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A Note on Harris Extended Generalized Exponential Distribution

Oseghale Osezuwa Innocent[1], Ayoola Femi Joshua[2], Oluwole Adegoke Nuga[3]

[1]Department of Statistics, Joseph Ayo Babalola University, Ikeji-Arakeji, Osun State, Nigeria.

[2]College of Business, Jackson State University, Jackson, MS, USA.

[3]Department of Physical Sciences, Bells University of Technology, Nigeria.

Correspondence author: Oseghale O. Innocent, Department of Statistics, Joseph Ayo Babalola University, Ikeji-Arakeji, Osun State, Nigeria. E-mail: innocentoseghale@gmail.com

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Abstract

We introduce a four-parameter extension of the exponential distribution, which has the Exponentiated exponential, Marshall-Olkin exponential, and exponential distribution as sub-models. The proposed distribution has two important properties: it involves more parameters than the baseline model to obtain more flexibility and the extra parameters have a clear interpretation and representation. The proposed model is more flexible than any of its sub-models. Its probability density function can be monotone increasing, decreasing, or unimodal and its associated hazard rate may be increasing, decreasing, unimodal or bathtub-shaped. Statistical expression is obtained for certain structural statistical properties such as ordinary and incomplete moments, moment generating function, order statistics, quantile function, reliability function, and Renyi entropy. The maximum likelihood estimation method is used to obtain the estimates of the model parameters. An application of the new model to two lifetime data demonstrates the flexibility of the model.

Keywords: reliability function, moments, Bathtub-shaped, Incomplete moments

1. Introduction

Exponential distribution forms the basis for which all other lifetime distribution were extensively developed in the life testing literature. In applied sciences, namely: engineering, medicine, reliability study, finance, and many others, the statistical modeling and analysis of lifetime data are very important. Many lifetime distributions have been employed in modeling lifetime data, this may include the exponential, gamma, Weibull, Gompertz, and Rayleigh distribution. Each distribution has its own peculiar characteristics specifically due to the shapes of its hazard rate function (hrf), which can be constant, monotonically increasing or decreasing, bathtub and unimodal.

An extension of the exponential distribution was study by Nadarajah (2011) called Exponentiated exponential (EE) distribution, the cumulative distribution function of EE distribution is given by

$$J(x;\lambda,\theta)=\left(1-e^{-\lambda x}\right)^{\theta}, \qquad x>0;\ \lambda,\theta>0 \tag{1}$$

The corresponding Probability density function (PDF) to (1), is given by

$$jj(x;\lambda,\theta)=\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\theta-1}, \qquad x>0;\ \lambda,\theta>0 \tag{2}$$

Where θ is a scale parameter and λ is a shape parameter. Gupta and Kundu (2001), in their work note that the two-parameter Exponentiated exponential distribution can be used effectively in modeling and analyzing several lifetime data, particularly in place of two-parameter gamma or two-parameter Weibull distribution. Taking θ to be one, then all the three distributions coincide with the one parameter exponential distribution. Therefore, it will be observed that the three-parameter distributions are generalizations of the exponential distribution in different ways. In this study, an extension of the exponentiated exponential distribution is to be developed by the addition of two shape parameters to the existing distribution, which will increase its scope of applications by inducing flexibility into the distribution in such a way that it can be used effectively in modeling data that exhibits different shapes of the hazard function, which may be increasing, decreasing, increasing-decreasing, decreasing-increasing, unimodal, or bathtub shapes.

2. Harris Extended Generalized Exponential Distribution

The development of a new lifetime model is pioneered using the general construction method developed by Aly and

Benkherouf (2011) on the bases of the distribution proposed in Harris (1948). Consider a sequence of $i.i.d.$ random variables $Z_1, Z_2, \ldots, Z_N$ with cumulative distribution function (CDF) $G(x)$, probability density function (PDF) $g(x)$, Reliability function (rf) $R(x)$ and hazard rate function (hrf) $h(x)$. Suppose $X = min(Z_1, Z_2, \ldots, Z_N)$, where N is a positive integer valued random variable having a probability generating function (pgf) given by $\psi(.,\varphi)$ for $\varphi > 0$. The reliability function $\bar{F}(x)$ of X is given by

$$\bar{F}(x) = \psi(\bar{G}(x), \varphi) \tag{3}$$

Where $\bar{J}(x) = 1 - J(x)$. Several properties of (1) were studied in Aly and Benkherouf (2011). This method adds two shape shape parameter functioning as a tilt parameter to an existing distribution which further increase the modeling potential of the base distribution. Aly and Benkherouf (2011) generalized the Marshall-Olkin (MO) distribution using the pgf of the Harris distribution (Harris, 1948) for the development of a new lifetime distributions. This pgf is represented by

$$\psi(f, c, \varphi) = \left(\frac{\varphi f^c}{1 - \bar{\varphi} f^c}\right)^{1/c}, \quad c > 0. \tag{4}$$

They considered φ to be in the $(0, \infty)$ interval, whereas Harris (1948) restricted φ to be in the (0, 1) interval. Plugging (4) into (3) gives

$$\bar{F}(x; c, \varphi) = \left(\frac{\varphi \bar{J}^c}{1 - \bar{\varphi} \bar{J}^c}\right)^{1/c}, \quad c > 0. \tag{5}$$

Equation (3) reverses to the MO distribution when $c = 0$. The corresponding PDF becomes

$$f(x) = \frac{\varphi^{1/c} j(x)}{[1 - \bar{\varphi} \bar{J}^c]^{\frac{1}{c}+1}}. \tag{6}$$

