Journal of Mathematics Research; Vol. 15, No. 2; April 2023
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Convergence for an Immersed Finite Volume Method for Elliptic and Parabolic Interface Problems

Champike Attanayake[1], Deepthika Senaratne[2]

[1] Miami University, Department of Mathematics, Middletown, OH 45042, USA

[2] Department of Mathematics and Computer Science, Fayetteville State University, 1200 Murchison road, Fayetteville, NC 28301, USA

Correspondence: Deepthika Senaratne, Department of Mathematics and Computer Science, Fayetteville State University, 1200 Murchison road, Fayetteville, NC 28301, USA

Received: June 21, 2022 Accepted: February 10, 2023 Online Published: April 4, 2023

doi:10.5539/jmr.v15n2p19 URL: https://doi.org/10.5539/jmr.v15n2p19

Abstract

In this article we analyze an immersed interface finite volume method for second order elliptic and parabolic interface problems. We show the optimal convergence of the elliptic interface problem in L^2 and energy norms. For the parabolic interface problem, we prove the optimal order in L^2 and energy norms for piecewise constant and variable diffusion coefficients respectively. Furthermore, for the elliptic interface problem, we demonstrate super convergence at element nodes when the diffusion coefficient is a piecewise constant. Numerical examples are also provided to confirm the optimal error estimates.

Keywords: immersed interface, finite volume method, optimal error estimates, parabolic, elliptic

1. Introduction

Elliptic and parabolic interface problems with discontinuous coefficients appear in a variety of disciplines, such as electromagnetism, fluid dynamics and material science. These problems can be solved by standard finite element methods (FEM) using carefully tailored meshes to resolve the interface. If the grid lines of the fitted finite element mesh are not conformed with the interface, the solution has low global regularity due to the discontinuity of the coefficients across the interface. However, high quality mesh generation is difficult and has more computational cost for some complicated geometries and interfaces. To overcome the limitations of standard fitted mesh methods many numerical methods have been developed in the past several decades.

One of the more commonly used such a method is the immersed finite element (IFE) method (Li, Z., 1998) based on Cartesian meshes. The method uses special basis functions for the interface elements while allowing the interface to immerse in the regular elements. The local interface basis functions are designed to satisfy the jump conditions at the interface while their meshes do not have to be conformed with the interface (Li, Z., 2011; Li, Z., Lin, T., Wu, X., 2003; Li, Z., Ito, K., 2006). The solution of the one dimensional interface problem is second order accurate in the infinity norm (Li, Z., 2011). Li, et al. (Li, Z., Lin, T., Rogers, R.C, 2004) analyzed the second order elliptic interface problem and their results showed IFE has approximating capability similar to that of standard FEM based on body fitting partitions. The Convergence of the IFE method for semi-linear parabolic interface problem was analyzed by Attanayake, et al. (Attanayake, C, Senaratne, D., 2011) and their results prove that the convergence of the semi-discrete solution was of the optimal order in L^2 and energy norms and the fully discrete scheme based on the backward Euler method has optimal order in L^2 norm. However, some numerical results demonstrate that IFE methods have larger point-wise error over the interface elements.

Finite volume method (FVM) is another numerical method based on Cartesian meshes. Since the method inherits local conservation of physical quantities such as, mass, momentum, and flux, FVM is used in solving problems in science and engineering (Cai, Z., 1991; Lin, Y., Liu, J., Yand, M., 2013). Ewing and his colleagues (Ewing, R., Li, Z., Lin, T., Lin, Y., 1991) are the first to investigate immersed finite volume methods (IFVM) on second order elliptic interface problems and they obtained optimal error estimates in the energy norm. Cao and his colleagues (Cao, W., Zhang, X., Zhang, Z., Zou, Q., 2017) studied the convergence of the one dimensional elliptic problem for any order finite volume schemes and produced some super convergence properties as well. A second order convergence in L^∞ norm is obtained for an elliptic interface problem using IFVM in (Wang, Q., Zhang, Z., Wang, L., 2021). However, theoretical analysis of immersed finite volume methods on parabolic interface problems remains sparse in the literature.

The goal of this article is to propose an immersed finite volume analysis for the elliptic and parabolic interface problems and obtain optimal error estimates for the approximated solution. More specifically, we use so called covolume method. This method uses two types of meshes, the primal and the dual partition, associated with the trial and test spaces respectively. The exact solution of the equation is approximated in the primal partition, while the equation is discretized in the dual partition. With a transfer operator from trial space to test space, we deduce the relation to the Galerkin finite element method. As a result, we can use the properties of the finite element analysis to estimate optimal convergence rates. This is one of the main advantages of the covolume method. The theoretical framework of the covolume method was developed by Chou and his colleagues in the articles (Chou, S.H., Li, Q., 1999; Chou, S.H., Ye, X., 2007) and references therein. We extend these convergence properties to the parabolic problem. To the best of the author's knowledge, this is the first study that demonstrates convergence properties of the parabolic interface problem via the immersed finite volume method.

