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A Note on Star-Prüfer Extensions

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Abstract

Let $\star$ be a star operation on a ring extension $R \subseteq S$. The ring extension $R \subseteq S$ is said to be a $\star$*-Prüfer* if $R_{[\mathfrak{p}]} \subseteq S$ is a Prüfer extension for each $\star$-prime ideal $\mathfrak{p}$ of R. We study properties of $\star$-Prüfer extensions. In particular, we investigate the transfer of star-Prüfer properties from the extension $R \subseteq S$ to the extension $R[X] \subseteq S[X]$ of polynomial rings, where X is an indeterminate over S.

Keywords: star operation, ring extension, Prüfer extension, $\star$-Prüfer

MSC Classification: 13A15; 13A18; 13B02

1. Introduction and Background

Throughout this article, we assume that all rings are commutative with identity. Let $R \subseteq S$ be a ring extension, and let A be an R-submodule of S. The R-submodule A is said to be *S-regular* if $AS = S$ (Knebusch & Zhang, 2002, Definition 1 in Chapter II, p. 84). The R-submodule A of S is called *S-invertible*, if there exists an R-submodule B of S such that $AB = R$ (Knebusch & Zhang, 2002, Definition 3 in Chapter II, p. 90). In this case, we write $B = A^{-1}$, and $A^{-1} = [R :_S A] = \{x \in S : xA \subseteq R\}$(Knebusch & Zhang, 2002, Remarks 1.10 in Chapter II, p. 90).

Let S be a ring, and let Γ be an additive totally ordered abelian group. Let $\Gamma \cup \infty = \Gamma \cup \{\infty\}$, where $\infty + g = g + \infty = \infty$ and $g < \infty$ for all $g \in \Gamma$. A *valuation* on S with values in Γ is a map $v : S \longrightarrow \Gamma \cup \infty$ such that:

(1) $v(xy) = v(x) + v(y)$ for all $x, y \in S$.

(2) $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in S$.

(3) $v(1) = 0$ and $v(0) = \infty$.

In this case, $V = \{x \in S : v(x) \geq 0\}$ is called a *valuation subring* of S. If $v(S) = \{0, \infty\}$, then v is said to be *trivial*, otherwise v is called *non-trivial* (Knebusch & Zhang, 2002, Definition 1, p. 10). The subgroup of Γ generated by $v(S) \setminus \{\infty\}$ is called the *value group* of v and is denoted by Γ_v. If $v(S) = \Gamma_v \cup \infty$, then v is called a *Manis valuation* on S, and $V = \{x \in S : v(x) \geq 0\}$ is called a *Manis subring* of S (Knebusch & Zhang, 2002, Definition 4, p. 12).

Let S be a ring, and let R be a subring of S. If there exists a Manis valuation $v : S \longrightarrow \Gamma \cup \infty$ such that $R = \{x \in S : v(x) \geq 0\}$, then the ring extension $R \subseteq S$ is called a *Manis extension*. In this case, $(R, \mathfrak{p})$ is called a *Manis pair* in S, where $\mathfrak{p} = \{x \in S : v(x) > 0\}$. For each R-submodule M of S, and for each multiplicative subset τ of R, we denote by $M_{[\tau]}$ the set of $x \in S$ such that $tx \in M$ for some $t \in \tau$. If $\mathfrak{p}$ is a prime ideal of R, and $\tau = R \setminus \mathfrak{p}$, then $M_{[\mathfrak{p}]}$ denotes the set of $x \in S$ such that $tx \in M$ for some $t \in \tau$.

Let $R \subseteq S$ be a ring extension. The ring S is called a *Prüfer extension* of R if $(R_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is a Manis pair in S for every maximal ideal $\mathfrak{p}$ of R. In this case, we say that R is Prüfer in S. The ring extension $R \subseteq S$ is said to be *tight* if for every $x \in S \setminus R$, there exists an S-invertible ideal I of R such that $Ix \subseteq R$ (Knebusch & Zhang, 2002, Definition 1, p. 94). More on Manis valuations and Prüfer extensions can be found in Knebusch and Zhang (2002).

Lemma 1.1. *Let $R \subseteq S$ be a ring extension, and let $\mathfrak{p}$ be a prime ideal of R. If $R_{[\mathfrak{p}]} \subseteq S$ is a Prüfer extension, then $\left(R_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]}\right)$ is a Manis pair in S.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R. Suppose that $R_{[\mathfrak{p}]} \subseteq S$ is a Prüfer extension. By (Paudel & Tchamna, 2021b, Remark 2.1(1)), $\mathfrak{p}_{[\mathfrak{p}]}$ is a prime ideal of $R_{[\mathfrak{p}]}$. Hence, by the definition of a Prüfer extension, $\left(\left(R_{[\mathfrak{p}]}\right)_{[\mathfrak{p}_{[\mathfrak{p}]}]}, \left(\mathfrak{p}_{[\mathfrak{p}]}\right)_{[\mathfrak{p}_{[\mathfrak{p}]}]}\right)$ is a Manis pair in S. But by (Knebusch & Zhang, 2002, Lemma 2.9 (c), p. 28), we have

$$(R_{[\mathfrak{p}]})_{[\mathfrak{p}_{[\mathfrak{p}]}]} = R_{[\mathfrak{p}]}$$

and

$$(\mathfrak{p}_{[\mathfrak{p}]})_{[\mathfrak{p}_{[\mathfrak{p}]}]} = \mathfrak{p}_{[\mathfrak{p}]}.$$

It follows that $\left(R_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]}\right)$ is a Manis pair in S. □

Remark 1.2. Let $R \subseteq S$ be a ring extension. For any prime ideal $\mathfrak{p}$ of R, if $A_{\mathfrak{p}} = B_{\mathfrak{p}}$, then $A_{[\mathfrak{p}]} = B_{[\mathfrak{p}]}$. In fact, if $x \in A_{[\mathfrak{p}]}$. Then there exists $t \in R \setminus \mathfrak{p}$ and $a \in A$ such that $tx = a$. Furthermore, we have $\frac{a}{t} \in A_{\mathfrak{p}} = B_{\mathfrak{p}}$. Hence $\frac{a}{t} = \frac{b}{t'}$ with $b \in B$ and $t' \in R \setminus \mathfrak{p}$. Thus $s(at') = s(bt)$ for some $s \in R \setminus \mathfrak{p}$. It follows that $s(tx)t' = (st)b$. Hence $(stt')x \in B$. This shows that $x \in B_{[\mathfrak{p}]}$ since $stt' \in R \setminus \mathfrak{p}$. Thus $A_{[\mathfrak{p}]} \subseteq B_{[\mathfrak{p}]}$. Similarly, $B_{[\mathfrak{p}]} \subseteq A_{[\mathfrak{p}]}$. Hence $A_{[\mathfrak{p}]} = B_{[\mathfrak{p}]}$.

For a ring extension $R \subseteq S$, a map $\star$: $\mathcal{J}(R, S) \longrightarrow \mathcal{J}(R, S)$, where $\mathcal{J}(R, S)$ is the set of all R-submodules of S, is called *star operation on* $R \subseteq S$ if the following conditions are satisfied for all $A, B \in \mathcal{J}(R, S)$.

(c_1) $A \subseteq A^\star$.

(c_2) If $A \subseteq B$, then $A^\star \subseteq B^\star$.

(c_3) $(A^\star)^\star = A^\star$.

(c_4) $AB^\star \subseteq (AB)^\star$.