In this paper, we shall consider the Harris extended generalized exponential (HEGE) Distribution developed by setting $\bar{J}(x) = 1 - \left(1 - e^{-\lambda x}\right)^{\theta}, \lambda, \theta > 0$. The motivation behind the use of this new distribution is that an extension of the EE distribution and the additional shape parameters c and φ may provide better adjustments to some datasets compared to the EE distribution. The extra shape parameters help to control the shape of the density function of the $HEGE$ distribution. In addition, when c is a positive integer and $0 < \varphi < 1$, the distribution has a well-established physical interpretation. It can be described as a distribution of the time until failure is observed in a device composed of N serial components with constant failure rate, where N is a random variable arising from a branching process as the one described in Harris (1948). The $HEGE$ distribution can be applied in several areas such as seismology, engineering, biology, epidemiology, and insurance among many others. This method of generalization of the baseline distribution has been explored by many authors which includes Chahkandi and Ganjali (2009) and Morais and Barreto- Souza (2011), Barreto-Souza and Bakouch (2011), Lu and Shi (2012), Ristic and Nadarajah (2011), Pinho et al. (2012), Pinho et al. (2015), Jose and Paul (2018), Jose et al.(2018). More recently, Sophia et al.(2021), and Ogunde et al. (2021), among many others.

The paper is organized as follows. In Section 2, we obtain the pdf, cdf, hrf and quantile function of the new distribution. In addition, we discuss the shape of the density and hazard rate functions. Section 3 is we derived the moments and the incomplete moments, mean residual life, Renyl entropy, and order statistics. in section 4, Maximum likelihood estimation is discussed, simulation study is carried out and an application to a real dataset is performed, where we compare the HEGE distribution to other models. Finally, concluding remarks are addressed in section 5.

2.1 The HEGE Distribution

Inserting $\bar{J}(x) = 1 - \left(1 - e^{-\lambda x}\right)^{\theta}$ in (5) defines the $HEGE$ reliability function by

$$R(x) = \frac{\varphi^{1/c}\left[1 - \left(1 - e^{-\lambda x}\right)^{\theta}\right]}{[1 - \bar{\varphi}[1 - (1 - e^{-\lambda x})^{\theta}]^c]^{\frac{1}{c}}}, \quad x > 0; c, \lambda, \varphi, \theta > 0 \tag{7}$$

The corresponding PDF to (7) is

$$f(x) = \frac{\varphi^{1/c} \lambda \theta e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\theta}}{[1 - \bar{\varphi}[1 - (1 - e^{-\lambda x})^{\theta}]^c]^{(1+1/c)}}, \tag{8}$$

And the hazard $(h(x))$, is given by

$$h(x) = \frac{\lambda\theta e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\theta-1}}{[1-(1-e^{-\lambda x})^{\theta}][1-\bar{\varphi}[1-(1-e^{-\lambda x})^{\theta}]^{c}]}. \tag{9}$$

The graph of the of the density and hazard functions is given in figures (1) and (2)

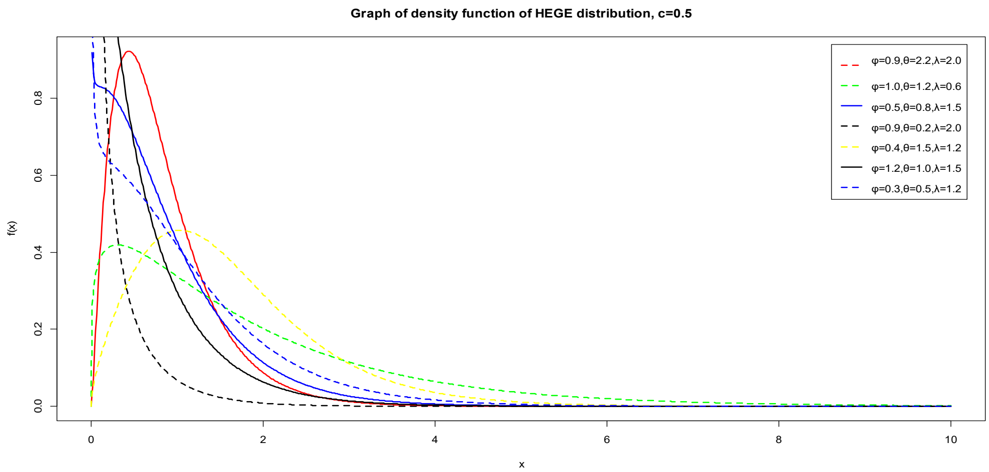

Figure 1. The graph of the density function of $\boldsymbol{HEGE}$ distribution

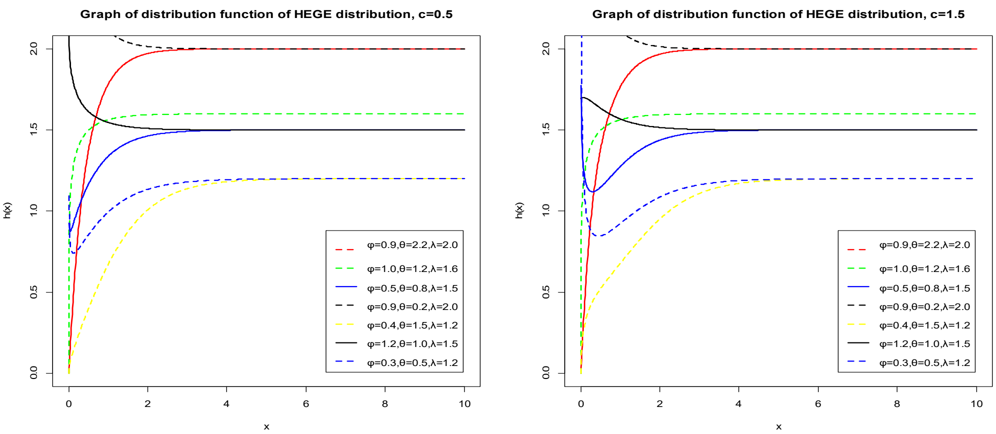

Figure 2. The graph of the hazard function of $\boldsymbol{HEGE}$ distribution

2.2 Quantile Function

For a random variable X with the PDF (8), the quantile function (u) is

$$Q(u) = inf(x \in R: F(x) \geq u), \text{ where } 0 < u < 1. \tag{10}$$

The relation in () is used to find the quantile function of $HEGE$ distribution. Then, we have