The rest of the paper is organized as follows. In Section 2, we introduce the immersed interface finite volume method and the convergence analysis for the elliptic interface problem. In particular, we show super convergence at the element nodes when the diffusion coefficient is a piecewise constant. The convergence for the parabolic interface problem is discussed in Section 3. Simulation of elliptic and parabolic interface problems are provided in Section 4 to confirm the theory. The conclusions are given in Section 5.

2. Elliptic Interface Problem

In this section we define an immersed interface finite volume method to solve a second order elliptic interface problem

$$-(\beta u')' = f(x) \text{ in } (0,1) \tag{1}$$

with boundary conditions

$$u(0) = 0, \quad u(1) = 0, \tag{2}$$

with the jump conditions on the interface $\alpha \in (0,1)$

$$[u]_\alpha = 0, \quad [\beta u']_\alpha = 0, \tag{3}$$

where $[v]_\alpha = \lim_{x\to\alpha^-} v(x) - \lim_{x\to\alpha^+} v(x)$. Here, the diffusion coefficient β has a finite jump across the interface. First, consider the primal partition independent of the interface as

$$0 = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = 1.$$

Denote $h_i = x_{i+1} - x_i$ and $h = \max_{0\le i<N} h_i$. We call the element $[x_k, x_{k+1}]$ an interface element which contains α, and the other elements $[x_i, x_{i+1}]$ $i \neq k$ are noninterface elements. For the corresponding trial space, basis functions on noninterface elements are defined as standard linear Lagrange nodal basis functions ϕ_i, $i \neq k$. On interface elements basis functions ϕ_k and ϕ_{k+1} are defined enforcing the jump conditions

$$[\phi_k]_\alpha = 0, \quad [\bar\beta^-\phi_k']_\alpha = 0 \quad [\phi_{k+1}]_\alpha = 0, \quad [\bar\beta^+\phi_{k+1}']_\alpha = 0,$$

where $\bar\beta^-$ and $\bar\beta^+$ are average of $\beta(x)$ over $[x_k, \alpha]$ and $[\alpha, x_{k+1}]$ respectively.

The basis functions for the interface elements are given by

$$\phi_k(x) = \begin{cases} 0 & 0 \le x < x_{k-1}, \\ \frac{x-x_{k-1}}{x_k-x_{k-1}} & x_{k-1} \le x \le x_k \\ \frac{x_k-x}{D}+1 & x_k \le x < \alpha \\ \frac{\rho(x_{k+1}-x)}{D} & \alpha \le x < x_{k+1} \\ 0 & x_{k+1} \le x \le 1 \end{cases} \tag{4}$$

$$\phi_{k+1}(x) = \begin{cases} 0 & 0 \le x < x_k, \\ \frac{x-x_k}{D} & x_k \le x \le \alpha \\ \frac{\rho(x-x_{k+1})}{D}+1 & \alpha \le x < x_{k+1} \\ \frac{(x_{k+2}-x)}{x_{k+2}-x_{k+1}} & x_{k+1} \le x < x_{k+2} \\ 0 & x_{k+2} \le x \le 1, \end{cases} \tag{5}$$

where

$$\rho := \frac{\beta^-}{\beta^+} \quad D := (x_{k+1} - x_k)\frac{\bar\beta^+ - \bar\beta^-}{\bar\beta^+}(x_{k+1} - \alpha).$$

So the trial space is chosen as,

$$V_h := \{v \in span\{\phi_i\}_{i=0}^{N} \cap H_0^1(0,1) : v(0) = 0 = v(1)\}.$$

In this space define the inner-product

$$(u, v)_h = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} u(x)v(x)dx \qquad \forall u, v \in V_h,$$

and the broken semi-norm and the norm for the above space V_h as

$$\begin{aligned} |u|_{1,h}^2 &= \sum_{i=0;i\neq k}^{N-1} |u|^2_{H^1(x_i,x_{i+1})} + |u|^2_{H^1(x_k,\alpha)} + |u|^2_{H^1(\alpha,x_{k+1})} \\ ||u||_{1,h}^2 &= ||u||^2 + |u|_{1,h}^2 \end{aligned}$$

respectively, where $||u|| = \sum_{i=0}^{N-1} |u|^2_{L^2(x_i,x_{i+1})}$. To construct the dual partition we choose the midpoints between nodes in the primal partition. We denote

$$x_{i-1/2} = \frac{x_{i-1} + x_i}{2}, \quad x_{i+1/2} = \frac{x_i + x_{i+1}}{2} \quad \text{for } i = 1, \dots, N-1.$$

and the dual partition is given by

$$[x_{i-1/2}, x_{i+1/2}], \ \ i = 1 \dots N-1.$$

The corresponding test space consists of the piecewise constant functions with respect to the dual partition

$$Q_h = \{q \in L^2(0,1) : q|_{[x_{i-1/2},x_{i+1/2}]} = const, \ i = 1 \dots N-1\}.$$