When $R^\star = R$, the star operation $\star$ is said to be *strict*. For more on star operations of ring extensions, see (Knebusch & Kaiser, 2014, pages 139 - 164).
A star operation $\star$ on a ring extension $R \subseteq S$ is said to be of finite type if for each R-submodule A of R, $A^\star = \bigcup K^\star$, where K ranges over all the finitely generated R-submodules of S contained in A (Knebusch & Kaiser, 2014, Definition 1 in Chapter 3, p. 156).

Remark 1.3. *((Knebusch & Kaiser, 2014, Proposition 6.3 in Chapter 3, p. 156))* For a star operation $\star : \mathcal{J}(R, S) \longrightarrow \mathcal{J}(R, S)$, and each R-submodule A of S, define $A^{\star_f} = \bigcup K^\star$, where K ranges over all the finitely generated R-submodules of S contained in A. Then the map $\star_f : \mathcal{J}(R, S) \longrightarrow \mathcal{J}(R, S)$ defined by $A \longmapsto A^{\star_f}$ is a star operation of finite type on the extension $R \subseteq S$.

Let I be an ideal of R. The ideal I is said to be a *$\star$-ideal if $I^\star = I$*. Following the terminology used in **?**, we call an ideal I of R a *$\star$-prime ideal* if I is both a $\star$-ideal and a prime ideal of R. A maximal element in the set of all $\star$-ideals of R is called *$\star$-maximal ideal*.

Lemma 1.4. *((Tchamna, 2020, Remark 2.4)) Let $R \subseteq S$ be a ring extension, and let $\star$ be a strict star operation of finite type on $R \subseteq S$. Then each proper $\star$-ideal of R is contained in a $\star$-prime ideal (which is also a $\star$-maximal ideal).*

Definition 1.5. *Let $R \subseteq S$ be a ring extension, and let $\star$ be a star operation on $R \subseteq S$. The ring extension $R \subseteq S$ is said to be a $\star$-*Prüfer extension *if $R_{[\mathfrak{p}]} \subseteq S$ is a Prüfer extension for each $\star$-prime ideal $\mathfrak{p}$ of R.*

In the next section, we study properties of $\star$-Prüfer extensions. In Proposition 2.7, we show that if $R \subseteq S$ is a $\star$-Prüfer extension, then each finitely generated S-regular $\star$-submodule of S is $\star$-invertible. We show that any $\star$-Prüfer extension is a Prüfer $\star$-multiplication extension (see Proposition 2.8).

In Paudel and Tchamna (2018), the notion of Prüfer $\star$-multiplication extension is defined and some properties are established. Let $\star : \mathcal{J}(R, S) \longrightarrow \mathcal{J}(R, S)$ be a star operation. The extension $R \subseteq S$ is called *(weak) Prüfer $\star$-multiplication extension (P$\star$ME)* if the pair $(R_{[\mathfrak{m}]}, \mathfrak{m}_{[\mathfrak{m}]})$ is Manis in S for every (S-regular) $\star$-maximal ideal $\mathfrak{m}$ of R. In Remark 2.9, we give a condition under which the notion of $\star$-Prüfer extension and P$\star$ME coincide.

In Example 2.10, we show that a $\star$-Prüfer extension is not always a Prüfer extension. In Example 2.12, we construct a ring extension to show that there exist $\star$-Prüfer extensions which are not Manis extensions. In Example 2.13, we give an example of a ring extension $R \subseteq S$ and a star operation $\star$ on $R \subseteq S$ such that the ring extension $R \subseteq S$ is not $\star$-Prüfer. In Theorem 2.16 and Theorem 2.18, we study $\star$-Prüfer extensions in commutative diagrams. In Theorem 2.24, we show that if the extension $R[X] \subseteq S[X]$ (respectively $R[X]_{[N(\star_1)]} \subseteq S[X]$) is $\star_2$-Prüfer, then $R \subseteq S$ is a $\star_1$-Prüfer extension, where $\star_1$ and $\star_2$ are two star operations on $R \subseteq S$ and $R[X] \subseteq S[X]$ respectively satisfying $A^{\star_1}R[X] = (AR[X])^{\star_2}$ for each R-submodule A of S.

2. Properties of $\star$-Prüfer Extensions

We start this section by a remark which shows that in Definition 1.5, when $\star$ is the identity, we get a Prüfer extension. This shows that the notion of $\star$-Prüfer extension is a generalization of the notion of Prüfer extension.

Remark 2.1. When the star operation in Definition 1.5 is the identity $d : \mathcal{J}(R, S) \longrightarrow \mathcal{J}(R, S)$, the d-Prüfer extension coincides with the definition of a Prüfer extension. In fact, if $R_{[\mathfrak{p}]} \subseteq S$ is a Prüfer extension for each prime ideal $\mathfrak{p}$ of R, then by Lemma 1.1, $\left(R_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]}\right)$ is a Manis pair in S. It follows from the definition of a Prüfer extension that $R \subseteq S$ is a Prüfer extension. Conversely, if $R \subseteq S$ is a Prüfer extension, then $R_{[\mathfrak{p}]} \subseteq S$ is a Prüfer extension. This is a direct consequence of (Knebusch & Zhang, 2002, Corollary 5.3, p. 50) and the fact that $R \subseteq R_{[\mathfrak{p}]} \subseteq S$ for each prime ideal $\mathfrak{p}$ of R.

Remark 2.2. Let $R \subseteq S$ be a ring extension, and let $\star$ be a strict star operation of finite type on $R \subseteq S$. The ring extension $R \subseteq S$ is $\star$-Prüfer if and only if $R_{[\mathfrak{m}]} \subseteq S$ is a Prüfer extension for each $\star$-maximal ideal $\mathfrak{m}$ of R.

Proof. Suppose that $R \subseteq S$ is a $\star$-Prüfer extension. Then by definition, $R_{[\mathfrak{m}]} \subseteq S$ is a Prüfer extension for each $\star$-maximal ideal $\mathfrak{m}$ of R, since each $\star$-maximal ideal of R is also a $\star$-prime ideal of R. Conversely, suppose that $R_{[\mathfrak{m}]} \subseteq S$ is a Prüfer extension for each $\star$-maximal ideal $\mathfrak{m}$ of R. Let $\mathfrak{p}$ be a $\star$-prime ideal of R. By Lemma 1.4, there exists a $\star$-maximal ideal $\mathfrak{n}$ of R containing $\mathfrak{p}$. Furthermore, $R_{[\mathfrak{n}]} \subseteq R_{[\mathfrak{p}]} \subseteq S$. But by hypothesis, the extension $R_{[\mathfrak{n}]} \subseteq S$ is Prüfer. It follows from (Knebusch & Zhang, 2002, Corollary 5.3, p. 50) that the extension $R_{[\mathfrak{p}]} \subseteq S$ is Prüfer. This shows that the extension $R \subseteq S$ is $\star$-Prüfer. □

A ring extension $R \subseteq S$ is said to be *weakly surjective* if for each prime ideal $\mathfrak{p}$ of R such that $\mathfrak{p}S \neq S$, the homomorphism $i_{\mathfrak{p}} : R_{\mathfrak{p}} \to S_{\mathfrak{p}}$ induced by the inclusion $i : R \hookrightarrow S$ is surjective (Knebusch & Zhang, 2002, Definition 1, p. 32).