$$X = -\frac{1}{\lambda} log\left\{1 - \left[1 - \frac{(1-u)}{[(1-u)\bar{\varphi}+\varphi]^{1/c}}\right]^{1/\theta}\right\}. \tag{11}$$

By using an expression for the quantile function given in (11), we can estimate the Bowley skewness; Kenney and Keeping (1962) and Moors kurtosis; Moor (1988) for $HEGE$ as

$$s_k = \frac{Q_{3/4} + Q_{1/4} - 2Q_{2/4}}{Q_{3/4} - Q_{1/4}},$$

and

$$k_u = \frac{Q_{3/8} - Q_{1/8} + Q_{7/8} - Q_{5/8}}{Q_{6/8} - Q_{2/8}}.$$

Table 2 illustrates the values of skewness and kurtosis for the $HEGE$ model for some values of λ, φ, θ and c. For a fixed value of c=8.5, $\theta = 3.5$.

Table 1. Skewness and kurtosis of $\boldsymbol{HEGE}$ for different values of $\boldsymbol{\lambda, \varphi, \theta}$ and $\boldsymbol{c}$

Quantiles	$\boldsymbol{\lambda = 1.1}$, $\boldsymbol{\varphi = 0.2}$	$\boldsymbol{\lambda = 1.5}$, $\boldsymbol{\varphi = 0.5}$	$\boldsymbol{\lambda = 2.1}$, $\boldsymbol{\varphi = 1.2}$	$\boldsymbol{\lambda\lambda = 2.5}$, $\boldsymbol{\varphi = 2.2}$
$Q_{1/4}$	1.8621	1.9523	1.9842	2.0917
$Q_{2/4}$	3.0714	3.2004	3.2325	3.3217
$Q_{3/4}$	4.7418	4.8978	4.9226	4.9829
$Q_{1/8}$	1.2458	1.3080	1.3338	1.4331
$Q_{3/8}$	2.4474	2.5589	2.5923	2.6934
$Q_{5/8}$	3.7980	3.9416	3.9706	4.0458
$Q_{7/8}$	6.2474	6.4141	6.4337	6.4791
s_k	0.1601	0.1525	0.1504	0.1491
k_u	1.2678	1.2640	1.2665	1.2775

3. Mixture Representation

The mixture representation of the density function is an important tool used when deriving an expression for the statistical properties of generalized distribution. In this section, the mixture representation of the $HEGE$ density function is derived. Using the series representation

$$(1-z)^{-m} = \sum_{i=0}^{\infty} (-1)^i \binom{m+i-1}{i} z^i \tag{12}$$

$m > 0$ and $|z| < 1$, the density of HEGE can be expressed as

$$f_{HEGE}(x) = \varphi^{1/c} \lambda\theta \sum_{i,j=0}^{\infty} \binom{\frac{1}{c}+i}{i} \binom{ci}{j} (-1)^{i+j} e^{-\lambda x} \left(1-e^{-\lambda x}\right)^{\theta(j+1)-1} \tag{13}$$

Therefore, (13) is an Exponentiated exponential distribution with shape parameter $\theta(j+1)$ and scale parameter λ. Further algebraic manipulation gives

$$f_{HEGE}(x) = \varphi^{1/c} \lambda\theta \sum_{i,j=0}^{\infty} \sum_{k=0}^{\theta(j+1)} \binom{\frac{1}{c}+i}{i} \binom{ci}{j} \binom{\theta(j+1)-1}{k} (-1)^{i+j+k} e^{-\lambda(k+1)x} \tag{14}$$

3.1 Moments

The moment is very useful when computing measures of central tendencies, shapes, and dispersion. The q^{th} non-central moment of the $HEGE$ random variable is

$$\mu_q' = \int_0^{\infty} x^q dF_{HEGE}(x) \tag{15}$$

$$= \varphi^{1/c} \lambda\theta \sum_{i,j,k=0}^{\infty} \binom{\frac{1}{c}+i}{i} \binom{ci}{j} \binom{\theta(j+1)-1}{k} (-1)^{i+j+k} \int_0^{\infty} x^q e^{-\lambda(k+1)x}\, dx$$

$$\mu_q' = \varphi^{1/c} \theta \sum_{i,j,k=0}^{\infty} \binom{\frac{1}{c}+i}{i} \binom{ci}{j} \binom{\theta(j+1)-1}{k} (-1)^{i+j+k} \lambda^{-q} (k+1)^{-(q+1)} \Gamma(q+1) \tag{16}$$

For $q = 1, 2, \ldots$, where $\Gamma(.)$ is the gamma function. Table 2 shows the first six moments, standard Deviation (σ)