As shown in (Chou, S.H., Ye, X., 2007) we define the transfer operator Π_h^* from the trail space V_h to the test space Q_h, defined by

$$\Pi_h^* v(x) = \sum_{i=0}^{N} v(x_i)\chi_{x_i}(x), \quad \forall x \in [0,1],$$

where χ_{x_i} is the characteristic function of the dual element $[x_{i-1/2}, x_{i+1/2}]$ associated with the primal node x_i. Moreover, the convergence of the approximated solution depends on the interpolation of the solution u given by

$$u_I(x) = \begin{cases} \frac{x_{i+1}-x}{h}u(x_i) + \frac{x-x_i}{h}u(x_{i+1}) & i \neq k, \ x_i \leq x \leq x_{i+1}, \\ u(x_k) + \kappa(x - x_k) & x_k \leq x \leq \alpha, \\ u(x_{k+1}) + \kappa\rho(x - x_{k+1}) & \alpha \leq x \leq x_{k+1}, \end{cases} \tag{6}$$

where

$$\kappa = \frac{u(x_{k+1}) - u(x_k)}{\alpha - x_k - \rho(\alpha - x_{k+1})}.$$

In (Chou, S.H., 2012), Chou proved that, $u_I(x)$ has the approximation property

$$||u - u_I|| + h||u - u_I||_{1,h} \leq Ch^2||u||_{2,\alpha}. \tag{7}$$

Here u_I is the usual Lagrange interpolation and $||u||_{2,\alpha} = ||u||_{H^2(0,\alpha)} + ||u||_{H^2(\alpha,1)}$.

Lemma 1.1. *For any $u \in H^2 \cap H_0^1$, on interface element $[x_k, x_{k+1}]$*

$$\int_{x_k}^{x_{k+1}} u_I - \Pi_h^* u_I \leq Ch^2||u||_{2,\alpha}$$

and on noninterface element $[x_i, x_{i+1}]$,

$$\int_{x_i}^{x_{i+1}} u_I - \Pi_h^* u_I = 0.$$

Proof. Consider an interface element $[x_k, x_{k+1}]$. For some $\xi \in [x_k, x_{k+1}]$

$$\begin{aligned}\int_{x_k}^{x_{k+1/2}} (u - \Pi_h^* u)dx &\leq \int_{x_k}^{x_{k+1/2}} h|u'(\xi)| \\ &\leq h^2 |u'|_{\infty,[x_k,x_{k+1/2}]} \\ &\leq h^2|u'|_{\infty,[0,1]} \\ &\leq Ch^2\|u\|_{2,\alpha}. \end{aligned} \tag{8}$$

We used Sobolev embedding theorem in the last step of the proof. Similarly

$$\int_{x_{k+1/2}}^{x_{k+1}} (u - \Pi_h^* u)dx \leq Ch^2\|u\|_{2,\alpha}. \tag{9}$$

Now with (7), (8) and (9), and using the fact that $\Pi_h^* u = \Pi_h^* u_I$ we have,

$$\begin{aligned}\int_{x_k}^{x_{k+1}} (u_I - \Pi_h^* u_I)dx &\leq \int_{x_k}^{x_{k+1}} |u_I - u|dx + \int_{x_k}^{x_{k+1}} |u - \Pi_h^* u_I|dx \\ &\leq Ch^2\|u\|_{2,\alpha}.\end{aligned}$$

By brute force calculation on noninterface elements, since u_I is linear

$$\int_{x_i}^{x_{i+1}} (u_I - \Pi_h^* u_I)dx = \int_{x_i}^{x_{i+1/2}} u_I'(x - x_i)dx + \int_{x_{i+1/2}}^{x_{i+1}} u_I'(x - x_{i+1})dx = 0$$

□

Now we present our immersed finite volume scheme. Integrating (1) over each control volume $[x_{i-1/2}, x_{i+1/2}]$ for $i = 1 \ldots N-1$ and multiplying by $\Pi_h^* v$, $v \in V_h$, we get,

$$-\sum_{i=1}^{N-1} \int_{x_{i-1/2}}^{x_{i+1/2}} (\beta u'(x))' \Pi_h^* v(x)dx = \sum_{i=1}^{N-1} \int_{x_{i-1/2}}^{x_{i+1/2}} f(x)\Pi_h^* v(x)dx,$$

$$\sum_{i=1}^{N-1} \left(\beta(x_{i-1/2})u'(x_{i-1/2})v(x_i) - \beta(x_{i+1/2})u'(x_{i+1/2})v(x_i)\right) = \sum_{i=1}^{N-1} \int_{x_{i-1/2}}^{x_{i+1/2}} f(x)v(x_i)dx.$$

Note that $u(x_0) = u(x_N) = 0$. We can define the finite volume bilinear form for any u and v in V_h as

$$a_h(u, v) = \sum_{i=1}^{N-1} \left((\beta u')(x_{i-1/2})v(x_i) - (\beta u')(x_{i+1/2})v(x_i)\right). \tag{10}$$

By writing the sum over the dual partition as a sum over the primal partition, the bilinear form (10) becomes

$$a_h(u, v) = \sum_{i=0}^{N-1} \left((\beta u')(x_{i+1/2})v(x_{i+1}) - (\beta u')(x_{i+1/2})v(x_i)\right) \tag{11}$$

Since $\Pi_h^* v$ is a piecewise constant in the dual partition, the sum in the equation (11) becomes

$$\begin{aligned} a_h(u, v) = \sum_{i=0}^{N-1} \Big(&(\beta u')(x_{i+1})v(x_{i+1}) - (\beta u')(x_i)v(x_i) \\ &- \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1/2}} (\beta u' \Pi_h^* v)' ds - \sum_{i=0}^{N-1} \int_{x_{i+1/2}}^{x_{i+1}} (\beta u' \Pi_h^* v)' ds \Big). \end{aligned}$$