Proposition 2.3. *Let $R \subseteq L \subseteq S$ be ring extensions such that $R \subseteq L$ is a weakly surjective extension with $\mathfrak{p}L \neq L$ for each prime ideal $\mathfrak{p}$ of R. Let $\star_1$ be a star operation on $R \subseteq S$, and $\star_2$ a star operation on $L \subseteq S$ such that $A^{\star_1}L = (AL)^{\star_2}$ for each R-submodule A of S. The ring extension $R \subseteq S$ is $\star_1$-Prüfer if and only if the ring extension $L \subseteq S$ is $\star_2$-Prüfer.*

Proof. Suppose that $R \subseteq S$ is $\star_1$-Prüfer. Let $\mathfrak{q}$ be a $\star_2$-prime ideal of L, and let $\mathfrak{p} = \mathfrak{q} \cap R$. Then

$$\begin{aligned} \mathfrak{p}^{\star_1} = (\mathfrak{q} \cap R)^{\star_1} &\subseteq (\mathfrak{q} \cap R)^{\star_1} L \\ &= ((\mathfrak{q} \cap R)L)^{\star_2} \text{ (by hypothesis)} \\ &\subseteq (\mathfrak{q}L)^{\star_2} \\ &= \mathfrak{q}^{\star_2} = \mathfrak{q}. \end{aligned}$$

Hence $\mathfrak{p}^{\star_1} \subseteq \mathfrak{q}$. On the other hand, we have $\mathfrak{p}^{\star_1} = (\mathfrak{q} \cap R)^{\star_1} \subseteq R^{\star_1} = R$. This shows that $\mathfrak{p}^{\star_1} \subseteq \mathfrak{q} \cap R = \mathfrak{p}$. Thus $\mathfrak{p}$ is a $\star_1$-prime ideal of R. It follows from the hypothesis that $R_{[\mathfrak{p}]} \subseteq S$ is a Prüfer extension. But by (Knebusch & Zhang, 2002, Theorem 3.13, p. 37), we have $R_{[\mathfrak{p}]} = L_{[\mathfrak{q}]}$. Hence $L_{[\mathfrak{q}]} \subseteq S$ is a Prüfer extension. This shows that the ring extension $L \subseteq S$ is $\star_2$-Prüfer.

Conversely, suppose that $L \subseteq S$ is $\star_2$-Prüfer. Let $\mathfrak{p}$ be a $\star_1$-prime ideal of R, and let $\mathfrak{q} = \mathfrak{p}L$. We have $(\mathfrak{p}L)^{\star_2} = \mathfrak{p}^{\star_1}L = \mathfrak{p}L$. Hence $\mathfrak{q} = \mathfrak{p}L$ is a $\star_2$-ideal of L. Furthermore, by hypothesis, $\mathfrak{q} = \mathfrak{p}L \neq L$. It follows from (Knebusch & Zhang, 2002, Theorem 4.8, p. 44) that $\mathfrak{q}$ is a prime ideal of L satisfying $\mathfrak{p} = \mathfrak{q} \cap R$. So, $\mathfrak{q}$ is a $\star_2$-prime ideal of L. Therefore, by hypothesis, the ring extension $L_{[\mathfrak{q}]} \subseteq S$ is Prüfer. But by (Knebusch & Zhang, 2002, Theorem 3.13, p. 37), we have $R_{[\mathfrak{p}]} = L_{[\mathfrak{q}]}$. Thus $R_{[\mathfrak{p}]} \subseteq S$ is a Prüfer extension. This shows that the ring extension $R \subseteq S$ is $\star_2$-Prüfer. □

Lemma 2.4. *((Paudel & Tchamna, 2021b, Lemma 3.2)) Let $\star$ be a strict star operation of finite type on a ring extension $R \subseteq S$, and let A be an R-submodule of S such that $A^\star = A$. Then for any multiplicatively closed subset τ of R, we have $(A_{[\tau]})^\star = A_{[\tau]}$. In particular, $(R_{[\tau]})^\star = R_{[\tau]}$.*

Lemma 2.5. *Let $\star$ be a strict star operation of finite type on a ring extension $R \subseteq S$, and let $\mathcal{P}(\star)$ be the set of all $\star$-prime ideals of R. Let A, B be two R-submodules of S.*

(1) $A^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} A^\star_{[\mathfrak{p}]} = \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} \left(A_{[\mathfrak{p}]}\right)^\star$ *for each R-submodule A of R.*

(2) *If $A_{[\mathfrak{p}]} = B_{[\mathfrak{p}]}$ for each $\star$-prime ideal $\mathfrak{p}$ of R, then $A^\star = B^\star$.*

Proof. (1) The containment $A^\star \subseteq \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} A^\star_{[\mathfrak{p}]}$ is true since the containment $A^\star \subseteq A^\star_{[\mathfrak{p}]}$ is always true for each $\mathfrak{p} \in \mathcal{P}(\star)$. Let $x \in \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} A^\star_{[\mathfrak{p}]}$, and let $\mathfrak{a} = \{t \in R : tx \in A^\star\}$. Then for each $\mathfrak{p}_0 \in \mathcal{P}(\star)$, we have $x \in A^\star_{[\mathfrak{p}_0]}$. So, there exists $t_0 \in R \setminus \mathfrak{p}_0$ such that $t_0x \in A^\star$. Hence $t_0 \in \mathfrak{a} \cap (R \setminus \mathfrak{p}_0)$. This shows that $\mathfrak{a} \cap (R \setminus \mathfrak{p}_0) \neq \emptyset$. It follows from Lemma 1.4 that $\mathfrak{a}^\star = R$. Since $\star$ is a star operation of finite type, there exist $t_1, \ldots, t_\ell \in \mathfrak{a}$ such that $R = (t_1, \ldots, t_\ell)^\star$. Thus $x \in xR = x(t_1, \ldots, t_\ell)^\star \subseteq (t_1x, \ldots, t_\ell x)^\star \subseteq A^\star$. This shows that $\bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} A^\star_{[\mathfrak{p}]} \subseteq A^\star$. Therefore $A^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} A^\star_{[\mathfrak{p}]}$.

Let $y \in \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} \left(A_{[\mathfrak{p}]}\right)^\star$. Then for each $\mathfrak{p}_0 \in \mathcal{P}(\star)$, we have $y \in \left(A_{[\mathfrak{p}_0]}\right)^\star$. Since the star operation $\star$ is of finite type, there exist $x_1, \ldots, x_\ell \in A_{[\mathfrak{p}_0]}$ such that $y \in (x_1, \ldots, x_\ell)^\star$. For $1 \leq i \leq \ell$, let $t_i \in R \setminus \mathfrak{p}_0$ such that $t_ix_i \in A$, and let $t = \prod_{i=1}^{\ell} t_i$. Then $ty \in t(x_1, \ldots, x_\ell)^\star \subseteq (tx_1, \ldots, tx_\ell)^\star \subseteq A^\star$. Thus $y \in A^\star_{[\mathfrak{p}_0]}$. This shows that $y \in \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} A^\star_{[\mathfrak{p}]}$. It follows that $\bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} \left(A_{[\mathfrak{p}]}\right)^\star \subseteq \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} A^\star_{[\mathfrak{p}]} = A^\star$. On the other hand, for each $\mathfrak{p} \in \mathcal{P}(\star)$, we have $A \subseteq A_{[\mathfrak{p}]}$. Thus $A^\star \subseteq \left(A_{[\mathfrak{p}]}\right)^\star$. Hence $A^\star \subseteq \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} \left(A_{[\mathfrak{p}]}\right)^\star$. Therefore, $A^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} A^\star_{[\mathfrak{p}]} = \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} \left(A_{[\mathfrak{p}]}\right)^\star$.