Coefficient of Variation (σ_v), coefficient of skewness (σ_{sk}), and coefficient of kurtosis (σ_{ku}). The values of σ, σ_v, σ_{sk}, and σ_{ku} are respectively, given by:

$$\sigma = \sqrt{\mu_2' - \mu^2}, \quad \sigma_v = \frac{\sigma}{\mu}, \quad \sigma_{sk} = \frac{\mu_3' - 3\mu\mu_2' - 2\mu^3}{{}^{3}\!/_{2}\sqrt{\mu_2' - \mu^2}} \; and \; \sigma_{ku} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 2\mu^4}{(\mu_2' - \mu^2)^2}$$

Table 2. First six moments, σ, $\boldsymbol{\sigma_v}$, $\boldsymbol{\sigma_{sk}}$, and $\boldsymbol{\sigma_{ku}}$

Moments	$\boldsymbol{\lambda = 5.5, \theta = 0.5}$	$\boldsymbol{\lambda = 1.5, \theta = 3.5}$	$\boldsymbol{\lambda = 3.5, \theta = 3.5}$	$\boldsymbol{\lambda = 3.5, \theta = 5.5}$
μ_1'	0.2634	0.6682	0.8898	0.5663
μ_2'	0.1218	0.6136	0.9753	0.3950
μ_3'	0.0762	0.6987	1.2569	0.3239
μ_4'	0.0599	0.9477	1.8622	0.3054
μ_5'	0.0568	1.4993	3.1342	0.3271
μ_6'	0.0634	2.7278	5.9469	0.3949
σ	0.2289	0.4088	0.6462	0.2726
σ_v	0.8692	0.6118	0.7262	0.4814
σ_{sk}	1.3750	0.9569	0.7937	0.7926
σ_{ku}	5.7781	4.5104	4.1906	4.2044

3.2 Incomplete Moment

The incomplete moment can be applied in different field of study. The first incomplete moment is used in calculating the Bonferroni and Lorenz curves which are very useful in demography economics, insurance, and medicine. The p^{th} incomplete moment of the $HEGE$ random variable is

$$\rho_q(t) = \int_0^t x^q dF_{HEGE}(x) \tag{17}$$

$$= \varphi^{1/c}\lambda\theta \sum_{i,j=0}^{\infty} \sum_{k=0}^{\theta(j+1)} \binom{\frac{1}{c}+i}{i}\binom{ci}{j}\binom{\theta(j+1)-1}{k}(-1)^{i+j+k}\int_0^t x^q e^{-\lambda(k+1)x}\,dx$$

$$\rho_q(t) = \varphi^{1/c}\theta \sum_{i,j=0}^{\infty} \sum_{k=0}^{\theta(j+1)} \binom{\frac{1}{c}+i}{i}\binom{ci}{j}\binom{\theta(j+1)-1}{k}\beta^{**} \tag{18}$$

Where $\Gamma(p,y) = \int_y^\infty v^{p-1}e^{-v}dv$ is the incomplete gamma function, and

$$\beta^{**} = (-1)^{i+j+k}\lambda^{-q}(k+1)^{-(q+1)}\Gamma\left(q+1; {}^{t}\!/_{\lambda(k+1)}\right)$$

3.3 Mean Residual Life and Mean Inactivity Time

The Mean Residual life (MRL) or the life expectancy at age t is the useful additional life length for a unit, which is alive at age t. It has several life applications min life testing, insurance, product quality control, economics and demography, and many others. The MRL is given by

$$\mathfrak{M}_X(t) = \frac{\mu - \rho_1(t)}{R(t)} - t,$$

Where $\mu = \mu_1', \rho_1(t)$ is the first incomplete moment and $R(t)$ is the reliability function. Thus the MRL of $HEGE$ distribution is:

$$\mathfrak{M}_X(t) = \frac{\mu - \varphi^{1/c}\theta \sum_{i,j=0}^{\infty} \sum_{k=0}^{\theta(j+1)} \binom{\frac{1}{c}+i}{i} \binom{ci}{j} \binom{\theta(j+1)-1}{k} (-1)^{i+j+k} \lambda^{-1} H^u}{R(t)} - t, \tag{19}$$

Where

$$H^u = (k+1)^{-2}\Gamma\left(2; {t}/{\lambda(k+1)}\right)$$

The Mean Inactivity Time (MIT) is a useful measure used in predicting the actual time at which the failure of the device occurs. The MIT of the HEGE random variable X is defined for $t > 0$ as:

$$\psi_X(t) = t - \frac{\rho_1(t)}{F_{HEGE}(t)}. \tag{20}$$

Substituting the first incomplete moment and the CDF of the $HEGE$ random variable into (20) gives its MIT as

$$\psi_X(t) = t - \frac{\varphi^{1/c}\theta \sum_{i,j=0}^{\infty} \sum_{k=0}^{\theta(j+1)} \binom{\frac{1}{c}+i}{i} \binom{ci}{j} \binom{\theta(j+1)-1}{k} (-1)^{i+j+k} \Gamma\left(2; {t}/{\lambda(k+1)}\right)}{(k+1)^2 \lambda F_{HEGE}(t)} \tag{21}$$

3.4 Entropy of $\boldsymbol{HEGE}$ *Distribution*

Entropy has been used in life sciences as a measure of variation of uncertainty of a system. According to Renyi (1961). The Renyi entropy is given as:

$$I_R(\gamma) = \frac{1}{1-\gamma} log[M^\gamma] \tag{22}$$

Where

$$M^\gamma = \int_0^\infty f^\gamma dx, \qquad \gamma > 0, and\ \gamma \neq 0 \tag{23}$$