Then integration by parts of $\sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (\beta u')' v dx$ brings the finite volume bilinear form to

$$\begin{aligned} a_h(u, v) &= (\beta u', v')_h \\ &+ \sum_{i=0}^{N-1} \left[\int_{x_i}^{x_{i+1/2}} (\beta u')' (v - \Pi_h^* v)dx + \int_{x_{i+1/2}}^{x_{i+1}} (\beta u')' (v - \Pi_h^* v)dx \right]. \end{aligned} \tag{12}$$

Furthermore,

$$\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1/2}}\left[(\beta u')'\,(v-\Pi_h^* v)dx+\int_{x_{i+1/2}}^{x_{i+1}}(\beta u')'\,(v-\Pi_h^* v)dx\right]$$
$$\leq \sum_{i=0;i\neq k}^{N-1} h\left[\int_{x_i}^{x_{i+1/2}}\beta' u' v' dx+\int_{x_{i+1/2}}^{x_{i+1}}\beta' u' v' dx\right]$$
$$+\int_{x_k}^{x_{k+1/2}} h\beta' u' v' dx+\int_{x_{k+1/2}}^{\alpha} h\beta' u' v' dx+\int_{\alpha}^{x_{k+1}} h\beta' u' v' dx$$

Thus from classical analysis $a_h(u, v)$ bounded

$$a_h(u, v) \leq C(\beta, \beta')\|u\|_{1,h}\|v\|_{1,h}. \tag{13}$$

Next we'll show the coercivity. From (11) for noninterface elements $[x_i, x_{i+1}]$, $(i \neq k)$

$$\left(\beta(x_{i+1/2})u'(x_{i+1/2})u(x_{i+1})-\beta(x_{i+1/2})u'(x_{i+1/2})u(x_i)\right)dx=\beta(x_{i+1/2})(u')^2 h \tag{14}$$

Similarly, using the fact that u is only piecewise linear, on the interface element $[x_k, x_{k+1}]$,

$$\begin{aligned}
&\beta(x_{k+1/2})u'(x_{k+1/2})(u(x_{k+1})-u(x_k))\\
&\quad=\beta(x_{k+1/2})u'^{+}[u^{+}(x_{k+1})-u^{+}(\alpha)+u^{-}(\alpha)-u^{-}(x_k)]\\
&\quad=\beta(x_{k+1/2})u'^{+}([u'^{+}(x_{k+1}-\alpha)+u'^{-}(x_k-\alpha)]\\
&\quad=\beta(x_{k+1/2})([(u'^{+})^2(x_{k+1}-\alpha)+u'^{+}u'^{-}(x_k-\alpha)]
\end{aligned} \tag{15}$$

Combining (14) and (15) and applying them on (11) we can see that $\beta(x_{k+1/2})u'^{+}u'^{-}(x_k-\alpha)$ is the only possible nonepositive term. Therefore, for the sufficiently small h, particularly on the interface element there is a constant C depend on β such that

$$a_h(u, u) \geq C(\beta)\|u\|_{1,h}^2. \tag{16}$$

Remark 1.2. *Suppose that the diffusion coefficient β is piecewise constant such that*

$$\beta(x)=\begin{cases}\beta^- & 0\leq x\leq\alpha\\ \beta^+ & \alpha\leq x\leq 1\end{cases} \tag{17}$$

then from (12) note that $a_h(u, v) = (\beta u', v')_h$. In other words when β is piecewise constant finite volume bilinear form is same as Galerkin finite element bilinear form.

2.1 Convergence of the Elliptic Problem

The immersed finite volume method to solve (1) is to find $u_h \in V_h$

$$a_h(u_h, v)=\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}} f\Pi_h^* v_h dx \quad \forall v\in V_h \tag{18}$$

and the true solution u satisfies

$$a_h(u, v)=\sum_{i=0}^{N-1}\int_{x_i}^{x_{i+1}} f\Pi_h^* v_h dx \quad \forall v\in V_h. \tag{19}$$

Subtracting (18) from (19) we obtain

$$a_h(u-u_h, v)=0 \quad \forall v\in V_h. \tag{20}$$

Theorem 1.3. *Let u and u_h be solutions of (1) and (18) respectively. Then*

$$\|u-u_h\|_{1,h}\leq Ch\|u\|_{2,\alpha}.$$

Proof. Using (20) with boundedness in (13) and coercivity in (16) for any $v \in V_h$ we have

$$\|u-u_h\|_{1,h}^2 \le a_h(u-u_h, u-u_h) = a_h(u-v, u-u_h) \le C\|u-v\|_{1,h}\|u-u_h\|_{1,h}.$$

Then using the triangular inequality and the approximation property of the interpolation (7), we can conclude

$$\|u-u_h\|_{1,h} \le C \inf_{v\in V_h} \|u-v\|_{1,h} \le C\|u-u_I\|_{1,h} \le Ch\|u\|_{2,\alpha}.$$