(2) Let A, B be two R-submodules of S such that $A_{[\mathfrak{p}]} = B_{[\mathfrak{p}]}$ for each $\mathfrak{p} \in \mathcal{P}(\star)$. Then $(A_{[\mathfrak{p}]})^\star = (B_{[\mathfrak{p}]})^\star$. It follows from part (1) that $A^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} \left(A_{[\mathfrak{p}]}\right)^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} \left(B_{[\mathfrak{p}]}\right)^\star = B^\star$. □

Let $R \subseteq S$ be a ring extension, and let X be an indeterminate over S. For any $g \in S[X]$, and any ring A between R and S, we denote by $c_A(g)$ the A-submodule of S generated by the coefficients of g. When $A = S$, $c_S(g)$ coincides with the ideal of S generated by the coefficients of g.

Let S be a ring, and let $T = \{g \in S[X] : c_S(g) = S\}$, where X is an indeterminate over S. Then the set T is a multiplicative system in $S[X]$ (Gilmer, 1992, Proposition 33.1, p. 410). Following the same notation in Gilmer (1992), we denote by $S(X)$ the quotient ring $(S[X])_T$.

Proposition 2.6. *Let $R \subseteq S$ be a ring extension. If the ring extension $R \subseteq S$ is $\star$-Prüfer for any strict star operation $\star$ on $R \subseteq S$, then $R \subseteq S$ is a Prüfer extension if and only if $R(X) \subseteq S(X)$ is a Prüfer extension.*

Proof. Suppose there exists a strict star operation on $R \subseteq S$ such that the ring extension $R \subseteq S$ is $\star$-Prüfer. We show that all the hypothesis of (Paudel & Tchamna, 2021a, Theorem 4.9) are satisfied. By Lemma 2.5, we have $R = R^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} R^\star_{[\mathfrak{p}]} = \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} R_{[\mathfrak{p}]}$. Furthermore, by hypothesis, $R_{[\mathfrak{p}]} \subseteq S$ is Prüfer for each $\mathfrak{p} \in \mathcal{P}(\star)$. Therefore, by Lemma 1.1, $\left(R_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]}\right)$ is a Manis pair in S. It follows from the equivalency (1) $\Leftrightarrow$ (4) of (Paudel & Tchamna, 2021a, Theorem 4.9) that $R \subseteq S$ is a Prüfer extension if and only if $R(X) \subseteq S(X)$ is a Prüfer extension. □

Proposition 2.7. *Let $\star$ be a star operation on a ring extension $R \subseteq S$. If $R \subseteq S$ is a $\star$-Prüfer extension, then each finitely generated S-regular $\star$-submodule of S is $\star$-invertible.*

Proof. Suppose that $R \subseteq S$ is a $\star$-Prüfer extension, and let A be an S-regular finitely generated R-submodule of S. Let $\mathfrak{p}$ be a $\star$-prime ideal of R. By (Paudel & Tchamna, 2021b, Remark 2.1(1)), $\mathfrak{p}_{[\mathfrak{p}]}$ is a prime ideal of $R_{[\mathfrak{p}]}$. The ring extension $R_{[\mathfrak{p}]} \subseteq S$ is Prüfer since $R \subseteq S$ is $\star$-Prüfer. If follows from Lemma 1.1 that the ring extension $\left(R_{[\mathfrak{p}]}\right)_{[\mathfrak{p}_{[\mathfrak{p}]}]} \subseteq S$ is Manis. But by (Knebusch & Zhang, 2002, Lemma 2.9, p. 28), we have $\left(R_{[\mathfrak{p}]}\right)_{[\mathfrak{p}_{[\mathfrak{p}]}]} = R_{[\mathfrak{p}]}$. Hence the ring extension $R_{[\mathfrak{p}]} \subseteq S$ is Manis. Thus $R_\mathfrak{p} \subseteq S_\mathfrak{p}$ is a Manis extension (see comments in (Knebusch & Zhang, 2002, p. 13) after the proof of (Knebusch & Zhang, 2002, Proposition 1.3, p. 13)). It follows from (Knebusch & Zhang, 2002, Remark 1.10(e), p. 90) that $A_\mathfrak{p}[R_\mathfrak{p} :_{S_\mathfrak{p}} A_\mathfrak{p}] = R_\mathfrak{p}$. Therefore, by (Knebusch & Zhang, 2002, Lemma 1.1(c)), we have $(A[R :_S A])_\mathfrak{p} = A_\mathfrak{p}[R :_S A]_\mathfrak{p} = R_\mathfrak{p}$ since A is finitely generated. It follows from Remark 1.2 that $(A[R :_S A])_{[\mathfrak{p}]} = R_{[\mathfrak{p}]}$. Thus $\left((A[R :_S A])_{[\mathfrak{p}]}\right)^\star = (R_{[\mathfrak{p}]})^\star$. But by Lemma 2.4, we have $(R_{[\mathfrak{p}]})^\star = R_{[\mathfrak{p}]}$. Thus $\left((A[R :_S A])_{[\mathfrak{p}]}\right)^\star = R_{[\mathfrak{p}]}$. Furthermore, by Lemma 2.5(2), we have $\left((A[R :_S A])_{[\mathfrak{p}]}\right)^\star \subseteq \left((A[R :_S A])^\star\right)_{[\mathfrak{p}]}$. Hence $R_{[\mathfrak{p}]} \subseteq \left((A[R :_S A])^\star\right)_{[\mathfrak{p}]} \subseteq (R^\star)_{[\mathfrak{p}]} = R_{[\mathfrak{p}]}$. This shows that $R_{[\mathfrak{p}]} = \left((A[R :_S A])^\star\right)_{[\mathfrak{p}]}$. It follows from Lemma 2.5(1) that $R = R^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} R_{[\mathfrak{p}]} = \bigcap_{\mathfrak{p}\in\mathcal{P}(\star)} \left((A[R :_S A])^\star\right)_{[\mathfrak{p}]} = (A[R :_S A])^\star$. This shows that A is S-invertible. □

Proposition 2.8. *Let $R \subseteq S$ be a ring extension, and let $\star$ be a star operation on $R \subseteq S$. If $R \subseteq S$ is a $\star$-Prüfer extension, then $R \subseteq S$ is P$\star$ME.*

Proof. Suppose that ring extension $R \subseteq S$ is $\star$-Prüfer. Let $\mathfrak{m}$ be a $\star$-maximal ideal. Then $R_{[\mathfrak{m}]} \subseteq S$ is a Prüfer extension. It follows from Lemma 1.1 that the pair $(R_{[\mathfrak{m}]}, \mathfrak{m}_{[\mathfrak{m}]})$ is Manis in S. This shows that the ring extension $R \subseteq S$ is P$\star$ME. □

Remark 2.9. Let $R \subseteq S$ be a ring extension, and let $Q(R)$ be the total ring of fractions of the ring R. Let $\star$ be a star operation on $R \subseteq S$. If $Q(R) \subseteq S$, then the notions of $\star$-Prüfer and P$\star$ME coincide.