Plugging (8) into (23), follow by algebraic manipulation result to

$$M^\gamma = \varphi^{\gamma/c} \lambda^{\gamma-1} \theta^\gamma \sum_{i,j,k=0}^{\infty} \binom{\gamma(1+1/c}{i} \binom{ic}{j} \binom{\theta(\gamma+j}{k} (-1)^{j+k} k^{-1}$$

Finally, we have

$$I_R(\gamma) = \frac{1}{1-\gamma} log\left[\varphi^{\gamma/c} \lambda^{\gamma-1} \theta^\gamma \sum_{i,j,k=0}^{\infty} \binom{\gamma(1+1/c}{i} \binom{ic}{j} \binom{\theta(\gamma+j}{k} (-1)^{j+k} k^{-1}\right] \tag{24}$$

3.5 Order Statistics of HEGE Model

Suppose $x_1, x_2, x_3, \ldots, x_p$ represent a random sample of size p from a distribution with PDF $f(x)$ and CDF $F(x)$ and $x_{1:P}, x_{2:P}, x_{3:P}, \ldots, x_{p:p}$ are the analogous order statistics.

The PDF and CDF of $x_{r:p}$, $1 \leqslant r \leqslant p$, are

$$f_{r:p}(x) = \frac{f(x)}{B(r, p-r+1)} [G(x)]^{r-1} [1-G(x)]^{p-r} \tag{25}$$

$$= \frac{f(x)}{B(r, p-r+1)} \sum_{i=0}^{p-r} \binom{p-r}{i} (-1)^i [G(x)]^{i+r-1}$$

Where, $B(r, p-r+1)$ is a beta function

Putting (8) and (25) in the expression above, follow by algebraic manipulation, we have

4. Maximum Likelihood Estimation

Let $x_1, x_2, x_3, \dots, x_n$ be a random sample of size n from $HEGE(\lambda, \theta, \varphi, c)$ distribution. The log-likelihood function is given by

$$l(\zeta) = logL = log\left[\prod_{i=1}^{n} f(x)\right]$$

$$l(\zeta) = {}^{n}/_{c}\, log(\varphi) + nlog(\lambda) + nlog(\theta) - \lambda\sum_{i=1}^{n} x_i + \theta\sum_{i=1}^{n} log\left(1 - e^{-\lambda x_i}\right)$$

$$-\left(1 + {}^{1}/_{c}\right)\sum_{i=1}^{n} log\left[1 - \hat{\varphi}\left[1 - \left(1 - e^{-\lambda x_i}\right)^{\theta}\right]^{c}\right] \tag{26}$$

The score vector is given by

$$V_n(\zeta) = \left(V_1(\zeta), V_2(\zeta), V_3(\zeta), V_4(\zeta)\right)^T = \left(\frac{\partial l(\zeta)}{\partial\lambda}, \frac{\partial l(\zeta)}{\partial\theta}, \frac{\partial l(\zeta)}{\partial\varphi}, \frac{\partial l(\zeta)}{\partial c}\right)^T$$

Contain the elements

$$V_1(\zeta) = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \theta\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{(1 - e^{-\lambda x_i})} - \left(1 + {}^{1}/_{c}\right)\sum_{i=1}^{n}\frac{\theta\hat{\varphi}e^{-\lambda x_i}\left(1 - e^{-\lambda x_i}\right)^{\theta-1}}{[1 - \hat{\varphi}[1 - (1 - e^{-\lambda x_i})^{\theta}]^{c}]} \tag{27}$$

$$V_2(\zeta) = \frac{n}{\theta} + \sum_{i=1}^{n} log\left(1 - e^{-\lambda x_i}\right) - \left(1 + {}^{1}/_{c}\right)\sum_{i=1}^{n}\frac{\left(1 - e^{-\lambda x_i}\right)^{\theta} log\left(1 - e^{-\lambda x_i}\right)}{[1 - \hat{\varphi}[1 - (1 - e^{-\lambda x_i})^{\theta}]^{c}]} \tag{28}$$

$$V_3(\zeta) = \frac{n}{c\varphi} - \left(1 + {}^{1}/_{c}\right)\sum_{i=1}^{n}\frac{\left[1 - \left(1 - e^{-\lambda x_i}\right)^{\theta}\right]^{c}}{[1 - \hat{\varphi}[1 - (1 - e^{-\lambda x_i})^{\theta}]^{c}]} \tag{29}$$

$$V_4(\zeta) = -{}^{n}/_{c^2}\, log(\varphi) + {}^{1}/_{c^2}\sum_{i=1}^{n} log\left[1 - \hat{\varphi}\left[1 - \left(1 - e^{-\lambda x_i}\right)^{\theta}\right]^{c}\right]$$

$$-\left(1 + {}^{1}/_{c}\right)\sum_{i=1}^{n}\frac{\hat{\varphi}\left[1 - \left(1 - e^{-\lambda x_i}\right)^{\theta}\right]^{c} log\left[1 - \left(1 - e^{-\lambda x_i}\right)^{\theta}\right]}{[1 - \hat{\varphi}[1 - (1 - e^{-\lambda x_i})^{\theta}]^{c}]} \tag{30}$$

