_0 \in C \cap Rz$. We conclude that $C \leq_e \bar{\pi}_C(M)$ by Proposition 31 and then $C = \bar{\pi}_C(M)$ by Proposition 39. Moreover, if $m \in M$, then $\bar{\pi}_C(m) \in C$, so $\bar{\pi}_C^2(m) = \bar{\pi}_C(m)$. This means that $\bar{\pi}_C^2 = \bar{\pi}_C$. Thus, $M = \bar{\pi}_C(M) \oplus \ker(\bar{\pi}_C)$ by Proposition 23. This shows that $C \leq_\oplus M$. By Theorem 43, M is an extending hypermodule. □

In modules, a submodule of an extending module may not be extending. It follows that a subhypermodule of an extending hypermodule may not be an extending hypermodule in general. However, it can be shown that every closed subhypermodule of an extending hypermodule is also an extending hypermodule.

Proposition 45. *Every closed subhypermodule of an extending hypermodule is also an extending hypermodule.*

Proof. Let C be a closed subhypermodule of an extending hypermodule M and let $K \leq_c C$. Thus, $K \leq_c M$ by Proposition 40. Then, by Theorem 43, there exists $K' \leq M$ such that $M = K \oplus K'$. Thus, $C = K \oplus (K' \cap C)$ since $K \leq C$. This means that $K \leq_\oplus C$. As a result of Theorem 43, C is an extending hypermodule. □

Next, we define and explore C_{11}-hypermodules.

Definition 46. An R-hypermodule M is called a C_{11}*-hypermodule* if for each $N \leq M$, there exists a complement K of N in M such that $K \leq_\oplus M$.

Note that every subhypermodule of an R-hypermodule always has a complement which is also a closed subhypermodule. This concludes that every extending hypermodule is always a C_{11}-hypermodule from Theorem 43 but the converse does not hold.

Next, we characterize C_{11}-hypermodules concerning closed subhypermodules and the essentiality of direct sums in hypermodules.

Theorem 47. *Let M be an R-hypermodule. The following statements are equivalent:*

1. *M is a C_{11}-hypermodule;*
2. *for every closed subhypermodule C of M, there exists a direct summand D of M such that D is a complement of C in M;*
3. *for every closed subhypermodule C of M, there exists a direct summand D of M such that $D \oplus C \leq_e M$;*
4. *for every subhypermodule N of M, there exists a direct summand D of M such that $D \oplus N \leq_e M$.*

Proof. (1)⇒(2) This follows directly from the definition of C_{11}-hypermodules.

(2)⇒(3) This is obtained from Proposition 34.

(3)⇒(4) Assume that (3) holds. Let $N \leq M$. By Proposition 41, there exists $C \leq M$ such that $N \leq_e C$ and $C \leq_c M$. Then there exists $D \leq_\oplus M$ such that $D \oplus C \leq_e M$. Hence, D is a complement of C in M from Proposition 35. Moreover, D is a complement of N in M from Proposition 36. We conclude from Proposition 34 that $D \oplus N \leq_e M$.

(4)⇒(1) This follows from Proposition 35. □

Next, we show that any direct sums of two C_{11}-hypermodules must be a C_{11}-hypermodule.

Proposition 48. *Let K_1 and K_2 be subhypermodules of an R-hypermodule M such that $M = K_1 \oplus K_2$. If K_1 and K_2 are C_{11}-hypermodules, then M is a C_{11}-hypermodule.*

Proof. Assume that K_1 and K_2 are C_{11}-hypermodules. Let $N \leq M$. Since K_1 is a C_{11}-hypermodule, by Theorem 47, there exists $D_1 \leq_\oplus K_1$ such that $D_1 \oplus (N \cap K_1) \leq_e K_1$. By the modular law,

$$K_1 \cap (D_1 \oplus N) = D_1 \oplus (N \cap K_1) \leq_e K_1.$$

Since K_2 is a C_{11}-hypermodule and $(D_1 \oplus N) \cap K_2 \leq K_2$, again by Theorem 47, there exists $D_2 \leq_\oplus K_2$ such that $D_2 \oplus [(D_1 \oplus N) \cap K_2] \leq_e K_2$. By the modular law,

$$K_2 \cap [D_2 \oplus (D_1 \oplus N)] = D_2 \oplus [(D_1 \oplus N) \cap K_2] \leq_e K_2.$$

Let $D = D_2 \oplus D_1$. Since $M = K_1 \oplus K_2, D_1 \leq_\oplus K_1$ and $D_2 \leq_\oplus K_2$, it follows that $D \leq_\oplus M$. In addition, $K_2 \cap (D \oplus N) \leq_e K_2$. Note that $K_1 \cap (D_1 \oplus N) \leq K_1 \cap (D \oplus N)$, but then $K_1 \cap (D_1 \oplus N) \leq_e K_1$, so $K_1 \cap (D \oplus N) \leq_e K_1$ by Proposition 30(1). Hence,

$$[K_1 \cap (D \oplus N)] \oplus [K_2 \cap (D \oplus N)] \leq_e K_1 \oplus K_2 = M$$

from Proposition 30(4). Moreover,

$$[K_1 \cap (D \oplus N)] \oplus [K_2 \cap (D \oplus N)] \leq (K_1 \oplus K_2) \cap (D \oplus N) = M \cap (D \oplus N) = D \oplus N.$$

Thus, $D \oplus N \leq_e M$ from Proposition 30(1). By Theorem 47, M is a C_{11}-hypermodule. □

A direct summand of a C_{11}-module may not be a C_{11}-module (Smith & Tercan, 2004). This implies that a direct summand of a C_{11}-hypermodule may not be a C_{11}-hypermodule. In this paper, we show that if a C_{11}-hypermodule can be decomposed as a direct sum of two subhypermodules, then the subhypermodules are also C_{11}-hypermodules when at least one of them is a projection invariant subhypermodule. To illustrate this statement, the next proposition is needed.