Proof. Suppose that the ring extension $R \subseteq S$ is P$\star$ME. Let $\mathfrak{p}$ be a $\star$-prime ideal of R. Then $\left(R_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]}\right)$ is a Manis pair in S. Since $Q(R) \subseteq S$, it follows from (Paudel & Tchamna, 2018, Remark 3.3) that $R_{[\mathfrak{p}]} \subseteq S$ is a Prüfer extension. This shows that the ring extension $R \subseteq S$ is $\star$-Prüfer. Furthermore, by the previous proposition, if $R \subseteq S$ is $\star$-Prüfer, then $R \subseteq S$ is P$\star$ME. Therefore, the notions of $\star$-Prüfer and P$\star$ME coincide when $Q(R) \subseteq S$. □

It is worth noting that any Prüfer ring extension $R \subseteq S$ is also a $\star$-Prüfer extension for any star operation $\star$ on $R \subseteq S$. In fact, if $\mathfrak{p}$ is a $\star$-prime ideal of R, then $R \subseteq R_{[\mathfrak{p}]} \subseteq S$. So by (Knebusch & Zhang, 2002, Corollary 5.3, P. 50), the extension $R_{[\mathfrak{p}]} \subseteq S$ is Prüfer.

In Example 2.10, we construct a ring extension to show that there exist $\star$-Prüfer extensions which are not Prüfer extensions.

Example 2.10. Let K be a field, X, Y be two indeterminates over K and let $R = K[X, Y]$ and $S = K(X, Y)$. Then $R \subseteq S$ is a t-Prüfer which is not Prüfer.

Proof. Note that S is the total ring of fractions of R. The ring R is a Krull domain which is a PvMD (El Baghdadi, Izelgue, & Tamoussit, 2020, Proposition 1.2 (*i*) $\Leftrightarrow$ (*iii*)). It follows from Remark 2.9 that $R \subseteq S$ is t-Prüfer. On the other hands, the extension $R \subseteq S$ is not Prüfer. In fact, for $\mathfrak{p} = (X, Y)K[X, Y]$, the ring extension $R_{[\mathfrak{p}]} \subseteq S$ is not a Manis. Notice that in this case, $R_{[\mathfrak{p}]} = R_\mathfrak{p}$. If $R_{[\mathfrak{p}]}$ was a Manis valuation, then $R_{[\mathfrak{p}]}$ will be a discrete valuation ring (DVR) since $R_{[\mathfrak{p}]}$ is a Noetherian ring. But we know that $R_{[\mathfrak{p}]}$ is not a DVR since $R_{[\mathfrak{p}]}$ has Krull dimension 2. □

Example 2.11. Let K be a field, X, Y be two indeterminates over K, $R = K[X, Y]$ and $S = K[X, \frac{1}{X}, Y]$. The extension $R \subseteq S$ is α-Prüfer, where α the restriction of the star operation t on $K[X, Y] \subseteq K(X, Y)$ to the ring extension $R \subseteq S$

Proof. Let $\mathfrak{p}$ be a prime ideal of R. With respect to the ring extension $R \subseteq S$, we have $R_{[\mathfrak{p}]} = \{u \in K[X, \frac{1}{X}, Y] \ : tu \in K[X, Y]$ for some $t \notin \mathfrak{p}$

We show that $R_{[\mathfrak{p}]} = R_{\mathfrak{p}}$. Let $u \in R_{[\mathfrak{p}]}$ and $t \in R \setminus \mathfrak{p}$ such that $tu \in R$. Then $tu = a$ for some $a \in R$. Thus $u = \frac{a}{t} \in K(X, Y)$. It follows that $u \in R_{\mathfrak{p}}$. Hence $R_{[\mathfrak{p}]} \subseteq R_{\mathfrak{p}}$. On the other hand, if $v \in R_{\mathfrak{p}}$, then $v = \frac{b}{s}$ with $b \in R$ and $s \in R \setminus \mathfrak{p}$. Hence $sv = b \in R$. It follows that $v \in R$ □

In Example 2.12, we construct a ring extension to show that there exist ⋆-Prüfer extensions which are not Manis extensions.

Example 2.12. Let K be a field, and let X, Y be two indeterminates over K. Let b be a nonzero element of K and let $V_1 = (X)K[X, Y]_{(X)} + K[Y]_{(Y)}$, $V_2 = (X)K[X, Y]_{(X)} + K[Y]_{(b+Y)}$, $R = V_1 \cap V_2$. Then the ring extension $R \subseteq K(X, Y)$ is ⋆-Prüfer for each star operation ⋆ on $R \subseteq K(X, Y)$. Furthermore, the ring extension $R \subseteq S$ is not Manis.

Proof. Let $S = K(X, Y)$. The ring R is not a local ring since it has two maximal ideals (see (Paudel & Tchamna, 2023, Example 3.2)). It follows that $R \subsetneq R_{[\mathfrak{p}]} \subseteq S$ for each ⋆-prime ideal $\mathfrak{p}$ of R. Furthermore, by the definition of a maximal non-Manis extension ((Paudel & Tchamna, 2023, Definition 2.1)), the ring extension $R_{[\mathfrak{p}]} \subseteq S$ is Manis. But in this case, $R_{[\mathfrak{p}]} = R_{\mathfrak{p}}$ is a local ring. It follows from (Knebusch & Zhang, 2002, Scholium 10.4, p. 147), that the ring extension is Prüfer. This shows that for each ⋆-prime ideal $\mathfrak{p}$, $R_{[\mathfrak{p}]} \subseteq S$ is Prüfer extension. Thus, the ring extension $R \subseteq K(X, Y)$ is ⋆-Prüfer. □

In Example 2.13, we give an example of a ring extension $R \subseteq S$ which is never ⋆-Prüfer for any star operation ⋆ on $R \subseteq S$. Recall that a ring extension $R \subseteq S$ is called *maximal non-Prüfer extension* if $R \subseteq S$ is not a Prüfer extension and $T \subseteq S$ is a Prüfer extension for any intermediate ring T between R and S.

Example 2.13. Let $\mathbb{C}$ be the field of complex, $\mathbb{R}$ be the field of real numbers. Let $R = \mathbb{R} + X\mathbb{C}[[X]]$, S the total ring of fractions of R. Then the ring extension $R \subseteq S$ is not ⋆-Prüfer for any star operation ⋆ on $R \subseteq S$.

Proof. By (JABALLAH, 2012, Example 34), R is a pseudo-valuation domain (see Hedstrom and Houston (1978) for the definition and some properties of a pseudo-valuation domain), and the ring extension $R \subseteq S$ is maximal non-Prüfer. Furthermore, by (JABALLAH, 2012, Theorem 5), the ring R as a unique maximal ideal M. It follows from (Hedstrom & Houston, 1978, Theorem 2.10), M^{-1} is a valuation overring with maximal ideal M.

Let v be the classical "v-operation" on the domain R (i.e. $I^v = \left(I^{-1}\right)^{-1}$ for each fractional ideal of R). We show that M is a v-ideal. By contradiction, suppose that M is not v-ideal. We have $M \subseteq M^v \subseteq R$. Then by the maximality of M, we have $M^v = R$. It follows from (Tchamna, 2020, Lemma 2.5(2)) that $M^{-1} = R$. This is a contradiction since by the previous paragraph M^{-1} is an overring of R. This shows that M is a v-ideal.

Let ⋆ be a star operation on the ring extension $R \subseteq S$. By (Knebusch & Kaiser, 2014, Proposition 3.6(c), p. 141), we have $M \subseteq M^\star \subseteq M^v = M$. This shows that M is a ⋆-ideal of R.