□

Theorem 1.4. *Let u and u_h be solutions of (1) and (18) respectively. Then*

$$\|u-u_h\| \le Ch^2(\|u\|_{2,\alpha} + \|f\|_\infty)$$

Proof. We prove this using the duality argument. Let $w \in H^2 \cap H_0^1(0,1)$ be the solution to dual problem

$$\begin{aligned} -(\beta w')' \quad &= \quad u-u_h \ \text{ in } (0,1) \\ w(0)=0 \quad &\quad\ \ w(1)=0, \\ [w]_\alpha = 0, \quad &\quad\ \ [\beta w']_\alpha = 0. \end{aligned} \tag{21}$$

Note that

$$\|w\|_{2,\alpha} \le C\|u-u_h\| \tag{22}$$

and from (7)

$$\|w-w_I\|_{1,h} \le ch\|w\|_{2,\alpha} \tag{23}$$

where $w_I \in V_h$ is the usual linear interpolant of w. From 21 we obtain

$$\|u-u_h\|^2 = -(u-u_h, (\beta w')')_h = (\beta(u-u_h)', w')_h. \tag{24}$$

At the same time, (12) implies that

$$\begin{aligned} a_h(u-u_h, w_I) &= (\beta(u-u_h)', w_I')_h \\ &+ \sum_{i=0}^{N-1}\left[\int_{x_i}^{x_{i+1/2}} (\beta(u-u_h)')' (w_I - \Pi_h^* w_I)dx + \int_{x_{i+1/2}}^{x_{i+1}} (\beta(u-u_h)')' (w_I - \Pi_h^* w_I)dx\right]. \end{aligned} \tag{25}$$

Subtracting (25) from (24),

$$\begin{aligned} \|u-u_h\|^2 &= (\beta(u-u_h)', (w-w_I)')_h \\ &- \sum_{i=0}^{N-1}\left[\int_{x_i}^{x_{i+1/2}} (\beta(u-u_h)')' (w_I - \Pi_h^* w_I)dx + \int_{x_{i+1/2}}^{x_{i+1}} (\beta(u-u_h)')' (w_I - \Pi_h^* w_I)dx\right]. \end{aligned} \tag{26}$$

To estimate the first term in the right we use Theorem 1.3, (23) and (22),

$$\begin{aligned} (\beta(u-u_h)', (w-w_I)')_h &\le C\|u-u_h\|_{1,h}\|w-w_I\|_{1,h} \\ &\le Ch\|u-u_h\|_{1,h}\|u-u_h\| \\ &\le Ch^2\|u\|_{2,\alpha}\|u-u_h\|. \end{aligned} \tag{27}$$

Due to the linearity of w_I, $\|w_I - \Pi_h^* w_I\| \le Ch$. Then the last two terms in (26) can be estimated as follows. Applying Lemma 1.1 we have

$$\begin{aligned} &\sum_{i=0}^{N-1}\left[\int_{x_i}^{x_{i+1/2}} (\beta(u-u_h)')' (w_I - \Pi_h^* w_I)dx + \int_{x_{i+1/2}}^{x_{i+1}} (\beta(u-u_h)')' (w_I - \Pi_h^* w_I)dx\right] \\ &= \sum_{i=0}^{N-1}\left(\int_{x_i}^{x_{i+1/2}} f(w_I - \Pi_h^* w_I)dx + \int_{x_{i+1/2}}^{x_{i+1}} f(w_I - \Pi_h^* w_I)dx\right) \\ &\le \sum_{i=0}^{N-1}\left(\int_{x_i}^{x_{i+1/2}} \|f\|_{L_\infty[x_i,x_{i+1/2}]}(w_I - \Pi_h^* w_I)dx + \int_{x_{i+1/2}}^{x_{i+1}} \|f\|_{L_\infty[x_{i+1/2},x_{i+1}]}(w_I - \Pi_h^* w_I)dx\right) \\ &\le \sum_{i=0}^{N-1}\|f\|_\infty\left(\int_{x_i}^{x_{i+1/2}} (w_I - \Pi_h^* w_I)dx + \int_{x_{i+1/2}}^{x_{i+1}} (w_I - \Pi_h^* w_I)dx\right) \\ &\le Ch^2\|f\|_\infty\|w\|_{2,\alpha}. \end{aligned} \tag{28}$$

Now using (22) in (27) and applying (27) and (28) in (26) we prove the theorem. □

The following theorem indicates super convergence at the nodal points. To prove it we use the properties of Galerkin finite elements.

Theorem 1.5. *Assume that the diffusion coefficient β is piecewise constant as in (17). Then*

$$u(x_i) - u_h(x_i) = 0$$

for all nodes x_i, $i = 0, \ldots, N$.