The MLE $\hat{\zeta}$ of ζ can be estimated by solving the system of non-linear equations $V_n(\zeta) = 0$ numerically. For interval estimation and tests of hypotheses on ζ, we require the observed information matrix of a random sample of size n from the $HEGE$ distribution, given by

$$J_n(\zeta) = -\begin{bmatrix} \frac{\partial^2 l(\zeta)}{\partial\lambda^2} & \frac{\partial^2 l(\zeta)}{\partial\lambda\partial\theta} & \frac{\partial^2 l(\zeta)}{\partial\lambda\partial\varphi} & \frac{\partial^2 l(\zeta)}{\partial\lambda\partial c} \\ \frac{\partial^2 l(\zeta)}{\partial\theta\partial\lambda} & \frac{\partial^2 l(\zeta)}{\partial\theta^2} & \frac{\partial^2 l(\zeta)}{\partial\theta\partial\varphi} & \frac{\partial^2 l(\zeta)}{\partial\theta\partial c} \\ \frac{\partial^2 l(\zeta)}{\partial\varphi\partial\lambda} & \frac{\partial^2 l(\zeta)}{\partial\varphi\partial\theta} & \frac{\partial^2 l(\zeta)}{\partial\varphi^2} & \frac{\partial^2 l(\zeta)}{\partial\varphi\partial c} \\ \frac{\partial^2 l(\zeta)}{\partial c\partial\lambda} & \frac{\partial^2 l(\zeta)}{\partial c\partial\theta} & \frac{\partial^2 l(\zeta)}{\partial c\partial\varphi} & \frac{\partial^2 l(\zeta)}{\partial c^2} \end{bmatrix}$$

Under mild regularity conditions, it could be observed the asymptotic distribution of the MLE $\hat{\zeta}$ is multivariate normal distribution with mean θ and variance covariance matrix $J_n(\zeta)$**.**

This calculated multivariate normal distribution can be used to construct approximate confidence intervals for the parameters and to test hypotheses about these parameters.

4.1 Simulation Study

Simulation study was performed for average MLEs, Mean Square Error (MSE) and Absolute Bias (AB). $M = 1000$ samples of size $n = 50,100,200,300, and\ 400$ were obtained from HEGE distribution. Random numbers were generated using the following expression

$$X = -\frac{1}{\lambda} log\left\{1 - \left[1 - \frac{(1-u)}{[(1-u)\bar{\varphi} + \varphi]^{1/c}}\right]^{1/\theta}\right\} \tag{31}$$

Where U represents the uniform random numbers with parameter $[0, 1]$. AB and MSE are calculated by

$$AB = \left|\frac{1}{N}\sum_{I=1}^{n}(\hat{f} - f)\right|$$

$$MSE = \frac{1}{N}\sum_{I=1}^{n}(\hat{f} - f)^2$$

Where $f = (\lambda, \varphi, \theta, c)$. Simulation results were obtained for different values of $\lambda, \varphi, \theta,$ and c. The average values of MSEs and AB are shown in Table 3. It is clearly shown that this estimates are and the values of the MSE decay to zero as the sample size increases. Therefore, it can be concluded that MLE process performs quite well in estimating the parameters of HEGE distribution.

Table 3. $\boldsymbol{AB}$, Variance, and $\boldsymbol{MSE}$ for the $\boldsymbol{HEGE}$ parameters

Sample size (n)	Parameters	AB	$Variance$	MSE
$n = 50$	c	1.1088	2.3722	3.6017
	φ	1.5537	0.0562	2.4702
	λ	0.4382	0.2708	0.4628
	θ	0.6650	0.4037	0.8459
$n = 100$	c	0.7052	1.8753	2.3726
	φ	1.5506	0.0494	2.4537
	λ	0.4284	0.3471	0.3040
	θ	0.4187	0.2816	0.4570
$n = 200$	c	1.0477	0.9756	2.0732
	φ	1.5546	0.0140	2.4308
	λ	0.4362	0.0672	0.2575
	θ	0.2832	0.0830	0.1632
$n = 300$	c	0.8329	1.2787	1.9724
	φ	1.4173	0.0306	2.0393
	λ	0.2404	0.0524	0.1102
	θ	0.1934	0.0568	0.1054
$n = 400$	c	0.6572	1.3941	1.8260
	φ	1.3150	0.0448	1.7741
	λ	0.1660	0.0463	0.0738
	θ	0.1692	0.0474	0.0760

4.2 Applications

We compare our proposed Harris Extended Generalized Exponential (HEGE) model with other competing models, namely, Marshall-Olkin Generalized Exponential distribution, see (Gupta and Waleed 2018), and Weibull-generalized Poisson (WGP), see (Gupta and Huang 2014). We also consider the three-parameter versions when $\theta = 1$, when the base distribution is exponential.

For model selection, we use Akaike information criterion (AIC) and Bayesian information criterion (BIC) where $AIC = 2z - 2\hat{l}$ and $BIC = 2\,log(n) - 2\hat{l}$ where z is the number of parameters in the model and $\hat{l}$ is the estimated log-likelihood function of this model. The model with the smallest AIC, BIC, CAIC, HQIC is considered the best model in the class of models considered.

4.3 Data Set I

The real data mentioned here are studied in Fuller et al. (1994), and they give the strength of glass for a sample of thirty-one aircraft windows. These data are listed below.

18.83, 20.8, 21.65, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.8, 26.69, 26.77, 26.78, 27.05, 27.67, 29.9, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

Table 4 shows the MLEs of the parameters of the various competing models, their estimated log-likelihood, AIC and BIC. This table also shows that the $HEGE$ model has the smallest AIC and BIC.

Data set II represents the actual data introduced here are offered by Mahmoud and Mandouh (2013) and they represents the simulated strengths for a sample of 63 glass fibers. These data are given as

1.014, 1.081, 1.082, 1.185, 1.223, 1.248, 1.267, 1.271, 1.748,1.288, 1.292, 1.304, 1.306, 1.355, 1.361, 1.364, 1.379, 3.197,1.481, 1.484, 1.501, 1.506, 1.524, 1.526, 1.535, 1.541, 1.276,1.602, 1.666, 1.670, 1.684, 1.691, 1.704, 1.731, 1.735, 1.459,1.867, 1.876, 1.878, 1.910, 1.916, 1.972, 2.012, 2.456, 1.581,1.278, 1.460, 1.591, 1.800, 1.286, 1.476, 1.593, 1.806, 1.757, 1.272, 1.409, 1.568, 1.747, 2.592, 1.275, 1.426, 1.579, 4.121. Figure 3 is the graph of Total Test on Time (TTT) plot for data set I and II, and the graph of the empirical distribution is given in Figure 4.