Furthermore, $R = R_{[M]} = R_M$. But by the definition of a maximal non-Prüfer, the extension $R = R_{[M]} \subseteq S$ is not Prüfer. Therefore, the ring extension $R \subseteq S$ is not ⋆-Prüfer. □

Proposition 2.14. *Let $R \subseteq S$ be a ring extension, and let $\Psi : S \longrightarrow T$ be a surjective ring homomorphism. If the ring extension $\Psi(R) \subseteq \Psi(S)$ is Manis and* $\ker \Psi \subseteq R$*, then $R \subseteq S$ is a Manis extension.*

Proof. Suppose that $\Psi(R) \subseteq \Psi(S)$ is a Manis extension. Then there exists a Manis valuation $v : \Psi(S) \longrightarrow \Gamma \cup \{\infty\}$ such that $\Psi(R) = \{t \in \Psi(S) : v(t) \geq 0\}$ and $v(\Psi(S)) = \Gamma_v \cup \{\infty\}$. Define the map $v' : S \longrightarrow \Gamma \cup \{\infty\}$ by $v'(x) = v(\Psi(x))$ for each $x \in S$. Then for two elements x, y of S, we have:
(i) $v'(x + y) = v(\Psi(x + y)) = v(\Psi(x) + \Psi(y)) \geq \min(v(\Psi(x)), v(\Psi(y))) = \min(v'(x), v'(y))$.
(ii) $v'(xy) = v(\Psi(xy)) = v(\Psi(x)\Psi(y)) = v(\Psi(x)) + v(\Psi(y)) = v'(x) + v'(y)$.
(iii) $v'(1) = v(\Psi(1)) = v(1) = 0$ and $v'(0) = v(\Psi(0)) = v(0) = \infty$.
This shows that v' is a valuation map. Furthermore, $v'(S) = v(\Psi(S)) = \Gamma_v \cup \{\infty\}$.

Let $\alpha \in \Gamma_{v'}$. Since $\Gamma_{v'}$ is the (additive) subgroup of Γ generated by $v'(S) \setminus \{\infty\}$, we have $\alpha = \sum_{i=1}^{n} v'(x_i) = \sum_{i=1}^{n} v(\varphi(x_i)) = v\left(\prod_{i=1}^{n} \Psi(x_i)\right) = v\left(\Psi(\prod_{i=1}^{n} x_i)\right) \in \Gamma_v$. Hence $\Gamma_{v'} \subseteq \Gamma_v$. On the other hand, if $\beta \in \Gamma_v$, then $\beta = \sum_{i=1}^{\ell} v(y_i)$, with $y_i \in \Psi(S)$ for $1 \leq i \leq \ell$. Write $y_i = \Psi(x_i)$ with $x_i \in S$ for $1 \leq i \leq \ell$. Then $\beta = \sum_{i=1}^{\ell} v(\Psi(x_i)) = v\left(\prod_{i=1}^{\ell} \Psi(x_i)\right) = v\left(\Psi\left(\prod_{i=1}^{\ell} x_i\right)\right) = v'\left(\prod_{i=1}^{\ell} x_i\right) \in \Gamma_{v'}$. This shows that $\Gamma_v \subseteq \Gamma_{v'}$. Hence $\Gamma_v = \Gamma_{v'}$. Therefore $v'(S) = v(\Psi(S)) = \Gamma_v \cup \{\infty\} = \Gamma_{v'} \cup \{\infty\}$. It follows that v' is a Manis valuation.

Let $x \in R$. Then $\Psi(x) \in \Psi(R)$. Hence $v'(x) = v(\Psi(x)) \geq 0$. On the other hand, let $s \in S$ such that $v'(s) \geq 0$. Then $v(\Psi(s)) \geq 0$. Thus $\Psi(s) \in \Psi(R)$. Therefore, $\Psi(s) = \Psi(r)$ for some $r \in R$. Hence $s - r \in \ker \Psi \subseteq R$. It follows that $s \in R$. This shows that $R = \{x \in S : v'(x) \geq 0\}$. Thus the extension $R \subseteq S$ is Manis. □

Lemma 2.15. *Let $R \subseteq S$ and $L \subseteq T$ be two ring extensions, and consider the following commutative diagram with* $\ker\Psi \subseteq \mathrm{Nil}(R)$*, where* $\mathrm{Nil}(R)$ *is the intersection of the prime ideals,* Ψ *is surjective and* α *is the restriction of* Ψ *to R.*

$$\begin{array}{ccc} R & \xrightarrow{\ \alpha\ } & L \\ \big\downarrow & & \big\downarrow \\ S & \xrightarrow{\ \Psi\ } & T \end{array}$$

If $\mathfrak{p}$ *is a prime ideal of R, then* $\Psi(R)_{[\Psi(\mathfrak{p})]} = \Psi(R_{[\mathfrak{p}]})$.

Proof. Let $\mathfrak{p}$ be a prime ideal of R. Since Ψ is surjective and $\ker\Psi \subseteq \mathrm{Nil}(R) \subseteq \mathfrak{p}$, $\Psi(\mathfrak{p})$ is a prime ideal of $\Psi(R)$. Let $y = \Psi(x) \in \Psi(R)_{[\Psi(\mathfrak{p})]}$ with $x \in S$. There exists $u \in \Psi(R) \setminus \Psi(\mathfrak{p})$ such that $uy \in \Psi(R)$. Let $t \in R$ such that $u = \Psi(t)$. Then $t \in R \setminus \mathfrak{p}$; otherwise we will have $u = \Psi(t) \in \Psi(\mathfrak{p})$. Which is a contradiction. Furthermore, $uy \in \Psi(R)$ implies that $\Psi(t)\Psi(x) = \Psi(r)$ for some $r \in R$. Thus $tx - r \in \mathrm{Ker}\,\Psi \subseteq R$. Hence $tx \in R$. It follows that $x \in R_{[\mathfrak{p}]}$. Therefore, $y = \Psi(x) \in \Psi(R_{[\mathfrak{p}]})$. This shows that $\Psi(R)_{[\Psi(\mathfrak{p})]} \subseteq \Psi(R_{[\mathfrak{p}]})$.

On the other hand, if $y \in \Psi(R_{[\mathfrak{p}]})$, then there exists $x \in R_{[\mathfrak{p}]}$ such that $y = \Psi(x)$. Let $t \in R \setminus \mathfrak{p}$ such that $tx \in R$. Then $\Psi(t)\Psi(x) \in \Psi(R)$. By contradiction, if $\Psi(t) \in \Psi(p)$, then there exists $s \in \mathfrak{p}$ such that $t - s \in \ker\Psi \subseteq \mathrm{Nil}(R) \subseteq \mathfrak{p}$. So, $t \in \mathfrak{p}$. Which is a contradiction to the choice of $\mathfrak{p}$. Thus $\Psi(\mathfrak{p}) \in \Psi(R) \setminus \Psi(\mathfrak{p})$. This shows that $y = \Psi(x) \in \Psi(R)_{[\Psi(\mathfrak{p})]}$. Hence $\Psi(R_{[\mathfrak{p}]} \subseteq \Psi(R)_{[\Psi(\mathfrak{p})]}$. This shows that $\Psi(R_{[\mathfrak{p}]} = \Psi(R)_{[\Psi(\mathfrak{p})]}$. □