Proof. Fix $y \in (0, 1)$ and due to Remark 1.2, let $G(x, y)$ be the Green's function satisfying

$$a_h(G(\cdot, y), v) =< \delta(x - y), v >, \quad v \in V_h.$$

By working out the closed form of G satisfying the classical formulation

$$-(\beta G')' = \delta(x - y), \quad [G]_\alpha = 0, \quad [\beta G']_\alpha = 0, \quad G(a, y) = G(b, y) = 0,$$

we see that the Green's function G, $y < \alpha$ takes the form [Chou, H.S, Attanayake, A, 2017)]

$$G(x, y) = \begin{cases} \dfrac{(1-y)x}{\beta^-}, & 0 < x \leq y, \\ \dfrac{y(1-x)}{\beta^-}, & y \leq x \leq \alpha, \\ \dfrac{y(1-x)}{\beta^+}, & \alpha \leq x \leq 1 \end{cases}$$

Now let $G = G(x, x_i)$ and use Galerkin orthogonality property, then

$$u(x_i) = u_h(x_i) = e(x_i) = (\delta(x - x_i), e) = a_h(G, e) = 0,$$

since $G \in V_h$ this proves the theorem. □

3. Parabolic Interface Problem

In this section we consider a second order semilinear parabolic interface problem of the form

$$u_t - (\beta u')' = f \ \text{ in } (0, 1) \times [0, T] \tag{29}$$

with initial and boundary conditions

$$u(\cdot, 0) = 0 \ \text{ in } (0, 1), \qquad u(0) = 0, \quad u(1) = 0 \tag{30}$$

with the jump conditions on the interface

$$[u]_\alpha = 0, \quad [\beta u']_\alpha = 0, \tag{31}$$

for $T > 0$. The semi-discrete immersed interface finite volume problem based on above weak formulation is, find $u_h : [0, T] \mapsto V_h$, such that

$$(u_{h,t}, \Pi_h^* v)_h + a_h(u_h, v) = (f, \Pi_h^* v)_h \quad \forall v \in V_h \tag{32}$$

We introduce the operator $R_h : H^2 \cap H_0^1 \mapsto V_h$ defined by

$$a_h(R_h u, v) = a_h(u, v) \quad \forall v \in V_h. \tag{33}$$

So by theorems (1.3) and (1.4), it follows that,

$$||R_h u - u|| + h|R_h u - u|_{1,h} \leq O(h^2). \tag{34}$$

We separate the error into two terms as

$$u_h(t) - u(t) = \theta(t) + \rho(t), \text{ where } \theta = u_h - R_h u, \rho = R_h u - u,$$

and from Theorem (1.4) we can see that

$$||\rho|| = ||R_h u - u|| \leq Ch^2(||u||_{2,\alpha} + ||f||_\infty) \tag{35}$$

$$||\rho_t|| = ||R_h u_t - u_t|| \leq Ch^2(||u_t||_{2,\alpha} + ||f_t||_\infty). \tag{36}$$

Lemma 1.6. *If the mesh size h is sufficiently small,*

$$\sum_{i=1}^{N-1}\int_{x_i}^{x_{i+1}} u\Pi_h^* u dx \geq C\|u\|^2.$$

Proof. Since u is piecewise linear for each noninterface elements, we can show that,

$$\begin{aligned}
\sum_{i=1;i\neq k}^{N-1}\int_{x_i}^{x_{i+1}} u\Pi_h^* u dx &= \sum_{i=0;i\neq k}^{N-1}\int_{x_i}^{x_{i+1/2}} u(x)u(x_i)dx + \sum_{i=0;i\neq k}^{N}\int_{x_{i+1/2}}^{x_{i+1}} u(x)u(x_{i+1})dx,\\
&= \frac{h}{4}\sum_{i=0;i\neq k}^{N-1}\Big(u(x_i)(u(x_i)+u(x_{i+1/2})) + u(x_{i+1})(u(x_{i+1/2})+u(x_{i+1}))\Big),\\
&= \frac{h}{4}\sum_{i=0;i\neq k}^{N-1}(u(x_i)^2+u(x_{i+1})^2) + \frac{h}{4}\sum_{i=0;i\neq k}^{N-1}u(x_{i+1/2})(u(x_i)+u(x_{i+1})),\\
&= \frac{h}{4}\sum_{i=0;i\neq k}^{N-1}(u(x_i)^2+2u(x_{i+1/2})^2+u(x_{i+1})^2) \geq 0.
\end{aligned}$$

And the Simpsons rule implies

$$\sum_{i=1}^{N-1}\int_{x_i}^{x_{i+1}} u\Pi_h^* u dx > \sum_{i=0}^{N-1}\frac{h}{6}(u(x_i)^2+4u(x_{i+1/2})^2+u(x_{i+1})^2) = C\|u\|^2.$$

However, on the interface element u is only piecewise linear. That is

$$\int_{x_k}^{x_{k+1}} u\Pi_h^* u dx > C\|u\|^2_{L_2[x_k,x_{k+1}]} + E_k$$

where E_k is a quadrature error depends on the u and $x_{k+1}-x_k$. Now for sufficiently small enough h we find that there is a constant C where,

$$\sum_{i=1}^{N-1}\int_{x_i}^{x_{i+1}} u\Pi_h^* u dx \geq C\|u\|^2. \tag{37}$$

□

The convergence of the immersed finite volume method in the $L2$ norm is derived in the following theorem.