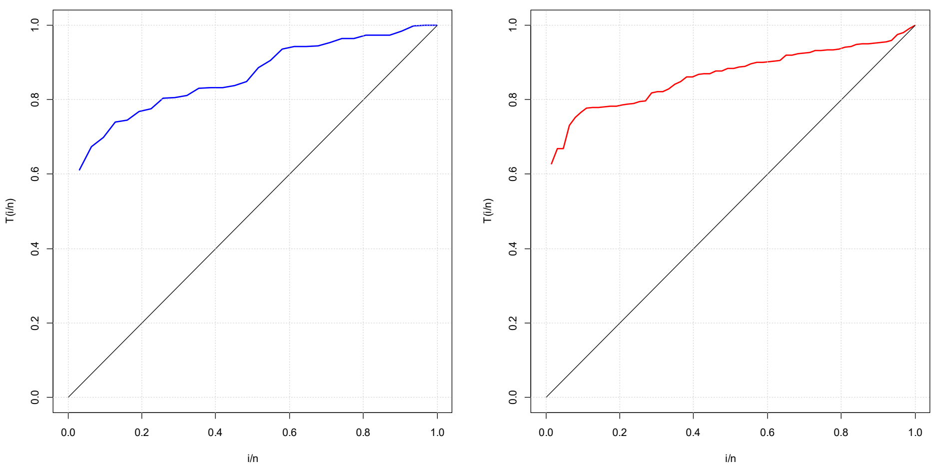

Figure 3. TTT plot for data sets I and II

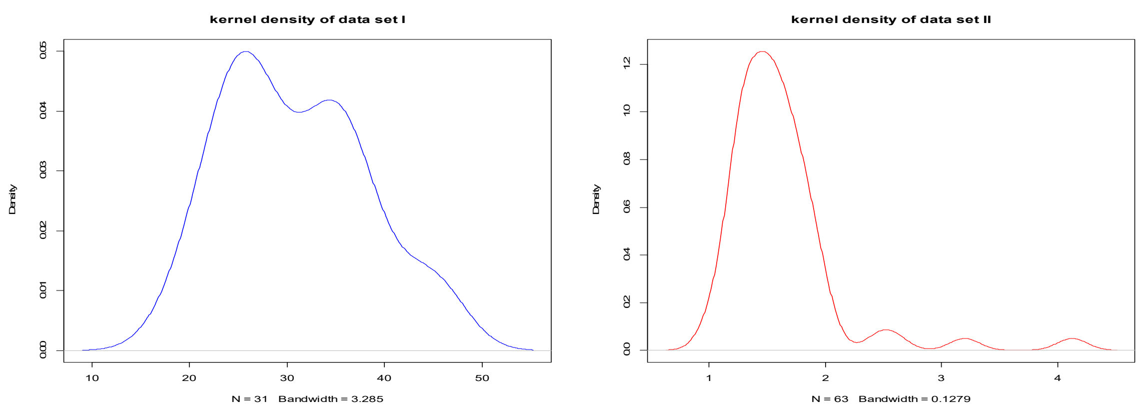

Figure 4. Graph of the empirical density function of data sets I and II

Table 4. MLEs and SEs (in Parenthesis), for data set I

$Model$	c	φ	λ	θ
$HEGE$	-0.1121 (0.3596)	0.0761 (0.4333)	0.1566 (0.0384)	5.0976 (2.1734)
$MOEE$	$-$ $(-)$	0.0044 (0.0032)	0.2276 (0.0300)	4.3376 (4.5712)
MOE	$-$ $(-)$	$-$ $(-)$	-12.5020 (4.8136)	-0.0027 (0.0008)
EE	$-$ $(-)$	$-$ $(-)$	0.0812 (0.0105)	6.8108 (1.8838)
E	$-$ $(-)$	$-$ $(-)$	$-$ $(-)$	0.0325 (0.0058)

Table 5. The AIC, BIC, CAIC, and HQIC for data set ***I***

$Models$	L	AIC	BIC	$HQIC$	$CAIC$
$HEGE$	104.093	216.186	221.922	218.056	217.724
$MOEGE$	105.776	217.552	221.854	218.954	218.441
GE	114.261	232.521	235.389	233.456	232.950
$MOEE$	150.110	304.220	307.088	305.155	304.649
E	137.264	276.529	277.963	276.996	276.667

The estimated variance-covariance matrix for the parameters of the HEGE distribution for the data set I is:

$$[J(\zeta)]^{-1} = \begin{pmatrix} 1.2421e-06 & 7.7253e-09 & -1.2414e-08 & -1.3860e-07 \\ 7.7253e-09 & 3.1534e-06 & 2.7643e-08 & -8.6499e-10 \\ -1.2414e-08 & 2.76436e-08 & 3.2788e-06 & 1.3899e-09 \\ -1.3860e-07 & -8.6499e-10 & 1.3899e-09 & 2.8502e-10 \end{pmatrix}$$