Theorem 2.16. *Consider the commutative diagram in Lemma 2.15. Let $\star$ be a star operation of finite type on $R \subseteq S$, and let $\star'$ be a star operation of finite type on $L \subseteq T$ such that $\Psi(A)^{\star'} = \Psi(A^{\star})$ for each R-submodule A of S. If the ring extension $R \subseteq S$ is $\star$-Prüfer, then the ring extension $L \subseteq T$ is $\star'$-Prüfer.*

Proof. Suppose that the ring extension $R \subseteq S$ is $\star$-Prüfer. Let $\mathfrak{q}$ be a $\star'$-maximal ideal of L, and let $\mathfrak{p} = \Psi^{-1}(\mathfrak{q})$. Then $\mathfrak{q} = \Psi(\mathfrak{p})$ since Ψ is surjective. We have $\mathfrak{q} = \mathfrak{q}^{\star'} = \Psi(\mathfrak{p})^{\star'} = \Psi(\mathfrak{p}^{\star})$. Hence $\mathfrak{p}^{\star} \subseteq \Psi^{-1}(\mathfrak{q}) = \mathfrak{p}$. It follows that $\mathfrak{p}^{\star} = \mathfrak{p}$ since the inclusion $\mathfrak{p} \subseteq \mathfrak{p}^{\star}$ is always true. Furthermore, if $x, y \in R$ such that $xy \in \mathfrak{p}$, then $\Psi(xy) \in \mathfrak{q}$. Thus $\Psi(x) \in \mathfrak{q}$ or $\Psi(y) \in \mathfrak{q}$ since $\mathfrak{q}$ is maximal (hence a prime ideal). Therefore, $x \in \Psi^{-1}(\mathfrak{q}) = \mathfrak{p}$ or $y \in \Psi^{-1}(\mathfrak{q}) = \mathfrak{p}$. This shows that $\mathfrak{p}$ is a $\star$-prime ideal of R.

The ring extension $R_{[\mathfrak{p}]} \subseteq S$ is Prüfer since by hypothesis, the ring extension $R \subseteq S$ is $\star$-Prüfer. Therefore, by (Knebusch & Zhang, 2002, Proposition 5.7, p. 51) and the fact that Ψ is surjective, the ring extension $\Psi(R_{[\mathfrak{p}]}) \subseteq T$ is Prüfer. It follows that $L_{[\mathfrak{q}]} \subseteq T$ is a Prüfer extension. This shows that the ring extension $L \subseteq T$ is $\star'$-Prüfer. □

Remark 2.17. A question worth investigating is the converse of the statement of Theorem 2.16. In other words, with the same hypothesis as in Theorem 2.16, when the ring extension $L \subseteq T$ is $\star'$-Prüfer, is the ring extension $R \subseteq S$ $\star$-Prüfer? This remains an open question. However, in the next theorem, we prove that the ring extension $L \subseteq T$ is $\star'$-Prüfer when $R \subseteq S$ is P$\star$ME.

Theorem 2.18. *With the same hypothesis as in Theorem 2.16, if the ring extension $L \subseteq T$ is $\star'$-Prüfer, then the ring extension $R \subseteq S$ is P$\star$ME.*

Proof. Suppose that the ring extension $L \subseteq T$ is $\star'$-Prüfer. Let $\mathfrak{m}$ be a $\star$-maximal ideal of R. Observe that $\Psi(\mathfrak{m})^{\star'} = \Psi(\mathfrak{m}^{\star}) = \Psi(\mathfrak{m})$. So, $\Psi(\mathfrak{m})$ is a $\star'$-ideal of L. Let $\mathfrak{n}$ be a $\star'$-maximal ideal of L containing $\Psi(\mathfrak{m})$, and let $\mathfrak{p} = \Psi^{-1}(\mathfrak{n})$. Then $\mathfrak{m} \subseteq \mathfrak{p}$ since $\Psi(\mathfrak{m}) \subseteq \Psi(\mathfrak{p}) = \mathfrak{n}$. It follows from the maximality of $\mathfrak{m}$ that $\mathfrak{m} = \mathfrak{p}$ or $\mathfrak{p} = R$. By contradiction, suppose that $\mathfrak{p} = R$. Then $\Psi(\mathfrak{p}) = \Psi(R)$. Thus $\mathfrak{n} = L$. Which is a contradiction since $\mathfrak{n}$ is a proper ideal of L. Hence $\mathfrak{m} = \mathfrak{p}$, and we have $\Psi(\mathfrak{m}) = \Psi(\mathfrak{p}) = \mathfrak{n}$. This shows that $\Psi(\mathfrak{m})$ is a $\star'$-maximal ideal of L. It follows from the hypothesis that the ring extension $L_{[\Psi(\mathfrak{m})]} \subseteq T$ is Prüfer. Hence, by Lemma 1.1, $L_{[\Psi(\mathfrak{m})]} \subseteq T$ is a Manis extension. But by Remark 2.15, we have $L_{[\Psi(\mathfrak{m})]} = \Psi(R)_{[\Psi(\mathfrak{m})]} = \Psi(R_{[\mathfrak{m}]})$. Hence the extension $\Psi(R_{[\mathfrak{m}]}) \subseteq \Psi(S)$ is Manis. It follows from Proposition 2.14 that $R_{[\mathfrak{m}]} \subseteq S$ is a Manis extension. This shows that $R \subseteq S$ is P$\star$ME. □

Lemma 2.19. *Let $R \subseteq S$ be a ring extension, and let X be an indeterminate over S. If $R[X] \subseteq S[X]$ is a Prüfer extension, then $R \subseteq S$ is a Prüfer extension.*

Proof. Suppose that $R[X] \subseteq S[X]$ is a Prüfer extension. Let B be an S-overring of R, and let $a \in B$. Then by (Knebusch & Zhang, 2002, Theorem 5.2, p. 47), we have $(R[X] :_{S[X]} ax)\, B[X] = B[X]$. But by (Tchamna, 2020, Lemma 2.5 (3)), we have $(R[X] :_{S[X]} ax) = (R :_S a)\, R[X]$. Therefore $(R :_S a)\, B[X] = B[X]$. Thus $(R :_S a)\, B = B$. It follows from (Knebusch & Zhang, 2002, Theorem 5.2, p. 47) that $R \subseteq S$ is a Prüfer extension. □

Proposition 2.20. *Let $R \subseteq S$ be a ring extension, and let X be an indeterminate over S. Let $\star_1$ be a star operation on $R \subseteq S$, and let $\star_2$ be a star operation on $R[X] \subseteq S[X]$ such that $A^{\star_1}R[X] = (AR[X])^{\star_2}$ for each R-submodule A of S. If $R[X] \subseteq S[X]$ is a $\star_2$-Prüfer extension, then the extension $R \subseteq S$ is $\star_1$-Prüfer.*

Proof. Suppose that $R[X] \subseteq S[X]$ is a $\star_2$-Prüfer extension. Let $\mathfrak{p}$ be a $\star_1$-prime ideal of R. Then $\mathfrak{p}R[X]$ is a prime ideal of $R[X]$, since $(R/\mathfrak{p})[X]$ is isomorphic to $R[X]/\mathfrak{p}R[X]$. Furthermore we have $(\mathfrak{p}R[X])^{\star_2} = \mathfrak{p}^{\star_1}R[X] = \mathfrak{p}R[X]$. This shows that $\mathfrak{p}R[X]$ is a $\star_2$-prime ideal of $R[X]$. Therefore, by hypothesis, the ring extension $(R[X])_{[\mathfrak{p}R[X]]} \subseteq S[X]$ is Prüfer. But by (Paudel & Tchamna, 2021b, Lemma 2.6), we have $(R[X])_{[\mathfrak{p}R[X]]} = R_{[\mathfrak{p}]}[X]$. Hence $R_{[\mathfrak{p}]}[X] \subseteq S[X]$ is a Prüfer extension. It follows from Lemma 2.19 that $R_{[\mathfrak{p}]} \subseteq S$ is a Prüfer extension. This shows that $R \subseteq S$ is $\star_1$-Prüfer extension. □