Proposition 49. *Every projection invariant subhypermodule of a C_{11}-hypermodule is also a C_{11}-hypermodule.*

Proof. Let P be a projection invariant subhypermodule of a C_{11}-hypermodule M. Let $N \leq P$. Then there exists $D \leq_\oplus M$ such that D is a complement of N in M, so $M = D \oplus D'$ for some $D' \leq M$. Thus, $N \oplus D \leq_e M$ from Proposition 34. By the modular law, $N \oplus (P \cap D) = P \cap (N \oplus D) \leq_e P$. This implies that $P \cap D \leq_\oplus P$ from Proposition 27. By Theorem 47, P is a C_{11}-hypermodule. □

Proposition 50. *Let K_1 and K_2 be subhypermodules of a C_{11}-hypermodule M such that $M = K_1 \oplus K_2$. If $K_1 \leq_p M$, then both K_1 and K_2 are C_{11}-hypermodules.*

Proof. Assume that $K_1 \leq_p M$. By Proposition 49, we obtain that K_1 is a C_{11}-hypermodule. It remains to show that K_2 is a C_{11}-hypermodule. Let $N_2 \leq K_2$. Since M is a C_{11}-hypermodule, by Theorem 47, there exists $D \leq_\oplus M$ such that $D \oplus (K_1 \oplus N_2) \leq_e M$. Then $M = D \oplus D'$ for some $D' \leq M$. By Proposition 27, $K_1 = (K_1 \cap D) \oplus (K_1 \cap D')$. Moreover, $K_1 = K_1 \cap D'$ since $K_1 \cap D = \{0\}$. Then $K_1 \leq D'$, so $D' = K_1 \oplus (K_2 \cap D')$. Hence, we can write $M = D \oplus K_1 \oplus (K_2 \cap D')$. Let $\pi_2 : K_1 \oplus K_2 \to K_2$ be the projection map on K_2. Recall that $K_1 \cap D = \{0\}$. By Proposition 25, $K_1 \oplus D = K_1 \oplus \pi_2(D)$. Therefore,

$$M = D \oplus K_1 \oplus (K_2 \cap D') = K_1 \oplus \pi_2(D) \oplus (K_2 \cap D').$$

Recall that $\pi_2(D) \leq K_2$. This concludes that $\pi_2(D) \leq_\oplus K_2$. Moreover,

$$K_1 \oplus \pi_2(D) \oplus N_2 = K_1 \oplus D \oplus N_2 \leq_e M.$$

Thus, $K_2 \cap (K_1 \oplus \pi_2(D) \oplus N_2) \leq_e K_2$. By the modular law,

$$K_2 \cap (K_1 \oplus \pi_2(D) \oplus N_2) = (\pi_2(D) \oplus N_2) \oplus (K_2 \cap K_1) = \pi_2(D) \oplus N_2.$$

Hence, $\pi_2(D) \oplus N_2 \leq_e K_2$. By Theorem 47, K_2 is a C_{11}-hypermodule. □

Proposition 51. *Let M be a C_{11}-hypermodule. Then for each $X \leq_p M$, there exist $K_1, K_2 \leq M$ such that $X \leq_e K_2$ and $M = K_1 \oplus K_2$.*

Proof. Let $X \leq_p M$. Then there exists $K_1 \leq_\oplus M$ such that K_1 is a complement of X in M. Thus, $M = K_1 \oplus K_2$ for some $K_2 \leq M$. Let $\pi_1 : K_1 \oplus K_2 \to K_1$ be the projection map on K_1. Then $\pi_1^2 = \pi_1 \in \mathrm{End}_0(M)$, $\pi_1(M) = K_1$ and $\ker(\pi_1) = K_2$. Claim that $X \leq K_2$. Let $x \in X$. Since $X \leq_p M$, we obtain $\pi_1(x) \in X$. Thus $\pi_1(x) \in K_1 \cap X = \{0\}$. This implies that $x \in \ker(\pi_1) = K_2$. Therefore, $X \leq K_2$. By Proposition 37, we conclude that $X \leq_e K_2$. □

From Proposition 51, for a C_{11}-hypermodule M and $X \leq_p M$, there exist $K_1, K_2 \leq M$ such that $X \leq_e K_2$ and $M = K_1 \oplus K_2$, but there is no conclusion to insist that K_1 and K_2 are C_{11}-hypermodules in the case that $X \neq K_2$, although M is a C_{11}-hypermodule.

4. Conclusions

In this article, we define and study extending hypermodules and C_{11}-hypermodules where hypermodules were investigated by Siraworakun in 2012. Note that extending hypermodules and C_{11}-hypermodules are extended from extending modules and C_{11}-modules, respectively. In module theory, extending modules and C_{11}-modules relate to some submodules consisting of essential submodules, complements and closed submodules. Therefore, we first define essential subhypermodules, complements (in hypermodules) and closed subhypermodules and then prove some properties of these subhypermodules. These properties are mainly used to characterize extending hypermodules and C_{11}-hypermodules. Finally, we show that any direct sums of two C_{11}-hypermodules must be a C_{11}-hypermodule. On the other hand, for the case that a C_{11}-hypermodule M can be decomposed as a direct sum of two subhypermodules K_1 and K_2, i.e., $M = K_1 \oplus K_2$, then each of K_1 and K_2 cannot be concluded whether it is a C_{11}-hypermodule or not; however, K_1 and K_2 are C_{11}-hypermodules if at least one of them is a projection invariant subhypermodule.

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