Table 6. MLEs and SEs (in Parenthesis), for data set II

$\boldsymbol{Model}$	c	φ	λ	θ
$HEGE$	-0.2233 (0.1745)	0.0861 (0.2803)	3.1111 (0.4991)	6.5122 (2.0031)
$MOGEE$	$-$ $(-)$	0.0098 (0.0116)	4.4085 (0.4635)	8.9747 (10.8038)
EE	$-$ $(-)$	$-$ $(-)$	2.6373 (0.2239)	43.3268 (13.5764)
MOE	$-$ $(-)$	0.0020 (0.0002)	$-$ $(-)$	3.9865 (0.1435)
E	$-$ $(-)$	$-$ $(-)$	$-$ $(-)$	0.6189 (0.0780)

Table 7. The AIC, BIC, CAIC, and HQIC for data set ***I***

$Models$	l	AIC	BIC	$HQIC$	$CAIC$
$HEGE$	21.912	51.825	60.399	55.197	52.515
$MOEE$	30.499	66.999	73.428	69.528	67.406
GE	26.356	56.712	60.998	58.397	56.912
MOE	31.870	67.739	72.026	69.425	67.939
E	93.223	188.446	190.589	189.289	188.511

The estimated variance-covariance matrix for the parameters of the HEGE distribution for the data set II is:

$$[J(\zeta)]^{-1} = \begin{pmatrix} 1.7475e-05 & 4.2959e-05 & -1.9229e-06 & -3.5757e-07 \\ 4.2959e-05 & 1.8306e-02 & 6.2554e-05 & -8.8387e-07 \\ -1.9229e-06 & 6.2554e-05 & 1.6807e-04 & 3.9557e-08 \\ -3.5756e-07 & -8.8387e-07 & 3.9557e-08 & 1.7563e-09 \end{pmatrix}$$

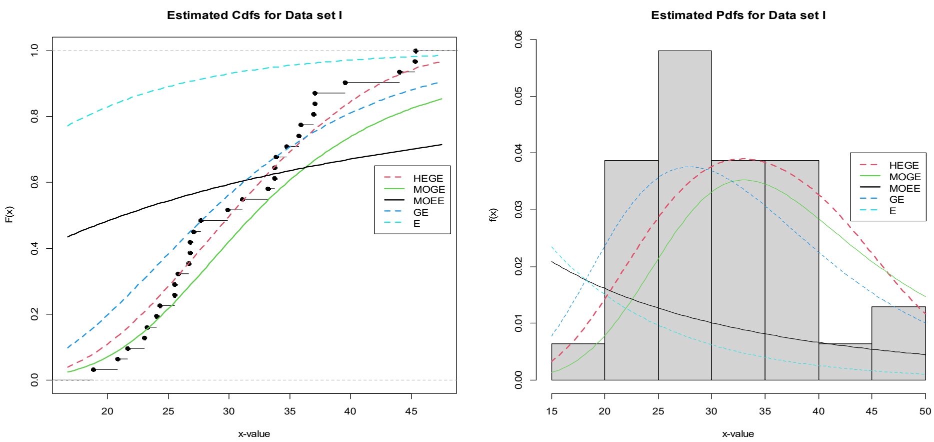

Figure 5. Graph of fitted densities for data set I

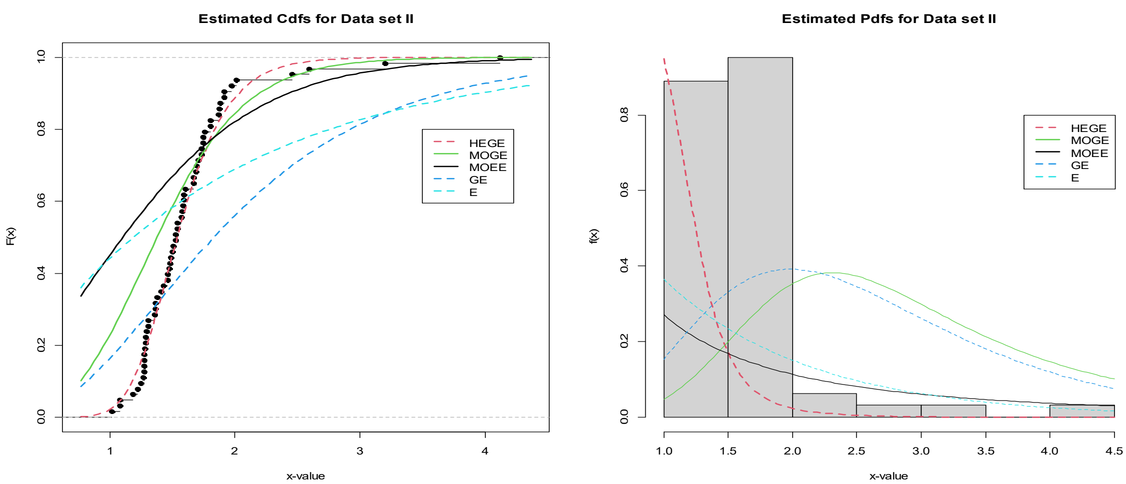

Figure 6. Graph of fitted densities for data set II

5. Concluding Remarks

The Harris Extended Generalized Exponential Distribution, which extends the exponential distribution to model the complex behavior encountered in the analysis of real-life data, is proposed in this study. Real-life data modeling is flexible and more tractable with the proposed distribution. Moments, incomplete moments, moment generating function, quantile function, and order statistics for some of the distributional properties described here. The maximum likelihood estimation technique is applied to estimate the model parameters, also obtain its information matrix. The reliability behavior, as well as the distributions of various order statistics, are also discussed. The likelihood ratio statistic is applied to compare the proposed model with its base model. In real-world applications, the Harris Extended Generalized Exponential distribution outperforms all other models considered in this study.

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