Theorem 1.7. *Let u and u_h be solutions of (1) and (18) respectively. Then, there exists a positive constant C independent of h such that*

$$\|u_h - u\|_{1,h} \leq O(h)$$

Proof. Since we have the error bound for ρ, we only need to obtain the error bound for θ. Then, it follows from (32) and (33) that

$$\begin{aligned}
\sum_{i=1}^{N-1}\int_{x_i}^{x_{i+1}} \theta_t\Pi_h^* v dx + a_h(\theta, v) &= \Big((u_h-\tilde{u})_t, \Pi_h^* v\Big)_h + a_h\left((u_h-\tilde{u}), v\right)\\
&= (u_{h,t}, \Pi_h^* v)_h + a_h(u_h, v) - (\tilde{u}_t, \Pi_h^* v)_h - a_h(\tilde{u}, v)\\
&= (f, \Pi_h^* v)_h - (\tilde{u}_t, \Pi_h^* v)_h - a_h(\tilde{u}, v)\\
&= (f, \Pi_h^* v)_h - (\rho_t, \Pi_h^* v)_h - (u_t, \Pi_h^* v)_h - a_h(\tilde{u}, v)\\
&= -(\rho_t, \Pi_h^* v)_h \quad \forall v \in V_h, \;\; t \in J.
\end{aligned}$$

If $v = \theta_t$ we have that

$$(\theta_t, \Pi_h^*\theta_t)_h + a_h(\theta, \theta_t) = -(\rho_t, \Pi_h^*\theta_t)_h. \tag{38}$$

Thus using (37) on (38)

$$\begin{aligned} (\theta_t, \Pi_h^*\theta_t)_h + a_h(\theta, \theta_t) &\leq -(\rho_t, \Pi_h^*\theta_t)_h \\ C\|\theta_t\|^2 + \frac{1}{2}\frac{d}{dt}a_h(\theta,\theta) &\leq -(\rho_t, \Pi_h^*\theta_t)_h + \frac{1}{2}\left(a_h(\theta_t,\theta) - a_h(\theta,\theta_t)\right) \end{aligned} \tag{39}$$

From the same argument as in (13) and by inverse inequality

$$\begin{aligned} (a_h(\theta_t,\theta) - a_h(\theta,\theta_t) &= \sum_{i=0}^{N-1}\left[\int_{x_i}^{x_{i+1/2}} (\beta\theta_t')'(\theta - \Pi_h^*\theta) + \int_{x_{i+1/2}}^{x_{i+1}} (\beta\theta_t')'(\theta - \Pi_h^*\theta)\right] \\ &\quad - \sum_{i=0}^{N-1}\left[\int_{x_i}^{x_{i+1/2}} (\beta\theta')'(\theta_t - \Pi_h^*\theta_t) + \int_{x_{i+1/2}}^{x_{i+1}} (\beta\theta')'(\theta_t - \Pi_h^*\theta_t)\right] \\ &\leq Ch\|\theta_t\|_{1,h}\|\theta\|_{1,h} \\ &\leq C\|\theta_t\|\|\theta\|_{1,h}. \end{aligned} \tag{40}$$

Similarly,

$$\begin{aligned} (\rho_t, \Pi_h^*\theta_t)_h &= (\rho_t, \theta_t)_h + (\rho_t, \Pi_h^*\theta_t - \theta)_h \\ &\leq \|\rho_t\|\|\theta_t\| + Ch\|\rho_t\|\|\theta_t\|_{1,h} \\ &\leq \|\rho_t\|\|\theta_t\| + C\|\rho_t\|\|\theta_t\| \end{aligned} \tag{41}$$

Now applying (40) and (41) on (39), with the inequality $ab < \epsilon a^2 + b^2/4\epsilon$ for $a, b, \epsilon > 0$ and choosing small enough ϵ to absorb $\|\theta_t\|^2$ term on the right hand into left hand, we get,

$$\frac{1}{2}\frac{d}{dt}a_h(\theta,\theta) \leq C(\|\rho_t\|^2 + \|\theta\|_{1,h}^2) \tag{42}$$

After integrating both sides with respect to t and using (16),

$$\|\theta(t)\|_{1,h}^2 \leq \|\theta(0)\|^2 + C\int_0^t \left(\|\theta\|_{1,h}^2 + \|\rho_t\|^2\right) ds,$$

and using Gronwalls lemma we obtain

$$\|\theta(t)\|_{1,h}^2 \leq C\|\theta(0)\|_{1,h}^2 + C(T)\int_0^t \|\rho_t\|^2 ds.$$

Since $\theta(0) = 0$,

$$\|\theta(t)\|_{1,h}^2 \leq C(T)h^2\|u_t\|_{2,\alpha}$$

and hence we obtain the desired result

$$\begin{aligned} \|u_h - u\|_{1,h} &\leq \|\theta\|_{1,h} + \|\rho\|_{1,h}, \\ &\leq O(h). \end{aligned}$$

□

Theorem 1.8. *Let u and u_h be solutions of (1) and (18) respectively. And β is a picewise constant function as defined in (17). Then,*

$$\|u_h - u\| \leq O(h^2)$$

Proof. According to the remark 1.2, the immersed finite volume bilinear form reduces to the Galerkin immersed finite element bilinear form. In (Attanayake, C., Senaratne, D., 2011), authors have proved the optimal convergence of a parabolic interface problem using immersed finite element method that has the same bilinear form. Thus the proof of this theorem follows from theorem 3.1 in (Attanayake, C., Senaratne, D., 2011). □

4. Simulation

In this section we present two numerical examples to confirm our theory.