In Proposition 2.21, we give some conditions under which the converse of the statement of Proposition 2.20 is true. More precisely, we study conditions under which the ring extension $R[X] \subseteq S[X]$ is $\star_2$-Prüfer when $R \subseteq S$ is a $\star_1$-Prüfer extension, where $\star_1$ and $\star_2$ are star operations on $R \subseteq S$ and $R[X \subseteq S[X]$ respectively, and X an indeterminate over S. We consider a ring extension $R \subseteq S$ for which $(R[X] :_{S[X]} Q) \neq R[X]$ for each proper ideal Q of $R[X]$. An example of such ring extensions is given in (Tchamna, 2020, Example 3.10).

Proposition 2.21. *Let $R \subseteq S$ be a tight ring extension for which $(R[X] :_{S[X]} Q) \neq R[X]$ for each proper ideal Q of $R[X]$, where X is an indeterminate over S. Let $\star_1$ be a star operation on $R \subseteq S$, and let $\star_2$ be a star operation on $R[X] \subseteq S[X]$ such that $A^{\star_1}R[X] = (AR[X])^{\star_2}$. If the ring extension $R \subseteq S$ is $\star_1$-Prüfer, then the ring extension $R[X] \subseteq S[X]$ is a weak P$\star_2$ME.*

Proof. Let $\mathfrak{q}$ be an $S[X]$-regular $\star_2$-maximal ideal of $R[X]$, and let $\mathfrak{p} = \mathfrak{q} \cap R$. There exist $f_1, \ldots, f_\ell \in \mathfrak{q}$ and $g_1, \ldots, g_\ell \in S[X]$ such that $1 = \sum_{k=1}^{\ell} f_k g_k$. Since $R \subseteq S$ is a tight extension, there exists an S-invertible ideal I_i of R such that $g_i I_i \subseteq R[X]$ for each i with $1 \leq i \leq \ell$. The ideal $I = \prod_{k=1}^{\ell} I_i$ is S-invertible, and for $1 \leq i \leq \ell$, we have $g_i I \in R[X]$. Furthermore, $I \neq 0$ since I is S-invertible. Let $0 \neq r \in I$. Then $r = \sum_{k=1}^{\ell} f_i(rg_i) \in q \cap R = \mathfrak{p}$. It follows from (Tchamna, 2020, Lemma 3.9) that $\mathfrak{p}$ is a $\star_1$-maximal ideal of R such that $\mathfrak{q} = \mathfrak{p}R[X]$. Furthermore, by hypothesis, the extension $R_{[\mathfrak{p}]} \subseteq S$ is Prüfer. Therefore, by Lemma 1.1, $\left(R_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]}\right)$ is a Manis pair in S. Thus $R_{[\mathfrak{p}]}[X] \subseteq S[X]$ is a Manis extension (Paudel & Tchamna, 2018, Remark 2.11). But by (Paudel & Tchamna, 2021b, Lemma 2.6), we have $R_{[\mathfrak{p}]}[X] = (R[X])_{[\mathfrak{p}R[X]]} = (R[X])_{[\mathfrak{q}]}$. Therefore, $(R[X])_{[\mathfrak{q}]} \subseteq S[X]$ is a Manis extension. It follows that $R[X] \subseteq S[X]$ is P$\star_2$ME. □

Lemma 2.22. *((Paudel & Tchamna, 2021b, Lemma 2.6)) Let $R \subseteq S$ be a ring extension, and let T be a multiplicatively closed subset of R.*

(1) *$R[X]_{[T[X]]} = R_{[T]}[X]$, where X is an indeterminate over S.*

(2) *If N is a multiplicatively closed subset of R such that $N \subseteq T$, and A is an R-submodule of S, then $T_{[N]}$ is a multiplicatively closed subset of $R_{[N]}$ and $(A_{[N]})_{[T_{[N]}]} = A_{[T]}$.*

Lemma 2.23. *((Paudel & Tchamna, 2021b, Proposition 3.4)) Let $\star_1$ be a star operation of finite type on a ring extension $R \subseteq S$, and let $\star_2$ be a star operation of finite type on $R[X] \subseteq S[X]$, where X is an indeterminate over S. Let T be a multiplicatively closed subset of $R[X]$. If I is an ideal of R such that $I^{\star_1}R[X] = (IR[X])^{\star_2}$, then $\left(I^{\star_1}[X]\right)_{[T]} = \left(I[X]_{[T]}\right)^{\star_2}$.*

For a star operation $\star$ on a ring extension $R \subseteq S$, consider the set

$$N(\star) = \left\{f \in R[X] : c_R(f)^{\star} = R^{\star}\right\}$$

where X is an indeterminate over S and $c_R(f)$ is the ideal of R generated by the coefficient of f. Proprieties of $N(\star)$ have been studied in Paudel and Tchamna (2021b). In particular, it was shown that $N(\star)$ is a multiplicatively closed subset of $R[X]$ (Paudel & Tchamna, 2021b, Lemma 3.5).

Theorem 2.24. *Let $R \subseteq S$ be a ring extension, and let X be an indeterminate over S. Let $\star_1$ be a star operation on $R \subseteq S$ and $\star_2$ a star operation on $R[X] \subseteq S[X]$ such that $A^{\star_1}R[X] = (AR[X])^{\star_2}$ for each R-submodule A of S. If $R[X]_{[N(\star_1)]} \subseteq S[X]$ is a $\star_2$-Prüfer extension, then $R \subseteq S$ is a $\star_1$-Prüfer extension.*

Proof. Let $\mathfrak{m}$ be a $\star_1$-maximal ideal of R. By assumption on $\star_2$, we have $(\mathfrak{m}R[X])^{\star_2} = \mathfrak{m}R[X]$. It follows from Lemma 2.23 that $\mathfrak{m}R[X]_{[N(\star_1)]}$ is a $\star_2$-ideal of $R[X]_{[N(\star_1)]}$. Furthermore, $\mathfrak{m}R[X]_{[N(\star_1)]}$ is a prime ideal of $R[X]_{N(\star_1)}$ since $\mathfrak{m}R[X]$ is a prime ideal of $R[X]$ (Paudel & Tchamna, 2021b, Proposition 3.7(2)). It follows from the hypothesis that the ring extension $\left(R[X]_{[N(\star_1)]}\right)_{[\mathfrak{m}R[X]_{[N(\star_1)]}]} \subseteq S[X]$ is a Prüfer. But by Lemma 2.22, we have

$$\left(R[X]_{[N(\star_1)]}\right)_{[\mathfrak{m}R[X]_{[N(\star_1)]}]} = R[X]_{[\mathfrak{m}R[X]]} = R_{[\mathfrak{m}]}[X].$$

So, the ring extension $R_{[\mathfrak{m}]}[X] \subseteq S[X]$ is Prüfer. It follows from Lemma 2.19 that $R_{[\mathfrak{m}]} \subseteq S$ is a Prüfer extension. This shows that ring extension $R \subseteq S$ is $\star_1$-Prüfer. □

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