Problem 1 Consider the elliptic interface problem

$$\begin{cases} (\beta u')' = 2x & x \in [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

with

$$\beta(x) = \begin{cases} x^2 + 1 & x \in [0,\alpha), \\ x^2 & x \in [\alpha, 1), \end{cases} \tag{43}$$

The exact solution is

$$u(x) = \begin{cases} -x + (1+d)\tan^{-1} x & x \le \alpha \in [0,\alpha), \\ -x + \frac{d}{x} + (1-d) & x \in [\alpha, 1), \end{cases}$$

where

$$d = \frac{\alpha \tan^{-1}\alpha - \alpha}{1 - \alpha + \alpha \tan^{-1}\alpha}$$

Let $\alpha = \pi/3$. In Table 1 we illustrate L^2 and energy norm errors are $O(h^2)$ and $O(h^1)$ respectively.

Table 1. L_2 and energy norm errors for problem 1

h	$\|\|u - u_h\|\|$	$\|\|u - u_h\|\|_{1,h}$
1/16 2	4.949e-2	5.519e-
1/32	1.252e-2	2.696e-2
1/64	3.222e-3	1.337e-2
1/128	8.178e-4	6.6758e-3
order	≈ 2	≈ 1

Problem 2 Consider following parabolic interface problem.

$$\begin{cases} u_t - (\beta u')' = f(x,t) & x \in [0,1] \times [0,2] \\ u(x,0) = 0 & x \in [0,1] \\ u(0,t) = u(1,t) = 0 & t > 0 \end{cases} \tag{44}$$

Here,

$$f(x,t) = \begin{cases} -\frac{x^2}{\beta^-} + \frac{t_2 x}{\beta^-} + x^2 + 2t & x \le \alpha \\ -\frac{x^2}{\beta^+} + \frac{t_2 x}{\beta^+} - \frac{t_2}{\beta^+} + \frac{1}{\beta^+} + x^2 + 2t & \alpha \le x. \end{cases}$$

where,

$$\begin{aligned} t_1 &= \left(\frac{1-\alpha^4}{12\beta^+} - Q\frac{1-\alpha}{\beta^+} + \frac{\alpha^4}{12\beta^-}\right)\frac{1}{((1-\alpha)/\beta^+ + \alpha/\beta^-)}; \\ t_2 &= \left(\frac{-\alpha^2}{\beta^-} + \frac{\alpha^2}{\beta^+} - \frac{1}{\beta^+}\right)\frac{1}{((\alpha-1)/\beta^+ - \alpha/\beta^-)}. \end{aligned}$$

Exact solution is given by

$$u(x,t) = \begin{cases} -\frac{x^4}{12\beta^-} + \frac{t_1 x}{\beta-} + t(-\frac{x^2}{\beta^-} + \frac{t_2 x}{\beta^-}) & x \le \alpha \\ \frac{-x^4}{12\beta^+} + \frac{t_1 x}{\beta^+} - \frac{t_1}{\beta^+} + \frac{1}{12\beta^+} + t(-\frac{x^2}{\beta^+} + \frac{t_2 x}{\beta^+} - \frac{t_2}{\beta^+} + \frac{1}{\beta^+}) & \alpha \le x. \end{cases}$$

Table 2. L_2 and energy norm errors for problem 2

h	$\|\|u-u_h\|\|$	$\|\|u-u_h\|\|_{1,h}$	h	$\|\|u-u_h\|\|$	$\|\|u-u_h\|\|_{1,h}$
1/16	2.347e-2	8.411e-2	1/16	1.053e-3	3.341e-2
1/32	5.822e-3	4.215e-2	1/32	2.817e-4	1.636e-2
1/64	1.444e-3	2.119e-2	1/64	6.928e-5	7.612e-3
1/128	3.609e-4	1.075e-2	1/128	1.7219e-5	3.773e-3
order	≈ 2	≈ 1	order	≈ 2	≈ 1
$\beta^- = 1, \beta^+ = 100$			$\beta^- = 100, \beta^+ = 1$		

The table 2 contain L^2 energy norm errors of the solutions of the parabolic problem for $\beta^- = 1$ $\beta^+ = 100$ and $\beta^- = 100$ $\beta^+ = 1$. Error values in Table 2 clearly indicate that solution u_h converges to u with optimal rate $O(h^2)$ and $O(h)$ in L^2 and energy norms for parabolic problem. We use Crank-Nicolson method to solve time discretized problem.

5. Conclusions

We considered an immersed interface finite volume method for second order elliptic and parabolic interface problems. By assuming the diffusion coefficient β has a finite jump across the interface and is piecewise constant,we obtained the optimal convergence in L^2 and energy norms. Further, we prove super convergence at the element nodes. We obtained optimal convergence in the L^2 norm for the parabolic interface problem with piecewise constant diffusion coefficient β. When a variable diffusion coefficient is present in the parabolic interface problem, we obtained the the optimal order in energy norm.

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